FULLY BAYESIAN ANALYSIS OF SVAR MODELS UNDER ZERO AND SIGN RESTRICTIONS

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First version: December 23, 2015
Very Preliminary

Abstract: The paper points to some methodological faults in existing literature on Structural VAR (SVAR) models under zero and sign restrictions. We also propose the methodologically sound method to deal with this kind of models. Moreover the issue of existence and path-connectedness of Impulse Response Functions is tackled. In this respect we offer complementary results to those available in the literature. Lastly we illustrate our approach with the help of two serious empirical examples. First of all we challenge the output puzzle obtained by Uhlig (2005). Second, we check the robustness of the results given by Beaudry et al. (2014), concerning impact of optimism shocks on economy.

I. INTRODUCTION

Following the work of Uhlig (2005), there have been more and more papers that apply sign restrictions in order to decide on most important problems is empirical macroeconomics. It seems that the methodology of sign restrictions is attractive for applied researchers because it is supposed to be robust with respect to particular identifying scheme imposed on Structural VAR (SVAR) model within the classical framework of exact identification. However recent papers points to some problems in appropriate application of the partially identified SVAR models under zero and sign restrictions.

If only sign restrictions are used, the method proposed by Uhlig (2005) largely survives the passing time test. The only problem with pure sign restrictions approach (still not completely resolved) is when and whether the given sign restrictions are consistent with the model. The obvious analogy is to the question when the given set of inequalities possesses nonempty solution. Some partial results were given in Moon et al. (2013), Giacomini and Kitagawa (2015) and Gafarov and Montiel Olea

(2015). One of the contribution of our paper are the complementary results with this respect.

On the other hand if zero or zero and sign restrictions are used simultaneously, the problem with the methodology of Uhlig (2005) was clearly pointed by Arias et al. (2014). This is one of the (not so?) many cases in economics when methodology really matters, and economists who do not pay much attention to the applied methodology could reach economic implications that are intriguing and economically significant yet are based on methodological faults. In particular Arias et al. (2014) by juxtaposing their results with that of Beaudry et al. (2014), show, that contribution of optimism shocks to the Forecast Error Variance Decomposition (FEVD), for whatever horizon and variable, regularly goes down two or even three times. Unfortunately we point to some deficiency in the methodology proposed by Arias et al. (2014). Our second contribution is the proposal how to fix the problems appearing in Arias et al. (2014).

The overall goal of the paper is to clean up existing methods available in the literature and propose the fully Bayesian methodology to deal with partially identified SVAR models. Having developed appropriate tool, we apply it to challenge output puzzle obtained in Uhlig (2005). Further we try to obtain reliable estimates of Impulse responses functions and forecast error variance due to optimism shocks in a model considered by Beaudry et al. (2014).

II. THE MODEL AND NOTATION

Our model framework is the standard SVAR model

$$A_0 y_t = A_1 y_{t-1} + A_2 y_{t-2} + \cdots + A_p y_{t-p} + c + \varepsilon_t \tag{1}$$

where A_0 is an $(n \times n)$ nonsingular matrix of coefficients measuring contemporaneous relations between $n \times 1$ vector of observations y_t, $c \in \mathbb{R}^{n \times 1}$ is vector of constants, and $A_1, \ldots, A_p$ are $(n \times n)$ matrices of coefficients on lagged data. We assume that structural shocks i.e. $\varepsilon_t : (n \times 1)$, are independently, identically normally distributed with zero expectation and identity covariance matrix i.e. $\varepsilon_t \sim i.i.d.\ N(0, \mathrm{I}_n)$. We need some additional notation. Let $B = [A_1\ A_2\ \ldots\ A_p\ c] \in \mathbb{R}^{n \times (np+1)}$, $y = [y_1\ y_2\ \cdots\ y_T] \in \mathbb{R}^{n \times T}$, T denotes the sample size that is effectively used at the estimation stage and $\underset{(np+1)\times T}{X'} = \begin{bmatrix} y_0 & y_1 & \cdots & y_{T-1} \\ y_{-1} & y_0 & \cdots & y_{T-2} \\ \vdots & \vdots & \cdots & \vdots \\ y_{-p+1} & y_{-p+2} & \ddots & y_{T-p} \\ 1 & 1 & \cdots & 1 \end{bmatrix}$. Reduced form coefficients induced by (1) will be denoted as $\Pi = A_0^{-1}B$ and the reduced form covariance as $\Sigma = A_0^{-1}A_0'^{-1}$. Let us denote by Ψ_h an $n \times n$ matrix of impulse responses after h periods of time, $\Psi_0 = A_0^{-1}$ being the instantaneous response. In particular let $\psi_{ij,h}$ denote the $i-$th row, $j-$th column generic entry of Ψ_h. This is simply the response of $i-$th variable to the $j-$th shock ε_{jt} after h periods. Prominent role in our paper play the orthogonal matrices. The space of those matrices is denoted as $O(n) = \{Q \in \mathbb{R}^{n\times n} \mid Q'Q = \mathrm{I}_n\}$. Sometimes the following partitioning of $Q \in O(n)$ will be used $Q = [q_1 q_2 \ldots q_n]$, where q_i is just the $i-$th column of Q. We will frequently use the *QR* decomposition of the instantaneous impulse response matrix i.e. $A_0^{-1} = LQ$, where L is an $(n \times n)$ lower triangular matrix with positive diagonal elements and $Q \in O(n)$. Note that $\Sigma = LL'$. Lastly let e_j denote the $j-$th column of I_n and $\Gamma_p(a)$ is the notation for the multivariate gamma function defined by $\Gamma_p(a) = \pi^{\frac{1}{4}p(p-1)}\prod_{i=1}^{p}\Gamma(a - \frac{i-1}{2})$.

Following Arias et al. (2014) let us define selections matrices for zero restrictions as Z_i $(i = 1,\ldots,k \leq n)$ and those for sign restrictions as S_i $(i = 1,\ldots,k \leq n)$. The rationale behind introduction of the notation involving Z_i and S_i was given by Rubio–Ramírez et al. (2010). Suffice it to say the notation is instrumental to write down all interesting restrictions appearing in SVAR literature, see also Giacomini and Kitagawa (2015). We assume that each Z_i and S_i is of full row rank, in particular $\mathrm{rank}(Z_i) = z_i$ and $\mathrm{rank}(S_i) = s_i$. Z_i captures all zero restrictions implicitly imposed on the $i-$th column of Q i.e. q_i, and S_i those sign restrictions implicitly imposed on q_i. For example zero restriction imposed on (i,j) element of A_0^{-1} may written as follows $0 = e_i'A_0^{-1}e_j = e_i'LQe_j = e_i'Lq_j$. Thus all z_j zero restrictions imposed on the $j-$th column of A_0^{-1} may be written as $Z_jLq_j = 0$, where Z_j is the selection matrix. Further zero restriction imposed on (i,j) element of A_0 reads as $0 = e_i'A_0e_j = e_i'Q'L^{-1}e_j = e_j'L'^{-1}Qe_i = e_j'L'^{-1}q_i$, hence all z_i restrictions imposed on the $j-$th row of A_0 may be written as $Z_iL'^{-1}q_i = 0$. Similar reasoning yields restrictions put on lag coefficients B. Lastly if Ψ_h^{red} is the reduced form impulse response matrix, then it is easy to show that $\Psi_h = \Psi_h^{red}LQ$, for $h = 0,1,\ldots$, including $h = \infty$. The latter relates to the long-run impulse response, provided the data are in differences. Hence all zero restrictions imposed on the $j-$th column of Ψ_h may be written as $Z_j\Psi_h^{red}Lq_j = 0$. In particular, this allows us to write all linear restrictions on either A_0, B or Ψ_h as $Z_jf(\Pi, L)q_j = 0$ for some $j = 1,\ldots,k \leq n$, where $f(\Pi, L)$ is a function of the reduced form parameters Π, L only[1] and $k < n$ holds if not all columns of Q are subject to linear restrictions. Repeating all the above reasoning in the context of sign restrictions one may write all these restrictions as $S_j f(\Pi, L) q_j \geq 0$, where $j = 1, \ldots, k \leq n$ and $k < n$ holds if not all columns of Q are subject to sign restrictions.

III. UNRESTRICTED POSTERIOR

The notion of the unrestricted posterior occupies important place in Arias et al. (2014). In this section we suggest slightly different way to produce it.

An unrestricted posterior is just the posterior without introducing any (sign or exact) restrictions. In our notation it will always be identified with the subscript "*ur*". In order to derive it one should take a stand on a prior distribution. Since there is no universal uninformative prior it is a good idea to state ignorance with respect to aspects of phenomenon that your model is intended to cope with. In our case this aspects are Impulse Response Functions (IRF's). That is why we think that unrestricted posterior should be derived under ignorance prior explicitly stated in the context of IRF's. In contrast, Arias et al. (2014) ignored this issue emphasizing that any prior will do, provided it possesses the conjugacy property. Unfortunately many popular priors formulated on the space of SVAR parameters may induce unintended and unwanted consequences when looked from the perspective of IRF's behavior. If you really want to be agnostic with respect to IRF's you should state it formally in the form of the prior on that space. We think this methodological stance addresses some points raised by Baumeister and Hamilton (2015), at least those that can be operationally solved[2].

Being consistent with the above insight let us start with the assumption of the flat prior for the first $p+1$ IRF's i.e. $p(\Psi_0, \Psi_1, \ldots, \Psi_p) \propto 1$. Given that the model is completely unidentified this induces the prior for structural parameters (see. e.g. Kocięcki (2010))

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[1] In what follows we refer interchangeably to both Π, L and Π, Σ as the reduced form. In fact for some theoretical reasons it is more appropriate to consider Π, L as the reduced form parameters.

[2] The main recommendation of Baumeister and Hamilton (2015) to state the prior that is an outcome of the well-thought intellectual process is the old utopian idea, which has its roots in subjective Bayesianism. The process of elicitation of the informative prior in complex models (like SVAR's of even moderate size) is extremely difficult. It is exactly for this reason why subjective Bayesian approach to complex modelling is still unrealized promise, which is the most attractive yet unattainable goal. Two variable example used in Baumeister and Hamilton (2015) hides all problems that do arise in, say, six variables SVARs.

$$p(A_0, B) \propto |\det(A_0)|^{-2n(p+1)} \tag{2}$$

which leads to the following unrestricted posterior of SVAR model

$$p_{ur}(A_0, B \mid y) \propto \left|\det(A_0)\right|^{T-2n(p+1)} \operatorname{etr}\{-\tfrac{1}{2}A_0 M A_0' - \tfrac{1}{2}(B-\hat{B})X'X(B-\hat{B})'\} \tag{3}$$

where $M = y[\mathrm{I}_T - X(X'X)^{-1}X']y'$; $\hat{B} = A_0\hat{\Pi}$; $\hat{\Pi} = yX(X'X)^{-1}$, $\operatorname{etr}\{\cdot\} := \exp\{trace\{\cdot\}\}$ and subscript *"ur"* signifies that the posterior under consideration is unrestricted. Following e.g. Arias et al. (2014), Moon et al. (2013) or Giacomini and Kitagawa (2015), let us decompose the impact response as $A_0^{-1} = LQ$, where L is lower triangular with positive diagonal elements and $Q \in O(n)$. However in contrast to those works, to proceed further we follow the logic of our (fully Bayesian) approach and in order to work with L, Q instead of A_0, we must take into account the Jacobian $J(A_0 \to L, Q) = J(A_0 \to A_0^{-1})J(A_0^{-1} \to L, Q)$. Moreover changing variables from B to the reduced form coefficients $\Pi = A_0^{-1}B$, it is easy to show that the joint posterior may be decomposed as follows[3]

$$p_{ur}(L, Q, \Pi \mid y) = p_{ur}(\Pi \mid L, y)p_{ur}(L \mid y)p_{ur}(Q) \tag{4}$$

where

$$p_{ur}(\Pi \mid L, y) \propto \det(LL')^{-\frac{1}{2}(np+1)} \operatorname{etr}\{-\tfrac{1}{2}(LL')^{-1}(\Pi - \hat{\Pi})X'X(\Pi - \hat{\Pi})'\}$$
$$p_{ur}(L \mid y) \propto \prod_{i=1}^{n} l_{ii}^{-(T-2np)+n-i} \operatorname{etr}\{-\tfrac{1}{2}(LL')^{-1}M\}$$
$$p_{ur}(Q) \propto (Q'dQ)$$

In the last expression, $(Q'dQ)$ denotes the product of elements of $Q'dQ$ below the diagonal and is known as the Haar measure on $O(n)$. It means that $p_{ur}(Q)$ is a flat pdf with respect to the Haar measure, see James (1954) or Muirhead (1982) for details. Further, l_{ii} denotes (i, i) generic element of L.

Equivalently, using Choleski decomposition of the reduced form covariance matrix $\Sigma = LL'$ one may rewrite (4) as

[3] Note that lack of conditioning on y in the last density on the right of (4) means that it is just the prior. Moreover there is no Q among conditioning set in the first density on the right of (4), which means that Π is independent of Q given L. This notation will be generic to our paper, so please be careful.

$$p_{ur}(\Pi,\Sigma,Q \mid y) = p_{ur}(\Pi \mid \Sigma,y)p_{ur}(\Sigma \mid y)p_{ur}(Q) \tag{5}$$

where

$p_{ur}(\Pi \mid \Sigma,y) \propto \det(\Sigma)^{-\frac{1}{2}(np+1)} \operatorname{etr}\{-\frac{1}{2}\Sigma^{-1}(\Pi-\hat{\Pi})X'X(\Pi-\hat{\Pi})'\}$

$p_{ur}(\Sigma \mid y) \propto \det(\Sigma)^{-\frac{1}{2}(T-2np+1)} \operatorname{etr}\{-\frac{1}{2}\Sigma^{-1}M\}$

$p_{ur}(Q) \propto (Q'dQ)$

Hence one arrives at $p_{ur}(\Pi \mid \Sigma,y)$ being matricvariate Normal and $p_{ur}(\Sigma \mid y)$ being inverted Wishart i.e. the framework adopted e.g. by Uhlig (2005), Arias et al. (2014) or Giacomini and Kitagawa (2015) and many others. Advantage of our approach is that the prior $p_{ur}(Q)$ is not imposed like e.g. in Uhlig (2005) or Arias et al. (2014), but retrieved from our basic postulates (being ignorant about IRF's). Moreover this also allows us to address some critiques made by Giacomini and Kitagawa (2015), Baumeister and Hamilton (2015) concerning assumption of the uniform prior for Q.

It is clear that in our framework the prior for Q is always the consequence of two things: a) particular prior for A_0 and b) adopted factorization for A_0 (involving orthogonal Q). At first sight this undermines our framework. However as long as the marginal prior for A_0 is left–spherical, whatever decomposition of A_0 one uses, the induced marginal prior for Q is always proportional to the Haar measure, see e.g. Fang and Zhang (1990)[4]. Does left–spherical family of distributions exhaust all sensible priors for A_0? Well, even in fully identified case, according to Sims and Zha (1998) and Rubio–Ramírez et al. (2010) they do. When the model is completely unidentified (i.e. unrestricted model) and bearing in mind the still existent normalization problem in SVARs under assumption $\varepsilon_t \sim N(0,\mathrm{I}_n)$, justification of the uninformative prior that is not left–spherical is difficult. It is so because the model itself is also invariant under pre-multiplication by orthogonal matrix. See also more fundamental arguments presented in Kocięcki (2012). This sets the assumption of the uniform prior (i.e. Haar measure) for Q in a new light, since the real challenge is the justification of a non-uniform prior for Q as a candidate for non-informative prior. Even when Q is subject to some restrictions, we later show that the prior for Q is just the uniform distribution on the identified set (given reduced form parameters). Hence this is consistent with the recommendation given by Moon and Schorfheide (2012). Though Baumeister and Hamilton (2015) criticized the assumption of the uniform

[4] Let $X:(m \times k)$ be any matrix. If X and HX have the same distribution, for all $H \in O(m)$, then we say that X possesses left–spherical distribution.

prior for Q, they do that in the context of its conditional properties given the reduced form parameters. In our framework such a critique does not apply. It is so because our joint prior was derived explicitly under assumption of the flat prior for IRF's. So even if the prior distribution of Q given L may look strange as argued by Baumeister and Hamilton (2015), when considered jointly (i.e. augmenting it with our marginal prior for L), it induces the flat prior for IRF's, including A_0^{-1}.

IV. WHAT IS WRONG WITH ARIAS et al. (2014) AND GIACOMINI AND KITAGAWA (2015) APPROACHES?

Short answer is that when there are zero restrictions imposed on parameters and/or IRF's, their algorithms do not draw from the true posterior. Let us explain this in detail. We should start by emphasizing that the below reasoning is valid irrespective of how you introduce the posterior for the reduced form model. That is it may be retrieved as in our approach or imposed as in Arias et al. (2014) or Giacomini and Kitagawa (2015). The very reason is that we shall all agree that we should obey the probability rules.

We showed that the unrestricted joint posterior induced by the flat prior for IRF's is given by (4) or (5). The main role in our argument is played by $p_{ur}(Q)$. It is well known that the normalizing constant connected with this density is given by $C_{ur}^{-1} = [\int_{O(n)} (Q'dQ)]^{-1} = 2^{-n}\pi^{-\frac{1}{2}n^2}\Gamma_n(\frac{n}{2})$. We note in passing that it is not a coincidence that C_{ur} is just the surface area along $Q'Q = \mathrm{I}_n$ i.e. $\int_{O(n)} (Q'dQ) = \int_{Q'Q=\mathrm{I}_n} dQ$.

The unrestricted posterior involves three arguments: Π, Σ and Q. To obtain the marginal for Π,Σ we should integrate the joint posterior out with respect to Q

$$p_{ur}(\Pi,\Sigma \mid y) = \int_{O(n)} p_{ur}(\Pi \mid \Sigma, y) p_{ur}(\Sigma \mid y) \tfrac{C_{ur}}{C_{ur}} (Q'dQ) = p_{ur}(\Pi \mid \Sigma, y) p_{ur}(\Sigma \mid y) C_{ur} \int_{O(n)} \tfrac{1}{C_{ur}} (Q'dQ)$$
$$\propto p_{ur}(\Pi \mid \Sigma, y) p_{ur}(\Sigma \mid y) \qquad (6)$$

So when there are no zero restrictions imposed on parameters and/or IRF's, the posterior of the reduced form parameters Π,Σ is not affected by the prior for Q. But Arias et al. (2014) and Giacomini and Kitagawa (2015) claim that this is also true when some zero restrictions are present. Unfortunately it does not hold. The key feature to recognize this fact is to evaluate the surface area along $Q'Q = \mathrm{I}_n$ subject to linear restrictions imposed on Q. Incidentally, this will be an inverse of the normalizing constant for the restricted prior for Q.

Suppose that the zero restrictions are confined only to one column of Q. W.l.o.g we assume that it is the first column of Q i.e. q_1. Let us define $R_1 = \{Q \in O(n) \mid Z_1 f(\Pi, L) q_1 = 0\}$. We have

Proposition 1: *Let* $C_r = \int_{R_1} (Q' dQ)$, *where* $R_1 = \{Q \in O(n) \mid Z_1 f(\Pi, L) q_1 = 0\}$. *Then*

$C_r = 2^n \pi^{\frac{1}{2}((n-1)^2 + n - z_1)} [\Gamma_{n-1}(\frac{n-1}{2})]^{-1} [\Gamma(\frac{n-z_1}{2})]^{-1} \mid Z_1 f(\Pi, L) f(\Pi, L)' Z_1' \mid^{-\frac{1}{2}}$

where $\Gamma_p(a)$ *is the multivariate gamma function defined by* $\Gamma_p(a) = \pi^{\frac{1}{4}p(p-1)} \prod_{i=1}^{p} \Gamma(a - \frac{i-1}{2})$

Proof: See appendix.

Important thing to notice is that C_r depends on reduced form parameters. Hence under zero restrictions the prior for Q becomes in fact the conditional prior given the reduced form parameters i.e. $p_r(Q \mid \Pi, L) = (Q' dQ) / C_r$, where the subscript "*r*" signifies that we face the restricted density. Thus under zero restrictions we have

$$p_r(\Pi, \Sigma, Q \mid y) = p_r(\Pi, \Sigma \mid y) p_r(Q \mid \Pi, L) \qquad (7)$$

In order to obtain $p_r(\Pi, \Sigma \mid y)$ we have to integrate the unrestricted posterior (4) or (5) with respect to Q along R_1

$$p_r(\Pi, \Sigma \mid y) = \int_{R_1} p_{ur}(\Pi \mid \Sigma, y) p_{ur}(\Sigma \mid y) \tfrac{C_r}{C_r} (Q' dQ) = p_{ur}(\Pi \mid \Sigma, y) p_{ur}(\Sigma \mid y) C_r \int_{R_1} \tfrac{1}{C_r} (Q' dQ)$$

$$\propto p_{ur}(\Pi \mid \Sigma, y) p_{ur}(\Sigma \mid y) \mid Z_1 f(\Pi, L) f(\Pi, L)' Z_1' \mid^{-\frac{1}{2}} \qquad (8)$$

Note the subtle thing. Though the heuristic argument presented in Arias et al. (2014), p. 16, that "*zero restrictions impose no constraints on the reduced-form parameters but will impose constraints on the orthogonal matrix* Q", is correct, it does not tell all the truth. Although these zero restrictions do not impose any constraints they do change the probabilities of obtaining specific reduced form parameters to be consistent with the restrictions. Hence the last factor on the right of (8). Interestingly you may think of $\mid Z_1 f(\Pi, L) f(\Pi, L)' Z_1' \mid^{-\frac{1}{2}}$ as the "additional" prior. Since the restriction $Z_1 f(\Pi, L) q_1 = 0$ must hold, the prior (quite rationally) favors reduced form parameter space that result in small values for $Z_1 f(\Pi, L)$. Hence it works towards shrinking of those functions of reduced form parameters that are directly connected with restrictions. On the other hand the pitfall in the approach of Giacomini and Kitagawa (2015) is that once

$p_r(Q \mid \Pi, L)$ is changing, $p_r(\Pi, \Sigma \mid y)$ changes too, since each $p_r(Q \mid \Pi, L)$ entails different normalizing constants and they usually depend on the reduced form parameters.

In the general case, when zero restrictions concern IRF's other than the instantaneous response and/or parameters other than A_0 e.g. the long–run IRF, the above posterior becomes very complicated. However when zero restrictions are confined to A_0 and/or A_0^{-1}, at least $p_r(\Pi \mid \Sigma, y)$ is not affected. However the marginal posterior of Σ i.e. $p_r(\Sigma \mid y)$, is still nonstandard and not distributed as Inverted Wishart.

Possible defense of the approach in Arias et al. (2014) is that under some specific prior it is the case that their algorithm draws from the true posterior. For example, when the restrictions are confined to q_1 only, then we showed that if we employ the prior proportional to $\mid Z_1 f(\Pi, L) f(\Pi, L)' Z_1' \mid^{\frac{1}{2}}$ then the marginal posterior of the reduced form parameters is of the Normal–Inverted Wishart type. However when restrictions involve many columns of Q, then we cannot pin down analytically such a prior. Leaving aside the rationale for a prior itself, the methodological desideratum that "we draw from the true posterior under some unknown prior" is hard to defend.

Unfortunately obtaining analytical formula for integrating constant in $p_r(Q \mid \Pi, L)$, when more than one column of Q is subject to linear restrictions is in general impossible (in the opinion of the author). Though for some special cases (when the dimension of a model does not exceed 3 and/or only two columns of Q are restricted) this is feasible, the resultant formula is quite complicated and not easy to work with[5]. Dealing with restrictions embracing more than two columns of Q seems to be condemned to failure. That is why, in general drawing from $p_r(Q \mid \Pi, L)$ directly is infeasible and the Gibbs sampling technique suggested by Arias et al. (2014) is one option. It also means that the steps 2.1 and 2.2 in algorithm 4.1 from Giacomini and Kitagawa (2015) do not provide the draws from $p_r(Q \mid \Pi, L)$. Unfortunately it follows that the approach aiming at obtaining $p_r(\Pi, \Sigma \mid y)$ has its constraints. However we note in passing that we can do without $p_r(\Pi, \Sigma \mid y)$ to draw from the joint posterior, which will be explained in section VI.

V. IS EFFICIENT SAMPLING UNDER ZERO RESTRICTIONS OUT OF REACH?

Fortunately in some circumstances normalizing constant for $p_r(Q \mid \Pi, L)$ may be evaluated and the exact sampling from $p_r(Q \mid \Pi, L)$ is possible. This is the case when zero restriction imposed on (i,l) element of $f(\Pi, L)$ implies that all elements

[5] The formula involves infinite series.

$(i,1),(i,2),\ldots,(i,l-1)$ in $f(\Pi,L)$ are also restricted to zero. Putting it differently, Z_i contains all rows that appear in Z_{i+1}, Z_{i+2}, ... Let us formalize this statement. Suppose that first k columns of Q are subject to linear restrictions, where $1 \le k \le n$. Assume $z_1 \ge z_2 \ge \ldots \ge z_k$ and for every $i = 1,\ldots,k$, $Z_i f(A_0, A_+) q_l = 0$ for $l \le i$. Denote $R_k = \{Q \in O(n) \mid Z_i f(\Pi, L) q_l = 0, \forall l \le i; i = 1,\ldots,k\}$

We have

Proposition 2:

Let us denote $R_k = \{Q \in O(n) \mid Z_i f(\Pi, L) q_l = 0, \forall l \le i; i = 1,\ldots,k\}$. *Then* $\int_{R_k} (Q'dQ) = K \times \prod_{i=1}^{k} | Z_i f(\Pi,L) f(\Pi,L)' Z_i' |^{-\frac{1}{2}}$, *where* K *is a constant that does not depend on* Π, L.

Proof: See appendix.

In what follows we will call the SVAR model, which is subject to the pattern of restrictions consistent with R_k as R_k −restricted SVAR.

It is useful to state the (fully Bayesian) algorithm to sample from R_k −restricted SVAR. Needless to say since $R_1 \subseteq R_k$, the below algorithm is necessarily valid for the case when only one column of Q is subject to linear restrictions. In what follows $Q_{i-1} \equiv [q_1 \ldots q_{i-1}]$, with the convention that Q_0 is empty. The unit sphere in $\mathbb{R}$ (or uniform distribution on $O(1)$) consists of two points i.e. $\{-1,1\}$, to which we attach equal probability $\frac{1}{2}$. Moreover we assume $z_i \le n - i$.

Algorithm 1:

1) *Draw from* $p_r(\Pi, L \mid y) \propto p_{ur}(\Pi \mid L, y) p_{ur}(L \mid y) \cdot \prod_{i=1}^{k} | Z_i f(\Pi,L) f(\Pi,L)' Z_i' |^{-\frac{1}{2}}$
2) *Set* $i = 1$
3) *Draw a column vector* $x_i : (n - z_i + 1 - i) \times 1$ *from the uniform distribution on the unit sphere in* $\mathbb{R}^{n-z_i+1-i}$
4) *Find any* $G_i : n \times (n - z_i + 1 - i)$ *matrix (of full column rank) such that* $G_i' G_i = \mathrm{I}_{n-z_i+1-i}$ *and* $\begin{bmatrix} Z_i f(\Pi, L)) \\ Q_{i-1}' \end{bmatrix} G_i = 0$
5) *Set* $q_i = G_i x_i$
6) *Set* $i = i + 1$, *go to 3), and repeat until* $i = k$ *to get* $Q_k \equiv [q_1 \ldots q_k]$
7) *Find any* $W : n \times (n-k)$ *matrix such that* $W'W = \mathrm{I}_{n-k}$ *and* $Q_k' W = 0$
8) *Draw* Q_{-k} *from the uniform distribution on* $O(n-k)$ *and set* $[q_{k+1} q_{k+2} \ldots q_n] = W Q_{-k}$

9) Stack $Q=[Q_k \vdots q_{k+1}q_{k+2}...q_n]$, *which is an exact draw from* $p_r(Q \mid \Pi, L)$

10) Go to 1), and repeat N *times*

Justification of algorithm 1 is given in appendix. The sampling in steps 3) and 8) could be made as explained e.g. in Arias et al. (2014). Note that algorithm 1 allows for exact sampling from $p_r(Q \mid \Pi, L)$, see appendix. On the other hand sampling in the step 1) could be accomplished using the Independence Metropolis Hastings (IMH) algorithm:

Algorithm 2:

0) *Take the starting values by setting* $\Pi^{(0)}=\hat{\Pi}$, $\Sigma^{(0)}=\frac{1}{T-2np-2n-1}M$ *and applying the Choleski decomposition* $\Sigma^{(0)}=L^{(0)}(L^{(0)})'$

Update $(\Pi^{(j)}, L^{(j)})$ *to* $(\Pi^{(j+1)}, L^{(j+1)})$ *as follows:*

1) *Draw* $(\Pi^{(*)}, \Sigma^{(*)})$ *from the Normal–Inverted Wishart posterior* $p_{ur}(\Pi \mid \Sigma, y)p_{ur}(\Sigma \mid y)$

2) *Obtain the Choleski decomposition* $\Sigma^{(*)}=L^{(*)}(L^{(*)})'$ *and compute*

$$\alpha=\min\left\{1, \prod_{i=1}^{k} \mid Z_i f(\Pi^{(*)}, L^{(*)}) f(\Pi^{(*)}, L^{(*)})' Z_i' \mid^{-\frac{1}{2}} \mid Z_i f(\Pi^{(j)}, L^{(j)}) f(\Pi^{(j)}, L^{(j)})' Z_i' \mid^{\frac{1}{2}}\right\}$$

3) *Take* $(\Pi^{(j+1)}, L^{(j+1)})=\begin{cases}(\Pi^{(*)}, L^{(*)}), \text{ with probability } \alpha \\ (\Pi^{(j)}, L^{(j)}), \text{ with probability } 1-\alpha\end{cases}$

A few comments are in order. First of all, the algorithm 1, in its part to draw from $p_r(Q \mid \Pi, L)$, essentially appears in the older version of Arias et al. (2014). We think it is useful to know when this algorithm is still correct (i.e. when the SVAR is R_k –restricted), since it is a) the exact sampling and b) much easier to implement than the Gibbs sampling from the latest version of Arias et al. (2014). Second, what algorithm proves in a constructive way is that provided that $z_i \leq n-i$, for almost all reduced form parameter values one can find an orthogonal matrix consistent with zero restrictions, see step 5). Hence we weakened the assumption made by Arias et al. (2014), that $z_i \leq n-k$; $\forall i$. When $z_i=n-i$; $\forall i$, the algorithm collapses to finding a unique orthogonal matrix (up to the sign of each column or row) that is consistent with the restrictions. This is just algorithm 1 in Rubio–Ramírez et al. (2010). However even in this case in order to be fully Bayesian we should start by drawing reduced form parameter using our step 1). In contrast, Rubio–Ramírez et al. (2010) suggest that those parameters may be sampled from standard Normal–Inverted Wishart posterior. Note that due to cross restrictions between Q and L, Π (in the case of exact restrictions) there will be Bayesian learning concerning Q i.e. $p_r(Q \mid y)$ will no longer be the uniform distribution. But of course the general caveat raised by Baumeister and Hamilton (2015) applies here.

On the other hand, the IMH algorithm is instructive. It collapses to the usual sampling of the reduced form parameters used by Arias et al. (2014) or Giacomini and Kitagawa (2015), when α is identically equal to 1. Since we cannot expect it in practice, the underlying distributions of the reduced form parameters in our approaches differ. In general, if functions of the candidate reduced form parameters involved in the restrictions i.e. $Z_i f(\Pi^{(*)}, L^{(*)})$, are closer to zero than that in the previous draw $Z_i f(\Pi^{(j)}, L^{(j)})$, then we always accept a candidate draw. This is consistent with our discussion on rationale of the "additional" prior. Hence during the sampling process we penalize reduced form parameters that are probably inconsistent with the restrictions. Lastly it should be clear that if restrictions concern only A_0 and/or A_0^{-1}, the step 1) in algorithm 1 should be modified. That is we should apply IMH algorithm only to L. Drawing Π should be made from the (exact) conditional posterior $p_{ur}(\Pi \mid L, y)$.

Algorithm 3:

1) *Draw* Π, L, Q *from the joint (restricted) posterior* $p_r(\Pi, L, Q \mid y)$ *using algorithm 1*
2) *Keep the draw if all sign restrictions are satisfied*
3) *Go to 1) and repeat* N *times*

It is clear that linear restrictions on Q are a consequence of exclusion restrictions put on parameters and/or IRF's. If more than one column of Q is restricted then analytical formula for $p_r(\Pi, L \mid y)$ (or $p_r(\Pi, \Sigma \mid y)$) is extremely involved (in general unobtainable analytically unless the SVAR is R_k −restricted). Do we need explicit sampling from $p_r(Q \mid \Pi, L)$? Sometimes we do not. If there are only linear restrictions imposed on the structural parameters of SVAR i.e. A_0, B (but sign restrictions may concern either parameters or IRF's or both) then you may apply the efficient algorithm of Waggoner and Zha (2003a) to sample from the posterior. If you are interested in the marginal posterior for L and/or Q you should only augment the algorithm with one more step. That is after drawing from marginal posterior of A_0 you should compute $X = A_0^{-1}$ and apply *QR* decomposition $X = LQ$.

VI. THE ALGORITHM IN THE MOST GENERAL CASE

If the SVAR is R_k −restricted (which comprises the case when only one column of Q is subject to zero restrictions), the algorithm 1 may be used. What about the case when SVAR is not R_k −restricted? The present section proposes a numerical method to deal with such a case[6]. It should not be surprising that working with SVAR which is not R_k −restricted will be a little more computationally demanding. In particular, unlike in the R_k −restricted case, we can no longer exactly sample from the conditional posterior of Q given the reduced form parameters (and zero restrictions). Now we can only sample from the joint posterior of Q and reduced form parameters using Monte Carlo methods.

Algorithm 4:

0) Take the starting values for reduced form parameters by setting $\Pi^{(0)} = \hat{\Pi}$, $\Sigma^{(0)} = \frac{1}{T-2np-2n-1} M$ *and apply the Choleski decomposition* $\Sigma^{(0)} = L^{(0)}(L^{(0)})'$. *As a starting value for* $Q^{(0)}$ *take any draw, which is made applying steps 2) to 9) in algorithm 1.*

Update $(\Pi^{(j)}, L^{(j)}, Q^{(j)})$ *to* $(\Pi^{(j+1)}, L^{(j+1)}, Q^{(j+1)})$ *as follows:*

1) Draw $(\Pi^{(*)}, \Sigma^{(*)})$ *from the Normal–Inverted Wishart posterior* $p_{ur}(\Pi \mid \Sigma, y) p_{ur}(\Sigma \mid y)$ *and obtain the Choleski decomposition* $\Sigma^{(*)} = L^{(*)}(L^{(*)})'$

2) Set $i = 1$

3) Draw a column vector $x_i^{(*)} : (n - z_i + 1 - i) \times 1$ *from the uniform distribution on the unit sphere in* $\mathbb{R}^{n-z_i+1-i}$

4) Find any $G_i : n \times (n - z_i + 1 - i)$ *matrix (of full column rank) such that* $G_i' G_i = \mathrm{I}_{n-z_i+1-i}$ *and* $\begin{bmatrix} Z_i f(\Pi^{(*)}, L^{(*)}) \\ Q_{i-1}' \end{bmatrix} G_i = 0$

5) Set $q_i^{(*)} = G_i x_i^{(*)}$

6) Set $i = i + 1$*, go to 3), and repeat until* $i = k$ *to get* $Q_k^{(*)} \equiv [q_1^{(*)} q_2^{(*)} \ldots q_k^{(*)}]$

7) Find any $W : n \times (n - k)$ *matrix such that* $W'W = \mathrm{I}_{n-k}$ *and* $(Q_k^{(*)})' W = 0$

8) Draw $Q_{-k}^{(*)}$ *from the uniform distribution on* $O(n-k)$ *and set* $[q_{k+1}^{(*)} q_{k+2}^{(*)} \ldots q_n^{(*)}] = W Q_{-k}^{(*)}$

9) Stack $Q^{(*)} = [Q_k^{(*)} \vdots q_{k+1}^{(*)} q_{k+2}^{(*)} \ldots q_n^{(*)}]$

[6] Unlike the rest of this paper, this section was written after seeing the most recent version of Arias et al. (2014), dated December 6, 2015, and greatly benefited from the long discussion with Juan Rubio-Ramírez. As a matter of fact, the most recent version of Arias et al. (2014) differs so much from the previous one that it necessitates the substantial revision of our paper, because it invalidated many comments made in the present version of our paper. So this version of our paper reflects the author's knowledge on the subject before the most recent version of Arias et al. (2014) had been known to him.

10) Compute

$$\alpha = \min\left\{1, \prod_{i=1}^{k} \left|\begin{pmatrix} Z_i f(\Pi^{(*)}, L^{(*)}) \\ (Q_{i-1}^{(*)})' \end{pmatrix} \begin{pmatrix} Z_i f(\Pi^{(*)}, L^{(*)}) \\ (Q_{i-1}^{(*)})' \end{pmatrix}'\right|^{-\frac{1}{2}} \left|\begin{pmatrix} Z_i f(\Pi^{(j)}, L^{(j)}) \\ (Q_{i-1}^{(j)})' \end{pmatrix} \begin{pmatrix} Z_i f(\Pi^{(j)}, L^{(j)}) \\ (Q_{i-1}^{(j)})' \end{pmatrix}'\right|^{\frac{1}{2}}\right\}$$

$$\textit{and take } (\Pi^{(j+1)}, L^{(j+1)}, Q^{(j+1)}) = \begin{cases} (\Pi^{(*)}, L^{(*)}, Q^{(*)}), \ \textit{with probability } \alpha \\ (\Pi^{(j)}, L^{(j)}, Q^{(j)}), \ \textit{with probability } 1-\alpha \end{cases}$$

11) Go to 1), and repeat N times

The justification of algorithm 4 is given in appendix. Suffice it to say that the step 10) stems from the IMH algorithm, when the candidate generating distribution is a product of Normal-Inverted Wishart marginal posterior for reduced form parameters and some $g(Q \mid \Pi, L)$ subject to zero restrictions. The (exact) drawing from the latter distribution is made using steps 2) to 9) in algorithm 1. Of course when sign restrictions are present (in addition to zero restrictions) then we use the algorithm 3 except that the first step in this algorithm should be made using algorithm 4.

VII. EXISTENCE AND PATH-CONNECTEDNESS OF IRF's

Unfortunately the algorithm 3 implicitly assumes the existence of orthogonal matrices that are consistent with zero/sign restrictions. That is that identified set is nonempty. The following section addresses this point. Moreover we tackle the problem of path-connectedness of IRF's. Overall we give complementary results to those that appeared in Moon et al. (2013), Giacomini and Kitagawa (2015) and Gafarov and Montiel Olea (2015).

If not specified differently, $x \in \mathbb{R}^n$ denotes a column vector in this section. Let us define the unit sphere in $\mathbb{R}^n$ as $\mathcal{S}^{n-1}=\{x \in \mathbb{R}^n \mid x'x = 1\}$. Let us define the fundamental object in this section as

$$\Gamma_i = \begin{bmatrix} Z_i f(\Pi, L) \\ Q_{i-1}' \\ S_i f(\Pi, L) \end{bmatrix} \qquad i = 1,\ldots,k \le n \tag{9}$$

Note that Γ_i is a $(z_i + s_i + i - 1) \times n$ matrix. Let us denote $\mathcal{C}_i = \{x \in \mathbb{R}^n \mid \Gamma_i x \ge 0\}$. Since $\mathcal{C}_i$ is the solution set of a finite number of homogenous equalities and inequalities it is the polyhedral convex cone, to be called the cone, see e.g. Schrijver (1986) p. 87. Although properties of $\mathcal{C}_i$ are fundamental for our considerations, the ultimate object of interest is the set being an intersection of the cone with a unit sphere in $\mathbb{R}^n$

i.e. $\mathcal{C}_i \cap \mathcal{S}^{n-1}$. This is just the set of unit length vectors q_i that are consistent with zero and sign restrictions and are orthogonal to all previous $q_{i-1},\ldots,q_1$. Immediate thing to notice is that although $\mathcal{C}_i$ is never empty (since it is a cone hence must contain the origin 0), $\mathcal{C}_i \cap \mathcal{S}^{n-1}$ may be empty since intersection of the origin with the unit sphere $\mathcal{S}^{n-1}$ is empty. However as long as $\mathcal{C}_i$ contains $x \neq 0$, $\mathcal{C}_i \cap \mathcal{S}^{n-1}$ will be non-empty, since $\mathcal{C}_i$ is the cone. That is if $x \neq 0 \in \mathcal{C}_i$ then $\lambda x \in \mathcal{C}_i$ for all $\lambda \geq 0$ so that $(x'x)^{-\frac{1}{2}} x \in \mathcal{C}_i \cap \mathcal{S}^{n-1}$. Hence the problem of existence of orthogonal matrix consistent with zero/sign restrictions amounts to checking whether each $\mathcal{C}_i$ contains at least one nonzero point. The following proposition gives sufficient conditions for the identified set being nonempty

Proposition 3: *Let the zero and/or sign restrictions be imposed on the first k columns of Q. Then for each $i = 1,\ldots,k$, in case $s_i = 0$, assume $z_i \leq n - i$ and in case $s_i \geq 1$ assume the matrix Γ_i is of full row rank i.e. $1 \leq \operatorname{rank}(\Gamma_i) = z_i + s_i + i - 1 \leq n$. Then $\mathcal{C}_i \cap \mathcal{S}^{n-1}$ is not empty for each $i = 1,\ldots,k$.*

First part of assumptions i.e. for the case $s_i = 0$, is quite standard in the literature e.g. Giacomini and Kitagawa (2015), Gafarov and Montiel Olea (2015). When $z_i = n - i$, for each $i = 1,\ldots,k$, we get exact identification for the first k equations, see Rubio–Ramírez et al. (2010).

On the other hand in case $s_i \geq 1$ the assumption that Γ_i is of full row rank (to be called the assumption in the rest of this paragraph) is new to the literature. It may be treated as restrictive, since it imposes the maximal number of equality and sign restrictions. For example the sum of equality and sign restrictions imposed on the first shock cannot be greater than n. In fact much of applied work violates this assumption including the seminal work by Uhlig (2005). But the point is that when the assumption does not hold, one can construct simple examples as those in Moon et al. (2013), Giacomini and Kitagawa (2015) (and many more), such that $\mathcal{C}_i \cap \mathcal{S}^{n-1}$ is empty and/or not path connected for a subset of Π, L that has positive posterior probability. In other words if the assumption is not satisfied this may impose sign restrictions on the reduced form itself i.e. Π, L, which may be awkward when working in partially identified environment.

On the other hand if the assumption is satisfied for one (e.g. randomly selected) Π, L, Q, then it holds for almost all Π, L and Q. The merit of the assumption is obvious: you can estimate your SVAR in a sound way without bothering about existence and path connectedness of unit length vectors consistent with the restrictions. Of course dealing with SVARs that violate the assumption is possible. We should only verify for given Π, L, whether each $\mathcal{C}_i$ contains non-zero element. As noted by Gafarov and Montiel Olea (2015), the main challenge is not to state the conditions *per se* but making it computationally efficient so as it could be easily verified in an algorithmic way[7].

Let us denote by $\psi_{ij,h}(\Pi, L)$ the impulse response of the $i-$th variable to the $j-$th shock after h periods, for given reduced form parameters Π, L. Let us denote the set of all admissible values (i.e. those consistent with all zero and sign restrictions) for given $\psi_{ij,h}(\Pi, L)$ as $IRF_{i,j}^{h}(\Pi, L)$. The second important issue when dealing with IRF's in partially identified environment is, whether $IRF_{i,j}^{h}(\Pi, L)$ is an interval (and e.g. not a disconnected sum of two intervals). It turns out that this question amounts to checking whether each $\mathcal{C}_i \cap \mathcal{S}^{n-1}$ is path connected (see Giacomini and Kitagawa (2015) or Gafarov and Montiel Olea (2015)). Practical importance of this issue follows from the fact that if $\mathcal{C}_i \cap \mathcal{S}^{n-1}$ were not path connected the uncertainty seen in the error bands may be substantially overestimated so as to prevent economic interpretations. In fact the problem of normalization in SVAR models treated comprehensively in Waggoner and Zha (2003b), is the special case of the problem of path connectedness of unit length vectors consistent with the restrictions. The following proposition gives sufficient conditions for each $IRF_{i,j}^{h}(\Pi, L)$ to be an interval

Proposition 4: *Let the zero and/or sign restrictions be imposed on the first k columns of Q. Then for each $i = 1,\dots,k$, in case $s_i = 0$, assume $z_i < n - i$ and in case $s_i \geq 1$ assume the matrix Γ_i is of full row rank i.e. $1 \leq \operatorname{rank}(\Gamma_i) = z_i + s_i + i - 1 \leq n$. Then $\mathcal{C}_i \cap \mathcal{S}^{n-1}$ is path connected for each $i = 1,\dots,k$.*

It is evident that assumptions in propositions 3 and 4 are identical except the treatment of the case $s_i = 0$. The assumption in proposition 4 is a little bit stronger in that case. On the other hand if $s_i \geq 1$ the conditions under which $\mathcal{C}_i \cap \mathcal{S}^{n-1}$ is nonempty and path connected are the same.

VIII. EFFECTS OF MONETARY POLICY ON OUTPUT

In the well-known article, Uhlig (2005) argues that the effects of monetary policy shock on real output are uncertain. Hence he challenged the whole literature on identified SVAR that reached the consensus that a contractionary monetary policy shock

[7] Giacomini and Kitagawa (2015) proposed very crude method to do that. If, for given reduced form parameter values, you cannot find Q satisfying sign restrictions among quite large number of draws from $p_r(Q \mid \Pi, L)$, say 10.000, you consider at least one $\mathcal{C}_i$ contains only the origin.

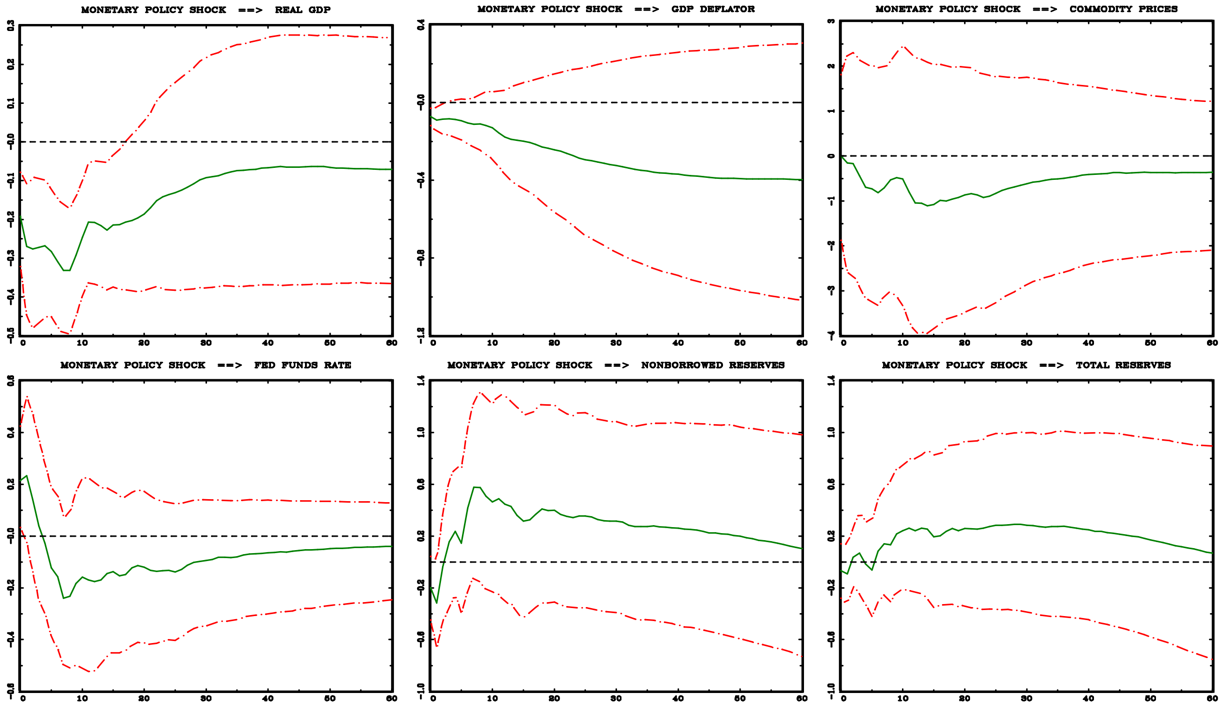
should lower the real output in a significant way. The conclusion was made on the basis of signed restricted SVAR. In a recent paper Arias et al. (2015) conclude that the pitfall in the approach of Uhlig (2005) was the fact that his sign restrictions imposed on a model to identify monetary policy shock, accommodated unreasonable systematic monetary policy behavior. That is that the induced probability that interest rates rise in response to an increase in output was not close to 1 at all (as one may conjecture), but only 0.34. According to Arias et al. (2015) this fact discredits the approach of Uhlig (2005) i.e. the shock identified by Uhlig (2005) is not the monetary policy shock. The suggestion of Arias et al. (2015) was to impose explicit zero and sign restrictions on "monetary policy" equation in SVAR. Quite naturally, since they challenged results of Uhlig (2005), they worked with the same 6-variable SVAR. W.l.o.g suppose the first equation in this SVAR is labeled as "monetary policy" equation. Since what matters are the coefficients of the contemporaneous relations matrix A_0, denoting by $lags$ all remaining terms and c_1 by a constant in the first equation, we get

$$a_{0,14}r_t = a_{0,11}y_t + a_{0,12}p_t + a_{0,13}p_{c,t} + a_{0,15}nbr_t + a_{0,16}tr_t + lags + c_1 + \varepsilon_{t,1} \qquad (10)$$

where r_t is the US federal funds rate, y_t is the real GDP, p_t is the GDP deflator, $p_{c,t}$ is the commodity price index, nbr_t denotes nonborrowed reserves and tr_t total reserves. In particular, in their baseline specification Arias et al. (2015) imposed the following restrictions: $a_{0,14} > 0$, $a_{0,11} \geq 0$, $a_{0,12} \geq 0$, $a_{0,15} = 0$ and $a_{0,16} = 0$. As a result they identified the structural shock connected with this equation as the monetary policy shock. Figure 1 presents IRF's to this shock[8].

[8] We used Uhlig's monthly data set available at https://estima.com/procs_perl/uhligjme2005.zip. The data we used to produce Figure 1 span 1965:01–2003:12. All computations in the paper are on the basis of 10.000 draws from the posterior. For monthly data we always set the number of lags in SVAR to 12.

Figure 1: The baseline specification in Arias et al. (2015). The data span is 1965:01–2003:12. Green line denotes the median response, and two dot-dashed lines restrict the area of 68% posterior error bands (pointwise).



Although our approaches slightly differ along many dimensions we roughly got the same picture as in Arias et al. (2015)[9]. The most important difference is related to IRF of real GDP. In our case the negative response of real GDP to the contractionary monetary policy shock is obtained with more confidence. In addition, according to our median results, the maximum response suggests lowering real GDP by about 0.35 percent, whereas Arias et al. (2015) estimated this affect as 0.2 percent. Moreover, it seems that our 68% error bands are a little bit wider than those given in Arias et al. (2015).

Although the rationale given by Arias et al. (2015) for imposing the restrictions on “monetary policy” equation seem to be compelling, one intriguing aspect of this methodology is worth mentioning. The IRF shape of real GDP does not resemble analogous IRF’s obtained in large number of works from 90’s that try to reach the consensus.

In the development of their arguments Arias et al. (2015) put much emphasize on probabilities of negative coefficients on real GDP and GDP deflator in “monetary policy” or “monetary feedback rule” equation i.e. $a_{0,11}$ and $a_{0,12}$, respectively. Large probability that $a_{0,11} < 0$ and $a_{0,12} < 0$ violates the systematic part of monetary policy

[9] We used slightly different prior, we included a constant in SVAR, the dataset is shorter by 4 years, we did not adopt likelihood preserving normalization of Waggoner and Zha (2003b) and the marginal posterior for the reduced form parameters is not of Inverted Wishart-Normal type.

behavior. Accordingly if sign and/or zero restrictions imply such probabilities we do not identify the monetary policy shock at all. As demonstrated by Arias et al. (2015) what drives the shape of IRF of real GDP is the restriction $a_{0,11} \geq 0$. The following equation presents median estimates (together with 68% credible interval) of the monetary feedback rule using Arias et al. (2015) baseline restrictions

$$r_t = \underset{(0.33,4.68)}{1.25} y_t + \underset{(0.87,13.13)}{3.35} p_t - \underset{(-0.38,0.24)}{0.04} p_{c,t} + lags + c_1 + \varepsilon_{t,1} \tag{11}$$

It is clear that imposing sign restrictions $a_{0,11} \geq 0$ and $a_{0,12} \geq 0$ implies that the set of possible monetary policy equations includes those highly responsive to the present state of the economy, see in particular the large upper bound in 68% credible interval for coefficient on the GDP deflator. Given that in fact the monetary feedback rule induced by SVAR contains many lags of prices and real GDP one may be skeptical whether such responsive feedback rules are consistent with the common sense. That is it may be that the set of monetary policy rules consistent with the restrictions contains some implausible models of monetary policy behavior. In addition, the median estimate of the coefficient on commodity prices is negative and 68% credible interval contains both negative and positive values. To some extent this undermines the well thought and successful route to include the commodity prices in the monetary policy function in order to avoid the price puzzle.

We asked ourselves the question whether the negative response of GDP to the contractionary monetary policy shock could be obtained 1) without explicit imposing sign restrictions for this IRF (as in Uhlig (2005)), 2) avoiding clever restrictions on monetary policy equation proposed by Arias et al. (2015), 3) without standard and (commonly criticized) restriction that monetary policy shock could not influence GDP and prices on impact. Using the whole data set from Uhlig (2005) (up to and including the year 2003) we could not produce such a response. But with data ending in the mid 90's, this was quite easy. In fact when Uhlig (2005) applied standard Choleski decomposition to the data up to year 2003 he obtained "price puzzle", which he commented as "*It may well be that the additional decade of data since 1992 has made this route* [i.e. introducing commodity prices] *to resolving the price puzzle more difficult*". Since the data set was prepared for the problem from the perspective of mid 90's we think that it is fair to play with the data constrained by that time. The clue how it may be accomplished was given by Uhlig (2005), since he wrote that "*the identification of additional shocks can help in principle, as orthogonality between the shocks provides an additional restriction for identifying the monetary policy shock*". Hence in addition to monetary policy shock we thought seriously about minimal zero restrictions for additional three shocks. Two of them may be identified as production shocks and the last one the information shock. Table 1 presents zero restrictions used by us[10]. The entries in this object may be read as elements of contemporaneous matrix A_0.

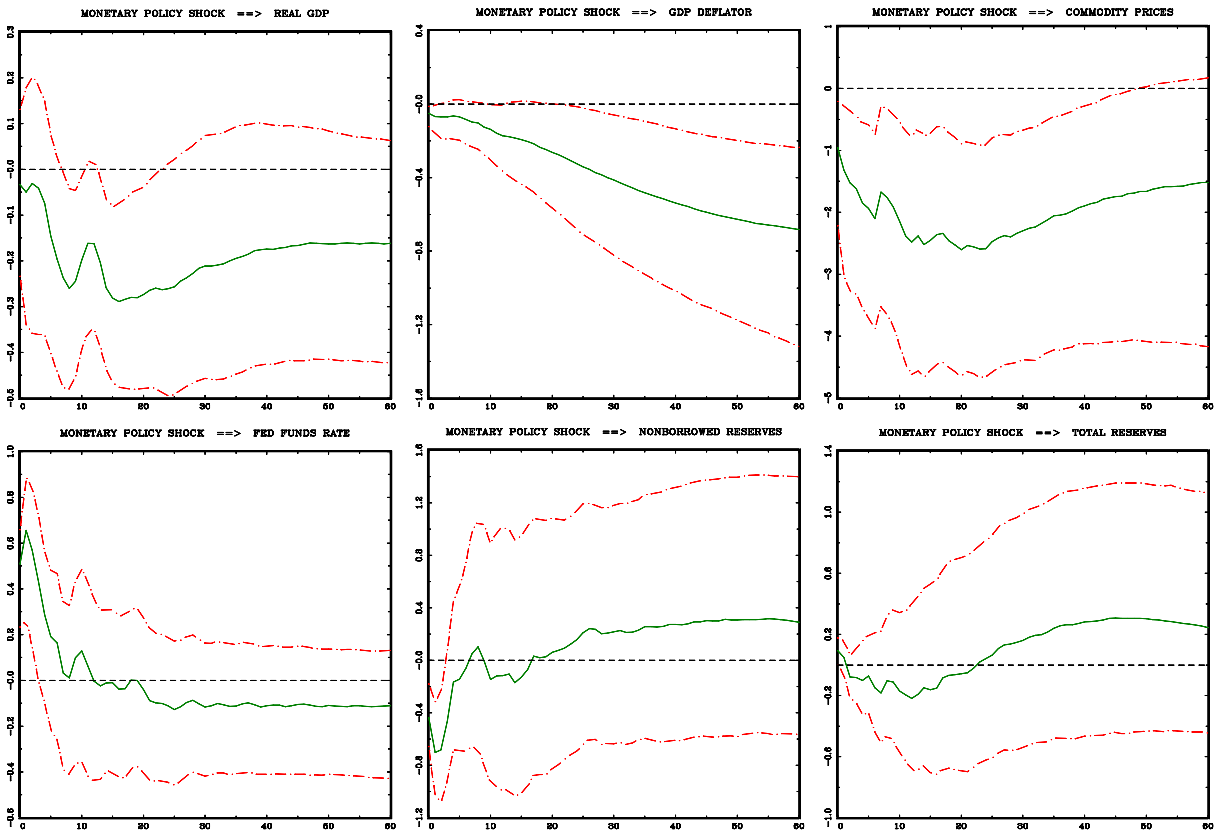
Table 1: Zero restrictions imposed on A_0

	y_t	p_t	$p_{c,t}$	r_t	nbr_t	tr_t
Prod	X	X	X	0	0	0
Prod	X	X	X	0	0	0
Inf	X	X	X	X	0	0
MP	X	X	X	X	0	0
Fin	X	X	X	X	X	X
Fin	X	X	X	X	X	X

"X" denotes unrestricted elements in A_0 and "0" those restricted to zero. The first column contains the reference names for the equation in SVAR. "Prod" refers to production sector, "Inf" refers to informational equation and "MP" is the monetary policy equation ("Fin" refers to financial sector but these equations play no role in our analysis). The main assumption underlying Table 1 is that nonpolicy variables do not respond contemporaneously to the policy variables (though instantaneous responses of these variables to the monetary policy shock are not restricted to zero). Clearly what makes the difference is the (3,4) element in A_0 i.e. unrestricted coefficient in the informational equation on the federal funds rate. The rationale for this is that commodity price world market should respond immediately to the main indicators of monetary policy in a very large economy like the US. In addition to the "zeros" induced by Table 1, we imposed Uhlig's sign restrictions but confined only to the instantaneous response $\Psi_0 = A_0^{-1}$. Specifically, responses of GDP deflator, commodity price index, nonborrowed reserves are nonpositive, and those of federal funds rate nonnegative on impact. Hence we substantially weakened sign restrictions used by Uhlig (2005), who imposed them for horizons from 0 to 5 months. Using dataset spanning 1965:01–1995:12, we obtained IRF's to the monetary policy shock identified by zero restrictions summarized in Table 1 and the sign restrictions confined to the impact responses, that are shown in Figure 2.

[10] It is easy to realize that SVAR under consideration is $R_k -$ restricted.

Figure 2: Monetary policy shock identified by zero restrictions summarized in Table 1 and the sign restrictions confined to the impact responses. The data span is 1965:01–1995:12. Green line denotes the median response, and two dot-dashed lines restrict the area of 68% posterior error bands (pointwise).



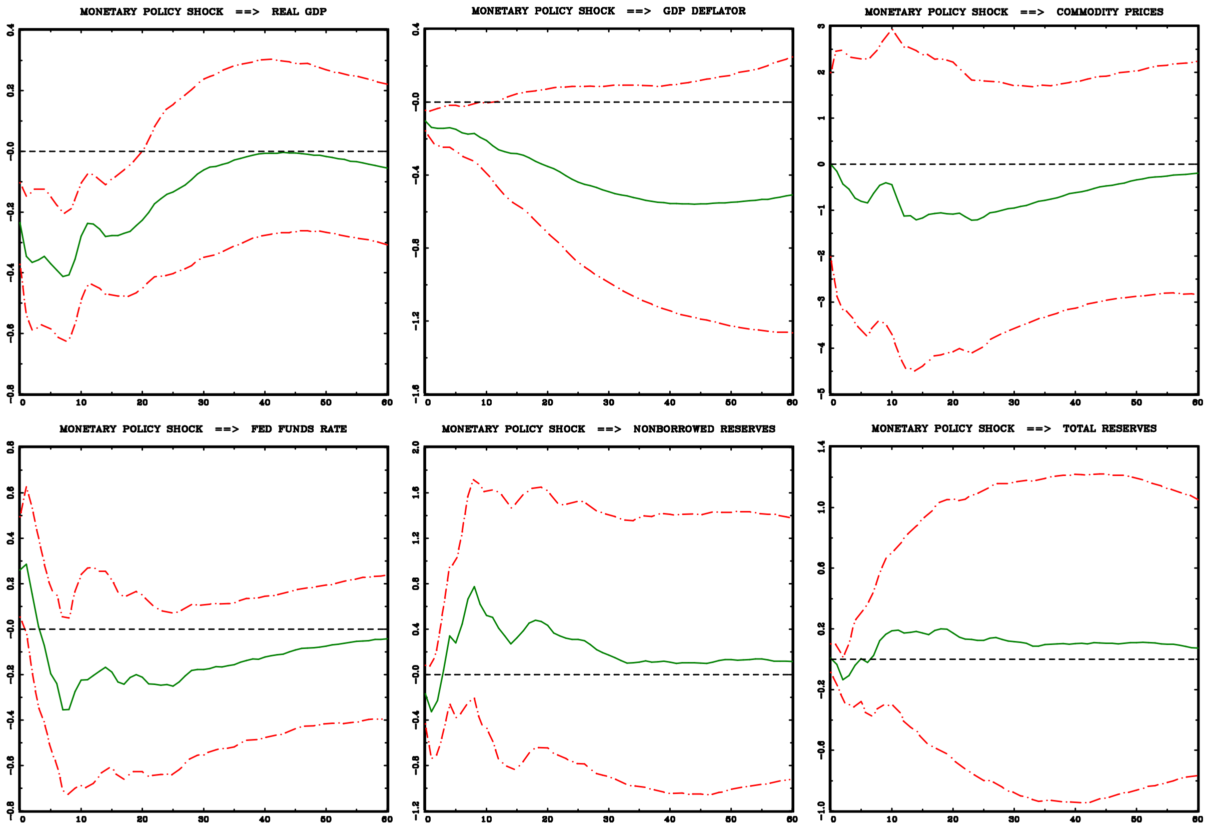
Now we do not observe the output puzzle. The one standard deviation monetary policy shock leads to the significant decrease in the real GDP by about 0,3% after one a half year. Overall all the remaining plots look more reasonable in comparison with those presented in Figure 1. Both GDP deflator and commodity price index respond negatively in a significant way for all horizons. Responses of federal funds rate are more sharp. As a consequence, the liquidity effect in responses of nonborrowed reserves is more visible. The analogous equation to (11) reads now

$$r_t = \underset{(-0.41,0.65)}{0.14}\, y_t + \underset{(0.008,2.72)}{0.81}\, p_t + \underset{(0.01,0.15)}{0.05}\, p_{c,t} + lags + c_1 + \varepsilon_{t,1} \tag{12}$$

Note that although we did not impose any sign restrictions on monetary policy equations, coefficients on prices and commodity prices are sharply constrained, i.e. the zero is outside the 68% error bands, and both coefficients admits the expected signs. The only problem is with coefficient on real GDP. Our import of this is that the assumed restrictions favor models that have moderate positive coefficients on the most recent real GDP. This is not irrational since the accumulated effect of coefficients on the lagged real GDP should be also taken into account. If this is the case then the non-negligible negative support of the underlying marginal posterior for the coefficient on the most recent real GDP may be accepted.

To be fair, we should repeat the exercise from Arias et al. (2015) for the dataset truncated at 1995:12. These results are presented in Figure 3

Figure 3: The baseline specification in Arias et al. (2015). The data span is 1965:01–1995:12. Green line denotes the median response, and two dot-dashed lines restrict the area of 68% posterior error bands (pointwise).



The monetary feedback rule is reestimated and looks as follows

$$r_t = \underset{(0.32,4.62)}{1.20}\, y_t + \underset{(0.71,10.75)}{2.92}\, p_t - \underset{(-0.45,0.23)}{0.06}\, p_{c,t} + lags + c_1 + \varepsilon_{t,1} \tag{13}$$

Coefficients are pretty much the same as those for the monetary policy equation estimated for the whole sample. In particular we still observe quite responsive feedback rule and the coefficient on the commodity price index is ambiguous. Hence even if inclusion of commodity prices was found essential in the literature from 90's (using the same dataset), the sign restrictions proposed by Arias et al. (2015) suggest something different. That is we could do without commodity prices to come up with reasonable IRF's. Finally we note that although our impulse responses presented in Figure 2 are quite different to those presented in Figure 3 (so as the estimated feedback rules), the forecast error variance decompositions (FEVD) in two models are remarkably similar[11].

IX. OPTIMISM SHOCKS

Beaudry et al. (2014) used zero and sign restrictions to identify so-called optimism shocks. They obtain quite sharp results in terms of IRF's and FEVD's, which were criticized by Arias et al. (2014). For example, the latter authors, using their methodology, claim that contribution of optimism shocks to FEVD's of many variables for whatever horizon was overestimated by Beaudry et al. (2014) by two or even three times. We decided to check the robustness of these results with respect to our methodology.

The benchmark model of Beaudry et al. (2014) contains seven variables: Total Factor Productivity (TFP), stock price index, consumption, the real interest rate, hours worked, investment and output. The variables were logged (but the real interest rate) and taken in levels. For obvious reasons we use the same dataset and adopt the same model specification (i.e. four lags) as in Beaudry et al. (2014)[12]. They considered three basic identification schemes. All of them amounted to putting one zero and several sign restrictions, but were confined to $\Psi_0 = A_0^{-1}$ only. They called them identification I, II and III. In all these identifications the optimism shock was assumed to have zero impact on TFP (in horizon "0"). In addition, in identification I, stock prices rise in response to optimism shock on impact, in identification II: stock prices and consumption rise in response to optimism shock on impact, and in identification III: stock prices, consumption and real interest rate increase in response to optimism shock on impact. Figures 4, 5, 6 present IRF's to optimism shock adopting identification I, II and III. They are pretty much similar to those presented in Arias et al. (2014), and strikingly different in terms of IRF's uncertainty to those showed in Beaudry et al. (2014). The only divergence between our results and those in Arias et al. (2014) is that in our case Identification III delivers slightly different results. Specifically, IRF's of TFP, stock prices and the real interest rate slightly differ. For example TFP goes down significantly after about a year, but except this short period, this shock is statistically insignificant. In contrast, in Arias et al. (2014) this shock becomes statistically significant after about 30 quarters. In Arias et al. (2014) stock prices response is statistically significant for all periods, whereas using our approach, this

[11] These are available from the author upon request.

[12] The author is extremely grateful to Jian Wang for making available the dataset used in Beaudry et al. (2014). The quarterly data span is 1955:1–2012:4. For detailed information about the sources and construction of this dataset we refer to Beaudry et al. (2014).

shock becomes statistically insignificant after 16 quarters. Lastly, the real interest rate is more responsive up to 8 quarters than it is the case in Arias et al. (2014).

Figure 4: Identification I. Green line denotes the median response, and two dot-dashed lines restrict the area of 68% posterior error bands (pointwise).

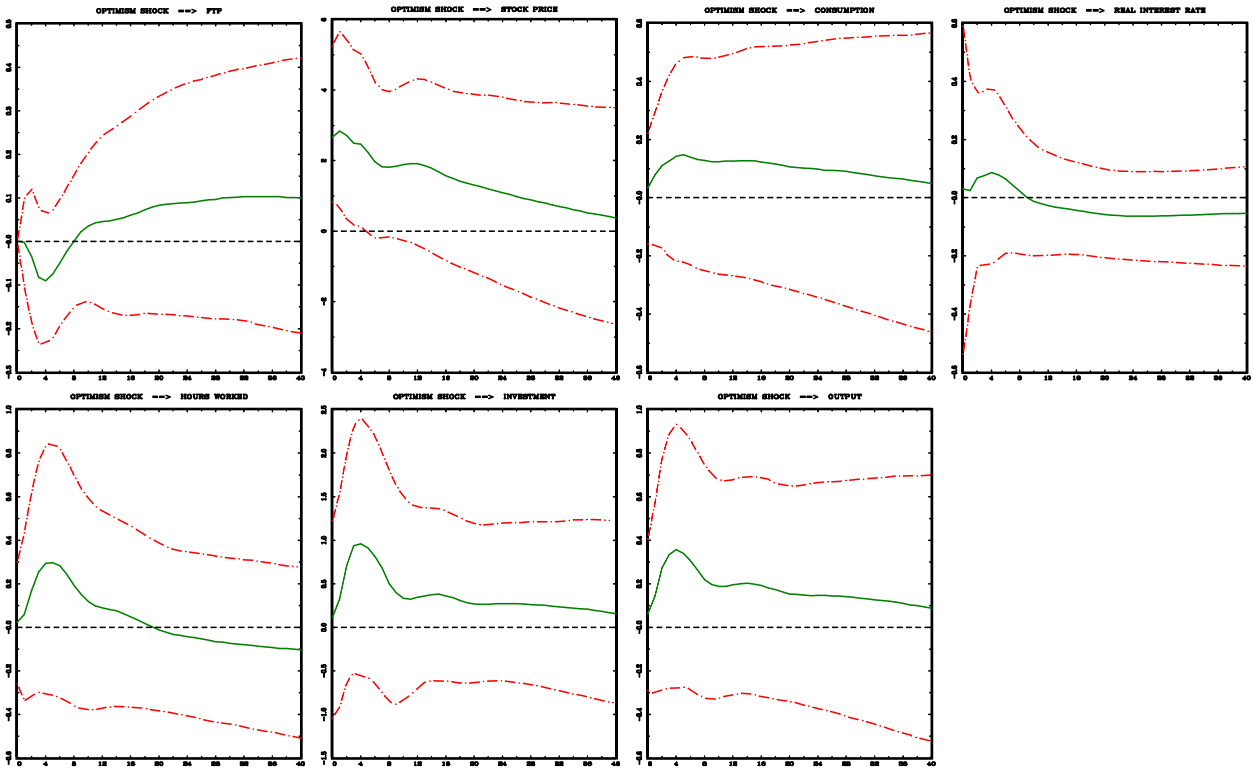


Figure 5: Identification II. Green line denotes the median response, and two dot-dashed lines restrict the area of 68% posterior error bands (pointwise).

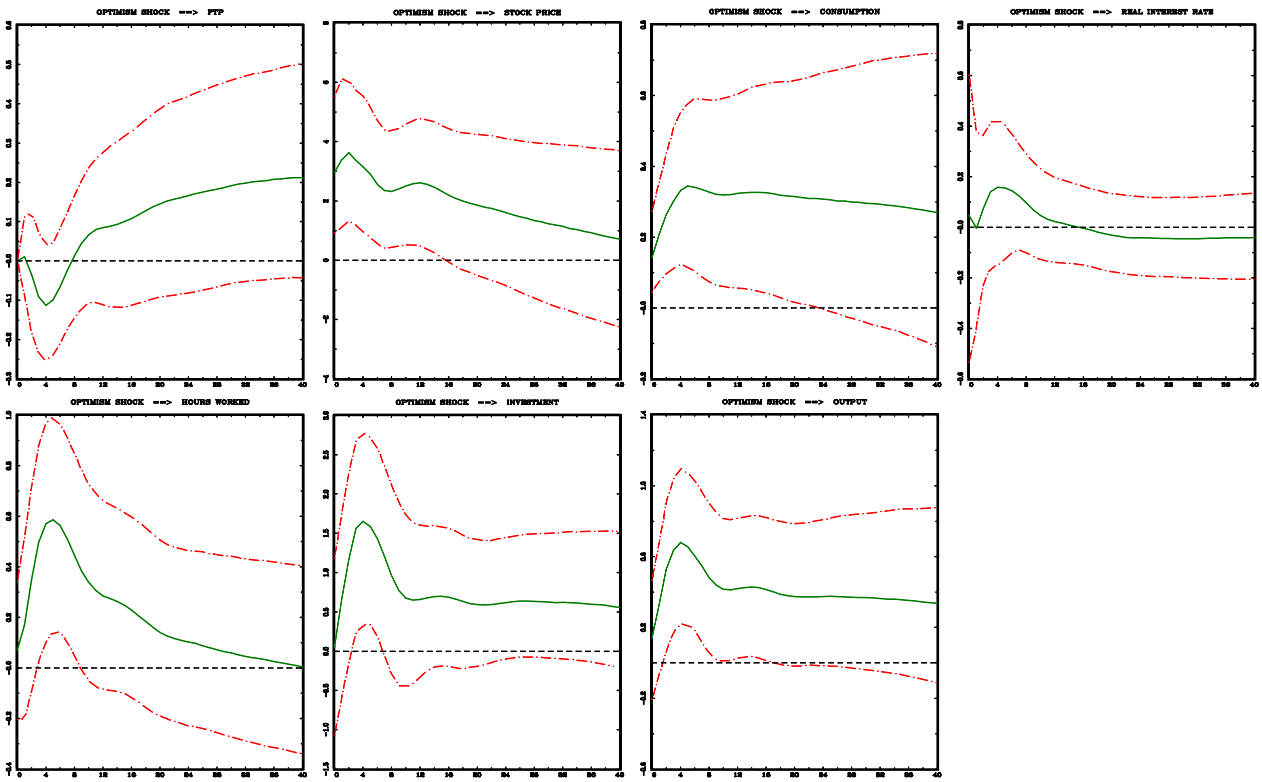


Figure 6: Identification III. Green line denotes the median response, and two dot-dashed lines restrict the area of 68% posterior error bands (pointwise).

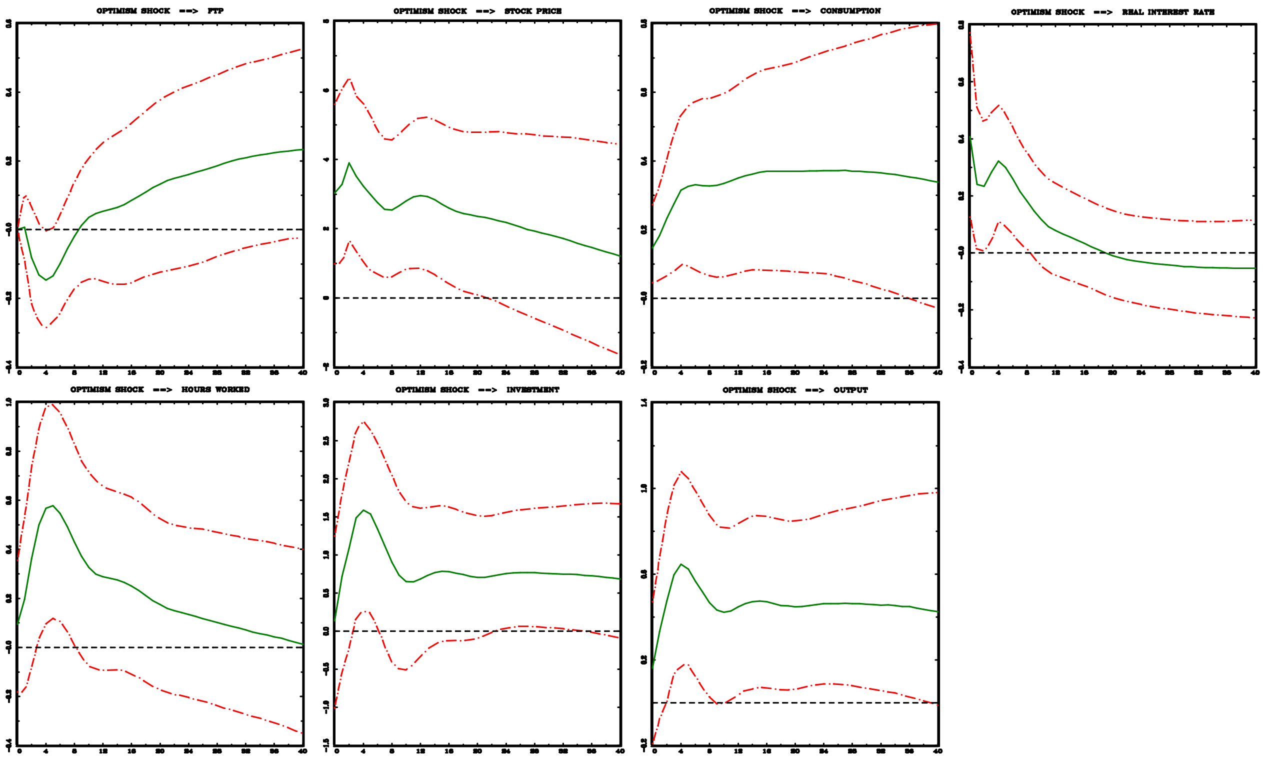


Table 2 contains median estimates of FEVD due to optimism shock (together with 68% credible set). Again, they are very similar to those given in Arias et al. (2014) and orthogonal to those presented in Beaudry et al. (2014). Slight differences between us and Arias et al. (2014) concern the long-run (i.e. h = 40) FEVD of stock prices, which echoes the results from IRF's.

Table 2: FEVD due to optimism shocks. Our methodology. Values in brackets show 68% posterior credible set.

	Identification I		Identification II		Identification III	
	h = 4 qtrs.	h = 40 qtrs.	h = 4 qtrs.	h = 40 qtrs.	h = 4 qtrs.	h = 40 qtrs.
TFP	0.02 [0.004, 0.05]	0.07 [0.02, 0.2]	0.02 [0.005, 0.05]	0.08 [0.02, 0.23]	0.02 [0.005, 0.06]	0.08 [0.02, 0.23]
stock prices	0.1 [0.02, 0.34]	0.09 [0.03, 0.25]	0.14 [0.03, 0.4]	0.12 [0.04, 0.28]	0.15 [0.03, 041]	0.13 [0.04, 0.31]
consumption	0.09 [0.01, 0.34]	0.1 [0.02, 032]	0.15 [0.03, 0.42]	0.12 [0.03, 0.36]	0.12 [0.02, 0.38]	0.15 [0.3, 0.39]
interest rate	0.1 [0.03, 0.3]	0.13 [0.05, 0.26]	0.1 [0.03, 0.3]	0.12 [0.05, 0.25]	0.12 [0.03, 0.33]	0.14 [0.06, 0.28]
hours worked	0.1 [0.02, 0.33]	0.11 [0.04, 0.26]	0.13 [0.03, 0.4]	0.13 [0.04, 0.29]	0.13 [0.03, 0.41]	0.12 [0.04, 0.29]
investment	0.1 [0.03, 0.29]	0.12 [0.04, 0.27]	0.13 [0.04, 0.35]	0.15 [0.06, 0.33]	0.12 [0.04, 0.35]	0.16 [0.06, 0.33]
output	0.09 [0.02, 0.31]	0.11 [0.03, 0.28]	0.15 [0.03, 0.39]	0.15 [0.05, 0.35]	0.13 [0.03, 0.37]	0.16 [0.05, 0.37]

We note quite striking uniformity of median estimates of FEVD for all variables (except TFP). One may say that each FEVD is about $1/7 \approx 0.14$. We interpret this as a complete lack of identification. Further identifying (zero and/or sign) restrictions are probably needed to obtain economically significant results.

X. CONCLUSIONS

The paper presents new methodology to deal with partially identified SVAR models under zero and/or sign restrictions. Our methodology is similar to that presented in Arias et al. (2014), however we differ in some details. We applied our methodology to challenge the output puzzle found in Uhlig (2005). Staying a priori agnostic about responses of output to monetary policy shock, we showed that it is not necessary to adopt sign restrictions on the systematic monetary policy equation proposed by Arias et al. (2015), to obtain significant real output drop as a result of contractionary monetary policy shock. In the second exercise, we largely confirm conclusions from Arias et al. (2014) concerning the results in Beaudry et al. (2014). Specifically the uncertainty in IRF's given by Beaudry et al. (2014) is highly underestimated, and the estimates of FEVD presented in Beaudry et al. (2014) should be divided by two, three or even four to be consistent with ours.

APPENDIX:

The proof of proposition 1:

Using lemma 9.5.3 in Muirhead (1982) and general results from James (1954) (in particular section 5) one has

$$\int_{R_1}(Q'dQ)=\int_{q_1'q_1=1;Z_1f(\Pi,L)q_1=0}\prod_{j=2}^{n}q_j'dq_1\int_{O(n-1)}(Q_*'dQ_*)=\int_{q_1'q_1=1;Z_1f(\Pi,L)q_1=0}dq_1\int_{O(n-1)}(Q_*'dQ_*)=$$

$$=2^{n-1}\pi^{\frac{1}{2}(n-1)^2}[\Gamma_{n-1}(\tfrac{n-1}{2})]^{-1}\int_{q_1'q_1=1;Z_1f(\Pi,L)q_1=0}dq_1$$

where $Q_*\in O(n-1)$. In order to evaluate the last integral, make the transformation $\begin{bmatrix}Z_1f(\Pi,L)\\ G\end{bmatrix}q_1=\begin{bmatrix}v_1\\ v_2\end{bmatrix}$, where $G:(n-z_1)\times n$ is any fixed (full row rank) matrix such that $GG'=\mathrm{I}_{n-z_1}$ and $G(Z_1f(\Pi,L))'=0$. Since $J(q_1\to v_1,v_2)=|\,Z_1f(\Pi,L)f(\Pi,L)'Z_1'\,|^{-\frac{1}{2}}$ and noting that $q_1'q_1=v_1'(Z_1f(\Pi,L)f(\Pi,L)'Z_1')^{-1}v_1+v_2'v_2$, we have

$$\int_{q_1'q_1=1}dq_1=J(q_1\to v_1,v_2)\int_{v_1'(Z_1f(\Pi,L)f(\Pi,L)'Z_1')^{-1}v_1+v_2'v_2=1}dv_1dv_2$$

Thus we can evaluate our original integral putting $v_1=0$ (and omitting dv_1). Hence

$$\int_{q_1'q_1=1;Z_1f(\Pi,L)q_1=0}dq_1=|\,Z_1f(\Pi,L)f(\Pi,L)'Z_1'\,|^{-\frac{1}{2}}\int_{v_2'v_2=1}dv_2$$

$$=2\pi^{\frac{1}{2}(n-z_1)}[\Gamma(\tfrac{n-z_1}{2})]^{-1}\,|\,Z_1f(\Pi,L)f(\Pi,L)'Z_1'\,|^{-\frac{1}{2}}$$

Ultimately

$$C_r=\int_{R_1}(Q'dQ)=2^n\pi^{\frac{1}{2}((n-1)^2+n-z_1)}[\Gamma_{n-1}(\tfrac{n-1}{2})]^{-1}[\Gamma(\tfrac{n-z_1}{2})]^{-1}\,|\,Z_1f(\Pi,L)f(\Pi,L)'Z_1'\,|^{-\frac{1}{2}}$$

Explaining the algorithms 4 and 1:

It turns out that the problem to propose the efficient algorithm to sample from the posterior subject to zero restrictions boils down to the evaluation of the integral $\int_{Q'Q=\mathrm{I}_n,Z_if(\Pi,L)q_i=0;i=1,\ldots,k<n}dQ$. Suppose that the first k columns of Q are subject to linear restrictions. Consider the transformation

$$\begin{bmatrix}Z_if(\Pi,L)\\ Q_{i-1}'\\ G_i'\end{bmatrix}q_i=\begin{bmatrix}\alpha_i\\ \beta_i\\ x_i\end{bmatrix}\text{, for each } i=1,\ldots,k \qquad \text{(A1)}$$

where $\alpha_i:(z_i\times 1)$, $\beta_i:(i-1)\times 1$ and $x_i:(n-z_i-i+1)\times 1$ are "new" variables. Note that in case $i=1$, β_1 is empty. Further, $Q_{i-1}\equiv[q_1\ldots q_{i-1}]$, with the convention that Q_0 is empty and $G_i:n\times(n-z_i-i+1)$ is any fixed (full column rank) matrix such that $G_i'G_i=\mathrm{I}_{n-z_i-i+1}$ and $\begin{bmatrix}Z_if(\Pi,L)\\ Q_{i-1}'\end{bmatrix}G_i=0$. We assume that $\begin{bmatrix}Z_if(\Pi,L)\\ Q_{i-1}'\\ G_i'\end{bmatrix}$ is nonsingular. We need the Jacobian underlying the transformation (A1). Due to recursiveness of the transformation one has

$J(Q_k \to \alpha_i, \beta_i, x_i; i = 1,\ldots,k) = \prod_{i=1}^{k} J(q_i \to \alpha_i, \beta_i, x_i)$, so that

$$J(Q_k \to \alpha_i, \beta_i, x_i; i = 1,\ldots,k) = \prod_{i=1}^{k} \left| \begin{pmatrix} Z_i f(\Pi, L) \\ Q'_{i-1} \end{pmatrix} \begin{pmatrix} Z_i f(\Pi, L) \\ Q'_{i-1} \end{pmatrix}' \right|^{-\frac{1}{2}}$$

Note that Q_{i-1} is the implicit function of the "new" variables. To be specific

$$q_1 = \begin{bmatrix} Z_1 f(\Pi, L) \\ G'_1 \end{bmatrix}^{-1} \begin{bmatrix} \alpha_1 \\ x_1 \end{bmatrix} = [* \vdots G_1] \begin{bmatrix} \alpha_1 \\ x_1 \end{bmatrix}$$

$$q_2 = \begin{bmatrix} Z_2 f(\Pi, L) \\ q'_1 \\ G'_2 \end{bmatrix}^{-1} \begin{bmatrix} \alpha_2 \\ \beta_2 \\ x_2 \end{bmatrix} = [* \vdots * \vdots G_2] \begin{bmatrix} \alpha_2 \\ \beta_2 \\ x_2 \end{bmatrix} \text{ and so on}$$

What the above formulas state is that the last $n - z_i - i + 1$ columns in the inverse of $\Lambda = \begin{bmatrix} Z_i f(\Pi, L) \\ Q'_{i-1} \\ G'_i \end{bmatrix}$ are necessarily equal to G_i, which will be of great importance in the sequel. The proof of this assertion is as follows. We must prove that $\Lambda X = \left[0 \vdots 0 \vdots \mathrm{I}_{n-z_i-i+1}\right]' \Rightarrow X = G_i$. By assumption, the system of equations in X is consistent (put $X = G_i$). Provided that Λ is nonsingular the system has one and only one solution.

The purpose of using the transformation (A1) should be evident. We use the above transformation because the restrictions explicitly show up among the "new" variables. So 1) $Z_i f(\Pi, L) q_i = 0; i = 1,\ldots,k \Leftrightarrow \alpha_i = 0; i = 1,\ldots,k$, 2) $Q'Q = \mathrm{I}_n \Leftrightarrow \beta_i = 0$ and $x'_i x_i = 1; i = 1,\ldots,k$ (the last one since $\alpha_i = 0, \beta_i = 0$ hence $q'_i q_i = x'_i x_i = 1$). Now we are in position to evaluate the original integral. We have

$$\int_{Q'Q=\mathrm{I}_n, Z_i f(\Pi,L) q_i = 0; i=1,\ldots,k} dQ \propto \int_{Q'_k Q_k = \mathrm{I}_k, Z_i f(\Pi,L) q_i = 0; i=1,\ldots,k} dQ_k =$$

$$= \int_{x'_i x_i = 1; i=1,\ldots,k} J(Q_k \to \alpha_i = 0, \beta_i = 0, x_i; i = 1,\ldots,k)\, dx_1 \ldots dx_k$$

where $J(Q_k \to \alpha_i = 0, \beta_i = 0, x_i; i = 1,\ldots,k)$ means the original Jacobian evaluated at $\alpha_i = 0, \beta_i = 0; i = 1,\ldots,k$ i.e.

$$J(Q_k \to \alpha_i = 0, \beta_i = 0, x_i; i = 1,\ldots,k) = \prod_{i=1}^{k} \left| \begin{pmatrix} Z_i f(\Pi, L) \\ Q'_{i-1} \end{pmatrix} \begin{pmatrix} Z_i f(\Pi, L) \\ Q'_{i-1} \end{pmatrix}' \right|^{-\frac{1}{2}} \quad \text{(A2)}$$

where $Q_{i-1} = [G_1 x_1 \vdots G_2 x_2 \vdots \ldots \vdots G_{i-1} x_{i-1}]$.

Note that the Jacobian evaluated at $\alpha_i = 0, \beta_i = 0$; $i = 1,...,k$, is the kernel of the conditional posterior of $(x_1,...,x_k)$ given the reduced form parameters and the restrictions. Unfortunately (as emphasized in the main text) we cannot provide the integrating constant of this density in the general case (which is equivalent to obtaining the marginal posterior for reduced form parameters given the restrictions). This would amount to analytical integration of the Jacobian with respect to $(x_1,...,x_k)$ with the support constrained so as $x_i'x_i = 1; i = 1,...,k$. However the fact that we derived conditional posterior of $(x_1,...,x_k)$ given the reduced form parameters and the restrictions will allow for sampling from the joint posterior of the reduced form and orthogonal matrices subject to exact linear restrictions. It is so because there is a 1-1 correspondence between $(x_1,...,x_k)$ and Q_k subject to zero and orthogonality restrictions (see the transformation (A1)), and having $(x_1,...,x_k)$ we get $Q_k = [G_1x_1 \vdots G_2x_2 \vdots \ldots \vdots G_kx_k]$.

Justification of algorithm 1 is as follows. If SVAR is R_k −restricted then $J(Q_k \to \alpha_i, \beta_i, x_i; i = 1,...,k) = \prod_{i=1}^{k} | Z_i f(\Pi, L) f(\Pi, L)' Z_i' |^{-\frac{1}{2}}$. Denote this Jacobian as J. Then changing variables in the unrestricted posterior (4) we arrive at the joint posterior $J \cdot p_{ur}(L, \Pi, \alpha_i, \beta_i, x_i; i = 1,...,k, q_{k+1}, \ldots, q_n \mid y)$. We are interested in decomposing the latter posterior into $p(q_{k+1}, \ldots, q_n \mid L, \Pi, \alpha_i, \beta_i, x_i; i = 1,...,k, y)$, $p(x_i; i = 1,...,k \mid L, \Pi, \alpha_i, \beta_i; i = 1,...,k, y)$ and $p(L, \Pi \mid \alpha_i, \beta_i; i = 1,...,k, y)$. If we manage to do that then putting $\alpha_i = 0, \beta_i = 0; i = 1,...,k$ we arrive at the decomposition of the restricted posterior. We note that $p(q_{k+1}, \ldots, q_n \mid L, \Pi, \alpha_i, \beta_i, x_i; i = 1,...,k, y)$ may be obtained using lemma 9.5.3 in Muirhead (1982). In particular its integrating constant does not depend on $L, \Pi, \alpha_i, \beta_i, x_i; i = 1,...,k$, hence w.l.o.g. we may set $\alpha_i = 0, \beta_i = 0$. Further $p(x_i; i = 1,...,k \mid L, \Pi, \alpha_i, \beta_i; i = 1,...,k, y) = \prod_{i=1}^{k} p(x_i \mid L, \Pi, \alpha_i, \beta_i, y)$, where $p(x_1 \mid L, \Pi, \alpha_1, \beta_1, y)$ is the uniform distribution on the sphere in $\mathbb{R}^{n-z_1}$ of radius $(1 - \alpha_1'(Z_1 f(\Pi, L) f(\Pi, L)' Z_1')^{-1} \alpha_1)^{\frac{1}{2}}$ and each $p(x_i \mid L, \Pi, \alpha_i, \beta_i, y)$ for $i = 2,...,k$, is the uniform distribution on the sphere in $\mathbb{R}^{n-z_i+1-i}$ of radius $(1 - \alpha_i'(Z_i f(\Pi, L) f(\Pi, L)' Z_i')^{-1} \alpha_i - \beta_i'\beta_i)^{\frac{1}{2}}$ (we used the fact that SVAR is R_k −restricted). Hence setting $\alpha_i = 0, \beta_i = 0; i = 1,...,k$, each $p(x_i \mid L, \Pi, \alpha_i = 0, \beta_i = 0, y)$ is the uniform distribution on the unit sphere in $\mathbb{R}^{n-z_i+1-i}$. Having $(x_1,...,x_k)$ and putting $\alpha_i = 0, \beta_i = 0; i = 1,...,k$ we get $q_i = G_i x_i$ for $i = 1,...,k$, by the 1-1 correspondence (A1). Integrating out $J \cdot p_{ur}(L, \Pi, \alpha_i, \beta_i, x_i; i = 1,...,k \mid y)$ with respect to $x_i; i = 1,...,k$, it may be shown that the joint posterior

$p(L,\Pi,\alpha_i,\beta_i; i=1,\dots,k \mid y) \propto J \cdot p_{ur}(\Pi \mid L,y) p_{ur}(L \mid y) \cdot (1-\alpha_1'(Z_1 f(\Pi,L) f(\Pi,L)' Z_1')^{-1}\alpha_1)^{\frac{1}{2}(n-z_1)-1} \cdot$
$\cdot \prod_{i=2}^{k} (1-\alpha_i'(Z_i f(\Pi,L) f(\Pi,L)' Z_i')^{-1}\alpha_i - \beta_i'\beta_i)^{\frac{1}{2}(n-z_i+1-i)-1}$
where the support is constrained so as $1 > \alpha_i'(Z_i f(\Pi,L) f(\Pi,L)' Z_i')^{-1}\alpha_i + \beta_i'\beta_i$ and $1 > \alpha_1'(Z_1 f(\Pi,L) f(\Pi,L)' Z_1')^{-1}\alpha_1$.
Since $p(L,\Pi \mid \alpha_i = 0, \beta_i = 0; i=1,\dots,k \mid y) \propto p(L,\Pi,\alpha_i = 0,\beta_i = 0; i=1,\dots,k \mid y)$ we get $p(L,\Pi \mid \alpha_i = 0,\beta_i = 0; i=1,\dots,k \mid y) \propto J \cdot p_{ur}(\Pi \mid L,y) p_{ur}(L \mid y)$.

The proof of proposition 3:

Recall that $\mathcal{C}_i \cap \mathcal{S}^{n-1} \neq \varnothing$ iff $\mathcal{C}_i \neq \{0\}$. Hence we must show that $\mathcal{C}_i$ contains at least one nonzero point. Suppose $s_i \geq 1$. Under the hypothesis that Γ_i is of full row rank, by Motzkin's theorem (see e.g. Mangasarian (1994), p. 29), $Z_i f(\Pi,L)x = 0$, $Q_{i-1}'x = 0$, $S_i f(\Pi,L)x > 0$, possesses a solution, which is nonzero solution (because $S_i f(\Pi,L)x > 0$). This solution must be also the solution of $\Gamma_i x \geq 0$. Hence $\mathcal{C}_i \neq \{0\}$, i.e. $\mathcal{C}_i \cap \mathcal{S}^{n-1}$ is non-empty. On the other hand suppose $s_i = 0$. Then $\mathcal{C}_i = \{x \in \mathbb{R}^n \mid \Gamma_i x = 0\}$. By assumption, $z_i \leq n-i$, so $z_i + i - 1 \leq n - i + i - 1 < n$. Thus $\operatorname{rank}(\Gamma_i) < n$. It follows that $\mathcal{C}_i$ contains at least one nonzero point i.e. $\mathcal{C}_i \cap \mathcal{S}^{n-1}$ is nonempty.

The proof of proposition 4:

In what follows if $x \in \mathbb{R}^n$ (a column vector) is fixed, let us define a line as $(x) := \{\lambda x \mid \lambda \in \mathbb{R}\}$ i.e. a line through 0 and x, and a ray as $(x)^+ := \{\lambda x \mid \lambda \geq 0\}$ i.e. a ray through 0 in the direction of x. The proof will be more transparent when we exploit the following properties of the polyhedral convex cone (to be called the cone). Each cone $\mathcal{C}_i = \{x \in \mathbb{R}^n \mid \Gamma_i x \geq 0\}$ can be decomposed as follows $\mathcal{C}_i = V + \mathcal{C}_i^*$, see e.g. Schrijver p. 100, where V is the largest linear subspace contained in $\mathcal{C}_i$ (so-called lineality space), $\mathcal{C}_i^*$ is a pointed cone[13] and $V + \mathcal{C}_i^* = \{x + y \mid x \in V, y \in \mathcal{C}_i^*\}$. Moreover let $\dim(\mathcal{C}_i)$ be the dimension of the smallest subspace that contains $\mathcal{C}_i$. Then $\dim(\mathcal{C}_i) = \dim(V) + \dim(\mathcal{C}_i^*)$. Hence the cone $\mathcal{C}_i$ itself will be pointed iff $\dim(V) = 0$ iff $\operatorname{rank}(\Gamma_i) = n$.

Suppose $s_i = 0$. Then $\mathcal{C}_i$ is a linear subspace i.e. $\mathcal{C}_i = V$ with $\dim(\mathcal{C}_i) = n - \operatorname{rank}(\Gamma_i)$.

Suppose $s_i \geq 1$. Let us denote the $j-$th row of S_i as S_i^j. An inequality in $S_i f(\Pi,L)x \geq 0$ is called an implicit equality if $S_i^j f(\Pi,L)x = 0$ for all $x \in \mathcal{C}_i$. Denote

[13] The cone is pointed if it contains no line. Geometrically, a pointed cone looks like what we would commonly call a cone.

the subsystem of implicit equalities as $S_i^{=} f(\Pi, L)x = 0$. Clearly if there exists at least one $x \in \mathcal{C}_i$ such that $S_i f(\Pi, L)x > 0$, then $S_i^{=}$ is empty. In fact under assumptions stated in proposition there is such an x (see proposition 3). Then $\dim(\mathcal{C}_i) = n - z_i - i + 1$. Moreover under assumption, $\dim(V) = n - z_i - i + 1 - s_i$. It follows that $\dim(\mathcal{C}_i^*) = s_i$.

Proof: Since assumptions are stronger than those in proposition 3, $\mathcal{C}_i \cap \mathcal{S}^{n-1}$ is not empty. All we have to do is to show that for each case specified in proposition 4, $\mathcal{C}_i \cap \mathcal{S}^{n-1}$ is path connected.

a) Suppose $s_i = 0$. Then Γ_i has $z_i + i - 1$ rows and by assumption $z_i < n - i$. But $z_i + i - 1 < n - i + i - 1 = n - 1$. Hence $\mathrm{rank}(\Gamma_i) \le n - 2$ and we have $n - (n-2) \le n - \mathrm{rank}(\Gamma_i) = \dim(\mathcal{C}_i)$. Hence $\mathcal{C}_i$ is a linear subspace with $\dim(\mathcal{C}_i) \ge 2$. Since $n - z_i - i \ge 1$, $\mathcal{C}_i \cap \mathcal{S}^{n-1}$ is $\mathcal{S}^{n-z_i-i}$, see e.g. Horn (1949). It is well known that the latter is path connected provided that $n - z_i - i \ge 1$, see e.g. Munkres (2000), p. 172.

b) Suppose $s_i \ge 1$. First assume that $\mathrm{rank}(\Gamma_i) = z_i + s_i + i - 1 = n$. Then we get $\dim(V) = 0$, hence $\mathcal{C}_i$ is a pointed cone with $\dim(\mathcal{C}_i) = s_i$. By Farkas–Minkowski–Weyl theorem, each element of a pointed cone $\mathcal{C}_i$ may be obtained as $x = \lambda_1 y_1 + \ldots + \lambda_{s_i} y_{s_i}$, where all $\lambda_i \ge 0$ and $y_i \in \mathbb{R}^n$ are fixed (i.e. $\mathcal{C}_i$ is generated by a finite combination of rays). In particular if there is only one ray i.e. $\mathcal{C}_i = (y_1)^+$, then $\mathcal{C}_i \cap \mathcal{S}^{n-1}$ is a singleton, hence trivially path connected. Otherwise, choose any two points in $\mathcal{C}_i$, $x = \lambda_1 y_1 + \ldots + \lambda_{s_i} y_{s_i}$ and $\overline{x} = \overline{\lambda}_1 y_1 + \ldots + \overline{\lambda}_{s_i} y_{s_i}$, where all $\lambda_i, \overline{\lambda}_i$ are nonnegative. Let us define the mapping $g : \mathbb{R}^n \setminus \{0\} \to \mathcal{S}^{n-1}$, $g(x) = \frac{x}{\|x\|}$ (we can ignore the origin since $\{0\} \cap \mathcal{S}^{n-1} = \varnothing$). Construct the mapping $f(t) = \frac{t}{\|tg(\overline{x}) + (1-t)g(x)\|} g(\overline{x}) + \frac{1-t}{\|tg(\overline{x}) + (1-t)g(x)\|} g(x)$, where $t \in [0,1]$. It is easy to observe that $f(t) \in \mathcal{C}_i \cap \mathcal{S}^{n-1}$, for each $t \in [0,1]$. The mapping is continuous since the cone is pointed i.e. if $\frac{x}{\|x\|} \in \mathcal{C}_i \cap \mathcal{S}^{n-1}$ then $-\frac{x}{\|x\|} \notin \mathcal{C}_i \cap \mathcal{S}^{n-1}$. Hence if $\mathrm{rank}(\Gamma_i) = z_i + s_i + i - 1 = n$, any two points in $\mathcal{C}_i \cap \mathcal{S}^{n-1}$ may be joined by a path i.e. $\mathcal{C}_i \cap \mathcal{S}^{n-1}$ is path connected.

Now let $1 \le \mathrm{rank}(\Gamma_i) = z_i + s_i + i - 1 < n$ (and $s_i \ge 1$). Note that we exclude the case $i = n$, since $\mathrm{rank}(\Gamma_n) = z_n + s_n + n - 1 < n \Rightarrow z_n = 0, s_n = 0$. Hence if there are any restrictions for $i = n$ we must have $\mathrm{rank}(\Gamma_n) = n$ i.e. the previous case. Since $\dim(V) = n - z_i - i + 1 - s_i \ge 1$, $\mathcal{C}_i$ must contain at least 1–dimensional linear subspace. Moreover, by assumption $s_i \ge 1$, $\mathcal{C}_i^*$ contains at least one nonzero ray. Hence both V and $\mathcal{C}_i^*$ contain nonzero elements. In such a case, all elements of the cone $\mathcal{C}_i$

may be obtained as $x = \gamma_1 r_1 + \ldots + \gamma_{n-z_i-i+1-s_i} r_{n-z_i-i+1-s_i} + \lambda_1 y_1 + \ldots + \lambda_{s_i} y_{s_i}$, where all $\lambda_i \geq 0$, $\gamma_i \in \mathbb{R}$, and $r_i, y_i \in \mathbb{R}^n$ are fixed. Choose any two nonzero points in $\mathcal{C}_i$, say $x = \gamma_1 r_1 + \ldots + \gamma_{n-z_i-i+1-s_i} r_{n-z_i-i+1-s_i} + \lambda_1 y_1 + \ldots + \lambda_{s_i} y_{s_i}$ and $\overline{x} = \overline{\gamma}_1 r_1 + \ldots + \overline{\gamma}_{n-z_i-i+1-s_i} r_{n-z_i-i+1-s_i} + \overline{\lambda}_1 y_1 + \ldots + \overline{\lambda}_{s_i} y_{s_i}$, where all $\lambda_i, \overline{\lambda}_i$ are nonnegative and $\gamma_i, \overline{\gamma}_i \in \mathbb{R}$. First assume that x and $\overline{x}$ do not lie on the same line i.e. $\overline{x} \notin (x)$. Then we can construct the same path as above. Now assume that x and $\overline{x}$ lie on the same line i.e. $\overline{x} \in (x)$. We can connect them with a path using the standard reasoning. Choose a third point, say $\overline{\overline{x}}$, such that $\overline{\overline{x}} \notin (x)$. We can construct the path f_1 from x to $\overline{\overline{x}}$, and the other one, say f_2, from $\overline{\overline{x}}$ to $\overline{x}$. Then there is a path f from x to $\overline{x}$, defined as $f(t) = f_1(2t)$; if $t \in [0, \frac{1}{2}]$, and $f(t) = f_2(2t-1)$; if $t \in [\frac{1}{2}, 1]$. Hence if $1 \leq \operatorname{rank}(\Gamma_i) = z_i + s_i + i - 1 < n$ and $s_i \geq 1$ any two points in $\mathcal{C}_i \cap \mathcal{S}^{n-1}$ may be joined by a path i.e. $\mathcal{C}_i \cap \mathcal{S}^{n-1}$ is path connected.

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