Market credit risk in the Eurozone area

Preliminary

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Abstract

In this paper, we model the sovereign debt default intensities for the Eurozone countries as self exciting point processes depending on two components: a systemic component and a country specific component. The first component is built as a simple weighted average of individual countries' default intensities. The weights are unknown parameters and confer information on the contribution of each country's default risk to the overall systemic risk. Moreover, we measure the impact of the systemic risk on individiual risk by introducing a special parameter reflecting the speed at which each country's default intensity adjusts to the changes in the default intensity of the Eurozone market as a whole. We show the above model is affine in the state variables and use the results in Duffie et al (2003) to obtain closed form CDS prices. We estimate the model based on CDS spread data for two maturities by minimizing the squared differences between inverted default intensities for the two maturities.

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1 Introduction

The recent economic and political turmoil related to Greece's debt crisis has raised the complex question of how vulnerable the European Union and especially the Eurozone are to shocks from individual members. While similar questions were previously addressed by the economic literature in relation to past various crises (Bekaert et al., 2014, see, for instance the literature on contagion), the strong interlinks between Eurozone countries dictated by the use of the same currency bring this question to the spotlight again with new and complex nuances.

In this paper, we explore the question of the vulnerability of the Eurozone to debt related risk by relying on the information content within sovereign credit default swap (CDS hereafter) spreads. The credit default swaps, which can be thought of as instruments providing insurance against credit default, are generally known to be sensitive to credit risk factors and to faithfully reflect the conditions of the economic environment. To the best of our knowledge, this is the first paper using a reduced form default model that concomitantly explores and measures the impact of systemic shocks on individual countries as well as the importance of country dynamics in generating systemic risk.

We model the intensity of defaults[1] for a portfolio of Eurozone sovereign debt as multivariate self-exciting (Hawkes) model with restricted parameters. These restrictions are established such that the default intensity of each country is modeled as the sum of the default intensity that would occur in the case of pure independence between countries and a systemic default intensity. This latter constituent captures the dynamics of the Eurozone CDS market. The systemic (market) component is modelled as a weighted sum of the default intensities of individual countries, where the weights come estimated and confer information about the importance of each country's risk into the systemic risk. In a similar fashion to classical finance theory, the intensity for each country has a different "sensitivity" to the systemic component, captured by a dedicated parameter.

We show that the above model is a particular case of an affine jump diffusion process with multiple jump components and use the results in Duffie et al. (2000) and At-Sahalia et al. (2014) to compute theoretical CDS spreads. We collect weekly spreads for the 5- year and 10-year CDS contracts for 12 countries in the Eurozone for the period April 2008 to August 2015. The model is subsequently estimated by exploiting the differences between observed and theoretical spreads. More precisely, for each time observation, maturity and country, we invert the spread function and obtain the corresponding

[1] The term default in this context refers to credit events in accordance to the definitions in the 2003 ISDA Credit Derivatives Definitions. This category includes for sovereign debt failure to make payments, restructuring, repudiation and moratorium.

default intensity. Then, to estimate the parameters, we minimize the sum of squares of the differences between the intensities for the two maturities. This procedure transfers the error from the spreads to the intensities and leads to a significant reduction in the parameter space, as intensities are no longer estimated.

Duffie and Singleton (2003) summarise the use of the doubly stochastic Poisson processes to model default intensities as a base for pricing debt and debt related instruments. This gave birth to several contributions to the literature where default intensities are modelled using these processes and the parameters of the processes are recovered by setting the observed CDS spreads equal to theoretical spreads (Pan and Singleton, 2008; Longstaff et al., 2005; Ang and Longstaff, 2013). From the existing literature using reduced form default models to explain the dynamics of CDS spreads, two more recent contributions are more relevant to the present paper. First, Ang and Longstaff (2013) consider an international set-up and incorporate in their model two default intensities, an idiosyncratic one and a systemic one. In their set-up, the above intensities are modeled as squared root processes driven by Brownian motions separately, without spillovers between the systemic and idiosyncratic component. Moreover, we relyon Hawkes point process rather than the Cox point processes in Ang and Longstaff (2013). The Hawkes class of processes is designed to create clusters of points, thus being able to incorporate feedback from previous events, which is a desirable feature from a practical point of view. Second, At-Sahalia et al. (2014) are the first to point out that previous models lack this event feedback characteristic. They estimate selfexciting bivariate models for some couples of Eurozone countries. While their new framework is valuable as it empowers the exploration of the transmission of shocks between countries, it does not explicitly take into account a systemic component and is subjected to the curse of dimensionality (the estimation can be performed only for couples of countries). Our work builds on the previous contributions stated above, but also concomitantly features three essential characteristics: 1- both idiosyncratic and systemic components are modelled; 2- default intensities are self- exciting; 3- a significant number of countries are included in the model and estimation takes place at the same time.

The rest of the paper is structured as follows. Section 2 presents the factor self exciting model we propose and the pricing of the CDS spreads. Section 3 presents the data we use in estimating the model. Section4 discusses the estimation method and the results obtained. Finally, section 5 concludes the paper.

2 Self exciting factor model for the default intensities

2.1 Model

Let N be the number of countries included in a portfolio of sovereign debt. Let $(\Omega, \mathscr{F}, \mathbf{P})$ be a probability space on which the default point processes are defined and $N_{i,t}$ the default counting process for country i, $i = 1, \cdots, N$, at time t. Let $\mathscr{F}_t \subseteq \mathscr{F}$ be the natural filtration for these processes. Then, the conditional intensity of occurrence (or simply, intensity) at time t for the default process of any country i, $i = 1, \cdots, N$, can be defined as:

$$\lambda_t^i = \lambda^i(t \mid \mathscr{F}_t) = \lim_{h \downarrow 0} \left[\frac{N_{i,t+h} - N_{i,t}}{h} \mid \mathscr{F}_t \right]$$

In the purely theoretical case where credit default events are non-transmittable between countries, we assume that the default for each country is a uni-variate self exciting (Hawkes, 1971) point process, where the intensity λ_t^i for country i is given by the following stochastic differential equation (SDE hereafter):

$$d\lambda_t^i = \alpha^i \left(\lambda_\infty^i - \lambda_t^i \right) dt + \beta^i dN_{i,t}, \tag{1}$$

with α^i and β^i country specific parameters. We denote the above SDE with $d\lambda_{t,\text{ind}}^i$ to underline that this equation is only valid in the case of no interdependence between countries.

In a world where credit default events cannot be isolated, to the process in equation 1 we add a second component that captures the dynamic of the market[2]. We re-write equation 1 as:

$$\begin{aligned} d\lambda_t^i &= \mathrm{d}_{t,\text{ind}}^i + \delta^i d\lambda_t^m \\ &= \alpha^i \left(\lambda_\infty^i - \lambda_t^i \right) dt + \beta^i dN_{i,t} + \delta^i d\lambda_t^m, \end{aligned} \tag{2}$$

where λ_t^m is the systemic/ market default intensity, while δ^i shows the sensitivity of each security to the market factor and can be thought of as a factor loading.

Similarly to classical finance theory, the market in this case will be represented by a weighted

[2]Here, the market refers to the market of sovereign debt and not what is known as market portfolio in finance theory

portfolio of its components. Thus, we obtain:

$$\mathrm{d}_t^m = \sum_{j=1}^{N} w_j d\lambda_t^j \tag{3}$$

Replacing above $d\lambda_t^j$ with its counterpart from equation (2), we have:

$$d\lambda_t^m = \sum_{j=1}^{N} w_j \left[\alpha^j \left(\lambda_\infty^j - \lambda_t^j\right) dt + \beta^j dN_{j,t}\right] + \sum_{j=1}^{N} w_j \delta^j d\lambda_t^m \Leftrightarrow$$

$$d\lambda_t^m \left(1 - \sum_{j=1}^{N} w_j \delta^j\right) = \sum_{j=1}^{N} w_j \left[\alpha^j \left(\lambda_\infty^j - \lambda_t^j\right) dt + \beta^j dN_{j,t}\right] \Leftrightarrow$$

$$d\lambda_t^m = \left(1 - \sum_{j=1}^{N} w_j \delta^j\right)^{-1} \sum_{j=1}^{N} w_j \left[\alpha^j \left(\lambda_\infty^j - \lambda_t^j\right) dt + \beta^j dN_{j,t}\right].$$

Given the above, (2) can be re-written as:

$$d\lambda_t^i = \alpha^i \left(\lambda_\infty^i - \lambda_t^i\right) dt + \beta^i dN_{i,t} + \delta^i \left(1 - \sum_{j=1}^{N} w_j \delta^j\right)^{-1} \sum_{j=1}^{N} w_j \left[\alpha^j \left(\lambda_\infty^j - \lambda_t^j\right) dt + \beta^j dN_{j,t}\right], \quad i = 1, \ldots, N \tag{4}$$

The model can be re-written in a more contracted way, using matrix notation:

$$d\lambda_t = \breve{\alpha}(\lambda_\infty - \lambda_t)dt + \breve{\beta} d\mathbf{N}_t, \tag{5}$$

where λ_t, λ_∞ and $\mathbf{N}_t$ are N-dimensional vectors where countries were stacked together, while

$$\breve{\alpha} = \begin{pmatrix} \alpha^1 \left[1 + \delta^1 \left(1 - \sum_{j=1}^{N} w_j \delta^j\right)^{-1} w_1\right] & \alpha^2 \delta^1 \left(1 - \sum_{j=1}^{N} w_j \delta^j\right)^{-1} w_2 & \cdots & \alpha^N \delta^1 \left(1 - \sum_{j=1}^{N} w_j \delta^j\right)^{-1} w_N \\ \vdots & \vdots & \ddots & \vdots \\ \alpha^1 \delta^N \left(1 - \sum_{j=1}^{N} w_j \delta^j\right)^{-1} w_1 & \alpha^2 \delta^N \left(1 - \sum_{j=1}^{N} w_j \delta^j\right)^{-1} w_2 & \cdots & \alpha^N \left[1 + \delta^N \left(1 - \sum_{j=1}^{N} w_j \delta^j\right)^{-1} w_N\right] \end{pmatrix} \tag{6}$$

and

$$\check{\beta}=\begin{pmatrix} \left[1+\delta^1\left(1-\sum_{j=1}^N w_j\delta^j\right)^{-1}w_1\right]\beta^1 & \delta^1\left(1-\sum_{j=1}^N w_j\delta^j\right)^{-1}w_2\beta^2 & \cdots & \delta^1\left(1-\sum_{j=1}^N w_j\delta^j\right)^{-1}w_N\beta^N \\ \delta^2\left(1-\sum_{j=1}^N w_j\delta^j\right)^{-1}w_1\ \beta^1 & \left[1+\delta^2\left(1-\sum_{j=1}^N w_j\delta^j\right)^{-1}w_2\right]\beta^2 & \cdots & \delta^2\left(1-\sum_{j=1}^N w_j\delta^j\right)^{-1}w_N\beta^N \\ \vdots & \vdots & \vdots & \vdots \\ \delta^N\left(1-\sum_{j=1}^N w_j\delta^j\right)^{-1}w_1\ \beta^1 & \delta^N\left(1-\sum_{j=1}^N w_j\delta^j\right)^{-1}w_2\beta^2 & \cdots & \left[1+\delta^N\left(1-\sum_{j=1}^N w_j\delta^j\right)^{-1}w_N\right]\beta^N \end{pmatrix} \quad (7)$$

To further simplify expressions, we introduce the following notations: $\mathbf{b}=\left(\beta^1,\ \cdots,\beta^N\right)'$, $\mathbf{a}=\left(\alpha^1,\ \cdots,\alpha^N\right)'$ and

$$\mathbf{W}=\begin{pmatrix} 1+\delta^1\left(1-\sum_{j=1}^N w_j\delta^j\right)^{-1}w_1 & \delta^2\left(1-\sum_{j=1}^N w_j\delta^j\right)^{-1}w_1 & \cdots & \delta^N\left(1-\sum_{j=1}^N w_j\delta^j\right)^{-1}w_1 \\ \delta^1\left(1-\sum_{j=1}^N w_j\delta^j\right)^{-1}w_2 & 1+\delta^2\left(1-\sum_{j=1}^N w_j\delta^j\right)^{-1}w_2 & \cdots & \delta^N\left(1-\sum_{j=1}^N w_j\delta^j\right)^{-1}w_2 \\ \vdots & \vdots & \ddots & \vdots \\ \delta^1\left(1-\sum_{j=1}^N w_j\delta^j\right)^{-1}w_N & \delta^2\left(1-\sum_{j=1}^N w_j\delta^j\right)^{-1}w_N & \cdots & 1+\delta^N\left(1-\sum_{j=1}^N w_j\delta^j\right)^{-1}w_N \end{pmatrix}.$$

As a result, we have $\check{\alpha}=\left(\text{diag}\left(\left(\alpha^i\right)_{i=1\ldots N}\right)*W\right)'$ and $\check{\beta}=\left(\text{diag}\left(\left(\beta^i\right)_{i=1\ldots N}\right)*W\right)'$. For the process to be well defined, we require that all elements in $\check{\alpha}$ and $\check{\beta}$ are strictly positive.

Stationarity Let us define $\Gamma=\left(\frac{\check{\beta}_{\text{ij}}}{\check{\alpha}_{\text{ij}}}\right)_{i,j=1\ldots N}$. Let $\rho(\Gamma)$ denote the spectral radius of Γ. Then, for the default processes to be stationary, a necessary and sufficient condition is $\rho(\Gamma)<1$.

Identification The identification of parameters in equation (5) amounts to the identification of the parameters in equation (6) and then identifying $\left(\beta^i\right)_{i=1\ldots N}$ as $\text{diag}\left(\left(\beta^i\right)_{i=1\ldots N}\right)=\Gamma\left(\text{diag}\left(\left(\alpha^i\right)_{i=1\ldots N}\right)\mathbf{1}_{(N,N)}\right)^{-1}$, where $\mathbf{1}_{(N,N)}$ is a (NXN) matrix of ones[3]. In equation (6), we have combinations of parameters that vary across matrix $\check{\alpha}$: α^i and w_i vary across columns, while δ^i vary across lines, $i=1\ldots N$. We conclude based on this that all N^2 terms of the matrix serve for the identification of the 3 sets of parameters: α^i, w_i and δ^i, $i=1\ldots N$. Thus the identification restriction is:

$$N^2\geq 3N.$$

This restriction is satisfied whenever the number of countries in the sample, N, is greater or equal to 3 ($N\geq 3$).

[3]This is equal to solving for identification in equation (7) and then recovering $\left(\alpha^i\right)_{i=1\ldots N}$.

2.2 CDS pricing and transform analysis

At-Sahalia et al. (2014) rely on a multivariate self-exciting model and derive the following theoretical CDS spreads:

$$s_{i,t}^{\tau} = \frac{r_i \int_t^{\tau} D(t,s)\,\mathrm{E}[\gamma_i \lambda_s^i (1-\gamma_i)^{N_{i,s}-N_{i,t}} \mid \mathscr{F}_t]}{\int_t^{\tau} D(t,s)\,\mathrm{E}[(1-\gamma_i)^{N_{i,s}-N_{i,t}} \mid \mathscr{F}_t]}, \tag{8}$$

where $s_{i,t}^{\tau}$ denotes the CDS spread at time t, for country i, $i = 1, \cdots, N$, and for maturity τ. In our application, τ is either 5 or 10 years. $1 - r_i$ is the recovery rate, $D(t,s)$ the discount factor at time t for a zero coupon bond with maturity s and $0 < \gamma_i \leq 1$ the probability of going into default upon the occurrence of a credit event.

It results that computing the theoretical spreads involves computing the two expectations present in equation (8). Duffie et al. (2000) derive closed form expressions for expectations of this type for the class of affine jump diffusions. We consider the vector of stacked counting processes and intensities denoted with X_t, which is given by the following SDE:

$$dX_t = \begin{pmatrix} dN_{1,t} \\ \vdots \\ dN_{N,t} \\ d\lambda_t^1 \\ \vdots \\ d\lambda_t^N \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \alpha^1\left(\lambda_\infty^1 - \lambda_t^1\right) + \delta^1\left(1 - \sum_{j=1}^N w_j\delta^j\right)^{-1}\sum_{j=1}^N w_j\alpha^j\left(\lambda_\infty^j - \lambda_t^j\right) \\ \vdots \\ \alpha^N\left(\lambda_\infty^N - \lambda_t^N\right) + \delta^N\left(1 - \sum_{j=1}^N w_j\delta^j\right)^{-1}\sum_{j=1}^N w_j\alpha^j\left(\lambda_\infty^j - \lambda_t^j\right) \end{pmatrix} dt+$$

$$\begin{pmatrix} 1 \\ \vdots \\ 0 \\ \left[1 + \delta^1\left(1 - \sum_{j=1}^N w_j\delta^j\right)^{-1} w_1\right]\beta^1 \\ \vdots \\ \delta^N\left(1 - \sum_{j=1}^N w_j\delta^j\right)^{-1} w_1\ \beta^1 \end{pmatrix} dN_{1,t} + \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ \delta^1\left(1 - \sum_{j=1}^N w_j\delta^j\right)^{-1} w_2\beta^2 \\ \left[1 + \delta^2\left(1 - \sum_{j=1}^N w_j\delta^j\right)^{-1} w_2\right]\beta^2 \\ \vdots \\ \delta^N\left(1 - \sum_{j=1}^N w_j\delta^j\right)^{-1} w_2\beta^2 \end{pmatrix} dN_{2,t} + \ldots +$$

$$\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \delta^1\left(1 - \sum_{j=1}^N w_j\delta^j\right)^{-1} w_N\beta^N \\ \delta^2\left(1 - \sum_{j=1}^N w_j\delta^j\right)^{-1} w_N\beta^N \\ \vdots \\ \left[1 + \delta^N\left(1 - \sum_{j=1}^N w_j\delta^j\right)^{-1} w_N\right]\beta^N \end{pmatrix} dN_{N,t}$$

This is a Markov process and, at the same time, is affine in the state variables. Preserving the notations in Duffie et al. (2000), we have the drift of X_t defined as $\mu(X_t) = K_0 + K_1 x$ where the two matrices can be defined as follows:

$$K_0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \lambda_\infty^1 \alpha^1 + \delta^1 \left(1 - \sum_{j=1}^N w_j \delta^j\right)^{-1} \sum_{j=1}^N w_j \alpha^j \lambda_\infty^j \\ \vdots \\ \lambda_\infty^N \alpha^N + \delta^N \left(1 - \sum_{j=1}^N w_j \delta^j\right)^{-1} \sum_{j=1}^N w_j \alpha^j \lambda_\infty^j \end{pmatrix}; \; K_1 = \begin{pmatrix} \mathbf{0}_{N\times N} & \mathbf{0}_{N\times N} \\ \mathbf{0}_{N\times N} & K_1^B \end{pmatrix} \text{ with}$$

$$K_1^B = \begin{pmatrix} -\alpha^1 \left[1 + \delta^1 \left(1 - \sum_{j=1}^N w_j \delta^j\right)^{-1} w_1\right] & -\alpha^2 \delta^1 \left(1 - \sum_{j=1}^N w_j \delta^j\right)^{-1} w_2 & \cdots & -\alpha^N \delta^1 \left(1 - \sum_{j=1}^N w_j \delta^j\right)^{-1} w_N \\ \vdots & \vdots & \ddots & \vdots \\ -\alpha^1 \delta^N \left(1 - \sum_{j=1}^N w_j \delta^j\right)^{-1} w_1 & -\alpha^2 \delta^N \left(1 - \sum_{j=1}^N w_j \delta^j\right)^{-1} w_2 & \cdots & -\alpha^N \left[1 + \delta^N \left(1 - \sum_{j=1}^N w_j \delta^j\right)^{-1} w_N\right] \end{pmatrix}$$

.

As we have no diffusion component in our process, $H_0 \equiv 0$ and $H_1 \equiv 0$. The vector of intensities satisfies the equation $\lambda(x) = l_0 + l_1 x$ with $l_0 = \mathbf{0}_{N\times 1}$ and $l_1^1 = (0, \cdots, 0, 1, \cdots, 0)'$; $l_1^N = (0, \cdots, 0, 0, \cdots, 1)'$. Further, we assume independence of the discount rate from the state variables, so that $\rho_0 = 0$ and $\rho_1 = \mathbf{0}$ (i.e. the discount rate is not affine in the state variables). Let $c = (c_1, \ldots, c_{2N})' \in \mathbb{C}^{2N}$. Then, the jump transform, $\theta(c)$, will be a vector of N components:

$$\theta^1(c) = \exp\left(c_1 + \left[1 + \delta^1 \left(1 - \sum_{j=1}^N w_j \delta^j\right)^{-1} w_1\right] \beta^1 c_{N+1} + \delta^2 \left(1 - \sum_{j=1}^N w_j \delta^j\right)^{-1} w_1 \beta^1 c_{N+2} + \ldots + \delta^N \left(1 - \sum_{j=1}^N w_j \delta^j\right)^{-1} w_1 \beta^1 c_{2N}\right)$$

$$\theta^2(c) = \exp\left(c_2 + \delta^1 \left(1 - \sum_{j=1}^N w_j \delta^j\right)^{-1} w_2 \beta^2 c_{N+1} + \left[1 + \delta^2 \left(1 - \sum_{j=1}^N w_j \delta^j\right)^{-1} w_2\right] \beta^2 c_{N+2} + \ldots + \delta^N \left(1 - \sum_{j=1}^N w_j \delta^j\right)^{-1} w_2 \beta^2 c_{2N}\right)$$

$$\ldots$$

$$\theta^N(c) = \exp\left(c_N + \delta^1 \left(1 - \sum_{j=1}^N w_j \delta^j\right)^{-1} w_N \beta^N c_{N+1} + \delta^2 \left(1 - \sum_{j=1}^N w_j \delta^j\right)^{-1} w_N \beta^N c_{N+2} + \ldots + \left[1 + \delta^N \left(1 - \sum_{j=1}^N w_j \delta^j\right)^{-1} w_N\right] \beta^N c_{2N}\right)$$

Proposition 1. *Let* $u = (0, \cdots, \log(1-\gamma_k), \cdots, 0, \mathbf{0}_{1\times N})^{'}$, $k \in \{1, \cdots, N\}$, *and consider all of the above definitions. Then,*

$$\mathrm{E}\left[(1-\gamma_k)^{N_{k,T}}\middle|\mathscr{F}_t\right] = \exp\left(\alpha(t) + \beta_k(t) N_{k,t} + \beta_{2N+1}(t)\lambda_t^1 + \ldots + \beta_{2N}(t)\lambda_t^N\right),$$

where the coefficients are solutions to the following ordinary differential equations (ODEs hereafter):

a.

$$\dot{\beta}(t) = -K1^{'}\beta(t) - \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} \theta(\beta(t)),$$

where $\beta(t) = (\beta_1(t), \cdots, \beta_{2N}(t))^{'}$ and $\beta(T) = u = (0, \cdots, \log(1-\gamma_k), \cdots, 0, 0, \cdots, 0)^{'}$, $k \in \{1, \cdots, N\}$.

To further simplify this system of ODEs, we introduce the following notations: $\beta_{\mathbf{N+1,2N}}(t) = (\beta_{N+1}(t), \cdots, \beta_{2N}(t))^{'}$, $\beta_{\mathbf{1,N}}(t) = (\beta_1(t), \cdots, \beta_N(t))^{'}$.

The above equation becomes, using the notations $\mathbf{a}$, $\mathbf{b}$ and W defined in the previous sub-section:

a.1.

$$\dot{\beta}_{\mathbf{N+1,2N}}(t) = \mathbf{a} \circ \mathbf{W}\beta_{\mathbf{N+1,2N}}(t) + 1 - \exp\left(\beta_{\mathbf{1,N}}(t) + \mathbf{b} \circ \mathbf{W}\beta_{\mathbf{N+1,2N}}(t)\right)$$

with $\beta_k(s) = \log(1-\gamma_k)$, and $\beta_{i\neq k}(s) = 0$ for all $i = 1, \ldots, N$, $k \in \{1, \cdots, N\}$ and $s \in [t, T]$.

b.

$$\dot{\alpha}(t) = -\begin{pmatrix} 0 \\ \vdots \\ 0 \\ \lambda_\infty^1\alpha^1 + \delta^1\left(1 - \sum_{j=1}^N w_j\delta^j\right)^{-1}\sum_{j=1}^N w_j\alpha^j\lambda_\infty^j \\ \vdots \\ \lambda_\infty^N\alpha^N + \delta^N\left(1 - \sum_{j=1}^N w_j\delta^j\right)^{-1}\sum_{j=1}^N w_j\alpha^j\lambda_\infty^j \end{pmatrix}^{'} \beta(t)$$

with $\alpha(T) = 0$. Using the previous notations, we have:

b.1.

$$\dot{\alpha}(t) = -(\lambda_\infty \circ \mathbf{a})' \mathbf{W} \beta_{\mathbf{N+1,2N}}(t)$$

with $\lambda_\infty = (\lambda_\infty^1, \ \ldots, \ \lambda_\infty^N)'$.

Before getting to the next result, let us define $\nabla\theta(c)$, the gradient of $\theta(c)$:

$$\nabla\theta(c) = \begin{pmatrix} \mathrm{diag}(\theta(c)) & \mathrm{diag}(\mathbf{b} \circ \theta(c))\,\mathbf{W} \end{pmatrix}$$

The extended version of this matrix is included in Appendix A.1.

Proposition 2. *Let* $u = (0, \ \cdots, \ \log(1-\gamma_k), \cdots, \ 0, \mathbf{0}_{1\times N})'$, *and* $v = (\mathbf{0}_{1\times N}, 0, \ldots, \gamma_k, \ldots, 0)'$, $k \in \{1, \ \cdots, N\}$, *and consider all of the above definitions. Then,*

$$\begin{aligned} \mathrm{E}\left[\gamma_k \lambda_{k,T} (1-\gamma_k)^{N_{k,T}} \middle| \mathscr{F}_t\right] = \exp\left(\alpha(t) + \beta_k(t) N_{k,t} + \beta_{N+1}(t)\lambda_t^1 + \ldots + \beta_{2N}(t)\lambda_t^N\right) \\ \left(A(t) + B_{N+1}(t)\lambda_t^1 + \ldots + B_{2N}(t)\lambda_t^N\right) \end{aligned}$$

where $\mathbf{B}(t) = (B_1(t) \cdots B_{2N}(t))'$ *and* $A(t)$ *are given by the following ODEs:*

a.

$$-\dot{\mathbf{B}}(t) = K1'\mathbf{B}(t) + l_1 \nabla\theta(\beta)\,\mathbf{B}(t)$$

with $\mathbf{B}(T) = v$ Defining $\mathbf{B}_{\mathbf{N+1,2N}}(t) = (B_{N+1}(t) \cdots B_{2N}(t))'$ and given that the upper N rows of K_1 an l_1 are zeros, we can re-write the above equation as:

a.1.

$$-\dot{\mathbf{B}}_{\mathbf{N+1,2N}}(t) = -\mathbf{a} \circ \mathbf{W}\mathbf{B}_{\mathbf{N+1,2N}}(t) + \nabla\theta(\beta)\,\mathbf{B}(t)$$

with $B_i(s) = 0$ for all $i = 1, \ldots, N$, and $s \in [t, T]$.

b.

$$-\dot{\mathbf{A}}(t) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \lambda_\infty^1 \alpha^1 + \delta^1 \left(1 - \sum_{j=1}^N w_j \delta^j\right)^{-1} \sum_{j=1}^N w_j \alpha^j \lambda_\infty^j \\ \vdots \\ \lambda_\infty^N {\alpha}^N + \delta^N \left(1 - \sum_{j=1}^N w_j \delta^j\right)^{-1} \sum_{j=1}^N w_j \alpha^j \lambda_\infty^j \end{pmatrix}^{'} \mathbf{B}(t)$$

with $A(T) = 0$.

Further simplifications lead to:

b.1.

$$\dot{\mathbf{A}}(t) = -(\lambda_\infty \circ \mathbf{a})^{'} \mathbf{W}\mathbf{B}_{\mathbf{N+1,2N}}(t)$$

To prove both propositions, we just applied Propositions 1 and 3 in Duffie et al. (2000).

3 Dataset

We collected weekly CDS spreads for the 5- and 10- year CDS contracts for 12 Eurozone members, namely: Germany, France, Greece, Ireland, the Netherlands, Belgium, Spain, Cyprus, Finland, Italy, Malta, Poland and Slovenia. The data ranges from the 10th of April 2008 to 20th of August 2015, adding up to 385 observations in time. Both the period and the cross-section of countries were chosen as to maximize the available data. Figure 3 shows the dynamics of teh CDS spreads for teh 12 countries considered in our sample in the considered period. Clearly, in teh given period teh Greek CDS spreads are much wider than for the rest of the Eurozone countries. Then, Greece's default occurs in March 2012. (more comments on the figure) Figure 3 shows the CDS spreads duynamics for the same countries but a shorter time window colse to the Greek default, i.e. 1st of June 2011 to 30th of March 2012.

The data is converted from the original semi-annual to continuously compounded[4].

Furthermore, our analysis requires computing the default free discount factors based on zero coupon bonds for up to a 10-year maturity. Following Longstaff et al. (2005) and At-Sahalia et al. (2014), we use a cubic spline interpolation algorithm and rely on a combination of LIBOR rates and Euro swap rates. More precisely, we collect LIBOR rates for Euro with maturities of 1-, 2-, 3-, 6- and 12- months

[4]We applied the formula: $s^{continuous} = 2log(1 + s^{semiannual}/2)$

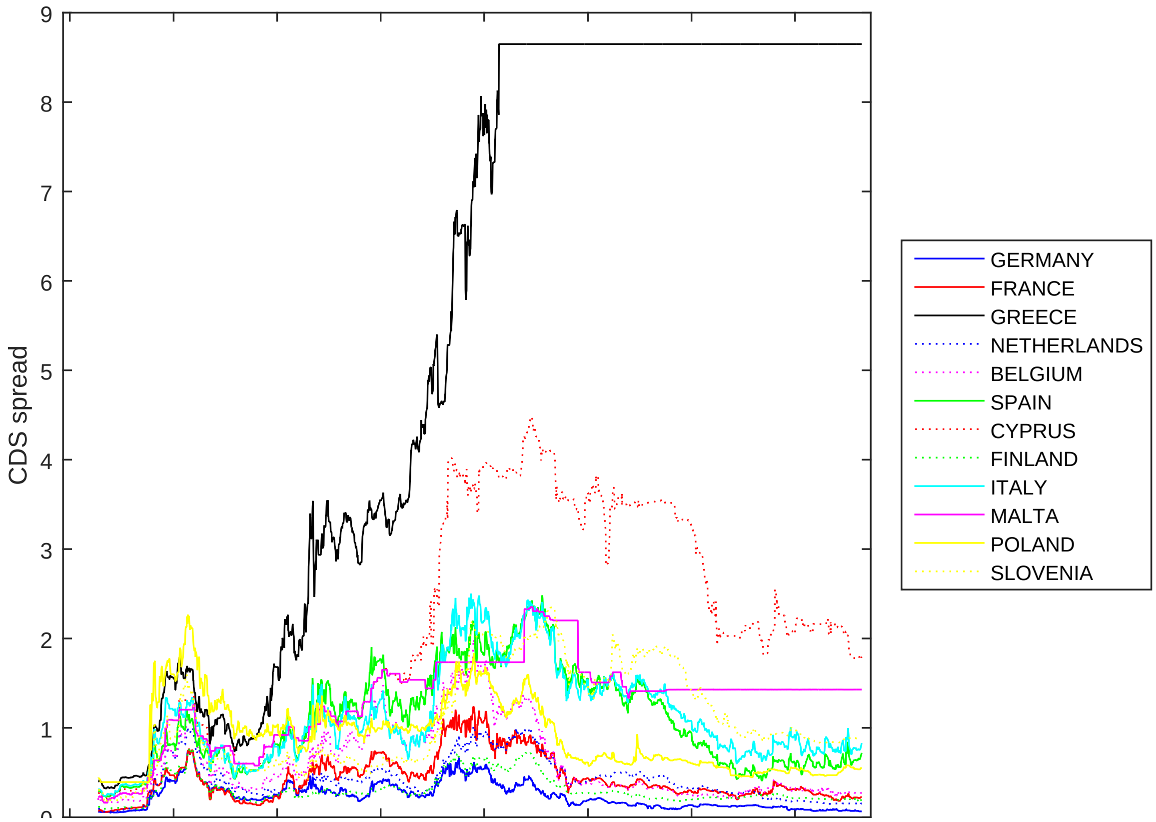

Figure 1: Log CDS spreads from the 10th of April 2008 to 20th of August 2015 for 12 Eurozone members

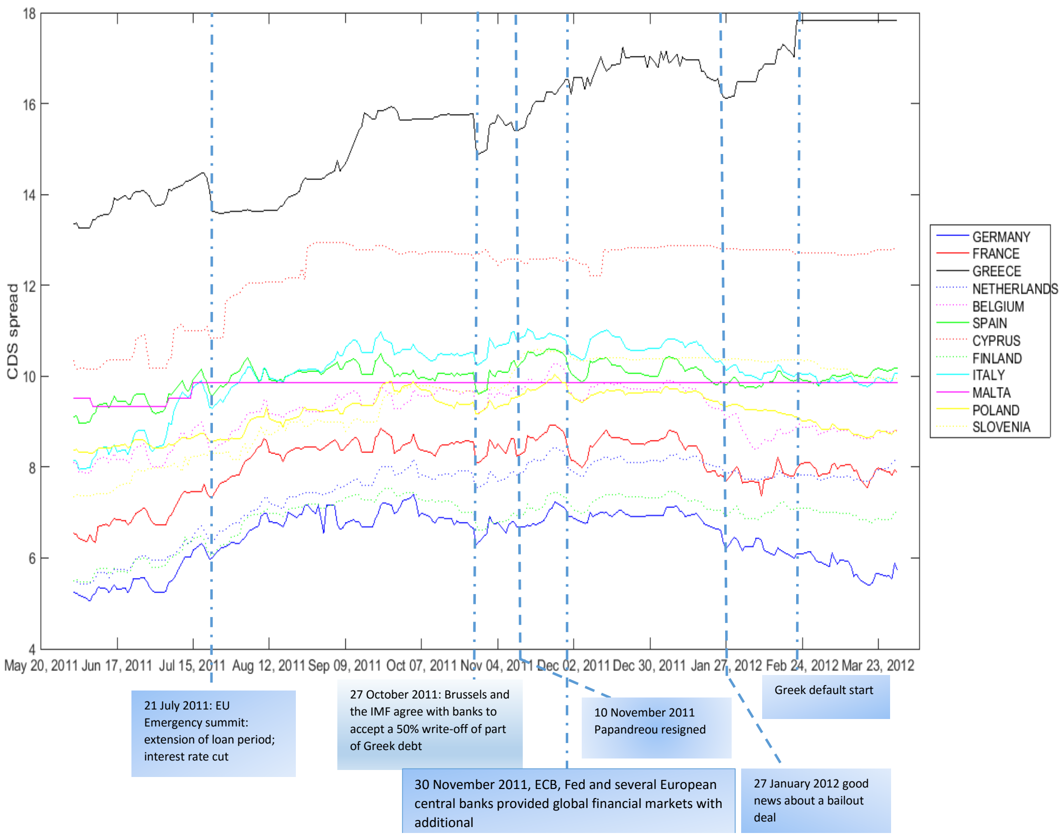

Figure 2: Log CDS spreads from the 1st of June 2011 to 30th of March 2012 for 12 Eurozone members

and Euro swap rates with maturities of 2-, 3-, 5-, 7- and 10- years.

4 Estimation and results

4.1 Estimation method

As already mentioned, the existing literature on modeling default process generally exploits the differences between observed and theoretical spreads to estimate via least squares (or equivalent GMM) the parameters of the employed model (Pan and Singleton, 2008; Longstaff et al., 2005; Ang and Longstaff, 2013; At-Sahalia et al., 2014). However, when the model is a multivariate Hawkes process, as in the present paper, the varying intensities for each country, λ_t^i, $i = 1, \cdots, N$, $t = 1, \cdots, T$, become new parameters to estimate. As a result, the parameter space becomes incredibly big and estimation is no longer feasible. In fact, for this very reason, At-Sahalia et al. (2014) opt to estimate only a bivariate Hawkes model and impose several other restrictions regarding some parameter constancy.

Here, the observed spread at time t for a certain country i and a certain maturity τ, $\tilde{s}_{i,t}^{\tau}$, can be written as the sum of the theoretical spread and an error, i.e. $\tilde{s}_{i,t}^{\tau} = s_{i,t}^{\tau} + \epsilon_{i,t}^{\tau}$. We assume this error is independent from the price formation process, i.e. $s_{i,t}^{\tau} \perp \epsilon_{i,t}^{\tau}$. Further, given the transforms in 2.2, equation (8) can be re-written as:

$$s_{i,t}^{\tau} = \frac{r_i \int_t^{\tau} D(t,s) \exp\left(\alpha\left(t\right) + \beta_{N+1}\left(t\right)\lambda_t^1 + \ldots + \beta_{2N}\left(t\right)\lambda_t^N\right)\left(A\left(t\right) + B_{N+1}\left(t\right)\lambda_t^1 + \ldots + B_{2N}\left(t\right)\lambda_t^N\right)\,\mathrm{d}s}{\int_t^{\tau} D(t,s) \exp\left(\alpha\left(t\right) + \beta_{2N+1}\left(t\right)\lambda_t^1 + \ldots + \beta_{2N}\left(t\right)\lambda_t^N\right)\,\mathrm{d}s}, \tag{9}$$

At time t, for a certain maturity and $i = 1, \cdots, N$, equation (9) becomes a system of N equations in N variables, $\lambda_t^i, \quad i = 1, \cdots, N$. In the case where $\epsilon_{i,t}^{\tau}$ is 0, this system of N equations can be (numerically) solved to get $\lambda_t^i, \quad i = 1, \cdots, N$. If we repeat this inversion operation for all considered maturities, the λ_t^i's obtained should be equal between maturities.

If $\epsilon_{i,t}^{\tau} \neq 0$, but the inversion is still performed, the error will be transferred, through some nonlinear function, from the observed spread to the $\tilde{\lambda}_t^i$'s. As a result, we have $\tilde{\lambda_t}^{i,\tau} = f^{\tau}\left(\lambda_t^i, \epsilon_{i,t}^{\tau}\right)$, where f^{τ} is some nonlinear continuous function in λ_t^i and $\epsilon_{i,t}^{\tau}$. Thus, in the presence of observation error, the $\tilde{\lambda}_t^i$'s are no longer equal between maturities. We exploit this difference to estimate the parameters of the model. This is feasible because in this set-up, the $\tilde{\lambda}_t^i$'s are no longer parameters to estimate, but become "quasi-observable", i.e. they are just deterministic functions of the original data.

The parameter set is defined as $\Omega = \{\delta^i, w_i, \alpha^i, \beta^i, \gamma_i, r_i, \quad i = 1, \cdots, N\}$ and can be estimated by minimizing the overall sum of squared differences between the $\tilde{\lambda}_t^i$'s for the 5- and 10- year maturities:

$$\hat{\omega} = \min_{\omega \in \Omega} \sum_{t=1}^{T} \sum_{i=1}^{N} \left(\tilde{\lambda_t}^{i,5} - \tilde{\lambda_t}^{i,10} \right)^2 \tag{10}$$

Inference for $\hat{\omega}$ is based on the usual asymptotic variance for m- estimators.

4.2 Results for 4 countries

Germany, France, Greece, Spain The estimated coefficients and their standard errors are included in table 1. $**$ denotes significance for a 0.1% significance level.

Table 1: Estimated parameters for the model including 4 countries

Country	δ^i	w_i	α^i	β^i	γ_i	λ_∞^i	r_i
Germany	0.0655**	0.5273**	1.5623**	0.0611**	0.8815**	0.0057**	0.9337**
France	0.0872**	0.0001	1.4223**	0.1292**	0.9903**	0.0079v	1.0000**
Greece	0.0002	0.2630**	0.5643**	0.1657**	1.0000**	0.0122**	0.9999**
Spain	0.0000	0.2097**	0.4065**	0.2339**	0.8999**	0.0097**	0.9998**

δ^i, the parameter that shows the sensitivity to the market, is slightly higher for Germany and France, indicating some degree of contagion coming from less stable countries. However, the parameter remains at low levels for all countries included. Looking at the data plots in figure 3, the path of of CDS spread for Greece is very different than for the other countries. While Greece went through turmoil and various negotiations concerning its debt, the rest of the European countries continued having their debt and debt related securities valued mostly based on country specific risk.Thus, while the variation of the above portfolio might be dominated by Greece, country specific variations are not and the coefficients, especially the δ^i's reflect this.

w_i can be interpreted as the weight of a country's default risk in the systemic default risk. The higher weights for Greece and Spain are related with the higher debt for this countries and higher default exposure. The relatively high values for the coefficients β^i show clear signs of self-excitability for all countries, but especially for Greece and Spain.

In equation (11), we also report the estimated $\breve{\alpha}$ and $\breve{\beta}$ matrices, which give a better understanding of how the model evolves.

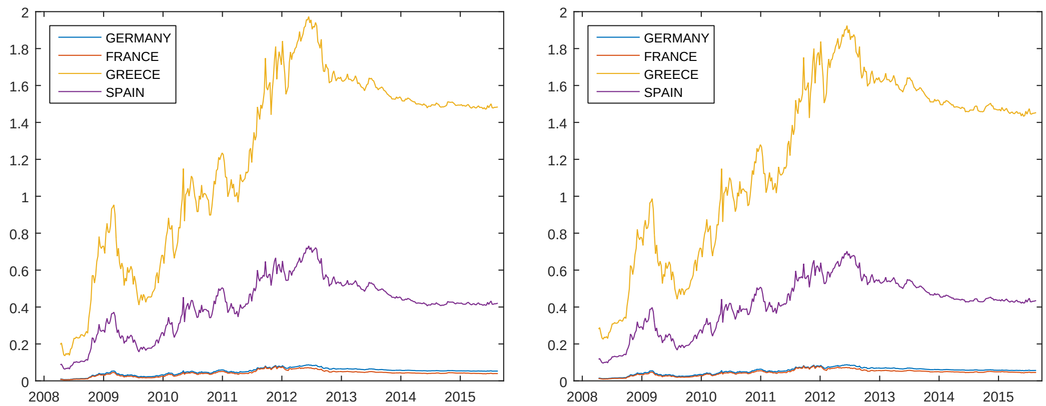

Figure 3: Default intensities for Germany, France, Greece and Spain from the 1st of June 2011 to 30th of March 2012

$$d\lambda_t = \begin{pmatrix} 1.6182 & 0.0000 & 0.0101 & 0.0058 \\ 0.0744 & 1.4223 & 0.0134 & 0.0077 \\ 0.0002 & 0.0000 & 0.5643 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.4065 \end{pmatrix} (\lambda_\infty - \lambda_t)dt + \begin{pmatrix} 0.0633 & 0.0000 & 0.0030 & 0.0033 \\ 0.0029 & 0.1292 & 0.0039 & 0.0044 \\ 0.0000 & 0.0000 & 0.1657 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.2339 \end{pmatrix} d\mathbf{N}_t, \tag{11}$$

The default intensities for the 5- and 10- year maturities are shown in figure 3. As mentioned before, these are obtained through teh inversion of spreads and not via estimation.

Germany, France, Spain, Italy The estimated coefficients and their standard errors are included in table 2. $**$ denotes significance for a 0.1% significance level

Table 2: Estimated parameters for the model including 4 countries

Country	δ^i	w_i	α^i	β^i	γ_i	λ^i_∞	r_i
Germany	0.2013	0.4053	0.7205	0.0011	0.5588	0.0070	0.9997
France	0.2048	0.0695	1.3483	0.0916	0.9968	0.0072	0.9747
Spain	0.0000	0.3375	0.3466	0.1629	0.9878	0.0084	0.9983
Italy	0.2888	0.1877	0.2814	0.1295	0.5879	0.0160	0.9994

Comparing the above results with the results in table 1, we observe that the δ^i's are higher for both Germany and France than before. This shows that this second "market" portfolio is somewhat more representative than the previous one including Greece. δ^i is close to 0 for Spain again, but combined with a high weight, showing that while it is an important part of the portfolio, it does not respond to the movements in the market much. In this portfolio, Germany seems to dominate again in terms of weights, followed by Spain and Italy. France has a low weight again, which could be explained by its very similar behaviour to Germany.

4.3 Results for 12 countries

Results for 12 countries are shown in table 3.

Table 3: Estimated parameters for the model including 12 countries

Country	δ^i	w_i	α^i	β^i	γ_i	λ^i_{infty}	r_i
Germany	2.0908	0.0062	3.9082	0.2254	0.9505	0.0059	0.9691
France	1.2651	0.0121	0.0443	0.0020	0.8380	0.0103	0.7612
Greece	2.0028	0.0114	0.5113	0.0325	0.9695	0.0176	0.9777
Netherlands	1.1121	0.0756	0.1964	0.0156	0.7703	0.0291	0.7642
Belgium	0.7206	0.3428	0.0404	0.0025	0.9249	0.0065	0.8563
Spain	1.2964	0.0460	0.1866	0.0122	0.7232	0.0647	0.5058
Cyprus	0.9814	0.0471	0.1193	0.0111	0.7974	0.0607	0.7441
Finland	1.2621	0.2473	0.9850	0.0159	0.1502	0.0927	0.9745
Italy	0.9314	0.0311	0.2912	0.0323	0.8678	0.0396	0.7844
Malta	0.9195	0.0978	0.0523	0.0066	0.8826	0.0980	0.8855
Poland	0.9666	0.0439	0.0903	0.0142	0.5609	0.0521	0.6244
Slovenia	0.9941	0.0387	0.1026	0.0109	0.7921	0.0285	0.7495

5 Conclusion

In this paper, we model the sovereign debt default intensities for the Eurozone countries as self exciting point processes with two components: the country-specific and the systemic default intensities. Each countrys default intensity adjusts to the market differently and a special parameter measures the speed of this adjustment. This structure confers feedback between occurring defaults and intensities and thus default risk and between the country specific and systemic components. We show the above model is affine in the state variables and use the results in Duffie et al (2003) to obtain closed form CDS prices. We estimate the model by inverting the spreads to obtain default intensities that vary with maturity and subsequently estimate the model by minimizing the differences between defaut intensities for 5 and 10 years. This new model presents various benefits over existing models for sovereign credit risk. First, we are able to measure systemic default intensity and thus risk at different points in time. This is of great relevance given the current tumultuous period for the Eurozone area. Second, we obtain a model that resembles classical finance theory. Default intensity for each component is a function of systemic default intensity and has a certain sensitivity, δ^i, to the systemic factor. Finally, this modeling approach substantially reduces the parameter space, which for a multivariate self exciting process (Hawkes) can be quite big.

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A Appendix

A.1 Gradient for $\theta(c)$

$$\nabla\theta(c) = \begin{pmatrix} \theta^1(c) & 0 & \dots & 0 & \left[1+\delta^1\left(1-\sum_{j=1}^N w_j\delta^j\right)^{-1} w_1\right]\beta^1\theta^1(c) & \delta^2\left(1-\sum_{j=1}^N w_j\delta^j\right)^{-1} w_1\ \beta^1\theta^1(c) & \dots & \delta^N\left(1-\sum_{j=1}^N w_j\delta^j\right)^{-1} w_1\ \beta^1\theta^1(c) \\ 0 & \theta^2(c) & \dots & 0 & \delta^1\left(1-\sum_{j=1}^N w_j\delta^j\right)^{-1} w_2\beta^2\theta^2(c) & \left[1+\delta^2\left(1-\sum_{j=1}^N w_j\delta^j\right)^{-1} w_2\right]\beta^2\theta^2(c) & \dots & \delta^N\left(1-\sum_{j=1}^N w_j\delta^j\right)^{-1} w_2\ \beta^2\theta^2(c) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \theta^N(c) & \delta^1\left(1-\sum_{j=1}^N w_j\delta^j\right)^{-1} w_N\beta^N\theta^N(c) & \delta^2\left(1-\sum_{j=1}^N w_j\delta^j\right)^{-1} w_N\beta^N\theta^N(c) & \dots & \left[1+\delta^N\left(1-\sum_{j=1}^N w_j\delta^j\right)^{-1} w_N\right]\beta^N\theta^N(c) \end{pmatrix}$$