Imperfect Credibility and Robust Monetary Policy*

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Abstract

In an effort to better anchor private-sector expectations and stabilize the economy, central banks are increasingly employing announcements about future policy. Although policy announcements can be a useful policy lever, concerns about credibility and about model uncertainty have the potential to undermine their usefulness. This paper studies the behavior of a central bank that seeks to conduct policy optimally while having imperfect credibility and harboring doubts about its model. Taking the Smets-Wouters model as the central bank’s approximating model, the paper’s main findings are as follows. First, policy announcements are useful both when the model is thought to be correctly specified and when it is suspected that it is not. Second, central banks that have low credibility can benefit from a desire for robustness because this desire motivates the central bank to follow through on policy announcements that would otherwise not be time-consistent. Third, even relatively small departures from perfect credibility produce big declines in policy performance. Finally, as a technical contribution, the paper develops a numerical procedure to solve the decision-problem facing an imperfectly credible policymaker that seeks robustness.

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1 Introduction

On August 9, 2011, against a background of heightened volatility in global financial markets, the Board of Governors of the Federal Reserve issued a monetary policy statement that said “The Committee currently anticipates that economic conditions—including low rates of resource utilization and a subdued outlook for inflation over the medium run—are likely to warrant exceptionally low levels for the federal funds rate at least through mid-2013.” This passage replaced the language in statements issued since December 16, 2008, which said “The Committee continues to anticipate that economic conditions—including low rates of resource utilization and a subdued outlook for inflation over the medium run—are likely to warrant exceptionally low levels for the federal funds rate for an extended period.” Although the precise language has changed, each passage is notable for presenting households, firms, and investors with forward-guidance about monetary policy, guidance provided in an effort to leverage credibility in order to stimulate current economic activity. The passages are also notable in that the forward guidance is conditioned on a forecast for inflation and resource utilization, or slack. As a consequence, the effectiveness of the policy announcements, the effectiveness of the forward-guidance, hinges on the Federal Reserve’s credibility and on the potential for the forecasting model to be misspecified.

We consider the decision problem facing a central bank that seeks robustness to model uncertainty and whose credibility is imperfect and explore the following questions. How does imperfect credibility affect the forward-guidance provided by a central bank? How is a central bank’s ability to pursue a robust policy affected by imperfect credibility? Does a central bank’s desire to pursue a robust policy serve to aid or undermine its credibility? The answers to these questions are important when central banks, such as the Federal Reserve System, are relying increasingly on their credibility and on forward-guidance to gain leverage over current economic outcomes, all-the-while model uncertainty remains an ongoing concern.

To model credibility, we adopt the quasi-commitment approach developed by Roberds (1987), Schaumburg and Tambalotti (2007), and Debortoli and Nunes (2010). According to this literature a policymaker’s credibility is associated with the probability that the promises it makes about future policy will be honored. Policymakers that have no credibility honor their promises with probability zero and conduct discretionary policy. Policymakers that have imperfect credibility honor their promises with probabilities between zero and one, with higher probabilities indicating higher credibility and a probability of one indicating commit-
ment. Central banks desire higher levels of credibility because a lack of credibility leads to a (time-consistent) equilibrium characterized by a discretionary inflation bias and/or a discretionary stabilization bias. Under the former, the central bank, faced with the goals of keeping unemployment close to the natural rate and inflation close to target, succumbs to a short-run incentive to create surprise inflation with permanently higher inflation and no reduction in the unemployment rate the equilibrium outcome (Kydland and Prescott, 1977). Under the latter, the central bank, seeking to stabilize output and inflation efficiently in response to supply shocks, has an incentive to promise future policy interventions that mitigate the size of today’s policy intervention, without having an incentive to subsequently deliver on those promises (Svensson, 1997; Clarida, Galí, and Gertler, 1999). The inefficiencies associated with both biases are overcome when credibility is perfect.

In addition to imperfect credibility, the central bank that we study is concerned with model misspecification. To model the central bank’s concern for model misspecification we adopt the robust control approach advanced by Hansen and Sargent (2008). According to the robust control literature, a policymaker that desires robustness against model misspecification will formulate policy in the context of a potentially distorted, or misspecified, approximating model so as to guard against the worst permissible misspecification. Through this mechanism the policymaker is able to conduct model-based policy while also expressing distrust in its model.

After developing the decision problem confronting an imperfectly credibility policymaker that seeks robustness to model uncertainty and presenting its solution, we use the Smets and Wouters (2007) model to examine the effects that imperfect credibility and robustness have on optimal policymaking. We employ the Smets and Wouters (2007) model for our analysis because it is widely used, it forms the basis for many other models, and it is thought to fit U.S. data well; in these respects it can usefully be viewed as the central bank’s approximating model. Moreover, the Smets-Wouters model contains a broad array of shocks whose presence provides ample cover for model misspecification, and it is forward-looking allowing policy announcements and central bank credibility to potentially play important roles. A further advantage to using the Smets-Wouters model for this exercise is that our qualitative findings are likely to generalize to the many related models.

The main lessons that emerge are the following. First, a central bank’s credibility gives it a powerful lever for managing private-sector expectations and for stabilizing the economy. This lever delivers optimal outcomes when the model is correctly specified, and it provides
robustness by ensuring private-sector expectations remain well-anchored even when the model is misspecified. As a consequence, a credible central bank can guard against a much wider array of specification errors than a central bank that lacks credibility can. Second, when a central bank has low credibility the economy can benefit from the central bank’s desire for robustness. Put differently, the central bank’s desire for robustness can act somewhat as a substitute for credibility when credibility is low. This result emerges because a robust central bank is directed to respond aggressively to stabilize inflation following shocks, pursuing a policy that would ordinarily be infeasible for a central bank that lacks credibility. Third, even relatively small departures from perfect credibility produce big declines in policy performance, giving rise to a form of discretionary stabilization bias. The over-riding lesson that emerges from this analysis is that credibility is extremely valuable for central banks, both when the model is known to be correctly specified and when it is suspected that it is not.

In addition to the work of Schaumburg and Tambalotti (2007), Debortoli and Nunes (2010), and Hansen and Sargent (2008), this paper is related to Bodenstein, Hebden, and Nunes (2010). However, where Bodenstein, Hebden, and Nunes (2010) focus on the interaction between quasi-commitment and the zero-bound on nominal interest rates, we focus on the interaction between quasi-commitment and model uncertainty. Nonetheless, our results are consistent with theirs in-so-much as we too find that policymakers tend to make more extreme policy announcements as their credibility declines.

The remainder of this paper is structured as follows. Section 2 describes the decision problem facing a central bank that seeks to guard against model misspecification while endowed with imperfect credibility. Section 3 analyzes a simple New Keynesian model. Section 4 summarizes and analyzes the Smets-Wouters model that serves as our main laboratory for analysis. Section 5 concludes.

2 Robustness and imperfect credibility

In this section we describe a linear-quadratic planning problem and characterize its solution. This planning problem involves constraints that contain non-predetermined variables and is related to the commitment problems that are solved routinely in the monetary policy literature, while differing in two important respects. First, the decisionmaking environment is one in which the policymaker has imperfect credibility. Imperfect credibility is modeled according to the quasi-commitment literature which allows the policymaker to stochastically default,
reoptimizing its plan at stochastic intervals. In this aspect, the analysis builds on work by Roberds (1987), Schaumburg and Tambalotti (2007), and Debortoli and Nunes (2010). Second, the decisionmaking environment is one in which the policymaker has doubts about its model and seeks a policy that is robust in the sense of Hansen and Sargent (2008). In this aspect, the analysis is related to work by Giordani and Söderlind (2004), Hansen and Sargent (2008, chapter 16), and Dennis (2008, 2010).

The two key parameters in the decision problem that we formulate are \( \alpha \), which governs the policymaker’s credibility, and \( \theta \), which governs the policymaker’s distrust in its model. Importantly, many standard decisionmaking problems emerge as special cases of this decision problem. Specifically, for different limiting values of \( \alpha \) and \( \theta \) the decision problem simplifies to non-robust commitment (\( \theta \uparrow \infty, \alpha \uparrow 1 \)), non-robust discretion (\( \theta \uparrow \infty, \alpha \downarrow 0 \)), robust commitment (\( \alpha \uparrow 1 \)), robust discretion (\( \alpha \downarrow 0 \)), and quasi-commitment (\( \theta \uparrow \infty \)).

2.1 The approximating model

The economy consists of households, firms, and a policymaker that in our application is a central bank. All agents are assumed to share an approximating model that they believe comes closest to describing the process governing economic outcomes. According to this approximating model, an \( n \times 1 \) vector of endogenous variables, \( \mathbf{z}_t \), consisting of \( n_1 \) predetermined variables, \( \mathbf{x}_t \), and \( n_2 \) \((n_2 = n-n_1)\) nonpredetermined variables, \( \mathbf{y}_t \), evolves over time according to

\[
\mathbf{x}_{t+1} = \mathbf{A}_{11}\mathbf{x}_t + \mathbf{A}_{12}\mathbf{y}_t + \mathbf{B}_1\mathbf{u}_t + \mathbf{C}_1\varepsilon_{x_{t+1}},
\]

\[
\mathbf{A}_0E_t\mathbf{y}_{t+1} = \mathbf{A}_{21}\mathbf{x}_t + \mathbf{A}_{22}\mathbf{y}_t + \mathbf{B}_2\mathbf{u}_t,
\]

where \( \mathbf{u}_t \) is a \( p \times 1 \) vector of policy control variables and \( \varepsilon_{x_t} \sim i.i.d. [0, \mathbf{I}] \) is an \( n_x \times 1 \) \((n_x \leq n_1)\) vector of white-noise innovations. The matrices \( \mathbf{A}_{11}, \mathbf{A}_{12}, \mathbf{A}_{21}, \mathbf{A}_{22}, \mathbf{B}_1, \) and \( \mathbf{B}_2 \) are conformable with \( \mathbf{x}_t, \mathbf{y}_t, \) and \( \mathbf{u}_t \) as necessary while the matrix \( \mathbf{C}_1 \) is constructed to ensure that \( \varepsilon_{x_t} \) has the identity matrix as its variance-covariance matrix. The operator \( E_t \) represents the private sector’s mathematical expectation operator conditional upon period \( t \) information. Equation (2) accommodates a leading matrix \( \mathbf{A}_0 \) that need not have full rank.

Equations (1) and (2) are standard constraints in linear-quadratic decision problems in which private agents are forward-looking and policy is conducted under either commitment or discretion (Currie and Levine, 1993) or under timeless-perspective commitment (Woodford,
2010; Svensson, 2010). Of course, when policy is conducted under discretion equations (1) and (2) must be augmented with an equation of the form

$$E_t y_{t+1} = HE_t x_{t+1},$$  \hspace{1cm} (3)

where $H$ is determined in equilibrium, to account to the fact that private-sector expectations depend only on the state variables in a Markov-perfect (and hence time-consistent) equilibrium (Oudiz and Sachs, 1985).

### 2.1.1 Introducing imperfect credibility

Building on Roberds (1987), Schaumburg and Tambalotti (2007) and Debortoli and Nunes (2010) analyze decision problems in rational expectations models that are characterized by what they call “quasi-commitment.” Quasi-commitment provides a workable approach to modeling imperfect credibility because it allows a separation between what policymakers do and what they say they are going to do. Moreover, quasi-commitment allows policy reoptimization to occur in equilibrium, making default on policy announcements more than just an out-of-equilibrium phenomenon.

Quasi-commitment views commitment and discretion as opposite ends of a unit-continuum of decision-problems. Each decision-problem on the unit-continuum is indexed by $\alpha \in [0, 1]$, where $\alpha$ denotes the mean of the random variable $\eta_t$, which obeys a Bernoulli distribution. The underlying environment can be interpreted several ways. One interpretation is that the environment is one in which the policymaker makes announcements about future policy with all agents (including the policymaker) making decisions knowing that the announced policy will only be implemented with probability $\alpha$. An alternative interpretation is that the environment is one in which policymakers can credibly commit to a state-contingent plan, or policy, for the duration of their tenure, but where each policymaker’s tenure is uncertain, governed by the outcome of a sequence of $i.i.d.$ draws of the random variable $\eta_t$. Accordingly, if $\eta_t = 1$, then the incumbent-policymaker’s tenure continues in period $t$, whereas if $\eta_t = 0$, then the incumbent-policymaker’s tenure ends at the beginning of period $t$. In the event that the incumbent-policymaker’s tenure ends, that policymaker is replaced by another with identical preferences, but that is not beholden to honor the policies announced by any of its predecessors. Under either interpretation, $\alpha = 1$ corresponds to commitment, $\alpha = 0$ corresponds to discretion, and $\alpha \in (0, 1)$ corresponds to a form of limited commitment or imperfect credibility.
At the start of every period a draw for \( \eta_t \) is received and is observed by all agents. In forming their period-\( t \) expectations of \( y_{t+1} \), therefore, private agents take into account uncertainty about the shocks hitting the economy and uncertainty about whether the incumbent or a new policymaker will be conducting policy in period \( t + 1 \). Assuming that the Bernoulli distribution that governs \( \eta_t \) is independent of the probability density that governs the innovations, \( \varepsilon_{xt} \), equation (2) can be written as

\[
A_0 E_t y_{t+1} = \alpha A_0 E_t [y_{t+1} | (\eta_{t+1} = 1)] + (1 - \alpha) A_0 E_t [y_{t+1} | (\eta_{t+1} = 0)],
\]

where the expectation \( E_t (y_{t+1} | \eta_{t+1} = 0) \) is governed by an expression that takes the form of equation (3).

2.1.2 Introducing model uncertainty

Following Hansen and Sargent (2008), the policymaker does not fully trust the approximating model, fearing that it may be misspecified. To accommodate this concern, distortions, or specification errors, \( v_{t+1} \), are introduced, disguised by the innovations, \( \varepsilon_{xt+1} \). A consequence of the specification errors is that equation (1) in the approximating model becomes

\[
x_{t+1} = A_{11} x_t + A_{12} y_t + B_1 u_t + C_1 (v_{t+1} + \varepsilon_{xt+1}),
\]

in the “distorted” model. Further, following Giordani and Söderlind (2004), Hansen and Sargent (2008, chapter 16), Dennis (2008), and Dennis, Leitemo, and Söderström (2009), we assume that private-sector expectations are formed using the approximating model (because private agents are not robust decisionmakers), and that the policymaker’s doubts about the approximating model do not lead it to also doubt how private-sector expectations are formed. With this modeling assumption, equation (2) in the approximating model becomes

\[
\alpha A_0 y_{t+1} | (\eta_{t+1} = 1) = [A_{21} - (1 - \alpha) A_0 HA_{11}] x_t + [A_{22} - (1 - \alpha) A_0 HA_{12}] y_t + [B_2 - (1 - \alpha) A_0 HB_1] u_t + \alpha A_0 C_2 (v_{t+1} + \varepsilon_{xt+1}),
\]

in the distorted model, where \( H \) and \( C_2 \) have yet to be determined. In equation (6), \( H \) characterizes the relationship between the non-predicted variables, \( y_t \), and the predicted variables, \( x_t \), in the event that a reoptimization occurs (\( \eta_t = 0 \)) while \( C_2 \) summarizes how errors in forecasting the non-predicted variables (i.e. \( y_{t+1} | (\eta_{t+1} = 1) - E_t [y_{t+1} | (\eta_{t+1} = 1)] \)) are related to the innovations, \( \varepsilon_{xt+1} \). More compactly, and in obvious notation, equation (6)
can be written as

\[ D_0 y_{t+1} | (\eta_{t+1} = 1) = D_1 x_t + D_2 y_t + D_3 u_t + D_4 v_{t+1} + D_4 \varepsilon_{x,t+1}. \] (7)

The sequence of specification errors, \( \{v_{s+1}\}_{s=t}^{\infty} \), is constrained to satisfy the boundedness condition

\[ \beta \mathbb{E} \sum_{s=t}^{\infty} \beta^{(s-t)} v'_{s+1} v_{s+1} \leq \omega, \] (8)

where \( \omega \in [0, \overline{\omega}] \). It is the satisfaction of this boundedness condition that defines the sense in which the approximating model, summarized by equations (1)---(2), is a “good” one. When \( \omega = 0 \), the policymaker trusts the approximating model and conducts policy as if the approximating model is correct. As \( \omega \) increases, however, the policymaker increasingly suspects that the approximating model is misspecified. For \( \omega > \overline{\omega} \), the policymaker’s doubts about the approximating model are such that it no longer views the approximating model to be a good representation of the data-generating process.

### 2.2 The robust decision problem with imperfect credibility

The policymaker’s objective function is given by the loss function

\[ \mathbb{E} \sum_{s=t}^{\infty} \beta^{(s-t)} L(x_s, y_s, u_s), \] (9)

where \( \beta \in (0, 1) \) is the discount factor and \( L(x_s, y_s, u_s) \) is quadratic and convex to the origin.

As noted earlier, in the event that \( \eta_t = 1 \), the incumbent policymaker’s tenure continues. However, in the event that \( \eta_t = 0 \), the period-\( t \) decision problem for the newly-appointed policymaker is to choose \( \{u_s\}_{s=t}^{\infty} \) to minimize and \( \{v_{s+1}\}_{s=t}^{\infty} \) to maximize equation (9) subject to equations (5), (7), and (8), and \( x_t \) known. According to this decision problem, to guard against the specification errors that it fears, the robust policymaker formulates policy subject to the distorted model with the mind-set that the specification errors will be as damaging as possible, a view operationalized via the metaphor that \( \{v_{s+1}\}_{s=t}^{\infty} \) is chosen by a fictitious evil agent whose objectives are diametrically opposed to those of the policymaker. Following Hansen and Sargent (2008), this constraint problem can be replaced with an equivalent multiplier problem, in which

\[ \mathbb{E} \sum_{s=t}^{\infty} \beta^{(s-t)} \left[ L(x_s, y_s, u_s) - \beta \theta v'_{s+1} v_{s+1} \right], \] (10)
we construct the "dual" loss function

\[ L (x_s, y_s, u_s) - \beta \theta y_{s+1} \]

Now defining \( \lambda_s \) and \( \gamma_s \), the nonrobust recursive saddle-point theorem (Marcet and Marimon, 2009), the robust decision problem for the newly-appointed policymaker can be expressed in terms of the Bellman equation

\[ \lambda_t = \mathbb{E} \sum_{s=t}^{\infty} \beta^{(s-t)} \left[ -2 \xi_s' (D_1 x_s + D_2 y_s + D_3 u_s + D_4 v_{s+1} + D_5 \xi_{x_{s+1}} - D_0 y_{s+1} \mid \eta_{s+1} = 1) - 2 \lambda_s' (A_{11} x_s + A_{12} y_s + B_1 u_s + C_1 (v_{s+1} + \xi_{x_{s+1}}) - x_{s+1}) \right] \]

subject to equations (5) and (7), and \( x_t \) known. The multiplier, or robustness parameter, \( \theta \), represents the shadow price of a marginal relaxation in the boundedness condition, equation (8). Larger values for \( \theta \), which correspond to smaller values of \( \omega \), signify greater confidence in the adequacy of the approximating model. Of course, in the limit as \( \theta \to \infty \), the nonrobust decision problem is restored.

From the Lagrange function

\[ \Lambda_t = \mathbb{E} \sum_{s=t}^{\infty} \beta^{(s-t)} \left[ -2 \xi_s' (D_1 x_s + D_2 y_s + D_3 u_s + D_4 v_{s+1} + D_5 \xi_{x_{s+1}} - D_0 y_{s+1} \mid \eta_{s+1} = 1) - 2 \lambda_s' (A_{11} x_s + A_{12} y_s + B_1 u_s + C_1 (v_{s+1} + \xi_{x_{s+1}}) - x_{s+1}) \right] \]

we construct the "dual" loss function

\[ \tilde{L} (x_s, \xi_{s-1}, y_s, u_s, \gamma_s, v_{s+1}) = L (x_s, y_s, u_s) - \beta \theta y_{s+1} \]

\[ -2 \gamma_s' (D_1 x_s + D_2 y_s + D_3 u_s + D_4 v_{s+1}) \]

\[ + 2 \xi_{s-1}' D_0 y_s \]

where \( \gamma_s = \xi_s \), allowing equation (11) to be expressed as

\[ \Lambda_t = \mathbb{E} \sum_{s=t}^{\infty} \beta^{(s-t)} \left[ -2 \lambda_{s-1}' (A_{11} x_s + A_{12} y_s + B_1 u_s + C_1 (v_{s+1} + \xi_{x_{s+1}}) - x_{s+1}) \right] \]

Now defining \( \tilde{X}_t = [x'_t \; \xi'_{t-1}]' \) and \( \tilde{u}_t = [y'_t \; u'_t \; \gamma'_t]' \) and employing Marcet and Marimon’s recursive saddle-point theorem (Marcet and Marimon, 2009), the robust decision problem for the newly-appointed policymaker can be expressed in terms of the Bellman equation

\[ \tilde{X}_t' V \tilde{X}_t + d = \max_{(\gamma_t)} \max_{(y_t, u_t, v_t)} \left[ \tilde{L} (\tilde{X}_t, \tilde{u}_t, v_t) + \beta \mathbb{E} _t \left( \tilde{X}_{t+1}' \tilde{V} \tilde{X}_{t+1} + d \right) \right] \]

in which

\[ \tilde{V} = \alpha V + (1 - \alpha) S'S^{-1} VS^{-1} S \]

with \( S = [I \; 0] \) (and \( S^{-1} \) representing the generalized left inverse of \( S \)), subject to

\[ \tilde{X}_{t+1} = \tilde{A} \tilde{X}_t + \tilde{B} \tilde{u}_t + \tilde{C} v_{t+1} + \tilde{C} \xi_{x_{t+1}}. \]

In equation (16), the system matrices are given by \( \tilde{A} = [A_{11} \; 0 \; 0], \tilde{B} = [A_{12} \; B_1 \; 0], \)

and \( \tilde{C} = [C_1 \; 0] \).
The decision problem described by equations (14)—(16) is essentially an optimal linear regulator problem that can be solved using standard methods. From its solution it is straightforward to recover updated terms for $H$ and $C_2$. Beginning with conjectured values for $H$ and $C_2$, iterating to convergence delivers the worst-case decision rules and the worst-case equilibrium law-of-motion. With the worst-case equilibrium in hand, it is straightforward to recover the approximating equilibrium. In the approximating equilibrium, although the policymaker employs its robust decision rule, the approximating model is taken to be correctly specified. The approximating equilibrium gives us an equilibrium law-of-motion

$$ \bar{X}_{t+1}(\eta_t = 1) = (\bar{A} + \bar{B}F) \bar{X}_t + \bar{C}\varepsilon_{t+1}, $$

and a collection of decision rules

$$ \bar{u}_t(\eta_t = 1) = F\bar{X}_t, $$

that describe behavior in the event that $\eta_t = 1$. In the event that $\eta_t = 0$, behavior differs from equations (17) and (18) only in as much as promises have no value and $\Xi_{t-1} = 0$.

### 3 A risk-sensitive formulation

For linear-quadratic infinite-horizon discounted stochastic models in which the constraints do not contain nonpredetermined variables, Hansen and Sargent (2008, chapter 2) show that the decision rule that solves the robust control problem also solves an alternative infinite-horizon discounted stochastic decision problem in which the policymaker does not fear model misspecification but instead has risk-sensitive preferences (Whittle, 1990). In this section we extend this result to models whose constraints contain nonpredetermined variables, focusing here on the case of pure discretion. The general case of quasi-commitment is treated in Appendix A.

The connection between the robust control formulation and the risk-sensitive preferences formulation is useful for several reasons. It links the robust control problem to ambiguity/uncertainty aversion and offers a more general interpretation of the decision problem as a consequence. Specifically, the policymaker’s doubts about the model lead to behavior that could equivalently be generated by additional sensitivity to risk (Whittle, 1990), Epstein-Zin-preferences (Epstein and Zin, 1989), or ambiguity aversion (Gilboa and Schmeidler, 1989).
With policy conducted under pure discretion ($\alpha = 0$), it follows from equation (6) that the aggregate reaction function for the nonpredetermined variables can be written as

$$y_t = Jx_t + Ku_t,$$

where

$$J = [A_{22} - A_0 HA_{12}]^{-1} [A_0 HA_{11} - A_{21}],$$

$$K = [A_{22} - A_0 HA_{12}]^{-1} [A_0 HB_1 - B_2].$$

Equation (19) applies under both the approximating model and the distorted model. Given equation (19), the law-of-motion for the predetermined variables in the approximating model and in the distorted model are

$$x_{t+1} = [A_{11} + A_{12}J] x_t + [B_1 + A_{12}K] u_t + C_1 \epsilon_{x,t+1},$$

and

$$x^*_{t+1} = [A_{11} + A_{12}J] x_t + [B_1 + A_{12}K] u_t + C_1 (v_{t+1} + \epsilon_{x,t+1}),$$

respectively.

The risk-sensitive formulation employs equation (20) (because the policymaker trusts the model), and leads to the Bellman-equation

$$x_t V x_t + d = \min_{u_t} \left[ L(x_t, Jx_t + Ku_t, u_t) + \frac{1}{\sigma} \ln \left( E_t \left( \exp \left( \sigma \beta \left( x^*_{t+1} V x_{t+1} + d \right) \right) \right) \right) \right],$$

where the risk-sensitivity parameter satisfies $\sigma \leq 0$ and $V$ is positive semi-definite. Employing a result from Jacobson (1973), equation (22) is equivalent to

$$x_t V x_t + d = \min_{u_t} \left[ L(x_t, Jx_t + Ku_t, u_t) + \beta E_t \left( x^*_{t+1} D(V) x_{t+1} + \hat{d} \right) \right],$$

where

$$D(V) = V + \sigma VC_1 \left( I + \sigma C_1 VC_1 \right)^{-1} C_1^1 V.$$

In contrast, the robust-control formulation employs equation (21) (because the policymaker distrusts the model), and leads to the Bellman-equation

$$x_t V x_t + d = \min_{u_t} \left[ L(x_t, Jx_t + Ku_t, u_t) - \beta \theta^t v_{t+1} v_{t+1} + \beta E_t \left( x^*_{t+1} V^* x_{t+1} + d \right) \right].$$

Performing the inner-maximization gives

$$v_{t+1} = \frac{1}{\theta} E_t \left[ \left( I - \frac{1}{\theta} C_1^1 VC_1 \right)^{-1} C_1^1 V x_{t+1} \right].$$
After some reorganization equation (24) can be written as

$$x_t'Vx_t + d = \min_{u_t} \max_{v_{t+1}} \left[ L (x_t, Jx_t + Ku_t, u_t) - \beta \theta v_{t+1}' \left( I - \frac{1}{\theta} C_1'VC_1 \right) v_{t+1} \right] \tag{26}$$

and substituting equation (25) into equation (26) results in

$$x_t'Vx_t + d = \min_{u_t} \left[ +\beta E_t \left( x_{t+1}'Vx_{t+1} + d \right) \right] \tag{27}$$

which, aside from the difference between $\hat{d}$ and $d$ (which does not affect the decision rules), is equivalent to equation (23) with $\theta = -\sigma^{-1}$. Importantly, it is the approximating model, equation (20), that constrains equation (27).

The treatment above parallels Hansen and Sargent’s (2008, chapter 2) treatment of the optimal linear regulator problem. The two problems are related because the aggregate reaction function (equation 19) allows the nonpredetermined variables to be eliminated from the system leading to a recursive problem in which the predetermined variables form the state vector. The connection between equation (27) and Epstein-Zin preferences comes from the fact that risk-sensitive preferences are a special case of Epstein-Zin preferences. Finally, the connection between equation (27) and ambiguity/uncertainty aversion follows from Hansen and Sargent (2007).

4 Robust policymaking with imperfect credibility in Smets and Wouters (2007)

To examine robust policymaking with imperfect credibility we use as our approximating model the DSGE model that Smets and Wouters (2007) estimate for the U.S. We use this model for several reasons. First, the model has been found to provide a reasonably good description of U.S. economic outcomes. Second, the model forms the basis for many related models and its widespread usage, together with its empirical support, make it a sensible choice. Third, the model’s structure accommodates many shocks, which from the robust control perspective, represent sources of potential misspecification. Fourth, private agents are forward-looking, allowing central-bank credibility to influence private-sector decisionmaking.

Because the Smets and Wouters (2007) model is widely known, we describe only its main characteristics here and refer interested readers to the original text. The economy is populated
by three types of agents: households, firms, and a central bank. Households own the capital stock and the equity in firms and receive income from dividends and from renting capital and supplying labor to firms. Households use their income to purchase goods that they allocate between consumption and investment in order to maximize expected lifetime utility. Goods allocated to investment augment the capital stock, subject to a resource cost associated with changing the investment-flow. Households gain utility from consumption (subject to an external consumption habit) and from leisure, and they are monopolistically competitive suppliers of their labor, setting their wage subject to a Calvo-style wage rigidity. Those households that are unable to change their wage in a given period are assumed to index their wage to lagged aggregate inflation. On the production side, firms are monopolistically competitive; they rent capital and hire labor and produce according to a constant-returns Cobb-Douglas production function. Firms choose how much capital and labor to employ and set prices in order to maximize the expected present discounted value of the firm, subject to a Calvo (1983) price rigidity and price indexation. Profits are returned to households in the form of a lump-sum dividend. Finally, the goods that firms produce are combined according to a Kimball (1995) technology to produce a final good that is sold to households in a perfectly competitive market.

Although Smets and Wouters (2007) characterize monetary policy in terms of an estimated Taylor-type rule, our focus is on optimal policymaking. Accordingly, we take the “primal” approach and replace their estimated policy rule with one chosen in order to minimize the following loss function

\[
L(x_t, y_t, u_t) = \pi_t^2 + \lambda(y_t - y_t^f)^2,
\]

where \(\pi_t\) denotes annualized quarterly inflation, \(y_t\) denotes output, \(y_t^f\) denotes flex-price output, and \(y_t - y_t^f\) denotes the output gap. The parameter \(\lambda \in [0, \infty)\) governs the weight assigned to stabilizing the output gap relative to stabilizing inflation. The model is log-linearized about a zero-inflation balanced growth path and is subject to six shocks, including those to the aggregate production technology, the investment-specific production technology, and to the price and wage markups. These six shocks obscure potential specification errors. In our main analysis we parameterize the model using the coefficient estimates provided by the posterior mean (Smets and Wouters, 2007, Tables 1A and 1B) and set \(\lambda = 0.25\). Subsequently, we consider a loss function in which output stabilization is weighted more highly than inflation stabilization and set \(\lambda = 4.00\).
Although there are six shocks in the model, we focus below on the effects of shocks to the price markup and to aggregate technology. In light of its policy objectives, the central bank always offsets the effects of the shock to the neutral interest rate, and the qualitative story that emerges regarding the effects of robustness and imperfect credibility on policymaking is consistent across the other three shocks.

4.1 The effects of imperfect credibility and robustness on the response to price-markup shocks

For a range of assumptions about credibility, Figure 1 displays the responses of inflation, the output gap, and the nominal interest rate to a one-standard-deviation price-markup shock. The panels in the left-most column of Figure 1 (panels A, D, and G), show the (nonrobust) responses under commitment and discretion and the announced and expected responses under imperfect credibility ($\alpha = 0.75$). The announced responses are constructed from the model’s solution along a path in which $\eta_t = 1$; they correspond to the forward-guidance released by the central bank, released in the form of a state-contingent forecast.

In response to the price markup shock, inflation rises (panel A) and a negative output gap opens up (Panel D). The negative output gap is somewhat larger when policy is conducted under commitment than under discretion, a consequence of the discretionary stabilization bias, which leads the discretionary policymaker to inefficiently trade-off movements in the output gap and inflation. Looking now at the equilibrium responses according to the announced policy, with $\alpha = 0.75$ the central bank announces a policy that tightens less rapidly (Panel G), but keeps interest rates higher for longer than the commitment policy. Thus, in order to stabilize inflation the imperfectly credible central bank attempts to leverage the credibility it has by announcing a policy that is tighter for longer than the commitment policy. This result is consistent with Bodenstein, Hebden, and Nunes (2010), who also found that imperfectly credible central banks seek to leverage their credibility by making more extreme policy announcements. Interestingly, the announced policy looks qualitatively much more like the discretionary policy than the commitment policy, with the shock leading to a permanent increase in the price level.
Having seen how imperfect credibility drives a wedge between the forward-guidance provided under commitment and discretion, we now investigate in more detail the effects that imperfect credibility has on the announced policy. To this end, the panels in the middle column of Figure 1 (panels B, E, and H) display, for varying values of $\alpha$, the announced policy responses following a one-standard deviation price markup shock. When $\alpha = 1$, and the
central bank is perfectly credible, the impact effect of the markup shock is to raise inflation (panel B) and lower the output gap (panel E), but the responses are muted relative to those under discretion. Because the central bank is committed to returning inflation to target (here the inflation target is a rate of zero), the commitment policy anchors long-run inflation expectations firmly on the target. With inflation expectations anchored on the target, monetary policy in the short-run can be directed at stabilizing the output gap. According to the commitment policy, the central bank raises interest rates in the short-run (panel H) while promising to subsequently lower rates as inflation declines. In contrast, under discretion inflation expectations are not well-anchored and in order to stabilize inflation the central bank must tighten monetary policy, raising interest rates for a sustained period. Whereas under commitment the promise of loose monetary policy in the future complementing tight monetary policy today is sufficient to stabilize inflation, under discretion the central bank must quickly embark on a policy of higher interest rates.

To see how imperfect credibility affects the policy that the central bank announces it will pursue in response to the price-markup shock, consider the responses associated with $\alpha = 0.95$. With its credibility high, the central bank is able to keep inflation expectations reasonably well-anchored. However, with default a possibility the central bank promises to keep interest rates higher for longer than the commitment policy, compensating for its lack of credibility with a more extreme policy announcement. Indeed, as $\alpha$ declines and the central bank’s credibility diminishes, its announced policies become more focused on stabilizing the output gap in the short-run with inflation stabilized by an announced policy of keeping policy tight for a sustained period, much like discretion.

Lastly, the panels in the third column of Figure 1 (panels C, F, and I) illustrate the effect that the central bank’s concern for robustness has on its policy announcements. With the robustness parameter, $\theta$, chosen such that $\omega = \bar{\omega}$, the main results that emerge are the following. First, for this model robustness has only very small effects on policy announcements. The effects of credibility on policy announcements are quantitatively much more important than those of robustness. Second, the policy announcements made by a perfectly-credible central bank are unaffected by its desire for robustness.
4.2 The effects of imperfect credibility and robustness on the response to technology shocks

Figure 2 displays the responses of inflation, the output gap, and the nominal interest rate to a one-standard-deviation aggregate-technology shock. Although the model, obviously, behaves quite differently following aggregate technology shocks than it does following price-markup shocks, the conclusions regarding the qualitative effects of imperfect credibility and robustness are similar. Looking that the responses associated with the perfectly credible central bank ($\alpha = 1$), because a rise in aggregate technology allows more goods to be produced from a given set of inputs the effects of the shock are to raise output and lower inflation (panel A). Because the central bank’s policy objective function assigns a large relative weight to stabilizing inflation (recall, $\lambda = 0.25$), the effect of the technology shock on monetary policy is to lower the nominal interest rate (panel G). With the nominal interest rate declining more that one-for-one with inflation, the real interest rate declines and this stimulates aggregate demand, opening up a positive output gap (panel D) and creating upward pressure on inflation. As inflation rises, monetary policy begins to tighten and the positive output gap begins to close.

The panels in the left-most column of Figure 2 (panels A, D, and G) reveal that the imperfectly credible central bank ($\alpha = 0.75$) responds to the shock with a policy announcement that is more extreme than that of the perfectly-credible central bank. Indeed, the imperfectly credible central bank sees inflation systematically below target and announces a policy path that has low interest rates for a considerable period. At the same time, the responses to the technology shock are all relatively small, a consequence of the fact that with a policy objective directed at stabilizing the economy about its flex-price equilibrium, policy largely accommodates shocks to technology. The panels in the middle column of Figure 2 (panels B, E, and H) show that changes in credibility have relatively muted effects on both inflation and the output gap while having larger effects on the announced path for the interest rate. The panels in the third column of Figure 2 (panels C, F, and I) further highlight that the central bank’s desire for robustness has no effect on the announced policy provided by the perfectly-credible central bank, while having a larger, but still small, effect on the announced policy provided by the discretionary central bank.
4.3 How detectable are the specification errors?

Our analysis of robust policymaking has assumed that the robustness parameter, $\theta$, equals the threshold value, $\theta$. This assumption expresses the idea that the central bank is as concerned as it can be about the approximating model while still holding the view that the approximating
model is a useful approximation of the actual data-generating process. A consequence of this assumption is that the effects of robustness on the impulse responses shown in Figures 1 and 2 cannot be made more damaging through a different—and more pessimistic—choice of $\theta$. The fact that the effects of model misspecification appear small even with $\theta = \theta$ may well imply that the Smets and Wouters (2007) model can be destabilized by relatively small specification errors, even under a robust policy rule. Such a result would not be unexpected because it is well-known that the performance of optimal policy rules, which exploit fully a model’s structure can be very poor when that structure is incorrect (Levin, Wieland, and Williams, 2003), providing a popular argument for the use of optimized simple rules, which exploit less structure (McCallum, 1988). With these issues in mind, here we ask the question of whether the central bank is likely to be able to detect the specification errors and how their detection is affected by credibility.

To explore this question we employ the notion of a detection-error probability promoted is a series of papers by Hansen and Sargent (see, for example, Anderson, Hansen, and Sargent, 2003). A detection-error probability is the probability that an econometrician observing equilibrium outcomes would make an incorrect inference about whether the approximating equilibrium or the worst-case equilibrium generated the data. The intuitive connection between $\theta$ and the probability of making a detection error is that when $\theta$ is small, greater differences between the distorted model and the approximating model (more severe misspecifications) can arise, which are more easily detected. Figure 3 displays the relationship between the (log of the) robustness parameter $\theta$ and the probability of making a detection error for discretion ($\alpha = 0.00$), perfect credibility ($\alpha = 1.00$), and imperfect credibility ($\alpha = 0.50$).
Figure 3. The relationship between detectability, robustness, and credibility ($\lambda=0.25$)

Because the relationship between robustness and the detection-error probability can only be constructed for $\theta \geq \theta^*_0$. Figure 3 informs us that the validity of the approximating model breaks down under discretion for higher values of $\theta$ than it does under either commitment or imperfect-credibility. At the same time, at the respective break-down thresholds (values for $\theta$), the specification errors can be more easily detected under discretion and are hardest to detect for the intermediate case in which credibility is imperfect with $\alpha = 0.50$. Imperfect credibility,
therefore, lowers the break-down threshold value, but it does so because the specification errors are difficult to detect, suggesting that the specification errors are also relatively small, which is consistent with Figures 1 and 2.

4.4 Is robustness a substitute for credibility?

It is well-known that commitment is superior to discretion and that higher credibility leads to better economic outcomes when the model is correctly specified. However, when the central bank has doubts about its model and implements a policy that robust in the Hansen-Sargent sense, then the model’s equilibrium with robust policy, the approximating equilibrium, will (invariably) differ from the model’s equilibrium with nonrobust policy, the rational expectations equilibrium. This raises the question of whether, for a given level of credibility, the central bank’s desire for robustness produces an improvement or a deterioration in policy performance. Is the policy performance, measured according to the policy objective function (equation 28), associated with the robust policy higher or lower than that associated with the nonrobust policy and how is the relative performance of these two policies affected by credibility? To address these questions, Figure 4 examines the relationship between relative policy performance and robustness, \( \theta \), for varying levels of credibility, \( \alpha \). If robust-loss relative to nonrobust-loss is greater than one, then robustness leads to a deterioration in policy performance. Conversely, if robust-loss relative to nonrobust-loss is less than one, then robustness leads to an improvement in policy performance.
Figure 4. The effectiveness of robustness as a substitute for credibility ($\lambda=0.25$)

With higher values of $\theta$ associated with less concern for model misspecification, and with policy performance for a given $\theta$ measured relative to the benchmark in which the model is known to be correctly specified, relative policy performance (robust-loss-over-nonrobust-loss) converges to one as $\theta$ rises to infinity for all levels of credibility. Figure 4 shows that whether the equilibrium outcomes associated with the approximating equilibrium are superior
or inferior to those associated with the nonrobust equilibrium depends on $\alpha$, the level of credibility. When credibility is perfect ($\alpha = 1$) the central bank’s desire for robustness leads to a slight decline in relative policy performance. Similarly, when the central bank has zero-credibility ($\alpha = 0$) the central bank’s desire for robustness improves relative policy performance. Each of these findings is consistent with Dennis (2010), who showed that a central bank’s desire for robustness can serve somewhat like a commitment mechanism even when policy is conducted under discretion. The results here extend Dennis (2010) by establishing that a desire for robustness can act as a substitute for credibility and improve relative policy performance for most levels of credibility, and not just for the case of pure discretion ($\alpha = 0$). Indeed, only when credibility is very high—close to perfect—does the central bank’s desire for robustness worsen policy performance.

4.5 The effect of a greater weight on output stabilization

The results above were obtained under the maintained assumption that the relative weight assigned to output stabilization in the policy objective function is $\lambda = 0.25$. To assess whether our results are qualitatively sensitive to this assumed value for $\lambda$ we repeated the analysis, but under the maintained assumption that $\lambda = 4.00$, i.e., that the weight assigned to output stabilization is four times the weight assigned to inflation stabilization. Although the nature of the impulse response functions do change, simply reflecting the greater importance that the central bank places on output stabilization, the qualitative results do not change. With the robustness parameter set to its threshold value, the effects of robustness on the impulse response functions is relatively small, considerably smaller than the effects of imperfect credibility. Further, the central bank’s lack of credibility continues to motivate it to leverage what credibility it has by announcing a policy that is more extreme than that associated with perfect credibility. In addition, the central bank’s desire for robustness continues to generate improved policy performance, except when credibility is close to perfect.

5 Conclusion

This paper has considered the decision problem facing an imperfectly credible central bank that seeks to conduct monetary policy using a model whose structure it has doubts about. Motivating this study is the increased use by central banks, including the Federal Reserve System, of policy announcements in the form of model-based forecast-contingent forward-guidance...
about future policy. In this paper, the central bank’s doubts about its model are modeled via the robust control literature, giving rise to a maxmin problem as per Hansen and Sargent (2008), while imperfect credibility is modeled according to the quasi-commitment literature, producing an equilibrium in which the central bank stochastically defaults, or reneges, on its policy promises. The resulting decision problem allows us to study separately, and in combination, the effects that robustness and imperfect credibility have on central bank behavior and economic outcomes, within a framework that accommodates commitment, discretion, quasi-commitment, robust control, and nonrobust control as special cases.

With the Smets and Wouters (2007) model providing the laboratory for analysis, our examination of robust policymaking with imperfect credibility offers the following main findings. First, a central bank’s credibility gives it a powerful lever for managing private-sector expectations and for stabilizing the economy. This lever delivers optimal outcomes when the model is correctly specified, and it provides robustness by ensuring private-sector expectations remain well-anchored even when the model is misspecified. Second, when a central bank has low credibility the economy can benefit from the central bank’s desire for robustness. In this regard, a desire for robustness can act somewhat as a substitute for credibility. Third, even relatively small departures from perfect credibility produce big declines in policy performance, giving rise to a form of discretionary stabilization bias.

A Appendix: Risk-sensitive preferences and robust control with quasi-commitment

We showed in section 3 that the solution to the robust-control problem under discretion could equivalently be obtained from a formulation with risk-sensitive preferences. In this appendix we extend that result to establish a risk-sensitive formulation that is equivalent to the robust-control decision problem with quasi-commitment. One simplifying assumption that we make is that \(A_0\) has full rank. Without loss of generality, then, we assume that \(A_0 = I\). With this assumption, and assuming \(\alpha \in (0, 1)\) (ruling out the discretionary case), the constraints according to the approximating model can be written as

\[
z_{t+1} =Az_t + Bu_t + C\varepsilon_{xt+1},
\]

while those according to the distorted model can be written as

\[
z^*_t = Az_t + Bu_t + C(v_{t+1} + \varepsilon_{xt+1}),
\]

where \(z_t = [x'_t, y'_t]'\). To show the connection between the robust-control problem and the risk-sensitive preferences problem it is convenient to utilize the solution strategy of Backus and Driffill (1986) which begins by treating \(z_t\), which contains nonpredetermined variables, as the state vector. Accordingly, the risk-sensitive preferences formulation takes the form

\[
z'_t Pz_t + p = \min_{ut} \left[ L(z_t, u_t) + \frac{1}{\sigma} \ln \left( E_t \left( \exp \left( \sigma \left( z'_{t+1} Pz_{t+1} + p \right) \right) \right) \right) \right],
\]

where \(\sigma\) is a constant, and \(P\) is the same covariance matrix as in the approximating model. The solution to this problem is given by

\[
z_t^{**} = Az_t + Bu_t + C(v_{t+1} + \varepsilon_{xt+1})
\]

which is the same as the solution to the robust-control problem under quasi-commitment.
where \( \sigma \leq 0 \),
\[
\tilde{P} = \alpha P + (1 - \alpha) S'S' - 1 P S - 1 S,
\]
with \( S = \begin{bmatrix} I & 0 \end{bmatrix} \), and the constraints are given by equation (A1). Employing Jacobson (1973), equation (A3) can be expressed as
\[
z_t' P z_t + p = \min_{u_t} \left[ L(z_t, u_t) + \beta E_t \left( z_{t+1}' D(\tilde{P}) z_{t+1} + \tilde{p} \right) \right], \quad (A4)
\]
where
\[
D(\tilde{P}) = \tilde{P} + \sigma \tilde{P}C \left( I + \sigma C' \tilde{P}C \right)^{-1} C' \tilde{P}.
\]
In contrast, the robust control formulation of the decision problem is
\[
z_t' P z_t + p = \min_{u_t} \max_{v_{t+1}} \left[ L(z_t, u_t) - \beta \theta v_{t+1}' v_{t+1} + \beta E_t \left( z_{t+1}' \tilde{P} z_{t+1} + p \right) \right], \quad (A5)
\]
where \( \theta \geq 0 \) and the constraints are given by equation (A2). Performing the inner maximization yields
\[
v_{t+1} = E_t \left[ \frac{1}{\theta} \left( I - \frac{1}{\theta} C' \tilde{P}C \right)^{-1} C' \tilde{P} z_{t+1} \right]. \quad (A6)
\]
Substituting equation (A6) back into equation (A5) gives
\[
z_t' P z_t + p = \min_{u_t} \left[ L(z_t, u_t) + \beta E_t \left( z_{t+1}' \tilde{P} \left( I - \frac{1}{\theta} C' \tilde{P}C \right)^{-1} C' \tilde{P} \right) z_{t+1} + \tilde{p} \right], \quad (A7)
\]
where the constraints are now given by the approximating mode, equation (A1).

With \( \theta = -\sigma^{-1} \) the solutions to equations (A4) and (A7) lead to the same decision rule
\[
u_t = F z_t.
\]

Following Backus and Driffill (1986) the next step is to transform the solution from one depending on predetermined and nonpredetermined variables to one depending on predetermined and costate variables, where the latter are the analogue of the multipliers \( \Xi_{t-1} \) in the text. Accordingly, let
\[
\Xi_{t-1} = P_{21} x_t + P_{22} y_t,
\]
where \( P_{21} \) and \( P_{22} \) are submatrices of \( P \). Then the solution has
\[
H = -P_{22}^{-1} P_{21},
C_2 = -P_{22}^{-1} P_{21} C_1. \quad (A8)
\]

Note that substituting equation (A8) into equation (A6) leads to
\[
v_{t+1} = E_t \left[ \frac{1}{\theta} \left( I - \frac{1}{\theta} C_1 \left( P_{11} - P_{21}' P_{22}^{-1} P_{21} \right) C_1 \right)^{-1} C_1 \left( P_{11} - P_{21}' P_{22}^{-1} P_{21} \right) x_{t+1} \right],
\]
which further establishes that the worst-case specification errors depend only on the the expected future predetermined variables (Dennis, 2008).
Appendix: Detection-error probability

Let \( A \) denote the approximating model and \( B \) denote the worst-case model; then, assigning equal prior weight to each model and assuming that model selection is based on the likelihood ratio principle, Hansen, Sargent, and Wang (2002) show that detection-error probabilities are calculated according to

\[
p(\theta) = \frac{\text{prob}(A|B) + \text{prob}(B|A)}{2},
\]

where \( \text{prob}(A|B) \) (\( \text{prob}(B|A) \)) represents the probability that the econometrician erroneously chooses \( A \) (\( B \)) when \( B \) (\( A \)) generated the data. Let \( \{z^B_t\}^T_1 \) denote a finite sequence of economic outcomes (the shocks, the shadow prices, the endogenous variables, and the followers’ and leader’s decision variables) generated by the worst-case equilibrium, and let \( L_{AB} \) and \( L_{BB} \) denote the likelihood associated with models \( A \) and \( B \), respectively; then the econometrician chooses \( A \) over \( B \) if \( \log(L_{BB}/L_{AB}) < 0 \). Generating \( M \) independent sequences \( \{z^B_t\}^T_1 \), \( \text{prob}(A|B) \) can be calculated according to

\[
\text{prob}(A|B) \approx \frac{1}{M} \sum_{m=1}^M I[\log(L_{\text{BB}}^m/L_{\text{AB}}^m) < 0],
\]

where \( I[\log(L_{\text{BB}}^m/L_{\text{AB}}^m) < 0] \) is the indicator function that equals one when its argument is satisfied and equals zero otherwise; \( \text{prob}(B|A) \) is calculated analogously using data generated from the approximating model.

Let

\[
\begin{align*}
  z_{t+1} &= H_A z_t + G \varepsilon_{t+1} \\
  z_{t+1} &= H_B z_t + G \varepsilon_{t+1}
\end{align*}
\]

are the inferred innovations in period \( t + 1 \) when model \( i \) is fitted to data \( \{z^i_t\}^T_1 \) generated from model \( j \), and let \( \hat{\Sigma}^{ij} \) be the associated estimates of the innovation variance-covariance matrices. Note that the Moore-Penrose inverse picks out the shock process from among the variables in \( z_t \).

Assuming that the innovations are normally distributed, it is easy to show that

\[
\begin{align*}
  \log \left( \frac{L_{AA}}{L_{BA}} \right) &= \frac{1}{2} \text{tr} \left( \Sigma^{B|A} - \hat{\Sigma}^{A|A} \right) \\
  \log \left( \frac{L_{BB}}{L_{AB}} \right) &= \frac{1}{2} \text{tr} \left( \Sigma^{A|B} - \hat{\Sigma}^{B|B} \right).
\end{align*}
\]

References


