Robust Self-Insurance

Abstract

We examine the problem of an agent who faces income risk and who insures himself by accumulating holdings of a riskless bond, subject to a no-borrowing constraint. The problem is novel since the agent is concerned that his benchmark model is misspecified. We use robust control analysis to model this concern. We consider two benchmark models of income risk - a simple one-shock case and a two-shock case. We characterize the ‘worst case’ models entertained by the agent and examine how their properties inform his actions. We show that the agent’s fears can be represented by processes featuring an adverse shift in the distribution of his income. In the two shock case the worst case model exhibits positive correlations between the income components - even though no such correlation exists under the benchmark. Overall the agent fears a world in which he is driven to low levels of wealth more rapidly than in his benchmark model. This enhances his desire to accumulate wealth, from a precautionary motive. Importantly, the distortions to the conditional distributions of innovations are more pronounced when the agent’s wealth level is low. Thus the pessimism of the robust agent is state dependent. This opens an interesting avenue to feedbacks between the agent’s pessimism and the decisions that are, in turn, influenced by this pessimism. Although this paper currently only deals with the partial equilibrium process of an individual agent, it hints at a promising approach to endogenizing disagreement in heterogeneous agent models and raises issues regarding how econometricians should identify orthogonal latent risk factors based on the behavior of robust households.
1 Introduction

Currently, one of the most active areas of research in macroeconomics is the development of a framework for agents’ behavior that acknowledges the influence of model uncertainty - a situation in which agents are unable to put a unique prior over alternative distributions implied by well defined models. The vigor with which this research is proceeding reflects the fact that existing treatments of choice under uncertainty have been shown to be inadequate in many dimensions by a large body of experimental evidence and can lead to estimates of parameters governing attitudes to well defined risks that seem highly implausible. The properties of the recent financial crisis and great contraction have also spurred research in this area since economic modeling is in a state of flux. This reflects the combination of unusual situations that agents now face, such as the attainment of the zero lower bound, the prosecution of unconventional policies and the questioning of hitherto trusted financial strategies.

One of the many approaches to modeling choice under model uncertainty or ambiguity is the use of Robust Control analysis, as advocated by Hansen and Sargent (2008). Robust control provides a formal approach to capturing an agent’s fear of model misspecification and how the agent behaves in this context. We will apply these methods within simple models of self-insurance in which an agent attempts to insure himself against sources of risk in the income process he confronts, using only a risk free bond and subject to a no borrowing constraint. The agent is endowed with multiplier preferences, expressing his distrust of his ‘benchmark’ model. We begin with a simple setup in which the agent faces only one source of risk in his income process and then also consider a two shock case. Although our analysis is currently only partial equilibrium, many of the insights obtained hint at possibly fruitful work in related general equilibrium models.

The agent expresses his doubts of his model by considering alternative distributions that
are distorted versions of the distribution implied by his benchmark model. In order to con-
struct a robust policy the agent considers adverse distributions and balances the damage
that an implicit misspecification would cause him, against the plausibility of the misspec-
ification. The distribution that emerges from this problem can be thought of as a ‘worst
case distribution’ that encodes these concerns and allows insight into the fears that guide the
agent’s decisions. Thus, the agent derives a policy that, while suboptimal in the context of
his benchmark model, protects him against suspected misspecifications in that benchmark.

We show that the worst case distribution has interesting properties. It indicates that
the agent fears misspecifications that would imply lower income on average, relative to his
benchmark model. Informed by this pessimistic scenario, the agent chooses to accumulate
a larger buffer of savings than he would, were he to trust his model fully. In addition, the
agent’s pessimism (as captured in the worst case distribution) is shown to be state dependent.
He is more pessimistic the lower is his current asset level. Intuitively, this is because the
consequences of a detrimental misspecification that implies lower income and assets holdings
relative to the benchmark are more severe in situations in which the agent is close to the
point where he will be required to make drastic reductions in his consumption if his asset
level falls further. This increased marginal cost to the agent of misspecification when wealth
is low is captured by increased curvature in the value function.

When we introduce a second risk component to the income process we find that the agent’s
worst case distribution implies correlation between the components, even though they are in-
dependent under the benchmark. This result hints at several interesting avenues. Firstly,
it suggests that an econometrician who ignores the possibility of robustness might struggle
to identify agents’ true aversion to well defined risks if the agent’s behavior is informed by
distributions that inject correlations among variables that imply reduced scope for diversifi-
cation. In addition, the results are suggestive that in related general equilibrium models a
robust agent might be guided by worst case distributions that feature correlation between
idiosyncratic and systemic risk components. For example, one would wish to avoid receiving an idiosyncratic shock that necessitates borrowing for consumption smoothing at the same time that the economy is being affected by an aggregate shock that makes borrowing more expensive. Furthermore, this suggests that an econometrician attempting to glean information about latent risk components from agents’ behavior might come to incorrect conclusions if he ignores the possibility of agents seeking robustness. At least qualitatively, this has cautionary implications for the practice of extracting idiosyncratic and systemic risk components in income, based upon consumption and earnings behavior, although the quantitative implications in our simple framework are likely small.

We base our characterizations of the worst case distribution on distorted transition matrices that are obtained by ‘twisting’ the conditional distributions implicit in the transition matrices governing the evolution of shocks under the benchmark model. This pessimistic change of measure is state dependent and based upon the agent’s perception of his welfare in different states of the world, as encoded in the properties of the value function. In earlier work, Bidder and Smith (2011a) and Bidder and Smith (2011b) it was shown how to capture worst case conditional distributions in the context of ‘smooth’ models solved with perturbation methods and with continuous shock processes. Although, much insight can be gained from such models, it seems important to consider models with discrete shock processes and non-smooth properties since these seem more likely to induce dramatic, yet plausible, worst case scenarios. In this paper, we adopt the popular self-insurance model to address these concerns. The presence of the no borrowing constraint also introduces an element of non-smoothness.

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1 See Boyarchenko (2010) for an application of essentially the same technique
2 We also are exploring extreme shock processes as a way of extending the models in this paper
2 Literature

There is a rapidly growing body of work on robust control analysis in economics, including applications to models of asset pricing (Kleshchelski and Vincent (2008), Barillas, Hansen, and Sargent (2009), Drechsler (2010), Bidder and Smith (2011a)), permanent income models (Hansen, Sargent, and Tallarini (1999)), monetary policy (Woodford (2010), Billi (2010), Cogley, Colacito, Hansen, and Sargent (2008), Ellison and Sargent (2009) and Orlik and Presno (2009)), fiscal policy (Karantounias (2009)) and RBC models (Bidder and Smith (2011b)).

We choose to study a partial equilibrium self-insurance problem in which an agent faces stochastic income fluctuations. In doing so we draw on a large literature characterizing such models (see Schechtman and Escudero (1977), Chamberlain and Wilson (2000), Deaton (1991) and Carroll (1997) for example). Our decision to analyze a model of self-insurance reflects the fact that we wish apply our techniques and discuss our insights within a familiar framework that is the basis for much work in modern macroeconomics. Although currently in partial equilibrium form, our ongoing work seeks to extend the insights of robust control to general equilibrium, ‘Bewley’ models such as those discussed in Bewley (1983), Aiyagari (1994), Krusell, Smith, and Jr. (1998) and Kaplan and Violante (2010).

3 Robust Control

A robust agent is endowed with a ‘benchmark’ model but fears that it is misspecified. He is concerned that the world is actually described by a model that is similar to the benchmark but distorted in some way. Given his doubts, the agent seeks a policy that performs well across a set of plausible models that express different forms of mis-specification in the benchmark. A useful way of deriving such a robust policy is to consider its performance in the worst case among the set of distorted models, thereby putting a lower bound on the agent’s welfare. We
now formalize this intuition in the context of an abstract model, following the methodology described in [Hansen and Sargent (2008)]. Once we have established the conceptual structure of a robust control problem, we will transfer our attention to a particular production economy.

3.1 Martingales, Martingale Increments and Distorted Distributions

Within the robust control literature it is convenient to have a formal language for characterizing alternative models in relation to the benchmark. Recognizing that the equilibrium of an economic model is a probability distribution, it is natural to specify alternative models in terms of distortions of the distribution implied by the agent’s benchmark model. In [Hansen and Sargent (2005)] it is proposed that the distortions be characterized in terms of Martingales. These Martingales act as Radon-Nikodym derivatives by twisting the measures implicit in the benchmark model so as to obtain absolutely continuous measures that represent alternative models considered by the agent. Under these twisted measures one can form objects interpretable as expectations taken in the context of the distorted alternative model.

Formally, let $\mathcal{I}_t$ be information available at $t$. Then define a non-negative $\mathcal{I}_t$ measurable function $M_t$ such that $E[M_t|\mathcal{I}_0] = 1$. This function can be used to derive a probability measure that is absolutely continuous with respect to the measure over $\mathcal{I}_t$ implied by the benchmark model. With respect to the undistorted measure, $M_t$ is a Martingale. Using this Martingale we can define distorted expectations as follows

$$\tilde{E}[W_t] = E[M_tW_t]$$

As a measure of how different the distorted measure is from the undistorted measure associated with the benchmark model, we use the concept of entropy, conditional on time-zero information $E[M_t log M_t | \mathcal{I}_0]$. Analytically and, as we shall see, computationally, it is convenient to factorize the Mar-
tingale, $M_t$, into a sequence of increments, $m_t$. Thus

$$m_{t+1} = \frac{M_{t+1}}{M_t} \quad (2)$$

Using this Martingale increment we can define a distorted conditional expectation for a $\mathcal{F}_{t+1}$-measurable random variable, $b_{t+1}$, given $\mathcal{F}_t$ and, more generally, use $m_{t+1}$ to capture the distortion of the conditional distribution of $b_{t+1}$ given $\mathcal{F}_t$

$$\frac{E[M_{t+1}b_{t+1}|\mathcal{F}_t]}{E[M_{t+1}|\mathcal{F}_t]} = \frac{E[M_{t+1}b_{t+1}|\mathcal{F}_t]}{M_t} = E[m_{t+1}b_{t+1}|\mathcal{F}_t] \quad (3)$$

Thus we define a distorted conditional expectation operator to be

$$\tilde{E}[b_{t+1}|\mathcal{F}_t] \equiv E[m_{t+1}b_{t+1}|\mathcal{F}_t] \quad (4)$$

### 3.2 Multiplier Preferences

Let us suppose that the robust agent entertains a benchmark model in which the state, $x_t$, is related to other variables in the economy by a possibly nonlinear law

$$x_{t+1} = g(x_t, u_t, \epsilon_{t+1}) \quad (5)$$

where $u_t$ is a vector of controls and $\{\epsilon_t\}$ is a sequence of random variates. In order to construct the problem of a robust agent we posit a particular two-player zero-sum game

$$\max_{\{u_t\}} \min_{\{m_{t+1}\}} \sum_{t} E \left[ \beta^t M_t \left( r(x_t, u_t) + \beta \theta E \left( m_{t+1} \log m_{t+1} | \epsilon^t, x_0 \right) \right) \right] \quad (6)$$

where $r(\cdot, \cdot)$ is the agent’s period payoff function, $\epsilon^t$ is the history of innovations through $t$ and the problem is subject to the state evolution equation, (5), and $M_{t+1} = m_{t+1} M_t$, 
\( E[m_{t+1}|\epsilon', x_0] = 1, m_{t+1} \geq 0 \) and \( M_0 = 1 \).

The agent’s desire for robustness is reflected in the minimization over the sequence of Martingale increments (chosen by a metaphorical ‘evil’ player) that twist the distributions used to evaluate continuation values towards realizations of the state that are painful to the agent. The degree of robustness is controlled by the penalty parameter \( \theta \) that enters in the objective by multiplying the term reflecting the entropy associated with a given distortion. The penalty reflects our earlier intuition that the agent considers models that, although different, are somehow ‘near’ the benchmark. For \( \theta > 0 \) the agent is penalized for considering distortions of his benchmark model. Thus, a particularly implausible model may imply dynamics that are painful for the agent but its negative effects on the objective is offset by a positive contribution reflecting its high entropy and, thus, the associated increments do not solve the inner minimization problem.

Before obtaining a recursive expression of the problem, it is convenient and to partition the state, \( x_t \) into elements unknown on entering the period, which we identify with \( \epsilon_t \), and those elements that are predetermined, denoted \( s_t \). We capture the dependence of \( s_t \) on the state prevailing in the previous period by the function \( f \), such that \( s_t = f(x_{t-1}) \). The function \( f \) is determined by the law of motion \(^5\) and the relationship between the control and the state that captures the agent’s behavior.

We seek a recursive expression of the problem and, invoking results in Hansen and Sargent (2008), obtain a value function of the following form

\[
V(\epsilon, s) = \max_u \min_{m(\epsilon', s')} r(x, u) + \beta \int (m(\epsilon', s')V(\epsilon', s') + \theta m(\epsilon', s') \log m(\epsilon', s')) p(\epsilon') d\epsilon' \tag{7}
\]

subject to \( \int m(\epsilon, s)p(\epsilon)d\epsilon = 1 \) for all values of \( s \). If one solves the inner minimization problem (interpretable as that of the ‘evil’ agent) one obtains the minimizing Martingale increment,
which has the form
\[
m^*(\epsilon', s') = \frac{e^{-\frac{V(\epsilon', x')}{\theta}}}{E\left[e^{-\frac{V(\epsilon', x')}{\theta}} | \epsilon, s\right]} \quad (8)
\]
If one imposes this solution to the minimization problem, then we obtain the following expression for the Bellman equation
\[
V(x) = \max_u r(x, u) - \beta \theta \log E \left[ \exp \left( -\frac{V(x')}{\theta} \right) | x \right] \quad (9)
\]

It is worth noting that this Bellman equation takes the same form as that of agent with risk sensitive preferences (see Hansen and Sargent (1995) and Whittle (1991)). Under the risk sensitivity interpretation, however, \( \theta \) reflects sensitivity to well defined, quantifiable risk whereas here it reflects the degree to which the agent fears model mis-specification.

### 3.3 Drawing from the Worst Case Distribution

Based on the above analysis one can characterize the (possibly state dependent) worst case conditional distributions using Monte Carlo methods. Characterizing these distributions allows insight into the concerns of the robust agent. In particular, by combining draws from the distorted conditional distributions with the agent’s policy functions and other laws of motion in the economy, we are able to construct draws from the worst case distribution over sequences entertained by the agent.

Firstly, note that the minimizing Martingale increment implies a distorted conditional distribution of \( \epsilon' \) as follows
\[
\tilde{p}(\epsilon'|x) \equiv m(x')p(\epsilon') \\
\equiv m(\epsilon', s)p(\epsilon') \\
\equiv m(\epsilon', f(x))p(\epsilon') \quad (10)
\]
where we recall that $p$ is the density of $\epsilon$ under the benchmark model. The conditional distribution $\hat{p}$ represents the twisted probability density function associated with the worst case distribution. In particular, note that the twisting by the Martingale increment can introduce dependence among the shocks and upon $x$, even if there is none in the benchmark.

While $\hat{p}$ is not directly interpretable as the ‘beliefs’ of the agent, the fact that it differs from $p$ emphasizes that, unlike under rational expectations, more than one distribution plays a role in the equilibrium. If the agent’s benchmark model happens to be correct then the agent’s behavior will seem inconsistent with the objective probabilities implied by the benchmark. Given the state dependence on $x$, these deviations from what might be expected under the benchmark may vary over time.

In certain frameworks, the distorted distribution may take a tractable form. In particular, if the model is linear, linear-quadratic or expressed in continuous time then one can often directly compute the distortions implied by the worst case (see Barillas, Hansen, and Sargent (2009) for a particularly simple case). This simplicity is typically lost in more general settings. Nevertheless, as described in Bidder and Smith (2011a), the ability to evaluate the distorted pdf is enough to (approximately) draw from the distorted distribution using Monte Carlo methods. However, evaluating the distorted pdf requires that the Martingale increment itself be evaluated. In general one will not have access to a closed form for the increment. In this case one must use an approximation based on the increment directly or on an approximation to the value function.

We substitute the function $m$ that relates the increment to the state, with an approximation $\tilde{m}$. Assuming that we also approximate the function $f$ with $\tilde{f}$, we arrive at an
approximate distorted pdf $\tilde{p}$ where

$$
\tilde{p}(\epsilon'|x) \equiv \tilde{m}(x') p(\epsilon') \\
\equiv \tilde{m}(\epsilon', s') p(\epsilon') \\
\equiv \tilde{m}(\epsilon', \tilde{f}(x)) p(\epsilon')
$$

(11)

If the shock process in the model is such that the conditional distribution of innovations
is discrete or such that the continuous distribution can be well approximated by discretized
version then we can represent the conditional distribution under the benchmark as $\{p(\epsilon_i')\}_{i=1}^{n_e}$. Clearly, a model with shocks following Markov chains would conform to this setup\(^3\) In this
case, it is natural to obtain a distorted distribution under the worst case as $\{\tilde{p}(\epsilon_i'|x)\}_{i=1}^{n_e}$
where

$$
\tilde{p}(\epsilon_i'|x) \equiv p(\epsilon_i') \frac{\exp \left\{ -\frac{V(\epsilon_i', s')}{\theta} \right\}}{E \left[ \exp \left\{ -\frac{V(\epsilon_i', s')}{\theta} \right\} | x \right]}
$$

Note that $s'$ will be known one period in advance so the $\{\tilde{p}(\epsilon_i'|x, k)\}_{i=1}^{n_e}$ can be easily
calculated since $p(\epsilon_i')$ will typically be known (obtained from a Tauchen or Rouwenhorst
approximation, for example), $\exp \left\{ -\frac{V(\epsilon_i', s')}{\theta} \right\}$ can be calculated using whatever solution or
approximation for the value function is available and $E \left[ \exp \left\{ -\frac{V(\epsilon_i', s')}{\theta} \right\} | x \right]$ can similarly be
easily calculated. We can then use this approximated distribution to simulate the worst case model and/or obtain conditional distorted moments of random variables, conditional on $x$.

### 3.4 Detection Error Probabilities

In order to discipline the choice of $\theta$, the parameter that controls the degree to which the agent doubts his model, we relate $\theta$ to the plausibility of the set of models expressing those doubts. In doing this we follow \[Hansen and Sargent (2008)\] who propose the use of de-

\(^3\)See \[Bidder and Smith (2011a)\] and \[Bidder and Smith (2011b)\] for applications with continuous shocks
tection error probabilities to characterize how dissimilar is the worst case model from the agent’s benchmark. Detection error probabilities express how difficult it is, with a limited amount of data, for an agent to distinguish between the worst case and benchmark model. If an agent has difficulty distinguishing between the two models then the mis-specification implicit in the distorted model is regarded as one that an agent might plausibly seek robustness against. If, however, the agent can easily distinguish between the two models then the implicit mis-specification is one against which an agent is unlikely to seek robustness. Formalizing ‘plausibility’ in this way restricts the modeling freedom arising from introducing the additional parameter, $\theta$. One can obtain detection error probabilities by applying likelihood ratio tests, combined with a prior over the benchmark and worst case model, and calculating the probability with which the agent will mis-identify the model that generated the data. We have not yet calculated detection error probabilities for the models below, but intend to use a variant of the approach discussed in Bidder and Smith (2011a).

4 Model

In this section we outline the model specifications we will consider. We begin with the most simple case, a simple self-insurance problem with a single source of risk in the income process and a constant bond price. In all cases the agent will be endowed with multiplier preferences.

4.1 Constraints and Preferences

Our agent derives income according to a random process $z$ with the interpretation of capturing income fluctuations, combined with a base wage, $w$. The resources gained from this income can be devoted to consumption, $c$ or to accumulation of holdings of a risk free discount bond at price $\frac{1}{1+r}$. The agent, starting with asset holdings of $a$ at the beginning of a period may choose to run down his assets, such that $a' \leq a$ but his asset level is bounded by zero - a
simple no borrowing constraint.

\[ \frac{a'}{1+r} + c = w \exp \{z\} + a \]
\[ c \geq 0 \]
\[ a \geq 0 \]

In the two shock case, the income process will feature two risk components (here assumed to be identically and independently distributed) such that the budget constraint becomes

\[ \frac{a'}{1+r} + c = w \left( \frac{\exp \{z_1\} + \exp \{z_2\}}{2} \right) + a \]

Both components are assumed to follow a Markov chain derived from the same AR(1) process, where that process has the same autoregressive parameter as in the one-shock case but with a standard deviation parameter that is scaled to equate the standard deviation of the average of the two components to that of the single-shock case.

We specify preferences over consumption sequences using a particular form of the indirect utility function (9) discussed in section 3, with the obvious extension in the two shock case.

\[ V(z, a) = \max_{c, a'} \left\{ c^{1-\varphi} - \frac{1}{1-\varphi} - \beta \theta \log E \left[ \exp \left\{ -\frac{V(z', a')}{\theta} \right\} \mid z \right] \right\} \]

4.2 Calibration

The calibration of the model is shown in table 1. The calibration partly follows Aiyagari (1994) in that the AR(1) parameterization that underpins our baseline discrete shock process
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Value</th>
<th>Parameter</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>Discount factor</td>
<td>0.96</td>
<td>$\sigma$</td>
<td>AR(1) condit. s.d.</td>
<td>0.19</td>
</tr>
<tr>
<td>$\varphi$</td>
<td>CRRA-EU</td>
<td>3</td>
<td>$r$</td>
<td>Net interest rate</td>
<td>0.034</td>
</tr>
<tr>
<td>$\theta$</td>
<td>Robustness</td>
<td>1.8</td>
<td>$w$</td>
<td>Base wage</td>
<td>1.22</td>
</tr>
<tr>
<td>$\rho$</td>
<td>AR(1) persistence</td>
<td>0.3</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 1: Calibration

adheres to one of the calibrations in that paper and the CRRA parameter is also comparable. The model considered here however is simply (for now) partial equilibrium and we choose a net interest rate that is exogenous to the system. Again, although this is partial equilibrium analysis, the wage rate choice is guided by appealing to a neoclassical production function and firm optimality conditions that would apply in an associated general equilibrium model.

We have not yet undertaken a formal calculation of detection error probabilities (see Hansen and Sargent (2008) and Bidder and Smith (2011a)) in order to assess the calibration of the robustness parameter, $\theta$. However, some insight into whether or not this calibration is plausible can be obtained from examining the benchmark and distorted transition matrices, shock and asset simulations to be discussed in section 5.

5 Results

5.1 One Shock Case

In table 3 we display the nodes obtained from applying the Tauchen (1986) discretization method to the AR(1) discussed in section 4. Under the benchmark model, these nodes are associated with a transition matrix described in table 2a. Along with the undistorted transition matrix, table 2 also contains distorted transition matrices associated with the

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4 We calculate an ‘aggregate’ or ‘representative’ labor supply by integrating $\exp z_i$ with respect to the stationary distribution and, using the FOCs associated with a perfectly competitive firm operating the technology $k^{\alpha (1-\alpha)}$ (a sort of non-stochastic steady state), derive the optimal capital level given the interest rate (and assumed depreciation, $\delta$) and the associated wage rate as the marginal product of labor. In this case, $\alpha = 0.36$ and $\delta = 0.08$. 

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agent’s wealth being at the 1st, 10th and 90th percentiles, where the percentiles are calculated according to simulations of the benchmark economy in which the agent pursues his robust policy.

![Value Function - 1 shock case](image)

Figure 1: Value Function - 1 shock case (lines correspond to shock realizations)

We observe that the worst case transition matrices all exhibit pessimistic reallocations of probability mass from the higher shock realizations to the lower, which reflects the fact that the agent prefers more income to less and, thus, is fearful of misspecifications that would imply greater probability of lower realizations. Notably, the distortions are greater at the lower wealth deciles, where the implications of such a misspecification would be more severe as it would drive the agent closer to the no-borrowing constraint and raise the probability that a drastic reduction of consumption will be necessary. Mechanically, one can relate this state dependence of distortions to the additional curvature of the value function in the vicinity of the zero borrowing constraint, as shown in figure 1 where slices of the value function (for each shock realization) are plotted against the asset grid. Thus one obtains feedback between a robust agent’s pessimism and endogenous elements of the state. Thus it is a form of state dependence distinct from that discussed in Bidder and Smith (2011a) and Bidder and Smith (2011b), where pessimism depended on an exogenous state that drove a stochastic volatility process. This suggests scope for modeling of interesting propagation and
amplification mechanisms based on robustness. Furthermore, although we are here dealing
only with the partial equilibrium problem of a single agent, the results above suggests that in a
general equilibrium extension with heterogeneous agents and idiosyncratic wealth, one might
be able to generate phenomena that would appear like disagreement by inducing different
conditional worst case distributions among the agents.\footnote{See \cite{cogley2012} for an interesting analysis of heterogeneous beliefs in a bond
economy, with a different concept of pessimism.}

Using the distorted transition matrices, we can simulate the worst case economy under the
robust policy and also derive the implied stationary distribution of shocks. Figure \ref{fig:stationary_distribution} illustrates
the stationary distribution across shock nodes under the benchmark and worst case models.
We observe clearly that the worst case stationary distribution is positively skewed, with more
mass at the lower nodes, which is unsurprising given the aforementioned transition matrices.

Since we work with a discretized asset space and because the agent’s policy function maps

\begin{table}
\centering
\begin{tabular}{cccc}
\hline
Node & 1 & 2 & 3 & 4 \\
Values & 0.54881 & 0.81873 & 1.2214 & 1.82212 \\
\hline
\end{tabular}
\caption{Shock Nodes - 1 Shock Case (exponentiated nodes obtained from AR(1) by Tauchen method)}
\end{table}

\begin{table}
\centering
\begin{tabular}{cccc}
\hline
(a) Benchmark & (b) W.C. (1\textsuperscript{st} %-ile) & (c) W.C. (10\textsuperscript{th} %-ile) & (d) W.C. (90\textsuperscript{th} %-ile) \\
\hline
0.1243 & 0.70284 & 0.17154 & 0.00118 & 0.17695 & 0.70482 & 0.1177 & 0.0052 & 0.03737 & 0.58606 & 0.36862 & 0.00795 & 0.05548 & 0.65004 & 0.29033 & 0.00415 & 0.00795 & 0.36862 & 0.58606 & 0.03737 & 0.01206 & 0.44221 & 0.52278 & 0.02295 & 0.00118 & 0.17154 & 0.70284 & 0.12443 & 0.00178 & 0.2157 & 0.69339 & 0.08912 \\
\hline
\end{tabular}
\caption{Transition Matrices - 1 Shock Case (benchmark and worst case at various wealth percentiles)}
\end{table}
from the product of the discretized shock and asset spaces into the asset space, one can define an augmented first order Markov chain capturing the evolution of the shock-asset pairs under the worst case. Using the associated transition matrix, one can easily simulate the worst case and calculate interesting properties. Using this matrix and results of Seneta (1981) discussed in Boyarchenko (2010) we can calculate the expected time it will take to reach a particular state or set of states, given an initial situation. In Table 4 we list the expected times taken to reach states where the agent’s asset level is in the lowest decile (where the decile is calculated from the benchmark model under the robust policy) under the benchmark model with the expected utility and robust policies and under the worst case with the robust policy. The initial wealth levels we consider are the median and the 90th percentile. It is clear that if the agent’s concerns for misspecification are unfounded and the benchmark is indeed generating the data, then he will accumulate a substantially higher level of capital than he would if using the expected utility policy, as seen by comparing Figures 3a and 3b, and the dispersion of the wealth distribution will be higher. If one compares Figures 3a and 3b, where the robust policy is used in both but the latter is under the worst case model we see a similar contrast, with the asset distribution under the robust policy in the worst case leading to a similar
distribution induced by using the expected utility policy under the benchmark. This explains why the expected waiting time for reaching a low wealth level is much longer in the robust case under the benchmark (to such an extent that one might be concerned about detection error probabilities and the plausibility of the calibration of θ). Under the worst case, however, the agent’s robust policy only succeeds in raising the expected waiting time slightly above the expected utility case under the benchmark.

5.2 Two Shock Case

We now briefly discuss properties of the two shock case. As aforementioned, the two shocks are chosen to be identically and independently distributed under the benchmark model. However, as we shall see, the worst case model implies a degree of correlation between these shocks. We will concentrate on the interrelation of these shocks under the distorted model, rather than discussing the (symmetric) marginal distortions that apply to each.

We begin by examining distorted transition matrices for $z_1$, conditional on assets being
Table 5: Transition Matrices for $z_1$- $2$ Shock Case (worst case at various wealth percentiles and at high and low $z_2$)

<table>
<thead>
<tr>
<th></th>
<th>$z_1$</th>
<th>$z_2$</th>
<th></th>
<th>$z_1$</th>
<th>$z_2$</th>
<th></th>
<th>$z_1$</th>
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<th></th>
<th>$z_1$</th>
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<th></th>
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<td>f) W.C. (90th %-ile) - High $z_2$</td>
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at the 1st, 10th and 90th deciles and also conditional on $z_2$ being at the lowest and highest nodes. Recall, that the undistorted transition matrices for $z_1$ and $z_2$ are the same as in the one shock case. We observe that the transition matrices are again distorted in pessimistic ways and the distortions are larger when wealth is low (for reasons discussed above) and when the $z_2$ realization is low. Thus there is correlation of the two shocks under the worst case. Intuitively this is because the agent is fearful of receiving a ‘double-whammy’ of adverse shock realizations. That is, when he is vulnerable from having had a bad realization of one of the components of his income, he is particularly concerned with model misspecifications that are represented by correlated realizations of the other component. It is as if the evil agent injects additional systemic risk into the mind of the agent - making the bad times worse (and the good times better). However, the effect seems small and relates primarily to the lowest realizations.

This insight is reinforced by figure 4, which compares the stationary distribution of $z_1$
in the benchmark and distorted models, where in the latter case the distribution is shown unconditionally and conditional on low and high realizations of $z_2$. We observe that the worst case distributions have mass shifted to the lower realizations and that this tendency is slightly, but noticeably, greater when $z_2$ is low.

One implication of the fact that the robust agent’s actions are underpinned by a distribution that involves correlation between shocks that are uncorrelated under the benchmark is that if an econometrician ignores the possibility of a concern for robustness and relies on the behavior of the agent to extrapolate the nature of uncorrelated components under the benchmark (which he takes to be the data generating process), then the extracted components are likely to be confounded. These observations also raise the possibility that in a more general model with systemic and idiosyncratic risk components, agents’ worst case models may feature correlation between idiosyncratic and systemic shocks even though, under the benchmark, they are by definition orthogonal. Thus, indirect inference based on income data and consumption-savings decisions might be somewhat misleading, although the quantitative effects identified in the model above are rather small.

In a related point, it is interesting that the worst case distribution encodes state dependent correlation structures. In the recent crisis, the breakdown of previously reliable correlation structures underpinning hedging strategies led to significant losses for many market participants. Although the question is beyond the scope of our very simple model, we see hints that a robust agent might have guarded against such shifts. In extensions to this paper we hope to explore this avenue further.

6 Conclusions

In this paper we apply the tools of robust control analysis to a pair of self-insurance problems of an agent who faces differing types of income risk, has access only to a risk free bond
Figure 4: Stationary Distributions of $z_1$ - Benchmark, Unconditional Distorted and Distorted Conditional on Low and High $z_2$

and who is subject to a no-borrowing constraint. The agent’s fear of model misspecification induces him to accumulate additional wealth in order to guard against misspecifications that would expose him to scenarios in which he is particularly vulnerable. We formulate worst case transition matrices to gain an insight into the distribution over sequences implied by the worst case model. The implied distortions take the form of lower average income and, in the multiple shock case, correlation among income components that, according to the benchmark, should be uncorrelated. We show that the pessimism exhibited by the agent is more intense when his wealth level is low, in that the distortions to the transition matrices are more intense when the agent is close to his no-borrowing constraint.

Beyond providing interesting insight into an individual robust agent’s decision making process this (ongoing) work aims to provide a rationale for endogenous disagreement among agents, capturing variation in the state dependent twisting of the benchmark probability distribution, according to the idiosyncratic elements of each agent’s state. In addition, the endogenous state dependence of pessimism perhaps hints at a powerful feedback and amplification mechanisms for crisis situations where agents who were previously relatively sanguine (despite being somewhat distrustful of their model) are thrust into situations where, due to
additional vulnerability to misspecification, their pessimism intensifies, possibly worsening the situation further. Finally, the work also provides a cautionary tale for econometricians trying to filter separate latent components of income processes, based on observed consumption and investment behavior. If agents are robust and their behavior is guided by worst case distributions that exhibit correlations among components, then attempts to extract components that are orthogonal under the benchmark may be confounded. Nevertheless, despite this qualitative insight, the quantitative implications of the models discussed above seem somewhat limited.
References


