The Optimum Quantity of Debt for Japan

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Abstract

Japan’s net government debt reached 130% of its GDP in 2013. This paper calculates the optimum amount of debt for Japan and the welfare loss that the current level of debt generates. We augment the incomplete asset markets model of Aiyagari and McGrattan (1998) with consumption tax and calibrate the extended model to Japan’s economy. We find that under the baseline parameterization, the welfare-maximizing level of debt is 40% of GDP. The current level incurs the average welfare cost that is 0.08% of the optimal consumption, whereas the cost is 1.8% for households with no wealth. If the interest rate rises by one percentage point from the baseline case, the optimal debt becomes −100% of GDP. The average welfare cost and the cost for zero-wealth households increase to 1.1% and 7.5%, respectively.

Keywords: Government debt, welfare, incomplete markets, consumption tax, Japan

JEL classification: E62, H63

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1 Introduction

Japan’s net government debt reached 130% of its GDP in 2013 and the debt-to-GDP ratio is the highest among major countries. There is growing concern regarding the fiscal condition of Japan and recent studies analyze various options to stabilize the debt.\footnote{See the next section for a review of the literature.} However, there has been little welfare analysis and it is not known whether the current level of debt is actually costly in terms of welfare, let alone the magnitude of the cost.

The present paper contributes to the literature by computing the optimum quantity of government debt for Japan and the welfare cost that the current level of debt generates. For this purpose, we augment the model of Aiyagari and McGrattan (1998) including consumption tax and calibrate it to Japan’s economy. The model is a version of Aiyagari (1994)’s incomplete asset markets model with exogenous growth and the government. There are a large number of households and those households face idiosyncratic labor income risk. Asset markets are incomplete and there are only two risk-free assets: government debt and physical capital. Hence, households cannot fully insure against idiosyncratic income risk and they partially self-insure through savings, leading to a distribution of wealth and labor income across households. In contrast, there is a representative firm with the neoclassical production function, whereas the government finances expenditures and transfers through debt, consumption tax, and proportional taxes on capital and labor income.

As Aiyagari and McGrattan (1998) show, increasing government debt in this environment affects welfare in several ways. First, since government debt can be used for savings in addition to capital, the return on savings increases, even after an increase in the tax rate is taken into account. The higher return improves welfare. Second, since capital is crowded out and the tax rate increases, the after-tax wage rate falls. The lower wage decreases welfare, although the reduced uncertainty on labor earnings improves welfare. Further, the combination of the higher interest rate and lower wage rate reduces the average welfare because it hurts households with high marginal utility, i.e., those holding low wealth and
primarily relying on labor income.\textsuperscript{2} Thus, there is a trade-off of increasing government debt
and the welfare-maximizing quantity of debt depends on the relative strengthens of the
above effects. Using this framework, Aiyagari and McGrattan (1998) find that the optimal
debt is about 67\% of GDP for the U.S. The optimal debt for Japan might be very different
since several parameters, including the growth rate and shares of government expenditures
and transfers in GDP, differ between Japan and the U.S. We also consider consumption tax
because it accounts for about 25\% of the national tax revenue in Japan.\textsuperscript{3}

We find that under the baseline calibration, the optimum quantity of debt is 40\% of GDP
for Japan. Hence, the current level of debt is too high in terms of welfare. However, the
average welfare cost is mild and it is only 0.08\% of the optimal consumption. There are
two caveats here. First, the cost is heterogeneous across households and the cost for poorer
households is much larger. For example, the cost is 1.76\% for households with zero wealth.\textsuperscript{4}
Second, the welfare cost increases substantially as the interest rate rises. If the interest rate
increases by one percentage point from the baseline scenario, the optimal debt is –100\% of
GDP, the average cost is 1.07\%, and the cost is 7.46\% for households with no wealth. Hence,
in view of welfare, the possibility of a rise in the interest rate rationalizes concerns regarding
Japan’s debt.

The rest of the present paper is organized as follows. Section 2 reviews the literature.
Section 3 describes the model of Aiyagari and McGrattan (1998) with consumption tax
and Section 4 calibrates the model to Japan’s economy. Section 5 presents results, whereas
Section 6 conducts robustness checks. Section 7 concludes.

\textsuperscript{2}See Gottardi, Kajii, and Nakajima (2014) for a discussion on the distribution and insurance effects.

\textsuperscript{3}Consumption tax increased to 8\% in April 2014 and is expected to increase further to 10\% in April 2017.
Nutahara (2015) shows that the Laffer curve for consumption tax does not have a peak in Japan. Trabandt
and Uhlig (2011) find the same result for the U.S. and EU countries.

\textsuperscript{4}The cost is for households currently having zero-wealth and the median productivity.
2 Related Literature

An increasing number of studies analyze the sustainability of Japan’s fiscal policy. However, the long-run level of debt is typically assumed, without considering the implication for welfare. For example, İmrohoroğlu and Sudo (2011b) use a standard growth model to analyze how TFP growth and fiscal adjustments affect the debt-to-GDP ratio of Japan. They find that in order to reduce the net debt to zero (0% of GDP), TFP must grow by more than 6% per year.5

Doi, Hoshi, and Okimoto (2011) analyze tax policies that stabilize the debt using the government budget constraint. They search for the tax revenue that stabilizes the debt to the initial level, which is 152.8% of GDP. They find that the tax revenue must increase to more than 40% of GDP.

İmrohoroğlu, Kitao, and Yamada (2013) construct a rich overlapping generation model describing Japan’s pension system in detail. They use the model to project the path of the government debt. Without any new changes in policy, they project that the net debt will be more than 375% of GDP in 2050. However, they do not analyze the welfare implication of the large debt.

Braun and Joines (2014) also develop a large-scale overlapping-generation model that describes the Japanese pension and health care systems. They use their model to analyze the future path of the debt-to-GDP ratio under several scenarios on fiscal adjustments. In doing so, they assume that the debt converges to 100% of GDP.

Hansen and İmrohoroğlu (2015) conduct a similar analysis using a representative-agent model. They set the long-run debt to 60% of GDP. Although they compare welfare under a few levels of long-run debt, they do not search for the optimal level.

5İmrohoroğlu and Sudo (2011a) analyze how the rise in consumption tax from 5% to 15% affects Japan’s primary surplus.
3 Model

We extend the model of Aiyagari and McGrattan (1998) by including consumption tax. We explain the model in the order of firms, households, and the government.

3.1 Firms

There is a representative firm. The firm rents capital $K$ and labor $N$ from households and produces the good $Y$. The production function is

$$ Y = K^\theta (zN)^{1-\theta}, \quad (1) $$

where $z$ is labor-augmenting productivity and $\theta \in (0, 1)$ is the capital share. Productivity grows at a constant rate of $g$, that is, $z' = (1 + g)z$, where a prime denotes a next-period value hereinafter.

The firm maximizes its static profit taking the rental rate of capital $r$ and the wage rate $w$ as given. The first-order conditions are

$$ r = \theta z^{1-\theta} K^{\theta-1} N^{1-\theta} - \delta \quad (2) $$

and

$$ w = (1 - \theta) z^{1-\theta} K^\theta N^{-\theta}, \quad (3) $$

where $\delta \in (0, 1)$ is the capital depreciation rate.

3.2 Households

There is a continuum of households (measure one). Households are endowed with one unit of time each period. The momentary utility function is $u(c, l) = (c^{\eta(1-\eta)} / (1 - \mu))$, where $c$ is consumption, $l$ is leisure, $\mu > 0$ ($\mu \neq 1$) is the risk aversion parameter, and $\eta \in (0, 1)$ is
the consumption share.\textsuperscript{6} Households differ in their labor productivity $e$. These idiosyncratic productivities are mutually independent and follow a AR(1) process: $\ln e' = \rho \ln e + \varepsilon', \varepsilon' \sim N(\mu_\varepsilon, \sigma_\varepsilon^2)$\textsuperscript{7}. This is the source of idiosyncratic earnings risk.

Households cannot fully insure against earnings risk because asset markets are incomplete and there is a borrowing constraint. Specifically, there are only two risk-free assets in the economy. One is physical capital, and the other is government debt. These two assets are perfect substitutes for households and both assets earn the interest rate $r$. There are no aggregate shocks and at the beginning of each period, households are distinguished by their current productivity $e$ and total asset holding $a$. The borrowing constraint requires $a \geq 0$.

To describe the optimization problem of households in a stationary form, define $\tilde{a} = a/Y$. The problem is written as

\begin{equation}
V(\tilde{a}, e) = \max_{\{\tilde{c}, l, \tilde{a}'\}} Y_0^n (1-\mu) \left\{ \frac{(\tilde{c}^{\eta} l^{1-\eta})^{1-\mu}}{1-\mu} + \beta (1+g)^n (1-\mu) E[V(\tilde{a}', e')|e] \right\} \tag{4}
\end{equation}

subject to $(1 + \tau_c)\tilde{c} + (1 + g)\tilde{a}' \leq [1 + (1 - \tau)r]\tilde{a} + (1 - \tau)\tilde{w}e(1 - l) + \chi$

\[\tilde{c} \geq 0, 0 \leq l \leq 1, \tilde{a}' \geq 0,\]

where $V(\tilde{a}, e)$ is the value function of households, $\tilde{c} = c/Y$, $\beta \in (0, 1)$ is the discount factor, and $E$ denotes conditional expectation. The second line is the budget constraint: $\tau_c$ is the consumption tax rate, $\tau$ is the common tax rate on capital and labor income, $\tilde{w} = w/Y$, and $\chi = TR/Y$, where $TR$ is exogenous transfers from the government to households.

\section*{3.3 Government}

The government budget constraint is

\textsuperscript{6}We also try a utility function that shows a constant Frisch elasticity of labor supply. The result is quite similar to that under a Cobb-Douglas utility function.

\textsuperscript{7}We adjust $\mu_\varepsilon$ so that the mean of $\varepsilon$ is 1.0.
\[ G + TR + rB = B' - B + \tau[wN + r(K + B)] + \tau_c C, \]  
\[ \text{(5)} \]

where \( G \) is exogenous government consumption, \( B \) is government debt, and \( C \) is aggregate consumption. Dividing (5) by \( Y \) leads to the constraint in a stationary form:

\[ \gamma + \chi + rb = (1 + g)b' - b + \tau[wN + r(k + b)] + \tau_c c_{agg}, \]  
\[ \text{(6)} \]

where \( b = B/Y, \, k = K/Y, \) and \( c_{agg} = C/Y. \) On the balanced growth path, \( b = b' \).

4 Benchmark Parameter Values

The column labeled ‘Baseline’ in Table 1 lists the benchmark parameter values. One period corresponds to one year. As for parameters concerning production and the government, we calibrate their value to the average value between 1995 and 2013 in Japan.\(^8\) Specifically, the growth rate of real GDP \( g \) is 1.0% per year.\(^9\) The share of government expenditures in GDP \( \gamma \) is 0.214, while the share of government transfers \( \chi \) is 0.126. The capital depreciation rate \( \delta \) is 0.071 and the capital share \( \theta \) is 0.412. The consumption tax rate \( \tau_c \) is 5%.

As for the utility function, the risk aversion parameter \( \mu \) is 1.5, which is standard in the literature. To determine the consumption share \( \eta \), we follow the strategy of Aiyagari and McGrattan (1998). First, (6) with \( 1 = [(r + \delta)k + \bar{w}N] \) implies

\[ \tau = \frac{\gamma + \chi + (r - g)b - \tau_c c_{agg}}{1 - \delta k + rb}, \]  
\[ \text{(7)} \]

where \( k = \theta/(r + \delta) \) by (2) and \( c_{agg} = 1 - (g + \delta)k - \gamma \) by the resource constraint: \( Y = C + K' - \delta K + G. \) As the benchmark, following İmrohoroğlu, Kitao, and Yamada (2013), we set the return on savings \( r \) to 3.0%.\(^{10}\) The equation (7) is solved for \( \tau \) at \( b = 0.79 \), which

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\(^8\)We adjust the data following Hayashi and Prescott (2002)’s method. Capital excludes government capital and includes foreign capital. GDP includes net factor payments.

\(^9\)The growth rate is close to the current estimate of the potential growth rate.

\(^{10}\)The average interest rate on government bonds, which account for more than 70% of government debt,
is the average net debt-to-GDP ratio between 1995 and 2013.

Second, the Frisch elasticity of labor supply is given by

\[
Frisch\ elasticity = [1 - \eta(1 - \mu)](1 - N)/(\mu N),
\]

where

\[
N = \frac{1}{1 + \frac{(1-\eta)(1+\tau\varepsilon)}{\eta(1-\tau)(1-\rho)}[1 - \gamma - \theta(g+\delta)/(r+\delta)]}.
\]

Given \(\tau\), we solve (8) for \(\eta\), assuming that the Frisch elasticity of labor supply is 1.0, which is in line with the estimate by Kuroda and Yamamoto (2008). The result is \(\eta = 0.5596\). We conduct sensitivity analyses with respect to \(\mu\), \(\eta\), and \(r\) in Section 6.

The stochastic process for idiosyncratic productivity \(\varepsilon\) is typically estimated using panel data of individual wages. Since such panel data is limited in Japan, we take the following strategy.\(^\text{11}\) First, we set \(\rho = 0.85\) because existing analyses using panel data in the U.S. typically find persistent idiosyncratic productivity. Second, given \(\rho = 0.85\), we choose \(\sigma_\varepsilon\) to match the variance of individual wages in Japan. According to Lise, Sudo, Suzuki, Yamada, and Yamada (2014), the variance of log of individual wages is 0.227 between 1995 and 2008. This indicates that \(\sigma_\varepsilon = 0.251\). These values of \(\rho\) and \(\sigma_\varepsilon\) are very similar to those estimated for the U.S.\(^\text{12}\) The AR(1) process is approximated with a 17-state Markov chain using Tauchen (1986)’s method.

Lastly, we choose the discount factor \(\beta\) so that the before-tax rate of return on savings \(r\) is 3.0% at the benchmark parameter values and the debt-to-GDP ratio \(b = 0.79\).
5 Results

The debt-to-GDP ratio $b$ is varied from –2.0 to 2.0 by 0.1 at one time and its effect on welfare is examined. Other parameters are fixed at their benchmark value. The welfare measure is the same as that in Aiyagari and McGrattan (1998):

$$\Omega = \int \int V(\tilde{a}, e)dH(\tilde{a}, e),$$

where $H(\tilde{a}, e)$ is the stationary distribution of wealth $\tilde{a}$ and productivity $e$ across households. The measure can be seen as a utilitarian welfare function that weights all households equally or the expected utility of a household before drawing the initial state. The welfare gain is expressed as a percentage of the consumption that would be enjoyed at the optimal level of debt.$^{13}$

Figure 1 shows the result. Under the benchmark parameter values, the optimum quantity of debt is 40% of GDP for Japan. Hence, the current level of 130% of GDP is higher than the welfare-maximizing level. On average, the current level of debt incurs a mild welfare cost, which is 0.08% of the optimum consumption. However, the cost is heterogeneous across households and households holding lower wealth face a larger cost. For example, the cost is 1.76% for households with no wealth. The results are summarized in Table 2.$^{14}$

As shown by Aiyagari and McGrattan (1998), government debt influences welfare in the present environment as follows. The increase in the debt-to-GDP ratio $b$ raises the return on savings $r$. At the same time, the higher $r$ crowds out capital and leads to a lower capital-to-GDP ratio $k$, reducing the wage rate $w$.$^{15}$

$^{13}$More specifically, the number shows how much the optimal consumption must increase at all states and dates in order to achieve the welfare $\Omega$ at a particular $b$.

$^{14}$The welfare costs of the current level of debt are comparable to costs of business cycles in an environment similar to the current one. For example, Mukoyama and Sahin (2006) estimate that the cost of business cycles is about 0.02–0.08% of consumption, Krusell, Mukoyama, Sahin, and Smith (2009) estimate that it is 0.08-0.10% under their benchmark scenario. These business-cycle costs consider the transition.

$^{15}$Since the return on savings in the model is lower than the return on capital in the actual data, the capital/output is higher in the model than in the data. One interpretation is that the model capital includes intangible capital.
When the debt-to-GDP ratio $b$ is sufficiently low, increasing $b$ improves welfare. The after-tax interest rate rises substantially, whereas the after-tax wage rate falls only mildly. The reason is that when $b$ is low, an increase in $b$ leads to an increase in the tax base, and this outweighs an increase in the interest payments (or outweighs the decrease in the interest receipts). Thus, the tax rate $\tau$ decreases. Further, the higher wage rate means lower wage risk, which improves welfare. Although the combination of the higher interest rate and lower wage rate hurts households with low wealth, who have high marginal utility, this distribution effect is dominated by the other effects when $b$ is low. Hence, welfare improves as $b$ increases.

As $b$ increases further, the increase in the tax base slows and the tax rate $\tau$ must increase substantially to cover the interest payments. The rise in the after-tax interest rate slows, while the after-tax wage rate falls rapidly. In addition, these changes in the prices especially hurt wealth-poor households, further reducing the average welfare. While the lower wage reduces wage risk, overall, welfare decreases.

6 Sensitivity Analyses

This section examines the robustness of the result in the last section. We change the risk aversion parameter $\mu$ and the consumption share $\eta$ for labor supply elasticity. We change one parameter at a time and keep other parameters to their benchmark value, except that the discount factor $\beta$ is adjusted to maintain the return on savings $r$ of 3.0% at the debt-to-GDP ratio of $b = 0.79$. We also consider a higher $r$. The parameter values and results are summarized in Tables 1 and 2, respectively.

6.1 Risk Aversion

The risk aversion parameter $\mu$ is increased from 1.5 to 2.5. Figure 2 shows the result. The optimum level of debt is 30% of GDP and it is slightly lower than the baseline case of 40%. There are two competing effects. First, higher risk aversion lowers the effective discount
factor and reduces households’ incentive to save. Second, since households become more risk averse, households prefer smooth consumption more and increase savings. Under the parameterization considered here, the first effect dominates and the optimal debt decreases. The welfare cost of keeping the current debt-to-GDP ratio $b = 1.3$ is 0.13% on average and is 2.06% for households with no wealth. The welfare loss increases more quickly than the benchmark case because the distribution effect is more severe when households are more risk averse. Nevertheless, the overall result is quite similar to the baseline result.

6.2 Labor Supply Elasticity

The Frisch labor supply elasticity is decreased from 1.0 to 0.5, which is in line with the estimate by Sugo and Ueda (2008). The result is shown in Figure 3. The optimum level of debt is 60% of GDP and it is slightly higher than the baseline case of 40%. The welfare loss being at the current debt-to-GDP ratio $b = 1.3$ is also smaller than the benchmark case: 0.04% of the optimal consumption on average and 1.27% for zero-wealth households. The cost increases more slowly than the benchmark case because with less elastic labor supply, the increase in the tax rate $\tau$ is less costly in terms of welfare. However, the result is largely the same as the benchmark case.

6.3 Return on Savings

We increase the return on savings $r$ by one percentage point from 3.0% to 4.0% by setting the discount factor $\beta = 0.9870$. Figure 4 shows the result. The optimal debt is –100% of GDP and is much lower than the baseline level and the current level of 130%. Crucially, the welfare cost of the current debt increases substantially and it is 1.07% of the optimal consumption on average and 7.46% for households with no wealth. A higher $r$ implies a lower discount factor and smaller demand for savings. Hence, the optimal amount of government

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16 Sugo and Ueda (2008) estimate a medium-scale New Keynesian model in the spirit of Christiano, Eichenbaum, and Evans (2005) for Japan using Bayesian methods and find that the elasticity is about 0.5.
debt decreases. Further, a higher $r$ increases the government’s interest payments and the tax $\tau$ increases quickly in $b$. As a result, the after-tax wage fall substantially, which especially hurts wealth-poor households. Hence, the cost of keeping the current level of debt is larger under a higher return on savings.

### 7 Conclusion

Japan’s net government debt reached 130% of its GDP in 2013. We have extended Aiyagari and McGrattan (1998)’s model including consumption tax and analyzed the welfare implication of Japan’s high debt/GDP ratio. Under the baseline parameterization, the optimal debt for Japan is 40% of GDP and the current level of debt generates a mild welfare cost on average. However, households with lower wealth tend to face more substantial costs and the welfare cost increases quickly as the interest rate rises.

### References


Christiano, L. J., M. Eichenbaum, and C. L. Evans (2005): “Nominal Rigidities and


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Note: ‘same’ indicates the baseline value.
Table 2: Results summary

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Note: The welfare cost is expressed as a percentage of the optimal consumption.
Figure 1: Benchmark result. The horizontal axis is the debt-to-GDP ratio \( b \). The welfare gain is expressed as a percentage of the optimal consumption. The wage rate \( w \) is the rate at aggregate productivity \( z = 1.0 \). Labor \( N \) shows a percent change from the optimal level.

Figure 2: Risk aversion \( \mu \). The horizontal axis is the debt-to-GDP ratio \( b \). The solid line is the benchmark case (\( \mu = 1.5 \)), whereas the dotted line is the case with higher risk aversion (\( \mu = 2.5 \)). The welfare gain is expressed as a percentage of the optimal consumption. The wage rate \( w \) is the rate at aggregate productivity \( z = 1.0 \). Labor \( N \) shows a percent change from the optimal level.
Figure 3: Elasticity of labor supply. The horizontal axis is the debt-to-GDP ratio $b$. The solid line is the benchmark case (elasticity of labor supply = 1.0), whereas the dotted line is the case with a lower elasticity (0.5). The welfare gain is expressed as a percentage of the optimal consumption. The wage rate $w$ is the rate at aggregate productivity $z = 1.0$. Labor $N$ shows a percent change from the optimal level.

Figure 4: Return on savings $r$. The horizontal axis is the debt-to-GDP ratio $b$. The solid line is the benchmark case ($r = 0.02$), whereas the dotted line is the case with a higher return ($r = 0.03$). The welfare gain is expressed as a percentage of the optimal consumption. The wage rate $w$ is the rate at aggregate productivity $z = 1.0$. Labor $N$ shows a percent change from the optimal level.