COMPETITION, INNOVATION, AND THE BUSINESS CYCLE

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ABSTRACT. This paper reports quantitative implications of a stochastic endogenous growth model in which growth displays a concave relationship with respect to competition. This model is consistent with unavoidable facts related to market structure, that other innovation-based endogenous growth models fail to replicate. This model shows promising empirical properties in both the short- and long-term. The current paper argues that this model might be better suited to measuring welfare implications of phenomena involving growth effects. An illustrative exercise reports and compares measurement of the welfare cost of imperfect competition and fluctuations. (Keywords: Innovation, Competition, Endogenous Growth, Real Business Cycles; JEL codes: D60, E32, 040.)

I. Introduction

In this paper, I present quantitative assessments of a canonical model of the real business cycle (RBC) in which output growth is determined endogenously and follows an inverted-U relationship with respect to competition. This quantitative analysis is stimulated by the emergence of a recent theoretical literature that relies on endogenous growth models to study the welfare impact of a set of general phenomena, following the idea that “the potential for welfare gains from better long-run, supply-side policies exceeds by far the potential from further improvements in short-run demand management” (Lucas, 2003). One of the strands of this literature aims at evaluating the welfare costs of fluctuations. For example, following the influential monograph of Lucas (1987), many authors including Jones, Manuelli, and Stachetti (2000), Lepaulard and Pommeret (2003), Barlevy (2004), to only cite a few, emphasize the idea that growth process may generate large welfare cost of business cycles.¹ Meanwhile, another line of this literature has focused on the growth and welfare effect of imperfect

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¹For a review on this topic, see Aghion and Banerjee (2003) and Jones and Manuelli (2005).
competition. For instance, the seminal quantitative work of [Harberger (1954)] has spurred many papers relying on endogenous growth models to assess the welfare cost of imperfect competition (e.g. [Matheron, 2002; Matheron and Maury, 2004]). Though extensively discussed in the literature as two distinct phenomena, the joint effects of aggregate fluctuations and competition on economic activity have been seldom studied simultaneously, at least in an up-to-date Schumpetarian endogenous growth framework. This paper aims at contributing to fill this gap.

The approach shared in most papers cited herein relies on accumulation of diverse type of capital to explain sustained long-term growth. By contrast, the present study defends the view that the Schumpetarian growth theory of innovation is better suited to analyze jointly volatility and competition. This idea is motivated by two reasons leastwise. The first motive is the belief that inventiveness and innovation play a larger part in the long-run growth process than efficiency and saving. Moreover, AK-type models do not permit accumulation and technological progress to be explicitly distinguished. Secondly, the importance of obsolescence and exit on growth has received recent empirical supports, that conforms the Schumpetarian growth theory in its ability to expound economic growth.

The core endogenous growth process studied in this paper derived from the step-by-step innovation framework subsequently analysed and expanded by [Vickers (1980), Budd et al. (1993), Aghion et al. (1997), and Aghion et al. (2001)]. Based on this corpus, Aghion et al. (2005) developed a model in which the link between competition and innovation is inverted-U shaped. De facto,

2 Fluctuations and competition apart, other studies making uses of endogenous growth models for welfare costs assessment are related, but not restricted, to inflation ([Dotsey and Ireland, 1996], taxation ([Jones, Manuelli, and Rossi, 1993]), and fiscal policy ([Jones and Manuelli, 2005]).

3 For early models of endogenous growth based on accumulation, see for instance [Romer (1986), Lucas (1988), and Rebelo (1991)].

4References on Schumpetarian, innovation-driven growth models include [Segerstrom et al. (1999), Grossman and Helpman (1991), and Aghion and Howitt (1992)].

5However, this view is adopted due to a lack of clear consensus on this issue, as illustrated by the debate on the causes of the Asian “Tigers” growth ([Young, 1995; Hsieh, 2002]).

6See Aghion and Howitt (2009) for a survey. Additionally, though they are based on innovation, models of product variety ([Romer, 1990]) seem less adapted for dealing with competition and ruled out on empirical grounds ([Broda et al., 2006]).
the model reconciles the Schumpeterian *disincentive effects* of competition on innovation (SCHUMPETER, 1942) that claims competition lessens rents and hence mitigates profit-seeking innovators, with the so-called *escape-from-competition effects* (ARROW, 1962) that asserts competitive firms innovate in order to differentiate and to reap expected profits.

In the original model, follower firms are deterred from innovating when competition increases, whereas leveled firms are incited to innovate. When the population of neck-and-neck and laggards firms is taken as endogenous, this combination generates an inverted-U shape relationship between competition and innovation; for lowest levels of competition, neck-and-neck firms are in the majority and hence aggregate innovation is an increasing function of competition. The opposite occurs when competitive pressures are the highest; laggards’ innovating behavior dominates and, as a result, aggregate innovation decreases with competition. Several econometric studies have bring empirical evidence to light backing this model, hereafter referred as the *inverted-U shape model* (e.g. AGHION et al., 2003; ASKENAZY et al., 2008). On top of that, the inverted-U shape model is able to make the facts tie in with the theory as far as the nexus between competition and innovation is concerned. 7

That aggregate fluctuations may have an influence on the main properties of the inverted-U shape model is investigated in CAHN (2011). Broadly speaking, two noticeable mechanisms are at play. The first one relies on the irreversibility of innovational efforts; an increase in uncertainty induces a rise in the opportunity cost of the aggregate innovation and hence a diminution of its level. The second mechanism derives from profits asymmetry. Current profits of leaders or even firms depend on aggregate output, whereas profits of followers do not. This gives rise to an unbalanced effect of fluctuations on innovation at the individual level, that depends on the market power. As a result, the inverted-U relationship between competition and innovation is substantively modified in a stochastic environment. Reciprocally, the issue under investigation here is in what extent the inverted-U shape assumption affects the quantitative implications of the standard RBC model.

This exercise is also motivated by empirical issues regarding the propagation mechanism in standard RBC models (COGLEY and NASON, 1995). Recent studies have shown that endogenous growth

7For a review on the link between competition and growth, see AGHION and GRIFFITH (2003).
feature might be a remedy to the failure of traditional RBC models to display sufficiently high persistence of output growth (Jones, Manuelli, and Siu, 2005). In this respect, one contribution of the present study is to quantitatively assess the power of the propagation mechanism induced by the inverted-U shape model.

In order to work through these matters, I incorporate the inverted-U shape model discussed above in a benchmark RBC model of the type pioneered by Kydland and Prescott (1982) and Long and Plosser (1983). The quantitative features of the resulting model are then gauged following an adaptation of the methodological approach instilled by Cooley and Prescott (1995). To begin with, I select an adequate solving method to work out the equilibrium decision rules. For this purpose, I resort on projection and orthogonal collocation methods (Judd, 1992) for its ability to deal sufficiently well with nonlinearities. Then I build a dedicated data set that bridge actual timeseries and their theoretical counterparts consistently. Along the lines of Cooley and Prescott (1995) and Gomme and Rupert (2007), this data set derives from the U.S. National Income and Product Accounts (NIPA) and is adjusted for several accounts, including home production, research and development, and government production. Finally, the model is parametrized and calibrated so that simulated variables mimic a selection of empirical moments. This calibration procedure differs from the widely-used one that consists in targeting deterministic steady state ratios. Instead this approach, descending from the simulated method of moment estimation (Lee and Ingram, 1991; Duffie and Singleton, 1993), allows the uncertainty surrounding timeseries ratios to be reckoned. Simulations from the estimated model are thus used to perform quantitative exercises.

The main empirical results may be summarized as follows. Broadly speaking, the present inverted-U shape model reproduces favorably the long- and short-term observations that one would make on a real economy. In particular, this model reproduces the inverted-U shape relationship between competition and aggregate innovation for a set of plausible parameter values. Moreover, average aggregate innovation and growth in the model decline with a rise in volatility, consistently with other empirical observations (Ramey and Ramey, 1995). Most notably, at the benchmark calibration the levels of competition that maximize innovation and

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8See Judd (1998) for a comprehensive review of numerical methods applied to economics.
growth differ and increase with volatility. As far as short-term properties are concerned, the model can explain about 60% of R&D expenses volatility. Also, the model succeeds in explaining about 37% of the first order autocorrelation of output growth. However, resources spent in research and development appears less volatile in the model than in the data. Carrying the inverted-U shape paradigm improves the benchmark RBC model in its ability to analyze the effects of fluctuations and competition on growth, in a theoretical framework that accords with observations. This fact is an important result in its own right.

The previous quantitative exercises show that both volatility and market structure have entangled effects on growth, that one should consider in welfare computations. Another contribution of this paper is to use this model to perform such computations. Based on these computations, the welfare cost of moving from the highest to the least competitive environment accounts for about 1.7% of annual consumption each period. Interestingly, this cost downs very close to zero, even becomes a gain, when competition reaches the level estimated on the adjusted data set. Additionally, the transition to a stochastic economy leads to a welfare cost of fluctuations lower than one tenth of percentage, that slightly increases with competition.

The paper is organized as follows. Section II develops a version of the canonical RBC model that encompasses the inverted-U shape concept discussed above. In Section III, the model is solved and calibrated, and simulation results are reported. Section IV offers a discussion about the welfare costs in this model of both competition and fluctuations. Section V concludes.

II. The Model

In this section I describe a version of the canonical real business cycle model that encompasses an endogenous growth process based on step-by-step innovation races, attributing explicitly a role to competition. This section presents all the agents’ problems and particular objects that one need know in order to compute the equilibrium. The complete system of equations is given in a technical appendix that the reader will find at the end of the paper.

Time is discrete and the horizon is infinite. Production takes place in two separate sub sectors. Innovation occurs in the first one which consists of a continuum of industries. These industries
supply a final good sector with intermediate wares through a combination of capital and labor furnished by a representative household. The final good can be consumed, invested, or used as research and development (R&D) expenditures. Finally the representative household owns all the intermediate industries.

A. The Production Sector

1. The Intermediate Goods Industries.— The intermediate goods production sector consists of a continuum of industries distributed uniformly on the unit interval. In each industry indexed by \( j \in [0, 1] \), two firms \( i \in \{1, 2\} \) produce an intermediate good assumed to be a perfect intra-industry substitute. These firms produce the input according to a new classical production function

\[
Y_{j,t}^i = z_t F \left( K_{j,t}^i, h_{j,t}^i, \Gamma_{j,t}^i \right), \quad i \in \{1, 2\}
\]

where \( h_{j,t}^i \) and \( K_{j,t}^i \) are respectively the level of labor paid at rate \( W_t \) and the level of capital stock paid at \( r_t \), and \( z_t \) a labor productivity shock that follows the law of motion

\[
z_t = \bar{z}^1 - \rho z_{t-1}^\rho e_t,
\]

with \( \rho \in (0, 1) \), \( e_t \sim \mathcal{N}(0, \sigma_e) \), and \( \ln(\bar{z}) = -\sigma_e^2/[2(1-\rho^2)] \) the mean preserving spread so that the unconditional mean \( \mathbb{E}[z_t] = 1 \) for any \( \sigma_e \) and \( \rho \).

At the beginning of the period, all industries share the same mean \textit{ex ante} level of labor productivity \( \Gamma_t \), that is, prior to innovation. However the intra-industry duopolists possibly differ in their productivity. For this reason a given firm is either a technological leader, a follower, or neck and neck with respect to its industry competitor. As a result of innovation activities, firms’ productivity evolves incrementally and randomly, though without leapfrogging between duopolists. When firm-specific innovation occurs at one point in time, the productivity of the successful firm increases by a factor \( \gamma > 1 \), which is a constant among firms and industries. However the technology gap between two contending firms (i.e. the ratio of their respective productivity) cannot exceed one increment of productivity and hence is bounded from above by \( \gamma \). Beyond this limit, technology spillovers are assumed to pervade at no cost in the industry (i.e. the follower’s productivity automatically increases by \( \gamma \) if the cutting-edge firm succeeds in innovating), thus leaving the
technology gap unchanged. It follows that

$$\Gamma_{j,t}^i = \begin{cases} \Gamma_t/\gamma & \text{if } i \text{ is a follower firm,} \\ \Gamma_t & \text{otherwise.} \end{cases}$$

Finally, firms seek to find the optimal allocation of capital and labor that minimize the total cost of production. As shown in the Appendix A, the leader and neck-and-neck firms have the same marginal cost, $\xi_t$, which is lower than a follower's by a factor $1+\omega$ (to be defined below) depends on $\gamma$ and the labor elasticity of $F(\cdot)$. Obviously, $\xi_t$ is a function of production factors' price.

2. Market Structure.— Bertrand competition holds in each unleveled industry $j$ (i.e., industries in which duopolists have different level of labor productivity). Consequently, a leader sets his price at his competitor's marginal cost $p_{j,t}^u = (1+\omega)\xi_t$ so that a technology leader captures the whole production $Y_{j,t}$ faced by the industry and its operating profit (excluding R&D costs) is given by

$$\Pi_{j,t}^u = p_{j,t}^u Y_{j,t} - \xi_t Y_{j,t} = \omega \xi_t Y_{j,t}. \tag{3}$$

By contrast, the follower makes no profit.

As far as neck and neck sectors—where duopolists stand at the same technological level—are concerned, firms are assumed to collude and they equitably share what a one-step-ahead monopoly would have earned. However, competition between duopolists is assumed to reduce these rents by a factor $Y(\epsilon)$, where $\epsilon$ is a parameter that represents competition—$\epsilon = 0$ stands for no competition and $\epsilon = 1$ when perfect competition prevails—and $Y$ a strictly decreasing function with $Y(0) = 1/2$ and $Y(1) = 0$. Consequently, the operating profit of a leveled firm is given by

$$\Pi_{j,t}^\alpha = Y(\epsilon)\Pi_{j,t}^u = Y(\epsilon)\omega \xi_t Y_{j,t}. \tag{4}$$

3. R&D activities.— Firms make decisions about innovation activities each period. R&D spending gives firms the opportunity to gain one increment of productivity $\gamma$ with a probability that depends on the firm's position in the industry. More precisely, allocating innovation efforts $\rho$ in R&D, a firm $i$ in industry $j$ may obtain the productivity increment with probability $\phi(\rho; \alpha_i)$. The function $\phi(\cdot; \alpha_i)$ is strictly increasing and concave, and $\alpha_i = \bar{\alpha} \in (0,1)$ if $i$ is follower, zero otherwise, $\bar{\alpha}$ representing the ease of imitation. Moreover, beyond the maximum technology gap, followers automatically catch up with leaders when the latter innovate. Finally, for a given level
of innovation efforts $\varphi$, firms pay R&D costs given by
\begin{equation}
\Delta_t(\varphi) = \delta(\varphi) \Gamma_t,
\end{equation}
where $\delta(\cdot)$ is an increasing, convex function.

4. Bellman Equations.— Let $\chi_{n_{1}^{0}|u}$ denote the probability of one unleveled sector at date $t$ becoming neck and neck at date $t + 1$. Incidentally, this reflects the probability of a follower firm becoming neck and neck (i.e. the follower innovates and the leader does not). The upper panel of Figure \ref{fig:bellman} shows different alternative outcomes for the R&D race that takes place in each unleveled industry. For instance, with probability $\chi_{u_{0}^{0}|u}$, the industry remains unleveled since no innovation occurs. Such a sector also stands in the unleveled state if either the leader innovates and the follower automatically catches up with probability $\chi_{u_{1}^{0}|u}$, or if both firms innovate with probability $\chi_{u_{1}^{1}|u}$. Clearly, for a given unleveled industry $j$, probabilities $\chi_{j|u}$ depend only on the R&D efforts of both the leader and the follower, $\varphi_{j,t}^{u}$ and $\varphi_{j,t}^{-u}$ respectively, according to
\begin{equation}
\begin{bmatrix}
\chi_{u_{1}^{1}|u} & \chi_{u_{0}^{0}|u} \\
\chi_{n_{1}^{0}|u} & \chi_{u_{1}^{0}|u}
\end{bmatrix}
\begin{bmatrix}
\varphi(\varphi_{j,t}^{u}, 0) \\
1 - \varphi(\varphi_{j,t}^{u}, 0)
\end{bmatrix}
\begin{bmatrix}
\varphi(\varphi_{j,t}^{-u}, \bar{\alpha}) \\
1 - \varphi(\varphi_{j,t}^{-u}, \bar{\alpha})
\end{bmatrix}^{\top}.
\end{equation}
Similarly, the innovation process leads to different outcomes for neck-and-neck industries as depicted in the lower panel of Figure \ref{fig:bellman}. Hence, such an industry remains leveled with probability $\chi_{n_{0}^{0}|n}$ if no innovation occurs, or with probability $\chi_{n_{1}^{1}|n}$ if both duopolists innovate. However, the neck-and-neck industry becomes unleveled with probability $\chi_{u_{0}^{0}|n}$ or $\chi_{u_{0}^{1}|n}$ if just one duopolist innovates while the other does not. It follows that these probabilities are functions of $\varphi_{j,t}^{n}$ and $\varphi_{j,t}^{-n}$, where $\varphi_{j,t}^{n}$ is related to the rival duopolist, according to
\begin{equation}
\begin{bmatrix}
\chi_{n_{1}^{1}|n} & \chi_{u_{1}^{1}|n} \\
\chi_{n_{0}^{0}|n} & \chi_{n_{1}^{0}|n}
\end{bmatrix}
\begin{bmatrix}
\varphi(\varphi_{j,t}^{n}, 0) \\
1 - \varphi(\varphi_{j,t}^{n}, 0)
\end{bmatrix}
\begin{bmatrix}
\varphi(\varphi_{j,t}^{n}, \bar{\alpha}) \\
1 - \varphi(\varphi_{j,t}^{n}, \bar{\alpha})
\end{bmatrix}^{\top}.
\end{equation}
Consequently, an expression of the present value of the different type of duopolist can be derived from the Bellman equations:
\begin{equation}
V_{j,t}^{u} = \max_{\varphi_{j,t}^{u}} \left\{ \Pi_{j,t}^{u} - \Delta_t(\varphi_{j,t}^{u}) + E_t \left[ \Phi_t \left( \chi_{n_{1}^{0}|u} V_{j,t+1}^{n} + \left( 1 - \chi_{n_{1}^{0}|u} \right) V_{j,t+1}^{u} \right) \right] \right\}
\end{equation}
(BL)
\begin{equation}
V_{j,t}^{-u} = \max_{\varphi_{j,t}^{-u}} \left\{ -\Delta_t(\varphi_{j,t}^{-u}) + E_t \left[ \Phi_t \left( \chi_{n_{1}^{0}|u} V_{j,t+1}^{n} + \left( 1 - \chi_{n_{1}^{0}|u} \right) V_{j,t+1}^{-u} \right) \right] \right\}
\end{equation}
(BF)
Unleveled industries

R&D race and innovation process

Neck and neck industries

R&D race and innovation process

Figure 1. Alternative innovation process outcomes for different types of industries
\[ V_{j,t}^n = \max_{\varphi_{j,t}} \left\{ \Pi_{j,t}^n - \Delta_t (\varphi_{j,t}) \right\} \]

\[ + E_t \left[ \Phi_t \left( \chi_{u_0|n} V_{j,t+1}^u + \chi_{u_0|n} V_{j,t+1}^u + (1 - \chi_{u_0|n} - \chi_{u_1|n}) V_{j,t+1}^n \right) \right] \]

where \( \Phi_t \) is the stochastic social discount factor between the current and next period and \( V_{j,t}^u, V_{j,t}^u, \) and \( V_{j,t}^n \) represent respectively the nominal value of a leader, follower and neck-and-neck firm in industry \( j \). Note that \( \varphi_{j,t}^n = \varphi_{j,t}^n \) would prevail at the ex post symmetric Nash equilibrium.

For interpretation's sake, I give below the derivation of these Bellman equations:

\[ E_t \left[ \Phi_t \varphi (\varphi_{j,t}^u, \tilde{\alpha}) S_{j,t+1}^u \right] = \frac{\delta' (\varphi_{j,t}^u)}{\varphi' (\varphi_{j,t}^u ; 0)} \Gamma_t, \tag{6} \]

\[ E_t \left[ \Phi_t \left( 1 - \varphi (\varphi_{j,t}^u) \right) S_{j,t+1}^n \right] = \frac{\delta' (\varphi_{j,t}^n)}{\varphi' (\varphi_{j,t}^n ; \tilde{\alpha})} \Gamma_t, \tag{7} \]

\[ E_t \left[ \Phi_t \left( 1 - \varphi (\varphi_{j,t}^n ; 0) \right) S_{j,t+1}^u + \varphi (\varphi_{j,t}^n ; 0) S_{j,t+1}^n \right] = \frac{\delta' (\varphi_{j,t}^n)}{\varphi' (\varphi_{j,t}^n ; 0)} \Gamma_t, \tag{8} \]

where \( S_{j,t}^n = V_{j,t}^n - V_{j,t}^u \) and \( S_{j,t}^u = V_{j,t}^u - V_{j,t}^n \), represent the firm’s surplus according to the state of industry \( j \), taking innovation efforts \( \varphi \) at their optimal values. Combining Bellman equations at the optimal level of innovative efforts gives the dynamics of surplus \( S_{j,t}^n \) and \( S_{j,t}^u \), as shown in Appendix A. Finally, surplus' dynamics and equations (6)–(8) define the entire dynamics of R&D decisions of firms belonging to neck and neck and unleveled sectors, that can be intuited as follows.

A rise in competition leads simultaneously through a direct, negative impact on \( \Pi_{j,t}^n \) to a decrease (resp. an increase) in the net present value of the gain to becoming a neck-and-neck firm \( S_{j,t}^n \). (resp. the one to become a leader \( S_{j,t}^u \)). Due to the curvature of \( \delta(\cdot) \) and \( \varphi(\cdot) \), equation (6) shows that an increase in competition implies a rise in the innovation efforts of the leader firm, which wants to save her rents.

In the meantime, it results from equation (7) that this increase in competition would mitigate the efforts of innovation of follower firms in unleveled sectors. This reveals the disincentive effect of competition on R&D.
Finally, equation (8) shows that an increase in competition has ambiguous implications. However, for low probability values, such an increase leads to an increase in the expected gain in capital for a leveled firm and results in an enhancement of innovational efforts in the neck and neck sectors. This reveals the escape from competition effect.

5. Final Good Production.— A representative agent sells a final good on a dedicated market that serves as numéraire in this economy. This aggregate good is made of a panel of intermediate goods $Y_{j,t}^d$ according to the production function

$$\ln Y_t = \int_0^1 \ln Y_{j,t}^d \, dj.$$ 

The representative firm buys the intermediate goods at the relative price $p_{j,t}$, so that the objective of the final good producer is

$$(\text{FGP}) \quad \max_{\{Y_{j,t}^d\}_{j\in[0,1]}} Y_t - \int_0^1 p_{j,t} Y_{j,t}^d \, dj,$$

subject to the technology constraint (9). Note that according to this specification, the total expenditure in value terms addressed to each industry $j$ does not depend on the type of industry and, hence, is identical among all $j \in [0,1]$.

B. Households

The economy is inhabited by a representative household that traditionally seeks to maximize its discounted flows of utility subject to a budget constraint and capital accumulation. The households' time endowment is normalized to unity and is shared between leisure $\ell_t$ and work $h_t = 1 - \ell_t$. Consumption $C_t$ and labor $h_t$ enter into the current utility function $U(C_t, h_t)$ which is assumed to be continuous and twice differentiable, with

$$U'_{c,t} = \frac{\partial U(C_t, h_t)}{\partial C_t} > 0, \quad U''_{cc,t} = \frac{\partial^2 U(C_t, h_t)}{\partial C_t^2} \leq 0,$$

$$U'_{h,t} = \frac{\partial U(C_t, h_t)}{\partial h_t} \leq 0, \quad U''_{hh,t} = \frac{\partial^2 U(C_t, h_t)}{\partial h_t^2} \leq 0.$$

The household supplies to the production input sector $h_t$ units of labor in period $t$ at real wage rate $W_t$ and $K_t$ units of capital stock at interest rate $r_t$. Capital is accumulated by investing $X_t$ and depreciates at rate $\mu \in (0, 1)$ so that

$$K_{t+1} = (1 - \theta)K_t + X_t, \quad \text{all } t.$$
Finally, the household receives dividends $\Pi_t$ from firms. The household's problem is then given by:

$$\max_{(C_t,h_t,K_t,X_t)_{t=0}} \mathbb{E}_0 \left[ \sum_{t=0}^{\infty} \beta^t U(C_t,h_t) \right]$$

(HP)

s.t. $C_t + X_t \leq W_t h_t + r_t K_t + \Pi_t$, all $t$;

$K_{t+1} = (1-\theta)K_t + X_t$, all $t$;

$K_0 > 0$ given

where $\beta \in (0,1)$ is the subjective discount factor.

C. Aggregation and Equilibrium Definition

1. Population dynamics.— Since industry profits only rely on the prevailing current technology gap, and not on past individual histories due to Bertrand competition, the dynamic problem faced by each duopoly has the Markovian property. So each industry is sufficiently referenced by the prevailing technology gap which is zero or $\gamma$. Meanwhile, let $\lambda_{u,t}$ and $\lambda_{n,t}$ denote respectively the mass of unleveled and neck-and-neck industries in the continuum during period $t$. We have

$$\lambda_{n,t} + \lambda_{u,t} = 1, \text{ all } t.$$  

Alternatively, $\lambda_{n,t}$ and $\lambda_{u,t}$ denote the firm's probability of belonging to a neck and neck or unleveled sector respectively in period $t$. Incidentally, note that $\lambda_n$ and $\lambda_u$ are both state variables and one can substitute the $j$ index with either $u$ or $n$. According to the probabilities for a firm belonging to both types of sectors to enter an unleveled sector, one could give the expected proportion of unleveled duopolies at the end of period $t$:

$$\lambda_{u,t+1} = \lambda_{n,t}(\chi_{u|^{|n}} + \chi_{u|^{|a}}) + \lambda_{u,t}(1-\chi_{n|^{|u}}), \text{ all } t.$$  

2. Aggregation.— Capital and labor markets clearing conditions give

$$K_t = \int_{0}^{1} \left( \sum_{i \in \{1,2\}} K_{j,t}^i \right) dj,$$

$$h_t = \int_{0}^{1} \left( \sum_{i \in \{1,2\}} h_{j,t}^i \right) dj.$$
3. Final Product Market.— Consistent with Walras Law, the equilibrium condition on the final good market can be derived from the set of other market equilibria. Recall the household’s budget constraint
\[ C_t + X_t = W_t h_t + r_t K_t + \Pi_t \]
with \( \Pi_t = \int_0^1 \left( \sum_{i \in \{1,2\}} \Pi_{j,t}^i \right) dj \) the aggregate profits of firms taking into account R&D expenses \( D_t = \int_0^1 \left( \sum_{i \in \{1,2\}} \Delta_t \left( \rho_{j,t-1}^i \right) \right) dj \). Using the homogeneity of production functions in input good sectors, we get the equilibrium condition on final product market
\[ (13) \quad C_t + X_t + D_t = Y_t. \]

4. Average level of productivity.— The next period level of productivity depends on the proportion of neck and neck and unlevelled sectors, which depends itself on the different possible evolutions of firms, given that each sector consists of two firms. As shown in Figure [1], the innovation process among the different industries leads us to distinguish between four types of industry at date \( t+1 \):

**Type 1:** unlevelled industry with the leader’s labor productivity given by \( \gamma \Gamma_t \);

**Type 2:** neck and neck industries where duopolists’ productivity equals \( \gamma \Gamma_t \);

**Type 3:** unlevelled industry with cutting edge productivity \( \Gamma_t \);

**Type 4:** neck and neck industries with productivity given by \( \Gamma_t \).

For each different type, we are able to count the number of firms once the outcomes of the R&D races are known. Let \( \lambda_{i,t+1} \) denote the measure of industries of type \( i = 1, \ldots, 4 \) in period \( t+1 \). Such \( \lambda_i \)'s depend on the previous distribution of sectors \( \lambda_{u,t} \) and \( \lambda_{n,t} \), as well as the probabilities of innovation occurrences \( \chi_{j,t} \left( \rho_t \right) \), where \( \rho_t \) is the vector of firms’ innovation efforts. Moreover, since we know how productivity in each sector evolves, we are able to compute the marginal costs for each duopolist and, hence, the price \( p_{i,t+1} \) charged by each industry \( i \), which depends necessarily on \( \Gamma_t \).

As the aggregation technology for final good production does not change and the expenses on each intermediate goods is the same, using conditions on relative demand for good \( i \) and output aggregation function \( (9) \), one can define the law of motion of aggregate productivity such as
\[ (14) \quad \Gamma_{t+1} = G \left( \Gamma_t, \lambda_{u,t}, \lambda_{n,t}, \chi_{j,t} \left( \rho_t \right) \right). \]

This aggregation procedure is the tricky part of the model, and hence deserves a few more explanations. For an individual viewpoint, the structure of the economy seems to evolve inconsistently.
At the beginning of period, the intermediate goods sector is made of two types of industry, namely neck and neck and unleveled. At the end of the innovation process, four differing sorts of sectors emerge, either by the market structure (unleveled vs neck and neck) or by the productivity level. It would seem more natural to consider each industry individually and track them along the time horizon but this would obviously lead to an intractable model.

However, we can circumvent this issue, at least intellectually, for two reasons. First, this way of counting aggregate productivity is consistent on average. Considering only two sectors or the four described in this section would lead to the same average productivity. So, since we are interested in a model for the aggregate economy, this way of reallocating average productivity is not problematic for our use. Second, given the market structure, an intermediate firm's profits do not depend on its own past history, but rather on aggregate demand $Y_t$, and is independent of the intra-industry productivity level. Once again, this leads us to disregard productivity individually.

5. **Definition.**— A dynamic general equilibrium is defined by sequences of aggregate variables $\{\Gamma_t, C_t, X_t, D_t, Y_t, h_t, K_t, z_t\}_{t=0}^{\infty}$, of price variables $\{W_t, r_t, t, \xi_t\}_{t=0}^{\infty}$, of industry related variables $\{\lambda_{u,t}, \lambda_{n,t}\}_{t=0}^{\infty}$, and of R&D related variables $\{\varphi_{u,t}, \varphi_{-u,t}, \varphi_{n,t}, S_{u,t}, S_{n,t}\}_{t=0}^{\infty}$ such as

(i) the system of Bellman equations (BL), (BF), (BN),
(ii) final good producer's problem (FGP), and
(iii) the representative household’s problem (HP) are solved,
(iv) equilibrium conditions (11) and (13) are satisfied,

given

(a) the state variables' laws of motion (2), (10), (12), and (14),
(b) functional forms $Y, \delta, \varphi, F, U$, and
(c) initial conditions $(K_0, \Gamma_0, \lambda_{u,0}, z_0)$.

6. **Stationary representation.**— In order to solve this model, one must transform the extensive variables into intensive ones. Indeed, the model features an economic growth process and balanced growth path that render major part of solution methods inefficient without stationarizing the model.

All extensive variables include a trend derived from productivity $\Gamma_t$. It is then convenient to remove the trend from growing variables by dividing them by $\Gamma_t$. Accordingly, the stationary representation of the model, as well as the steady state equations are given in appendix.
III. Results

The previous section has presented a dynamic stochastic general equilibrium model of endogenous growth in which competition plays a non trivial role. In this section I put forward the different steps leading to analyze the model's dynamic and static properties on the one hand, and how volatility and competition jointly affect them on the other hand.

Loosely speaking, one of the main tasks involved in the simulation process implies to clarify the functional forms of the model and to give specific and plausible values for its settings. Among the different available techniques, some consist in picking out parameters' values and generating simulations so as to mimic selected empirical moments based on actual data. However, the simulation step calls for methods to solve the model, that is methods able to return the modeled quantities as a function of a set of state variables, also known as decision rules.

Except in few cases, it is rarely possible to work out an analytical expression for such decision rules. That is why one generally relies on numerical methods to approach such functions. Nevertheless, looking into the macroeconomist’s toolbox, some methods are more convenient than others for the problem at hand. For instance, local approximations, which covers among others loglinearization or Taylor series expansions at higher degrees, are well adapted to problem which are concentrated in a close neighborhood of the approximation point. Hence, this excludes model in which an economy is submitted to large shocks or when decision rules show relatively large curvatures.

Global solution methods are interesting alternatives to local approximation procedures when the latter fail to meet the precision requirements. Generally, these methods consist in postulating some parametrized form of decision rules, e.g., a linear combination of polynomials, and use iterative schemes converging towards a numerically satisfying function over a given domain, i.e., reaching a certain degree of approximation all over the desired interval. Without any priors on the size of shocks nor on the curvature of the present model's decision rules, I consider global methods in the calibration process, more precisely the projection over a basis of Chebyshev polynomials identified by orthogonal collocation.

\[\text{Exceptions include, e.g., the log utility and full depreciation case. See }\text{Stokey and Lucas (1989).}\]
Moreover, since I am ultimately interested in the link between competition and innovation for different degrees of volatility, local approximations would render the simulations less accurate and potentially divergent.

Finally, the model is calibrated with parameters that engender simulations mimicking some empirical moments computed from U.S. data, following the Simulation Moments Method (SMM.) This entails the construction of a collection of time series reconcilable with their theoretical counterparts, as described further below in this section and in a detailed appendix.

I perform two simulation exercises with the calibrated version of the model. In the first one, I investigate long run properties of the replicated variables for different level of competition, and when competition and volatility jointly vary. In the second, I look into the dynamics the model features by drawing moments comparisons between actual and model data, and by showing dynamic responses of the model following an unexpected shock of productivity.

A. Model Solving, Data, and Calibration

1. Orthogonal Collocation.— For the reasons stated above, I rely on global approximation methods in the calibration process, rather than local approximation techniques. In particular, I use Chebyshev polynomials projection (see Judd (1992, 1998). I therefore choose to express the different decision rules as a linear combination of elementary functions, namely the orthogonal Chebyshev polynomials. Such class of polynomials is particularly useful with respect to its orthogonality properties intensely used in some computational stages.\footnote{See Judd (1998).}

It is shown in Section 11 and Appendix A that the core model’s dynamics could be described by five Euler’s equations with respect to innovation efforts of the leader firm (6), to the follower (7), to the neck-and-neck firm (8), to consumption (A.11), and to labor (A.9).

According to our general definition, equilibrium is established for a given set of initial conditions \((\Gamma_0, K_0, \lambda_{n,0,z0})\) that represents the state of the economy. Working out a stationary version of the model leads to only consider the triplet \((k_t, \lambda_{n,t}, z_t)\) as the state in period \(t\). For computational convenience, I use a discretization method to approximate the autoregressive technology shocks \(z_t\)
by a discrete Markov chain $\tilde{z}_t$ taking $n_{\tilde{z}}$ values in $\mathcal{Z} \equiv \{\tilde{z}_1, \ldots, \tilde{z}_{n_{\tilde{z}}}\}$.\footnote{See \cite{Kouwenhov\textsc{et~al}.} (1995).}

Then, as the model features two predetermined variables, namely household’s capital stock $k_t$ and the share of neck and neck sectors $\lambda_{n,t}$, the approximated decisions rules are defined over the cartesian product $\mathcal{K} \times \Lambda$ where $\mathcal{K} \equiv [k, k]$ and $\Lambda \equiv [0, 1]$, with $k$ and $k$ defining the two boundaries in which capital stock is most likely to evolve.

As a consequence, we have to find five times $n_{\tilde{z}}$ decisions rules $c_t(\tilde{z}_i)$, $h_t(\tilde{z}_i)$, $\varrho_{n,t}(\tilde{z}_i)$, $\varrho_{u,t}(\tilde{z}_i)$, and $\varrho_{-u,t}(\tilde{z}_i)$. Each decision rule are approximated by a linear combination of Chebyshev polynomials products, $T_{jk} T_{j\lambda}$, referenced by a collection of coefficients $\omega_{\#}(\tilde{z}_i)$. Let $n_k$ and $n_{\lambda}$ denote the highest degree of polynomials with respect to capital stock and neck and neck industries proportion. Hence, we are working out decision rules of the form

$$c_t(\tilde{z}_i) \simeq \Theta(k_t, \lambda_{n,t}; \omega_c(\tilde{z}_i))$$
$$\equiv \exp \left( \sum_{j_k=0}^{n_k} \sum_{j_{\lambda}=0}^{n_{\lambda}} \omega_c(j_k, j_{\lambda}|\tilde{z}_i) T_{jk} \left( \psi_k(\log(k)) \right) T_{j\lambda} \left( \psi_{\lambda}(\lambda_{n,t}) \right) \right)$$

$$h_t(\tilde{z}_i) = \Theta(k_t, \lambda_{n,t}; \omega_h(\tilde{z}_i))$$
$$\equiv \exp \left( \sum_{j_k=0}^{n_k} \sum_{j_{\lambda}=0}^{n_{\lambda}} \omega_h(j_k, j_{\lambda}|\tilde{z}_i) T_{jk} \left( \psi_k(\log(k)) \right) T_{j\lambda} \left( \psi_{\lambda}(\lambda_{n,t}) \right) \right)$$

$$\varrho_{n,t}(\tilde{z}_i) \simeq \Theta(k_t, \lambda_{n,t}; \omega_{n}(\tilde{z}_i))$$
$$\equiv \exp \left( \sum_{j_k=0}^{n_k} \sum_{j_{\lambda}=0}^{n_{\lambda}} \omega_{n}(j_k, j_{\lambda}|\tilde{z}_i) T_{jk} \left( \psi_k(\log(k)) \right) T_{j\lambda} \left( \psi_{\lambda}(\lambda_{n,t}) \right) \right)$$

$$\varrho_{u,t}(\tilde{z}_i) \simeq \Theta(k_t, \lambda_{n,t}; \omega_{u}(\tilde{z}_i))$$
$$\equiv \exp \left( \sum_{j_k=0}^{n_k} \sum_{j_{\lambda}=0}^{n_{\lambda}} \omega_{u}(j_k, j_{\lambda}|\tilde{z}_i) T_{jk} \left( \psi_k(\log(k)) \right) T_{j\lambda} \left( \psi_{\lambda}(\lambda_{n,t}) \right) \right)$$

$$\varrho_{-u,t}(\tilde{z}_i) \simeq \Theta(k_t, \lambda_{n,t}; \omega_{-u}(\tilde{z}_i))$$
$$\equiv \exp \left( \sum_{j_k=0}^{n_k} \sum_{j_{\lambda}=0}^{n_{\lambda}} \omega_{-u}(j_k, j_{\lambda}|\tilde{z}_i) T_{jk} \left( \psi_k(\log(k)) \right) T_{j\lambda} \left( \psi_{\lambda}(\lambda_{n,t}) \right) \right)$$

where the $\psi$’s are linear functions that map the state spaces to $[-1, 1]$, the domain of Chebyshev polynomials.
The use of exponential ensure positivity of consumption, labor, and innovation efforts. Since the model is very close to the standard real business cycle model, it seems plausibly nearly logarithmic. That is why capital stock enters in log. No need a priori for similar transformation as regards $\lambda_n$. Appendix C provides a description of the algorithm that allows the $\omega_\#'$s to be estimated.

To complete the description of the solution method, I discuss the choice of the number of Markov chain’s states, $n_\hat{z}$, as well as the degrees of approximating polynomials, $n_k$ and $n_\lambda$.

First, I set $n_\hat{z} = 2$, so as the productivity shock takes only two values. From the one hand, it greatly simplifies the computational task by lowering the number of coefficient to estimates. On the other hand this number is sufficient in order to study the effects of second order moments, such as volatility, on average variables.

Second, I rely on some accuracy check criteria in order to choose the degree of Chebyshev polynomials. One of the options suggests to select the degree of the polynomials so as to make the decision error of an agent about some quantity less than a certain small value. I therefore use the criterion of Bounded Rationality Measure as discussed by [Judd (1998), Chap. 17].

This criterion is obtained by measuring the discrepancy between the present consumption computed from the approximated decision rule, and what the agent would have consumed if tomorrow consumption was computed with the same approximation, this difference being related to the contemporary consumption level. More formally, noting $R$ the residual of the Euler’s equation on consumption, i.e.,

$$R(k, \lambda_n; \omega_c(\hat{z}_i)) \equiv \Theta(k, \lambda_n; \omega_c(\hat{z}_i)) - (U_c')^{-1}\{\Xi_c(k, \lambda_n, \bar{z}_i), h(\hat{z}_i)\}$$

where $\Xi_c(k, \lambda_n, \bar{z}_i)$ is the right hand side of Euler's equation over optimal choice of consumption

$$\Xi_c(k, \lambda_n, \bar{z}_i) \equiv \beta \mathbb{E}\left[ U_c' \{ \Theta(k', k, \lambda_n, \bar{z}_i), \lambda_n'(k, \lambda_n, \bar{z}_i); \omega_c(\bar{z}_s), h'(\bar{z}_s) \} \times (r_{t+1} + 1 - \theta)\left| \bar{z}_t \right. \right].$$

Then, the Bounded Rationality Measure is given by

$$E(k, \lambda_n; \omega_c(\hat{z}_i)) = \frac{R(k, \lambda_n; \omega_c(\hat{z}_i))}{\Theta(k, \lambda_n; \omega_c(\hat{z}_i))}.$$
Then, when $E(k, \lambda; \omega_{c}(\tilde{z}_i))$ is less on average than a small quantity, e.g., $10^{-6}$, this means that agents make a $1$ mistake for every million they spent, that would appear as an acceptable error according to Jude (1998).

The previous quantity is then computed across an equally spaced grid over the state space $\mathcal{K} \times \Lambda$. Table 1 shows both the mean and the supremum of the resulting set of measures. One can see that the higher the degree of polynomial, the lesser the max or average error on consumption optimization condition. One might be tempted to favor the degree that achieved the highest accuracy, i.e., the lowest error criteria displayed in the table. This would leads to select the triplet $(n_\lambda, n_k, n_z) = (2, 6, 2)$, as it will be the case for the forthcoming simulation results. However, the method of calibration I shall use requires to solve for the parameters $\omega_\#$ for an unknown, but *ex ante* huge, number of times. Consequently, considering the degree of approximating polynomials leads to a tradeoff between accuracy and time computing efficiency. For this purpose, the last column of Table 1 presents measures of relative speed with respect to the different computations of each variety of polynomials composition. For instance, computing the highest accurate explored combination takes about 25 times the computation of the product of two degree-one polynomials. For the calibration step, I rely on an in-between solution by choosing the triplet $(n_\lambda, n_k, n_z) = (2, 3, 2)$.

2. Building the Data Set.— A particularly critical step in calibration is to set up a data set consistent with the theoretical counterpart of the model. For instance, for a calibration based on U.S. data, taking directly the NIPA’s tables as given leads to inconsistencies since the model does not feature a government nor household production sectors. Along the line of Cooley and Prescott (1995), Gomme and Rupert (2007) improve the calibration methodology in several dimensions in order to extract model-consistent data, and most notably by explicitly considering home production. However, the process developed by these authors do not allow time series for labor and capital non market income to be constructed. Landfield, Fraumeni, and Vojtech (2009) give a prototype treatment of household production accounting, maintaining the double-entry national account procedure that preserves the whole data equilibrium.

12See NIPA (2010).
Table 1. Accuracy checks of collocation method

<table>
<thead>
<tr>
<th>((n_\lambda, n_k, n_\varepsilon))</th>
<th>(|E(k, \lambda_n)|)</th>
<th>(\sup |E(k, \lambda_n)|)</th>
<th>relative speed</th>
</tr>
</thead>
<tbody>
<tr>
<td>((1, 1, 2))</td>
<td>3(−2)</td>
<td>1(−1)</td>
<td>25.1</td>
</tr>
<tr>
<td>((1, 2, 2))</td>
<td>3(−3)</td>
<td>9(−3)</td>
<td>25.8</td>
</tr>
<tr>
<td>((1, 3, 2))</td>
<td>2(−4)</td>
<td>4(−4)</td>
<td>12.9</td>
</tr>
<tr>
<td>((1, 4, 2))</td>
<td>6(−5)</td>
<td>2(−4)</td>
<td>7.2</td>
</tr>
<tr>
<td>((1, 5, 2))</td>
<td>3(−5)</td>
<td>2(−4)</td>
<td>4.6</td>
</tr>
<tr>
<td>((1, 6, 2))</td>
<td>3(−5)</td>
<td>1(−4)</td>
<td>3.0</td>
</tr>
<tr>
<td>((2, 1, 2))</td>
<td>4(−2)</td>
<td>1(−2)</td>
<td>25.8</td>
</tr>
<tr>
<td>((2, 2, 2))</td>
<td>3(−3)</td>
<td>9(−3)</td>
<td>9.6</td>
</tr>
<tr>
<td>((2, 3, 2))</td>
<td>2(−4)</td>
<td>3(−4)</td>
<td>4.6</td>
</tr>
<tr>
<td>((2, 4, 2))</td>
<td>5(−5)</td>
<td>9(−5)</td>
<td>2.6</td>
</tr>
<tr>
<td>((2, 5, 2))</td>
<td>2(−5)</td>
<td>5(−5)</td>
<td>1.5</td>
</tr>
<tr>
<td>((2, 6, 2))</td>
<td>1(−6)</td>
<td>3(−6)</td>
<td>1.0</td>
</tr>
</tbody>
</table>

Note.— This table presents accuracy checks for the Chebyshev orthogonal collocation method presented in the text. Column 2 and 3 provides respectively the mean and maximum of the relative consumption discrepancy computed on the corresponding Euler’s equation. In general, \(a(m)\) represents \(a \times 10^m\). Last column represents the computing time incurred by solving the model with different combination of polynomials, relative to the case \((2, 6, 2)\).

 Basically, our starting point is the NIPA accounts from 1953 to 2008, due to data availability on R&D. We distinguish households sector, the public sector, and the private excluding households sector. Since R&D is also performed by the public sector, we take into account national income, including public sector, as a whole. Since the NIPA excludes public capital income, I shall add it. Households are assumed to provide non market labor income and capital income. Capital income comes from use of non durable goods and housing. Labor income comes from non market hours spent by households. In oder to compute them, I use data from RAMEY (2009) spliced with recent American Time Use Surveys. Then, following LANDFIELD, FRAUMENI, and VOJTECH (2009), I value non-market hours worked by occupation wage related to housekeeping.

\(^{13}\) See BLS (2010).

\(^{14}\) Resulting imputations are quite similar to LANDFIELD, FRAUMENI, and VOJTECH (2009) which report non market households services amounts of 222.4
Public R&D is considered as expenses by the NIPA, so we exclude the corresponding amount to service and non durable goods and create a specific category.\textsuperscript{15} Private R&D expenses are considered as intermediate input so they are not accounted in the NIPA. We impute them in the new category and split labor and capital incomes so as to replicate the labor share in other sources of income.

**Table 2. NIPA & Adjusted GDP/GDI Accounts—2007**

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Gross Domestic Product</td>
<td>14,062</td>
<td>4,076</td>
<td>1,262</td>
<td>247</td>
<td>—</td>
<td>19,647</td>
</tr>
<tr>
<td>Consumption</td>
<td>9,806</td>
<td>2,893</td>
<td>3,378</td>
<td>—</td>
<td>16,077</td>
<td></td>
</tr>
<tr>
<td>Investment</td>
<td>2,295</td>
<td>1,159</td>
<td>457</td>
<td>(714)</td>
<td>3,197</td>
<td></td>
</tr>
<tr>
<td>Net Exports</td>
<td>(714)</td>
<td>714</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td></td>
</tr>
<tr>
<td>Gov.’ Expenditures</td>
<td>2,674</td>
<td>(2,674)</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td></td>
</tr>
<tr>
<td>R&amp;D Expenditures</td>
<td>—</td>
<td>24</td>
<td>102</td>
<td>247</td>
<td>373</td>
<td></td>
</tr>
<tr>
<td>Compensation for Emp.</td>
<td>7,863</td>
<td>2,606</td>
<td>174</td>
<td>1,413</td>
<td>12,056</td>
<td></td>
</tr>
<tr>
<td>Other Incomes excl. CFC</td>
<td>4,410</td>
<td>613</td>
<td>1,262</td>
<td>(1,392)</td>
<td>4,967</td>
<td></td>
</tr>
<tr>
<td>Cons. of Fixed Capital</td>
<td>1,768</td>
<td>857</td>
<td>—</td>
<td>—</td>
<td>2,624</td>
<td></td>
</tr>
<tr>
<td>Gross Domestic Income</td>
<td>14,041</td>
<td>4,076</td>
<td>1,262</td>
<td>247</td>
<td>—</td>
<td>19,647</td>
</tr>
<tr>
<td>Statistical Discrepancy</td>
<td>21</td>
<td>(21)</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td></td>
</tr>
</tbody>
</table>

**Note.**— This table presents a bridge from NIPA to Adjusted Gross Domestic Product and Gross Domestic Income that summarizes the different adjustments and imputation to build the dataset. These modifications include Households production, Government imputations and transfers, business R&D imputations, and other transfers including statistical discrepancy and ambiguous capital breakdown. Additional information are given in the text and in appendix.

Appendix \[\text{D}\] gives in greater details the different assumptions to build up the database.

3. **Choice of Functional Forms.**— In order to characterize quantitatively some features of the artificial economy described by the set of conditions given in the previous section, I start the calibration process by picking out some functional forms and by choosing an assignment of values for the model’s parameters. To begin with, let and 2,219.5 billions of current dollars, respectively in 1965 and 2004, against our own calculation of 203 and 2,398 billions.

\textsuperscript{15}See NIPA (2010) and NFS (2010).
the technology of production of intermediate goods be given by
\[ F(K, h, \Gamma) \equiv K^v (\Gamma h)^{1-v} \]
which is a Cobb-Douglas production function, with \( v \in (0, 1) \) usually referred to as the capital share in value added in a competitive framework. However, since the intermediate firms can charge a markup, \( \nu \) does not express the share of capital in total firm’s income as the interest rate does not match the marginal productivity of capital. Moreover, productivity \( \Gamma \) represents a labor-augmenting (Harrod-neutral) technological progress in order to be consistent with the existence of a balanced growth path—this precision not being crucial in the Cobb-Douglas case.

I set the R&D cost of undertaken innovation effort \( x \) to be
\[ \delta(x) \equiv \bar{\delta} x^2 / 2, \]
with \( \bar{\delta} \) a positive parameter. On top of being the simplest convex function, this corresponds to the cost function used by Aghion et al. (2005). Then I set the elementary probability function to
\[ \varphi(x; \alpha, \kappa) \equiv \frac{\kappa x + \alpha}{1 + \kappa x} \]
where \( \alpha \in (0, 1) \) is the parameter of imitation, and \( \kappa > 0 \) the effectiveness of investment. This kind of probability function is bounded between 0 and 1, stochastically increasing in \( x \), and is used in applied firm dynamics studies in IO, following Pakes and McGuire (1994).16

16See also Doraszelski and Pakes (2007).

Competition in this model is characterized by the parameter \( \epsilon \) through the function \( Y(\cdot) \), which measures the effects of a move away from collusion towards independent behavior between rivals as defined by Vickers (1995). I follow Aghion et al. (2005) and use a linear relationship between competition and the share of what a monopoly would earn in a given industry, so that
\[ Y(\epsilon) \equiv \frac{1 - \epsilon}{2}. \]

Finally, I use the following CRRA momentary utility function
\[ U(C_t, 1 - h_t) \equiv \frac{C_t^\eta (1 - h_t)^{1-\eta})^{1-\sigma} - 1}{1 - \sigma} \]
with the limiting case \( U(C_t, 1 - h_t) \equiv \eta \ln C_t + (1 - \eta) \ln (1 - h_t) \) when \( \sigma = 1 \). As shown by King, Plosser, and Rebelo (1988), this sort of function is balanced-growth-path admissible since income and substitution effects of wage on labor supply are exactly offset.
4. Simulated Method of Moments.— From now, the model is in need of value assignment to the vector of parameters
\[ \Psi \equiv (v, \omega, \epsilon, \eta, \sigma, \beta, \theta, \delta, \alpha, \kappa). \]
Many papers followed the calibration process described by C\-LEY and PRESCOTT (1995): key parameters are set such as some deterministic steady state ratio or other variables of interest match average empirical counterparts. In fact, this is consistent only if the model is quasilinear since otherwise, deterministic steady state and average values must differ due to Jensen's inequality. With no a priori assumptions about the curvature of the model, we need an other calibration method.

As an alternative to the calibration based on steady state ratios, one could use simulated moments estimation method (SMM) pioneered by M\-CFADDEN (1989) and PAKES and POLLARD (1989). This estimation technique, based on general method of moments (GMM) (HANSEN, 1982) and extended to time series (LEE and INGRAM, 1991; DUFFIE and SINGLETON, 1993), leads to pick up parameters values that minimize a distance measure between observed and simulated moments computed from the series.

Let \( x = (x_1, \ldots, x_t, \ldots, x_T) \) be a sequence of time observations \( x_t = (x_{1,t}, \ldots, x_{n_o,t}) \), where \( n_o \) is the number of observable variables and \( T \) the length of the period of observation. For instance, \( x_{1,t} \) could be the Gross Domestic Product at date \( t \) and \( x_{2,t} \) the Gross Investment. Let \( h(x_t) \) be a function mapping \( \mathbb{R}^{n_o} \rightarrow \mathbb{R}^{n_m} \). For instance, let the first coordinate of \( h(x_t) \), \( h_1(x_t) \), be the investment to GDP ratio
\[ h_1(x_t) = \frac{x_{2,t}}{x_{1,t}}. \]
Finally, let
\[ H_T^o(x) = \frac{1}{T} \sum_{t=1}^{T} h(x_t) \]
be the statistics calculated as a time average of some function of observations. Let \( \Omega \) be the variance matrix of the empirical moments.

Assume now that we have a solution method that adequately provides us with sequences of length \( \mathcal{T} \) of \( n_s \) simulated variables, \( y(\beta, \eta) = (y_1, \ldots, y_t, \ldots, y_{\mathcal{T}}) \), with \( y_t = (y_{1,t}, \ldots, y_{n_s,t}) \), for a given set of parameters \( \Psi \), and some sequence of stochastic i.i.d. innovations \( \eta = (\eta_1, \ldots, \eta_{\mathcal{T}}) \). It may be possible to build a function \( \tilde{h} : \mathbb{R}^{n_s} \rightarrow \mathbb{R}^{n_m} \) so that \( \tilde{h}(y_t) \) is the model counterpart of \( h(x_t) \).\(^{17}\) We can therefore

\(^{17}\)Note that we do not require that \( n_s = n_o \) since, for instance, it is possible to simulate directly \( h(x_t) \) rather than the \( x_{i,t}'s. \)
compute the vector of statistics based on simulation

\[
H^s_{\mathcal{F}}(y(\Psi, \eta)) = \frac{1}{\mathcal{F}} \sum_{t=1}^{\mathcal{F}} \tilde{h}(y_t).
\]

The SMM consists in finding the \(\Psi\) that reduces a measure of the distance between \(H^o_T\) and \(H^s_{\mathcal{F}}\). But three main issues arise. First, some moment computation in \(H^o_T\) may suffer from small-sample biases, that could generate artificial discrepancies with simulated moments. Second, in the simulation process, the simulated series are not drawn from the ergodic, i.e., time invariant distribution, since we arbitrarily fix the initial values. Third, the value of \(H^s_{\mathcal{F}}\) depends critically of the sequence of stochastic innovation \(\eta\).

To overcome the first problem, I should set \(T = \mathcal{F}\) so that the estimation take into account such a bias. But, as regard the second issue, we set \(\mathcal{F}\) sufficiently large as compared to \(T\) in order to reach the time invariant distribution of the \(y\)'s and drop the first observations to keep \(T\)-period long simulations. Finally, to get rid of the last point, I draw \(\mathcal{S}\) sequences of innovations and compute the mean in order to lessen the dependence based on stochastic draws.\(^{18}\)

Accordingly, the SMM estimator is

\[
\hat{\Psi}_{T,\mathcal{F}} = \arg\min_{\Psi} \left[ H^o_T(x) - H^s_{\mathcal{F}}(y(\Psi)) \right]^\top W \left[ H^o_T(x) - H^s_{\mathcal{F}}(y(\Psi)) \right]
\]

where

\[
H^s_{\mathcal{F}}(y(\Psi)) = \frac{1}{\mathcal{F}} \sum_{c=1}^{\mathcal{F}} H^s_{\mathcal{F},T}(y(\Psi, \eta^c))
\]

and

\[
H^s_{\mathcal{F},T}(y(\Psi, \eta^c)) = \frac{1}{T} \sum_{t=\mathcal{F}-T+1}^{\mathcal{F}} \tilde{h}(y_t).
\]

The choice of the weight matrix \(W\) can follow different criteria. For instance, \(W = I_{nm}\) where \(I\) is the identity matrix and \(nm\) the number of moments, imply the same weight for each moments. This could leads to estimation problems if the moments are not comparable in unit size. As shown by Duffie and Singleton (1993), an appropriate alternative is to choose the optimal weight matrix given by

\[
W^* = [(1 + 1/\mathcal{F})\Omega]^{-1},
\]

\(^{18}\)In practice, I use a drop a pre-sample period of length \(\mathcal{F} - T = 100\) and averaging length \(\mathcal{F} = 100.\)
for which one can use a \textit{Newey and West} (1987) estimate for the variance matrix $\Omega$. This matrix is optimal in the sense that it provides the smallest possible asymptotic variance among the class of positive-definite matrices. Finally, we can chose between the two by selecting the diagonal of $W^*$, $\text{diag}(W^*)$. This last choice has the benefit to be unit-free. Identity and diagonal cases leads to select the main economic important moments conditions rather than statistically meaningful conditions.\footnote{See \textit{Cochrane} (2005) and \textit{Ruge-Murcia} (2010).}

However, giving a picture of the uncertainty surrounding the parameters' estimates needs to compute some measure of the variance-covariance matrix of the parameters $\Sigma_\psi$. In the general case, that is for a non optimal given matrix $W$, $\Sigma_\psi$ is given by

$$\Sigma_\psi = (D^\top WD)^{-1} D^\top W \Omega WD (D^\top WD)^{-1}$$

where

$$D = \left. \frac{\partial H^s(y(\Psi))}{\partial \Psi^\top} \right|_{\Psi_T^y}.$$ 

Choosing the optimal weight matrix greatly simplifies the expression of the matrix

$$\Sigma_\psi = (D^\top WD)^{-1}.$$ 

In the final step, one have to select the key moments that would serves as targets for the calibration. It seems quite reasonable to use the average ratios used in \textit{Cooley and Prescott} (1995), namely the labor share of total revenues, the investment to GDP ratios, the capital to GDP ratio, average total hours worked, and average growth of per capita real GDP.\footnote{To be precise, \textit{Cooley and Prescott} (1995) use investment to capital rather than GDP ratio, but the former seems less preferable due to frequent change in capital stock measures as in \textit{Comme and Rupert} (2007).} I use also the share of corporate profits in GDP and the R&D to GDP ratio that would identify competition level and the R&D cost parameter.\footnote{It is worth mentioning that reference to \textit{identification} is use for illustration only and might be improper here, since parameters are jointly estimated.} Finally, I compute a measure of the Solow’s residual $\zeta_t$ as

$$\zeta_t = \log(y_t) - \bar{v} \log(k_t) - (1 - \bar{v}) \log(h_t)$$

where $1 - \bar{v}$ is computed as the mean of labor share over the sample. Cyclical component $\hat{\zeta}_t$ is then extracted with the use of Hodrick-Prescott filter with parameter of 6.35, as in \textit{Ravn and Uhlig} (2002). I then add to the target moments the variance and first order correlation of $\hat{\zeta}_t$. Figure 2 plots these moments as well as actual and simulated mean data over the sample.
5. Calibration of Restricted Parameters.— Due to lack of information on aggregate data, I decide to restrict some parameters based on extra model information. As in Cooley and Prescott (1993) I assume the log-log case for the utility function, i.e. $\sigma = 1$. Moreover, efficiency of innovation $\kappa$ is normalized to one and $\bar{a} = 0.067$ which broadly correspond to a number of ten years elapsing before half of the non innovative firms has introduced innovation.
The imitation parameter can be intuited as follows. If we assume that for a given economy, innovation efforts are inefficient, i.e. $\kappa = 0$, and the firms are not allowed to innovate. Then, it implies that if at least one unleveled sector exists, we could observe a diffusion process due to imitation. According to the law of motion of the population given in appendix, when $\kappa$ is set to zero, we have

$$\lambda_{u,t+1} = \lambda_{u,t} - \bar{\alpha}\lambda_{u,t}.$$ 

Then, the population decrease according to

$$\lambda_{u,t} = \lambda_{u,0}(1 - \bar{\alpha})^t.$$ 

Let $T$ be the number of years for which the number of unleveled sectors has been divided by two, meaning that the technological gap has been filled by the half of the existing unleveled industries. It comes

$$\frac{\lambda_{u,t+T}}{\lambda_{u,t}} = \frac{1}{2} \Rightarrow T = -\frac{\ln 2}{\ln(1 - \bar{\alpha})}.$$ 

For the chosen value of the imitation parameter, innovation spans into half the remaining sectors every ten years at a given time.  

6. Estimates of other Parameters.— Finally, Table 4 shows resulting parameter values for the calibration process.

Generally speaking, standard key parameters related to the RBC literature are quite similar. In particular, depreciation rate and subjective discount factor are in the range of standard values. The share of labor is lower than in Cooley and Prescott (1995) since the data take into account labor income from non market house work. The productivity increment parameter’s value leads to a productivity gain $\gamma \equiv (1+\varpi)^{1/(1-\nu)} = 1.14$, which is quite in line with previous work. For instance, Stokey (1995) studies R&D models of economic growth and set a range of interesting values of $[1.02, 1.60]$ for what could be interpreted as the parameter $\gamma$. Moreover, this value leads also to a aggregate markup of about 10%, which is consistent with aggregate estimates for the whole U.S. economy as measured by Basu and Fernald (1997).

A first look at figures in parentheses of Table 4 gives a broad assessment of the uncertainty surrounding parameter estimates. They are computed as the squared root of the diagonal of variance.

---

22 Mansfield (1961) estimated this period to varied from 0.9 to 15 years, the average being 7.8. In Mansfield (1985), median time to diffusion is estimated to 8.2 years. In their medium run model, Comin and Gertler (2006) use an average time to adoption of ten years.

23 See King and Rebelo (2009).
### Table 3. Parameter Values

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td><strong>Calibrated parameters</strong></td>
<td></td>
</tr>
<tr>
<td>( \omega )</td>
<td>Productivity increment</td>
<td>0.1038</td>
</tr>
<tr>
<td>( \nu )</td>
<td>Coefficient of capital</td>
<td>0.3151</td>
</tr>
<tr>
<td>( \theta )</td>
<td>Depreciation rate</td>
<td>0.0727</td>
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<tr>
<td>( \beta )</td>
<td>Subjective discount factor</td>
<td>0.9576</td>
</tr>
<tr>
<td>( \eta )</td>
<td>Share parameter for consumption</td>
<td>0.5510</td>
</tr>
<tr>
<td>( \delta )</td>
<td>R&amp;D cost parameter</td>
<td>1.8173</td>
</tr>
<tr>
<td>( \sigma_e )</td>
<td>Standard deviation of the shock</td>
<td>0.0181</td>
</tr>
<tr>
<td>( \rho )</td>
<td>Persistence parameter</td>
<td>0.8938</td>
</tr>
<tr>
<td></td>
<td><strong>Restricted parameters</strong></td>
<td></td>
</tr>
<tr>
<td>( \sigma )</td>
<td>Relative risk aversion</td>
<td>1</td>
</tr>
<tr>
<td>( \kappa )</td>
<td>Effectiveness of R&amp;D investment</td>
<td>1</td>
</tr>
<tr>
<td>( \bar{\alpha} )</td>
<td>Imitation parameter</td>
<td>0.0670</td>
</tr>
<tr>
<td>( \epsilon )</td>
<td>Competition level</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Note.—This table presents estimation results based on method of simulated moments (SMM.) Figures in parentheses are the root squared of the diagonal element of matrix \( \Sigma_{\Psi} \) defined in the text.

Covariance matrix \( \Sigma_{\Psi} \). If traditional real business cycle related parameters are quite well identified (coefficient of capital, depreciation rate, and so forth) uncertainty over new parameters seems very large. However, it is worth noting that this criteria is based on gaussian approximation of distribution of the parameter (Duffie and Singleton, 1993) and might be hard to understand for other kind of distribution.

Another check of the quality of the minimization process in the SMM step consists of the draw of the objective function

\[
J(\Psi) = \left[ H^{\Theta}_T(x) - H^{\Psi}(y(\Psi)) \right]^{\top} W \left[ H^{\Theta}_T(x) - H^{\Psi}(y(\Psi)) \right],
\]

in each parameters direction, taken fixed the others at the optimum values. Such draws are given in the Figure 3 and confirm that the minimization process has converged to a minimum. All in all, these estimates are quite similar to the one obtained by traditional
**Figure 3. Check Plots**

![Graphs of various parameters](image)

**Objective function** $J(\Psi)$  
**Point Estimates** $\hat{\Psi}_{T,\varphi}$

**Note.**—This figure plots the SMM objective function $J(\Psi)$ in the parameter direction. Minimizing parameter value in dotted line should attain minimum of the objective function.

calibration procedure. Based on the deterministic steady state ratios, one would obtain very close estimates to the one obtained by SMM.\(^{24}\) This leads to conclude that the model is very close to a log linear dynamic system.

\(^{24}\)See Table 6 in Appendix B
B. *Long run Properties*

1. *The Mean Effect of Competition.*— The panel in Figure 4 shows average variables simulated from the benchmark calibration given in Table 4 when competition $\epsilon$ varies. The top left figure displays the innovation efforts undertaken by leader, follower, and neck and neck firms. It shows that when competition increases, reducing rents opportunities in neck and neck industries, follower firms decrease their innovation level since the disincentive effect is prominent for this type of firm. By contrast, neck and neck firms foster their innovational activities to gain monopoly position and escape competition. One can note that leader firms keep maintaining their innovation efforts, yet in a lesser extent.

It follows that the population of sectors across the economy evolves accordingly. Since innovation in neck and neck sectors increases with competition, the probability of one neck and neck sector to become a unlevled one increases too. Then, unlevled industries population grows. Conversely, less innovation by followers firms reduce their chance to become neck and neck with their competitor, and population of neck and neck industries diminishes, as plotted in the top right panel of Figure 4.

Based on the two previous observations, it comes out that aggregate innovation exhibits an inverted-U shape with respect to competition as depicted in the bottom left figure. For lowest competitive situations, neck and neck firms are sufficiently numerous to drive aggregate innovation up when competition increases. For highest level of competition, unlevled sectors are highly plentiful and prompt aggregate innovation down. As a consequence, a particular level of competition exists for which disincentive and fostering innovation effects are jointly neutralized, indicating the top of the inverted-U shape.

Finally, since the long term output growth is driven by innovation in this model, one can observe the same patterns for growth as shown in the bottom right picture of Figure 4. It appears then that given the benchmark calibration, the maximum gain in terms of growth of increasing competition would represent about one tenth of percentage point.

On top of providing an illustration of the mechanics at stakes in the model, Figure 4 raises some concerns on the ambiguity of targeting an optimal level for competition. Indeed, though they display a similar pattern, it seems that the maximizing aggregate innovation level of competition do not necessarily corresponds exactly
to the one that maximize growth. Looking at Research & Development expenses and Consumption in Figure 5, that do not display the inverted-U shape, so that maximizing competition level is the highest.

In the next section, I investigate how these relationships are affected by different level of volatility.
2. The joint effect of competition and volatility on Innovation and growth.— Figure 6 shows how aggregate innovation is affected by competition and volatility. One can see on the left figure that average aggregate innovation is a decreasing function of volatility, reflecting the irreversible investment effect of innovation efforts: higher volatility implies higher cost of opportunity to invest in innovation activities since a firm cannot get money back from future productivity gain.

Another interesting feature of Figure 6 is that the inverted-U shape is preserved for higher degree of volatility. Nevertheless, the maximum of the inverted-U relationship moves toward higher levels of competition, as depicted by the square marked line on the right picture. This reveals the effect of fluctuations on innovation decisions.

The same pictures can be drawn for growth. Surface and contour plots of Figure 7 feature the same kind of inverted-U shape with respect to competition, and growth declines with higher volatility rates. Furthermore, one can see that for higher degree of volatility, the level of competition that maximize growth increases.

From the Figures 6 and 7, one can draw at least three main observations:
Figure 6. Competition, Volatility, and Innovation

Note.—These pictures show the combined effect of volatility and competition on aggregate innovation. The left figure shows 3D surface where contour lines are projected and depicted on the right figure. Solid line marked with squares (■) indicates the innovation maximizing level of competition. Solid line marked with plain bullets (●) indicates the innovation maximizing level of growth.

1. Aggregate innovation and growth display an inverted-U relationship with respect to competition.
2. Average aggregate innovation and growth decline when volatility increases.
3. For the benchmark calibration, the levels of competition that maximize innovation and growth are distinct and increase when volatility increases.

C. Model Dynamics

1. Correlations.— In this section, I show some statistical features of the model. Table 5 presents a set of summary statistics.

   The first two columns of Table 5 report statistics computed from adjusted U.S. data, namely standard deviations (a) and correlation with output (b). Investigations covers the 1959 to 2007 period. Before moments calculations, actual time series were logged and

25See D.
Figure 7. Competition, Volatility, and Growth

Note.—These pictures show the combined effect of volatility and competition on growth. The left figure shows 3D surface where contour lines are projected and depicted on the right figure. Solid line marked with squares (●) indicates the innovation maximizing level of competition. Solid line marked with plain bullets (■) indicates the innovation maximizing level of growth.

trends were extracted using the filter of Hodrick and Prescott (1997). To filter annual frequency data, I follow Ravn and Uhlig (2002) and set the filter’s parameter to 6.25. According to these transformations, standard deviations can be interpreted as mean deviations from the trend express in percentage.

The model is then simulated hundred times and artificial time series are generated with the same sample periods than actual data. The same procedure previously described is applied to compute the simulated statistics. Figures in parentheses are standard deviations of these simulated statistics.

As far as standard deviation of output, consumption, investment, capital stock, hours, and productivity are concerned, the comparison of actual and simulation based statistics leads to common analyses of RBC models. As regards additional variables, the model succeeds in explaining about two third of the volatility of R&D expenses. However, the R&D expenses seems to be more correlated with output in the model than in the dataset.

Table 4. Business Cycle Statistics Comparison between U.S. and model data.

<table>
<thead>
<tr>
<th></th>
<th>Annual U.S. data</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(a)</td>
<td>(b)</td>
</tr>
<tr>
<td>Output</td>
<td>1.3589</td>
<td>1.2264 (0.46)</td>
</tr>
<tr>
<td>Consumption</td>
<td>0.9003</td>
<td>0.5585 (0.21)</td>
</tr>
<tr>
<td>Investment</td>
<td>3.5336</td>
<td>4.2052 (1.59)</td>
</tr>
<tr>
<td>Capital stock</td>
<td>0.3012</td>
<td>0.3682 (0.14)</td>
</tr>
<tr>
<td>Hours</td>
<td>0.5260</td>
<td>0.3523 (0.13)</td>
</tr>
<tr>
<td>Productivity</td>
<td>1.0388</td>
<td>0.8843 (0.33)</td>
</tr>
<tr>
<td>R&amp;D expenses</td>
<td>1.5697</td>
<td>0.9512 (0.35)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Annual U.S. data</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(a)</td>
<td>(b)</td>
</tr>
<tr>
<td>Output growth</td>
<td>2.1554</td>
<td>2.0142 (0.71)</td>
</tr>
<tr>
<td>Prod. growth</td>
<td>1.7236</td>
<td>1.4907 (0.51)</td>
</tr>
<tr>
<td>Inv./Output</td>
<td>1.2756</td>
<td>1.3000 (0.39)</td>
</tr>
<tr>
<td>R&amp;D/Output</td>
<td>0.1338</td>
<td>0.0160 (0.00)</td>
</tr>
</tbody>
</table>

Note.—This table presents summary statistics for actual U.S. and simulated data, namely standard deviations (a), correlation with output (b), and first order autocorrelation (c). Statistics related to actual data are computed with the dataset presented in the current section and Appendix D. Standard deviations, correlations, and autocorrelations are sample average over 100 simulations. Sample standard deviations are given in parentheses.

Table 5 presents also the standard deviation (a) of the growth rates of output, labor productivity, the investment–output ratio, and the R&D to output ratio, both for actual and simulated data. This table demonstrates the ability of the model to imitate the variability of these variables, except for the R&D to output ratio that seems less volatile in the model than in the data.

The table reports also first order autocorrelation (c) properties of both the actual U.S. and simulated data. The results shows that the model can account for 40% of the autocorrelation of output.
growth, which seems to be more than one could expect from a standard RBC model.\footnote{For instance, \cite{COGLEY and NASON (1995)} show how traditional RBC models fail to display propagation mechanisms. \cite{JONES, MANUELLE, and SIU (2005)} show how endogenous growth component can partially improve this ability.}

2. **Responses to a productivity shock.**— Figure 8 presents a panel of impulse response functions (IRF) of a log-linearized version of the model for different variables. It illustrates how the model behaves following a single perturbation $\varepsilon_0$ at date 0 of one standard deviation. Three cases are reported, namely for no competition (dashed line), for benchmark value (solid line), and for full competition (dotted line.)

Following a positive technology shock, output increases instantaneously, leading to higher profits for intermediate firms. This in turn leads to higher innovation activities for all type of firms and hence for aggregate innovation. It follows that productivity growth increases. Since innovation in neck and neck sectors is higher than the one in unleveled industries, the proportion of unleveled increases. Accordingly, aggregate markup increases with the productivity shock. Finally, aggregate R&D expenses increases but less than output. Consequently, the R&D to output ratio experiences an immediate drop and returns to its long term value from above.

All in all, among the previous analysis of the model’s dynamics, one can retain the following properties.

(1) The model can explain about 60% of R&D expenses volatility.
(2) The model succeeds in explaining about 37% of the first order autocorrelation of output growth.
(3) R&D expenses to output ratio is contra-cyclical, but appears less volatile in the model than in adjusted NIPA data.
(4) The model displays pro-cyclical aggregate markups.

**IV. Discussion: The welfare cost of fluctuations revisited**

**A. Concepts and Definitions**

The model studied in the previous sections, its solution method, and its calibration procedure, engenders a lab economy in which it is possible to assess the welfare cost of fluctuations, i.e., the loss in terms of economic welfare that an agent incurs when living in a fluctuating economy rather than in a risk-less, or deterministic,
Figure 8. Impulse Response Functions

Output

Leader

Follower

Neck and neck

Innovation

Productivity growth

Unleveled

Markup

R&D ratio

\[ \text{-- no competition (} \epsilon = 0 \text{) -- benchmark (} \epsilon = 0.496 \text{) -- full competition (} \epsilon = 1 \text{)} \]

Note.—These figures plot the response of the model following a unexpected shock of one standard deviation (see Table 4). IRF are obtain from a second order approximation of the model around the deterministic balanced growth path. The software Dynare has been used (Dynare, 2011).

ones, and to see how it depends on competition. Equivalently, one can use this model to analyses the welfare cost of competition, i.e., the loss in terms of economic welfare that an agent incurs when living in an economy where collusion between intermediate firms occurs rather than in more competitive ones, and to see how it depends on the volatility of shocks that hit the economy.

1. Compensation Parameter.— In order to compute such costs, one need to start by defining the agent’s economic welfare \( W \). I take as such a measure the, presumably convergent, sum of discounted
flows of utility drawn from consumption and leisure, as it define the household's objective function to be maximized as in (HP). It follows that

\[ W(s_0) = E_0 \left[ \sum_{t=0}^{\infty} \beta^t U(C_t, h_t) \right], \]

when \( C_t \) and \( h_t \) are optimally chosen. Note that this definition of welfare uses conditional expectation operator

\[ E_0[\cdot] = E[\cdot|s_0], \]

so that this measure of welfare depends on the initial state, \( s_0 \) from which streams of future consumption are assessed. As we will see below, it is convenient to use conditional rather than unconditional expectations since in the latter, initial level of productivity, \( \Gamma_0 \), is not defined.

When the household evolves in a deterministic environment, the welfare is similarly given by

\[ W^*(s'_0) = \sum_{t=0}^{\infty} \beta^t U(C^*_t, h^*_t), \]

with \( C^*_t \) and \( h^*_t \) are optimally chosen, and where expectation has been removed since there is no uncertainty in this world. Again, this measure depends on the initial state \( s'_0 \) from with optimal trajectories for hours and consumption are established.

We can also define the deterministic balanced growth path welfare value as

\[ W^*(s^*_0) = \sum_{t=0}^{\infty} \beta^t U(c^* \Gamma_0(1 + g^*)^t, h^*), \]

with \( c^* \), \( h^* \), and \( g^* \) are the long term stationary values of trend removed consumption, hours worked, and productivity growth. Appendix E presents a way of computing the welfare value accordingly.

According to the given definition, the welfare cost of fluctuations could be computed as the difference between \( W \) and \( W^* \) across all possible initial states \( s_0 \). But that the cost of fluctuations equals −11 or .6 is not easily understood. One would prefer an interpretation in terms of compensation parameter of consumption as described by Lucas (1987). The compensation parameter gives a measure of the fraction of consumption \( \omega \) that, being uniformly and equally added to future consumption streams across all possible states \( s \in \mathcal{S} \), would let the representative household indifferent between the two environments, i.e., leads to the same level of welfare.
An alternative way of computing such a cost consists in working out the fraction of consumption \( \omega(\epsilon) \) that an agent evolving on the balanced growth path of a full competitive and deterministic world need to receive in order to be indifferent between living this world or going into a less competitive, stochastic one. On top of being equivalent to the previous definition, this formulation has the advantage of resulting in simpler computations, as we will see below.

Finally, let the function \( W^*(\omega) \) denotes the welfare value of an agent evolving on the balanced growth path in the full competitive deterministic economy and receive a fraction \( \omega \) each period, i.e.,

\[
W^*(\omega) \equiv \sum_{t=0}^{\infty} \beta^t U \left( (1 + \omega) \bar{c}^* \Gamma_0 \left( 1 + \bar{g}^* \right)^t, \bar{h}^* \right),
\]

where \( \bar{c}^*, \bar{h}^*, \) and \( \bar{g}^* \) are the deterministic stationary values in the full competitive model.

2. Trend Normalization and Averaging.— One particularly important feature of welfare comparison between two alternative worlds is to compare the related measures from the same starting point. As described in the previous sections, we use a stationary version of the model, that is a version in which extensive, or growing, variables are deflated by the labor productivity \( \Gamma_t \). In the solution process of the stationary model, we identified two predetermined endogenous variables, namely the per capita capital stock, \( k \), and the measure of neck and neck sectors, \( \lambda_n \), and one predetermined exogenous variable, the productivity shock \( z \). Accordingly, the value of \( W(s_0) \) is understood for a given value of the quadruplet \( (\Gamma_0, k_0, \lambda_{n,0}, z_0) \).

It becomes clear now why I do not refer to unconditional mean in computing the economic welfare. As for capital and measure of neck and neck industries, standard theorems of dynamic programming tell us that they are drawn in ergodic distributions in a sufficient long period of time.\(^{28}\) This is also true for the technology shock, by construction of the Markov chain. However, this is not the case for \( \Gamma \) which is a trend. Consequently, one cannot integrate over the all possible states for \( \Gamma_0 \) and welfare measure using unconditional expectation operator makes no sense.

Finally, one should use an average of the sum of discounted utility flows when \( k \) and \( \lambda_n \) are taken in a distribution sufficiently close to the ergodic one which is defined on \( \mathcal{S} = \mathcal{K} \times \Lambda \times \mathcal{I} \), and set \( \Gamma_0 \) to an arbitrary value.

However, according to our alternative definition of the compensation parameter, we restrict attention to an agent that lives on the

\(^{28}\)See \textcite{Stokey1989}.
deterministic balanced growth path, so that the state of this agent is partly defined by the unique pair \((k^*, \lambda_n^*)\). Turning into the stochastic world at date 0, this agent could faces a productivity shock \(z_0\) that is then assumed to be drawn from the stationary distribution of the Markov chain, \(\mathcal{Q}(z)\).

We are now able to give formal definitions of the compensation parameter, depending on the conditions it is computed.

**Definition 1.** The welfare cost of competition, without transition effects, is measured by the static deterministic compensation parameter \(\omega_{NT}^*(\epsilon)\), that solves

\[
\overline{W}^* (\omega_{NT}^*(\epsilon)) = W^* (s^*(\epsilon); \epsilon)
\]

where \(s^*(\epsilon)\) is the deterministic stationary values in the model with competition parameter \(\epsilon\).

**Definition 2.** The welfare cost of competition accounting of transition effects is measured by the dynamic deterministic compensation parameter \(\omega_T^*(\epsilon)\), that solves

\[
\overline{W}^* (\omega_T^*(\epsilon)) = W^* (\bar{s}^*; \epsilon)
\]

where \(\bar{s}^*\) the deterministic stationary values in the model with full competition.

**Definition 3.** The joint welfare cost of competition and fluctuations is measured by the stochastic compensation parameter \(\omega^*(\epsilon)\), that solves

\[
\overline{W}^* (\omega^*(\epsilon)) = \int_{\mathcal{Z}} W(\bar{s}^*, z; \epsilon) \mathcal{Q}(z) dz.
\]

I show in Appendix \[\square\] how to solve for the compensation parameters in the definitions above.

**B. Results**

In order to compute the different compensation parameters, one need to compute the steady state values in the deterministic model for different values of competition and the value of the welfare for a given initial state \((k, \lambda_n, z)\).

First, steady state values are obtained by simulating the deterministic model for a sufficiently long period of time \(T\) (I use in practice \(T = 300\) periods.) Second, I rely again on orthogonal collocation to compute the value function. Basically, I work out a function of
the form

$$W(k, \lambda_n, z_i) = \Theta(k, \lambda_n; \omega_W(z_i))$$

$$= \exp \left( \sum_{j_k = 0}^{n_k} \sum_{j_\lambda = 0}^{n_\lambda} \omega_W(j_k, j_\lambda | z_i) T_{j_k} \left( \psi_k (\log(k)) \right) T_{j_\lambda} \left( \psi_\lambda (\lambda_n) \right) \right)$$

and I compute the residual

$$R_W(k_i, \lambda_n, j, z_l) = W(k_i, \lambda_n, j, z_l) - U(c_t(k_i, \lambda_n, j, z_l), h_t(k_i, \lambda_n, j, z_l))$$

$$- \beta \sum_{s=0}^{n_s} \pi(z_s | z_l) \Xi_W(z_s | k_i, \lambda_n, j, z_l)$$

with $\pi(z_s | z_l)$ the transition probability and

$$\Xi_W(z_s | k_i, \lambda_n, j, z_l)$$

$$= \begin{cases} W(k_{t+1}(\cdot), \lambda_{n,t+1}(\cdot), z_s) + \eta \frac{1}{1-\beta} \log(1 + g_t) & \text{if } \sigma = 1, \\ (1 + g_t)^{(1-\sigma)} W(k_{t+1}(\cdot), \lambda_{n,t+1}(\cdot), z_s) & \text{otherwise,} \end{cases}$$

for all $i = 1, \ldots, n_k$ and $j = 1, \ldots, n_\lambda$.

Figure 9 presents the three definitions for the compensation parameter for different value of competition. The observation of the figure leads to the following findings.

First, either in a deterministic or stochastic environment, the lack of competition represents a welfare cost. For instance, moving instantaneously form a full competitive economy to a full collusion one implies a cost of about 1.7% of average consumption. Interestingly, this cost downs very close to zero, even becomes a gain, when competition reaches between $\epsilon \approx 0.6$ and $0.7$.

Second, taking into account the transition in the deterministic case slightly decreases the cost of competition. This decline has a maximum value of two tenth of a percentage for the lowest level of competition.

Finally, the transition to a stochastic framework leads to assess a welfare cost of fluctuations of the magnitude of less than one tenth of percentage point. This law level of welfare cost of fluctuations is observed all over the range of the competition parameter. As an explanation why this kind of model fails at giving higher welfare cost of fluctuations, Barlevy (2007) invokes the absence of diminishing returns of investment: lower fluctuations lead to higher average resources devoted to innovative activities which are substituted for consumption. So higher future growth is partially compensated by a lower current level of consumption.
V. Conclusion

In this paper I have presented preliminary analyses of an stochastic endogenous growth model where competition plays an ambiguous role on innovation. Using model consistent data taken from U.S. national accounts, a calibrated version of the model is simulated to generate time series. Moments comparisons show that the model is very close to the canonical RBC model. It then features quite good business cycle properties. The model displays also long run characteristic that match recent empirical facts on the relationship between innovation and growth.

The model is also used to compute the joint welfare costs of competition and fluctuations. It comes that these costs are relatively small for the calibration at hand. However, the analysis shows that competition and volatility play a joint role on welfare: affecting one modifies the impact of the other.

The present analysis suggests two main implications. First, that far from being independent, short term and long run considerations are likely to entangled. For instance, when quantifying the effect of short run oriented policies, such as stabilization ones, one should take into account how other more structural policies may
affect the assessments. This may be of a particular interest considering an economy in which a money authority seeks at stabilizing inflation and where government works towards increasing competition. Second, the simulations indicates that for a set of plausibly calibrated parameters, increasing competition to promote innovation do not systematically lead to substantive welfare gains.

For future research, it would be interesting use an adapted and improved version of this model in order to assess the joint effect of a monetary policy and structural one that seeks at increasing competition as aimed in the Lisbon agenda in Europe.

References


A. The Production Sector

1. The Intermediate Goods Industries.— Production function of intermediate goods firms is given by

\[ Y_{j,t}^i = z_t \left( K_{j,t}^i \right)^{\Gamma_{j,t}^i h_{j,t}^i} \left( 1 - \Gamma_{j,t}^i h_{j,t}^i \right)^{1-v}, \quad i \in \{1, 2\} \]

with

\[ \Gamma_{j,t}^i = \begin{cases} \Gamma_t / y & \text{if } i \text{ is a follower firm}, \\ \Gamma_t & \text{otherwise}. \end{cases} \]

Cost minimization program for each firm writes

\[
\min_{K_{j,t}^i, h_{j,t}^i} \left[ r_t K_{j,t}^i + W_t h_{j,t}^i \right]
\]

s.t. \[ z_t \left( K_{j,t}^i \right)^{\Gamma_{j,t}^i h_{j,t}^i} \left( 1 - \Gamma_{j,t}^i h_{j,t}^i \right)^{1-v} \geq Y, \]

that leads to the total cost function

\[ (C_T)^{i,j,t}(Y) = \begin{cases} (1 + \varpi) \xi_t Y & \text{if } i \text{ is a follower firm}, \\ \xi_t Y & \text{otherwise}. \end{cases} \]

with

\[ \xi_t = z_t^{-1} \left( r_t \right)^{\Gamma_t} \left( W_t \right)^{1-v}, \]

and \( \varpi = \gamma^{1-v} - 1 \). Accordingly, marginal cost is given by

\[ (C_m)^{j,t}(Y) = \begin{cases} (1 + \varpi) \xi_t & \text{if } i \text{ is a follower firm}, \\ \xi_t & \text{otherwise}. \end{cases} \]

2. R&D Dynamics.— Rewriting Bellman equations \((BL)-(BN)\) when innovative efforts are optimally chosen, we have

\[ V_{u,t} - E_t \left[ \Phi_t V_{u,t+1} \right] = \Pi_{u,t} - \Delta_t (\theta_{u,t}) - E_t \left[ \Phi_t \chi_{n_0} | u S_{u,t+1} \right], \]

\[ V_{-u,t} - E_t \left[ \Phi_t V_{-u,t+1} \right] = -\Delta_t (\theta_{-u,t}) + E_t \left[ \Phi_t \chi_{n_0} | u S_{n,t+1} \right], \]

\[ V_{n,t} - E_t \left[ \Phi_t V_{n,t+1} \right] = \Pi_{n,t} - \Delta_t (\theta_{n,t}) + E_t \left[ \Phi_t \chi_{n_0} | n S_{u,t+1} - \chi_{u_0} | n S_{n,t+1} \right]. \]

Then, subtracting \((A.3)\) to \((A.3)\), and \((A.4)\) to \((A.5)\) leads to

\[ S_{u,t} = \Pi_{u,t} - \Pi_{n,t} - \Delta_t (\theta_{u,t}) + \Delta_t (\theta_{n,t}) + E_t \left[ \Phi_t \left( 1 - \chi_{u_0} | n S_{u,t+1} - \chi_{u_0} | n S_{n,t+1} \right) S_{u,t+1} + \chi_{u_0} | n S_{n,t+1} \right]. \]
\[ S_{n,t} = \Pi_{n,t} - \Delta_t \left( \rho_{n,t} \right) + \Delta_t \left( \rho_{-u,t} \right) + \mathbb{E}_t \left[ \Phi_t \left( x_{u|n} S_{u,t+1} + \left[ 1 - x_{u|n} - x_{n|u} \right] S_{n,t+1} \right) \right]. \]

3. Final Good Production.--- The final good firm’s problem write
\[
\max_{\left( Y_{j,t}^d \right)_{j \in [0,1]}} Y_t - \int_0^1 p_{j,t} Y_{j,t}^d \, dj,
\]
\[
s.t. \quad \ln Y_t = \int_0^1 \ln Y_{j,t}^d \, dj.
\]

First order conditions leads to the relative demand for intermediate goods
\[ Y_{j,t}^d = Y_t (p_{j,t})^{-1}. \]

B. Households

Recall the household’s problem:
\[
\max_{\left( C_t, h_t, K_t, I_t \right)} E_t \left[ \sum_{t=0}^{\infty} \beta^t U(C_t, h_t) \right]
\]
\[
s.t. \quad C_t + X_t \leq W_t h_t + r_t K_t + \Pi_t, \text{ all } t ;
K_{t+1} = (1 - \theta) K_t + X_t, \text{ all } t ;
K_0 > 0 \text{ given.}
\]

First order conditions then give
\[ W_t = -\frac{U'_{h,t}}{U'_{c,t}}, \]
\[ \mathbb{E}_t [\Phi_t (r_{t+1} + 1 - \theta)] = 1, \]
with
\[ \Phi_t = \beta \frac{U'_{c,t+1}}{U'_{c,t}}. \]

C. Aggregation and Equilibrium Conditions

1. Production input market.--- Recall that the leader’s price is \( p_{u,t} = (1 + \omega) \xi_t \) and the relative demand is given by \( Y_{u,t} = Y_t / p_{u,t} \). Hence we have
\[ Y_{u,t} = \frac{Y_t}{(1 + \omega) \xi_t} = z_t \left( K_{u,t} \right)^v (\Gamma_t h_{u,t})^{1-v}. \]

Using optimality condition in production factors allocation
\[ \frac{h_{j,t}}{K_{j,t}} = \frac{r_t}{W_t} \frac{1-v}{v} = \frac{h_{j,t}}{K_{j,t}}. \]
we can consequently write
\[ K_{u,t} = \frac{Y_t}{(1 + \omega)\xi_t} \left( \frac{W_t}{r_t} \right)^{1 - \nu} = \frac{\nu}{1 + \omega} \frac{Y_t}{r_t} \]
and
\[ h_{u,t} = \frac{Y_t}{(1 + \omega)\xi_t} \left( \frac{r_t}{W_t} \right)^{\nu} = \frac{1 - \nu}{1 + \omega} \frac{Y_t}{W_t}. \]

Recall that the profit of a neck and neck firm is given by
\[ \Pi_{n,t} = Y(e)\Pi_{u,t} \]
with
\[ \Pi_{u,t} = \omega \xi_t Y_{u,t}. \]

and write alternatively
\[ \Pi_{n,t} = p_{n,t}\left(\frac{Y_{n,t}^d}{2}\right) - \xi_t Y_{n,t} = Y_t/2 - \xi_t Y_{n,t}. \]

Combining these expressions leads to
\[(A.14) \quad Y_{n,t} = \frac{Y_t}{2(1 + \omega)\xi_t} [1 + \omega(1 - 2Y(e))]. \]

and
\[(A.15) \quad p_{n,t} = \frac{(1 + \omega)\xi_t}{1 + \omega(1 - 2Y(e))}. \]

The production of a neck and neck firm can also be written as
\[ Y_{n,t} = z_t (K_{n,t})^{\nu} (\Gamma_t h_{n,t})^{1 - \nu}. \]

This leads to the demand for capital and labor for a neck and neck firm
\[ K_{n,t} = \frac{\nu}{2(1 + \omega)} \frac{Y_t}{r_t} [1 + \omega(1 - 2Y(e))] \]
and
\[ h_{n,t} = \frac{1 - \nu}{2(1 + \omega)} \frac{Y_t}{W_t} [1 + \omega(1 - 2Y(e))]. \]

On the market of capital input, we have
\[ K_t = \int_0^1 \left( \sum_{i \in [1,2]} K_{j,t}^i \right) dj = \lambda_{n,t}K_{u,t} + \lambda_{n,t}2K_{n,t}, \]
then
\[ K_t = \frac{\nu}{1 + \omega} \frac{Y_t}{r_t} [1 + \lambda_{n,t}\omega(1 - 2Y(e))] \]
or alternatively
\[(A.16) \quad \mu(\lambda_{n,t}; e, \omega) r_t = \frac{Y_t}{K_t} \]
where
\[(A.17) \quad \mu(\lambda_{n,t}; e, \omega) = \frac{1 + \omega}{1 + \lambda_{n,t}\omega(1 - 2Y(e))} \]
defines the economy's aggregate markup.
Similarly, we have on the labour market
\[ h_t = \int_0^1 \left( \sum_{i \in [1,2]} h_{j,t}^i \right) \, dj = \lambda_{u,t} h_{u,t} + \lambda_{n,t} 2h_{n,t}, \]
then
\[ \mu(\lambda_{n,t}; \epsilon, \omega) W_t = (1 - \nu) \frac{Y_t}{h_t}. \]  

2. Zero Profit Condition for the Final Good Producer.— Combining the aggregation function and the first order condition for the representative firm (A.8), we have
\[ \lambda_{u,t} \ln p_{u,t} + \lambda_{n,t} \ln p_{n,t} = 0 \]
or
\[ p_{u,t}^{\lambda_{u,t}} p_{n,t}^{\lambda_{n,t}} = 1. \]
Using expression of \( p_{u,t} \) and \( p_{n,t} \) depending on \( \xi_t \), we obtain
\[ \xi_t = \frac{1 + \omega (1 - 2Y(\epsilon))}{1 + \omega}^{\lambda_{n,t}}. \]

3. Final Product Market Equilibrium.— Recall the household's budget constraint and substitute expressions for \( \Pi_t \)
\[ \Pi_t = \lambda_{u,t} \Pi_{u,t} + \lambda_{n,t} 2\Pi_{n,t} - D_t, \]
with
\[ D_t = \lambda_{u,t} \left[ \Delta_t (\rho_{u,t}) + \Delta_t (\rho_{-u,t}) \right] + \lambda_{n,t} 2\Delta_t (\rho_{n,t}), \]
we have
\[ C_t + X_t = W_t h_t + r_t K_t + \lambda_{u,t} \Pi_{u,t} + \lambda_{n,t} 2\Pi_{n,t} - D_t. \]
Combining with equations (A.16) and (A.18), as well as expressions for firms profit, we get
\[ C_t + X_t + D_t = \frac{(1 - \nu) Y_t}{\mu(\lambda_{n,t}; \epsilon, \omega)} + \frac{\nu Y_t}{\mu(\lambda_{n,t}; \epsilon, \omega)} + \frac{\omega}{1 + \omega} Y_t (\lambda_{u,t} + \lambda_{u,t} 2Y(\epsilon)) = Y_t. \]

4. Average level of productivity.— Let’s recall the four different industries that emerge following the R&D race:

**Type 1**: unlevelled industry with the leader’s labor productivity given by \( \gamma T_t \);

**Type 2**: neck and neck industries where duopolists’ productivity equals \( \gamma T_t \);

**Type 3**: unlevelled industry with the cutting edge productivity \( T_t \);

**Type 4**: neck and neck industries with productivity given by \( T_t \).

with \( \gamma = (1 + \omega)^{1/(1 - \nu)} \).
For each different type, let \( \lambda_{i,t} \) denotes the measure of industries of type \( i \). We have
\[
\begin{align*}
\lambda_{1,t+1} &\equiv \lambda_{u,t} \left( X_{u_t}^i | u + X_{u_t}^i | u \right) + \lambda_{n,t} \left( X_{n_t}^i | n + X_{n_t}^i | n \right), \\
\lambda_{2,t+1} &\equiv \lambda_{n,t} X_{n_t}^i | n, \\
\lambda_{3,t+1} &\equiv \lambda_{u,t} X_{u_t}^i | u, \\
\lambda_{4,t+1} &\equiv \lambda_{u,t} X_{n_t}^i | u + \lambda_{n,t} X_{n_t}^i | n,
\end{align*}
\]
with \( \sum_{1 \leq i \leq 4} \lambda_{i,t} = 1 \). Incidentally, note that \( \lambda_{1,t+1} + \lambda_{3,t+1} = \lambda_{u,t+1} \) and \( \lambda_{2,t+1} + \lambda_{4,t+1} = \lambda_{n,t+1} \).

We can calculate the marginal cost for each type of industries. First, we define
\[
(A.22) \quad \psi_{t+1} = z_{t+1}^{-1} \left( \frac{r_{t+1}}{\nu} \right)^\nu \left( \frac{W_{t+1}/T_t}{1 - \nu} \right)^{1 - \nu}.
\]
Following equation (A.2), \( t + 1 \) period marginal cost in industry of type 1 is given by
\[
(C_m)_{1,t+1} = \left\{ \begin{array}{ll}
\psi_{t+1} & \text{for the follower firm,} \\
\psi_{t+1} / (1 + \omega) & \text{for the leader.}
\end{array} \right.
\]
It follows for the other types of industry
\[
(C_m)_{2,t+1} = \psi_{t+1} / (1 + \omega),
\]
\[
(C_m)_{3,t+1} = \left\{ \begin{array}{ll}
(1 + \omega) \psi_{t+1} & \text{for the follower firm and} \\
\psi_{t+1} & \text{for the leader,}
\end{array} \right.
\]
\[
(C_m)_{4,t+1} = \psi_{t+1}.
\]
Then, we compute the price charged by each industry, given that \( p_{n,t} = p_{u,t} [1 + \omega (1 - 2Y(\epsilon))]^{-1} \) from (A.1.9) and Bertrand competition applies. It comes
\[
\begin{align*}
p_{1,t+1} &\equiv \psi_{t+1}, \\
p_{2,t+1} &\equiv \psi_{t+1} [1 + \omega (1 - 2Y(\epsilon))]^{-1}, \\
p_{3,t+1} &\equiv (1 + \omega) \psi_{t+1}, \\
p_{4,t+1} &\equiv (1 + \omega) \psi_{t+1} [1 + \omega (1 - 2Y(\epsilon))]^{-1}.
\end{align*}
\]
As the final good firm still has the same production function, (A.8) still holds and we have
\[
Y_{i,t+1} = Y_{t+1} / p_{i,t+1}, \quad 1 \leq i \leq 4
\]
for all \( i \)-industry (note that for neck and neck sectors, \( Y_{i,t+1} \) includes the both duopolists.) Finally, we can write the \( t + 1 \) period aggregate output as
\[
\ln Y_{t+1} = \int_0^1 \ln \left( \sum_{i \in \{1, 2\}} Y_{j,t+1}^i \right) dj = \sum_{1 \leq i \leq 4} \lambda_{i,t} \ln Y_{i,t+1} = \ln Y_{t+1} - \sum_{1 \leq i \leq 4} \lambda_{i,t} \ln p_{i,t+1}.
\]
Then using expressions of the $p_{i,t+1}$’s, we obtain

$$\psi_{t+1} = \frac{[1 + \bar{\omega}(1 - 2Y(\varepsilon))]^{\lambda_{n,t+1}}}{(1 + \bar{\omega})^{\lambda_{n,t+1} + \lambda_{i,t+1}}}.$$ 

According to the definition of $\xi$ and $\psi$, we have

$$\frac{\psi_{t+1}}{\xi_{t+1}} = \left(\frac{\Gamma_{t+1}}{\Gamma_t}\right)^{1-v}.$$ 

Hence, with (A.19)

$$\frac{\Gamma_{t+1}}{\Gamma_t} = \gamma^{\lambda_{i,t+1} + \lambda_{2,t+1}}$$

which define the function $G(\cdot)$ in equation (14).

**Appendix B. Stationary and Steady State Equations**

**A. Stationary Equations**

In the following, we present the model’s equations in their stationary version. We define stationary variables depending on their extensive counterpart by deflating with the productivity $\Gamma_t$. We define also the productivity growth $g_t = \Gamma_{t+1}/\Gamma_t - 1$.

(B.1) \[ z_t = \bar{z}^{1-\beta} z_{t-1}^\beta e^\varepsilon_t \]

(B.2) \[ \frac{1}{z_t} \left(\frac{r_t}{\nu}\right)^\nu \left(\frac{w_t}{1-\nu}\right)^{1-\nu} = \frac{[1 + \bar{\omega}(1 - 2Y(\varepsilon))]^{\lambda_{n,t}}}{1 + \bar{\omega}} \]

(B.3) \[ \mu(\lambda_{n,t}; \varepsilon, \bar{\omega}) \equiv \frac{1 + \bar{\omega}}{1 + \lambda_{n,t}\bar{\omega}(1 - 2Y(\varepsilon))} \]

(B.4) \[ \mu(\lambda_{n,t}; \varepsilon, \bar{\omega}) r_t = \nu \frac{y_t}{k_t} \]

(B.5) \[ \mu(\lambda_{n,t}; \varepsilon, \bar{\omega}) w_t = (1 - \nu) \frac{y_t}{h_t} \]

(B.6) \[ w_t = \frac{1 - \eta}{\eta} \frac{c_t}{1 - h_t} \]

(B.7) \[ E_t[\Phi_t(r_{t+1} + 1 - \theta)] = 1 \]

(B.8) \[ \Phi_t(1 + g_t) = \beta(1 + g_t)^{\eta(1-\sigma)} \left(\frac{c_{t+1}}{c_t}\right)^{\eta(1-\sigma)-1} \left(\frac{1 - h_{t+1}}{1 - h_t}\right)^{(1-\eta)(1-\sigma)} \]

(B.9) \[ (1 + g_t)k_{t+1} = (1 - \theta)k_t + x_t \]
The stationary equations taken when all variables indexed by $t$ have the steady state values $x_t^*$ in Section 1 are:

\[(B.10)\]
\[c_t + d_t + x_t = y_t\]

\[(B.11)\]
\[d_t = \lambda_{u,t} \frac{\delta}{2} (\rho^{2} + \rho^{2}) + \lambda_{n,t} \delta \rho_{n,t}\]

\[(B.12)\]
\[\lambda_{n,t} + \lambda_{u,t} = 1\]

\[(B.13)\]
\[\lambda_{u,t+1} - \lambda_{u,t} = \lambda_{n,t} 2 \frac{\kappa \rho_{n,t}}{(1 + \kappa \rho_{n,t})^2} - \lambda_{u,t} \frac{\kappa \rho_{u,t} + \bar{\alpha}}{(1 + \kappa \rho_{u,t})(1 + \kappa \rho_{u,t})}\]

\[(B.14)\]
\[E_t \left[ \Phi_t (1 + g_t) \left( \frac{\rho_{u,t} + \bar{\alpha}}{1 + \kappa \rho_{u,t}} s_{u,t+1} \right) \right] = \frac{\delta}{\kappa} \rho_{u,t} (1 + \kappa \rho_{u,t})^2\]

\[(B.15)\]
\[E_t \left[ \Phi_t (1 + g_t) \left( \frac{s_{n,t+1}}{1 + \kappa \rho_{n,t}} \right) \right] = \frac{\delta}{\kappa} \rho_{n,t} (1 + \kappa \rho_{n,t})^2\]

\[(B.16)\]
\[E_t \left[ \Phi_t (1 + g_t) \left( \frac{s_{u,t+1}}{1 + \kappa \rho_{n,t}} + \frac{\kappa \rho_{n,t} s_{n,t+1}}{1 + \kappa \rho_{n,t}} \right) \right] = \frac{\delta}{\kappa} \rho_{n,t} (1 + \kappa \rho_{n,t})^2\]

\[(B.17)\]
\[s_{u,t} - E_t \left[ \Phi_t (1 + g_t) s_{u,t+1} \right] = (1 - Y(\epsilon)) \frac{\omega}{1 + \omega} y_t - \frac{\delta}{2} (\rho_{u,t}^2 - \rho_{n,t}^2)\]

\[(B.18)\]
\[s_{n,t} - E_t \left[ \Phi_t (1 + g_t) s_{n,t+1} \right] = Y(\epsilon) \frac{\omega}{1 + \omega} y_t - \frac{\delta}{2} (\rho_{n,t}^2 - \rho_{u,t}^2)\]

\[(B.19)\]
\[\ln(1 + g_t) = \left( 1 - \frac{\lambda_{u,t}}{1 + \kappa \rho_{u,t}} - \frac{\lambda_{n,t}}{(1 + \kappa \rho_{n,t})^2} \right) \ln \gamma\]

**B. Steady State of the Stationary Model**

In this section, we present the equations that characterize the deterministic (i.e., $\sigma_z = 0$) steady state balanced growth path, which corresponds to the stationary equations taken when all variables indexed by $t$ as constant.

\[(B.20)\]
\[z = 1\]

\[(B.21)\]
\[\left( \frac{\rho}{1 - \rho} \right)^{1-\nu} = \frac{[1 + \omega (1 - 2 Y(\epsilon))]^{\lambda_n}}{1 + \omega}\]
\[ (B.22) \quad \mu = \frac{1 + \bar{\omega}}{1 + \lambda_n \bar{\omega}(1 - 2\gamma(e))} \]

\[ (B.23) \quad \mu r = \nu \frac{y}{h} \]

\[ (B.24) \quad \mu w = (1 - \nu) \frac{y}{h}. \]

\[ (B.25) \quad w = \frac{1 - \eta}{\eta} \frac{c}{1 - h} \]

\[ (B.26) \quad \Phi(r + 1 - \theta) = 1 \]

\[ (B.27) \quad \Phi(1 + g) = \beta(1 + g)^{\eta(1 - \sigma)} \]

\[ (B.28) \quad (g + \theta)k = x \]

\[ (B.29) \quad c + d + x = y \]

\[ (B.30) \quad d = \lambda_u \frac{\bar{\delta}}{2} (\varphi_u^2 + \varphi_{-u}^2) + \lambda_n \bar{\delta} \varphi_n^2 \]

\[ (B.31) \quad \lambda_n + \lambda_u = 1 \]

\[ (B.32) \quad \lambda_n \frac{2}{(1 + \kappa \varphi_n)^2} = \lambda_u \frac{\kappa \varphi_{-u} + \bar{\alpha}}{(1 + \kappa \varphi_u)(1 + \kappa \varphi_{-u})} \]

\[ (B.33) \quad \Phi(1 + g) \frac{(\kappa \varphi_{-u} + \bar{\alpha})s_u}{1 + \kappa \varphi_{-u}} = \frac{\bar{\delta}}{\kappa} \varphi_u (1 + \kappa \varphi_{-u})^2 \]

\[ (B.34) \quad \Phi(1 + g) \frac{s_n}{1 + \kappa \varphi_u} = \frac{\bar{\delta}}{\kappa(1 - \bar{\alpha})} \varphi_{-u} (1 + \kappa \varphi_{-u})^2 \]

\[ (B.35) \quad \Phi(1 + g) \left( \frac{s_u}{1 + \kappa \varphi_u} + \frac{\kappa \varphi_n s_n}{1 + \kappa \varphi_n} \right) = \frac{\bar{\delta}}{\kappa} \varphi_n (1 + \kappa \varphi_n)^2 \]

\[ (B.36) \quad s_u (1 - \Phi(1 + g)) = (1 - \gamma(e)) \frac{\bar{\omega}}{1 + \bar{\omega}} y - \frac{\bar{\delta}}{2} (\varphi_u^2 - \varphi_n^2) \]

\[ + \Phi(1 + g) \left( \frac{\kappa \varphi_n (s_n - s_u)}{(1 + \kappa \varphi_n)^2} - \frac{(\kappa \varphi_{-u} + \bar{\alpha}) s_u}{(1 + \kappa \varphi_u)(1 + \kappa \varphi_{-u})} \right) \]
\begin{align*}
\Phi(1 + g) &= \frac{\partial}{1 + \partial} y - \frac{\delta}{2}(\varphi_n^2 - \varphi_u^2) \\
&\quad + \Phi(1 + g) \left( \frac{\kappa \varphi_n(s_u - s_n)}{(1 + \kappa \varphi_n)^2} - \frac{(\kappa \varphi_u + \bar{\alpha})s_n}{(1 + \kappa \varphi_u)(1 + \kappa \varphi_u)} \right)
\end{align*}

\begin{align*}
\ln(1 + g) &= \left(1 - \frac{\lambda_u}{1 + \kappa \varphi_u} - \frac{\lambda_n}{(1 + \kappa \varphi_n)^2}\right) \ln \gamma
\end{align*}

C. Steady State Ratios Used for Calibration

In the calibration process, I use some particular steady state ratios to set values to some parameters. Here are the theoretical counterpart based on the previous steady state conditions.

\begin{align*}
\frac{x}{\bar{k}} &= g + \theta \\
\frac{1 + g}{\beta} &= \frac{v \, y}{\mu \, \bar{k}} + 1 - \theta \\
\frac{\eta}{1 - \eta} &= \frac{h}{1 - h} \left(1 - \frac{d}{y} \frac{x}{\bar{k} \, y}\right)
\end{align*}

The following table gives a comparison between the SMM and deterministic ratio calibration method.

Appendix C. Algorithm of the Orthogonal Collocation

The algorithm works as follows

1. Choose an order of approximation \( n_k \) and \( n_\lambda \) for each dimension and compute the \( n_k + 1 \) and \( n_\lambda + 1 \) roots of the Chebyshev polynomial of order \( n_k + 1 \) and \( n_\lambda + 1 \) respectively

\begin{align*}
z_k^i &= \cos \left( \frac{(2i - 1)\pi}{2(n + 1)} \right) \text{ for } i = 1, \ldots, n_k + 1 \\
z_\lambda^i &= \cos \left( \frac{(2i - 1)\pi}{2(n + 1)} \right) \text{ for } i = 1, \ldots, n_\lambda + 1
\end{align*}

and formulate an initial guess for \( \omega_c, \omega_{\varphi_n}, \omega_{\varphi_u}, \text{ and } \omega_{\varphi_{-u}} \).

2. Compute \( k_i \) as

\[
k_i = \exp \left( \log(k) + \left( z_k^i + 1 \right) \frac{\log(\bar{k}) - \log(k)}{2} \right) \text{ for } i = 1, \ldots, n_k + 1
\]

to map \([-1, 1]\) into \([\bar{k}, k]\) and

\[
\lambda_{n,i} = \frac{1}{2} \left( z_\lambda^i + 1 \right) \text{ for } i = 1, \ldots, n_\lambda + 1
\]
Table 5. Comparison of Calibration Methods

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>SMM</th>
<th>deterministic</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \phi )</td>
<td>Productivity increment</td>
<td>0.1038</td>
<td>0.1024</td>
</tr>
<tr>
<td>( \epsilon )</td>
<td>Competition</td>
<td>0.4960</td>
<td>0.4712</td>
</tr>
<tr>
<td>( \nu )</td>
<td>Capital’s share</td>
<td>0.3150</td>
<td>0.3152</td>
</tr>
<tr>
<td>( \theta )</td>
<td>Depreciation rate</td>
<td>0.0723</td>
<td>0.0719</td>
</tr>
<tr>
<td>( \delta )</td>
<td>R&amp;D cost parameter</td>
<td>1.7575</td>
<td>1.6334</td>
</tr>
<tr>
<td>( \beta )</td>
<td>Subjective discount factor</td>
<td>0.9578</td>
<td>0.9576</td>
</tr>
<tr>
<td>( \eta )</td>
<td>Share parameter for consumption</td>
<td>0.5530</td>
<td>0.5523</td>
</tr>
<tr>
<td>( \sigma_\epsilon )</td>
<td>Standard deviation of the shock</td>
<td>0.0182</td>
<td>—</td>
</tr>
<tr>
<td>( \rho )</td>
<td>Persistence parameter</td>
<td>0.8901</td>
<td>—</td>
</tr>
<tr>
<td>( \sigma )</td>
<td>Relative risk aversion</td>
<td>1</td>
<td>—</td>
</tr>
<tr>
<td>( \kappa )</td>
<td>Effectiveness of R&amp;D investment</td>
<td>1</td>
<td>—</td>
</tr>
<tr>
<td>( \bar{\alpha} )</td>
<td>Imitation parameter</td>
<td>0.0670</td>
<td>—</td>
</tr>
</tbody>
</table>

Restricted parameters

(3) Compute

\[
c_t(k_i, \lambda_{n,j}, z_t) = \exp \left( \sum_{j_h=0}^{n_h} \sum_{j_l=0}^{n_l} \omega_c(j_k, j_\lambda | z_t) T_{j_h} \left( \psi_k(\log(k_i)) \right) T_{j_\lambda} \left( \psi_\lambda(\lambda_{n,j}) \right) \right)
\]

\[
h_t(k_i, \lambda_{n,j}, z_t) = \exp \left( \sum_{j_h=0}^{n_h} \sum_{j_l=0}^{n_l} \omega_h(j_k, j_\lambda | z_t) T_{j_h} \left( \psi_k(\log(k_i)) \right) T_{j_\lambda} \left( \psi_\lambda(\lambda_{n,j}) \right) \right)
\]

\[
\varrho_{n,t}(k_i, \lambda_{n,j}, z_t) = \exp \left( \sum_{j_h=0}^{n_h} \sum_{j_l=0}^{n_l} \omega_{\varrho_n}(j_k, j_\lambda | z_t) T_{j_h} \left( \psi_k(\log(k_i)) \right) T_{j_\lambda} \left( \psi_\lambda(\lambda_{n,j}) \right) \right)
\]

\[
\varrho_{u,t}(k_i, \lambda_{n,j}, z_t) = \exp \left( \sum_{j_h=0}^{n_h} \sum_{j_l=0}^{n_l} \omega_{\varrho_u}(j_k, j_\lambda | z_t) T_{j_h} \left( \psi_k(\log(k_i)) \right) T_{j_\lambda} \left( \psi_\lambda(\lambda_{n,t}) \right) \right)
\]

\[
\varrho_{-u,t}(k_i, \lambda_{n,j}, z_t) = \exp \left( \sum_{j_h=0}^{n_h} \sum_{j_l=0}^{n_l} \omega_{\varrho_{-u}}(j_k, j_\lambda | z_t) T_{j_h} \left( \psi_k(\log(k_i)) \right) T_{j_\lambda} \left( \psi_\lambda(\lambda_{n,t}) \right) \right)
\]

and evaluate the residual

\[
R_h(k_i, \lambda_{n,j}, z_t) = \frac{1 - h_t(k_i, \lambda_{n,j}, z_t)}{h_t(k_i, \lambda_{n,j}, z_t)^v} \cdot \frac{1 + \phi \cdot 1 - \eta}{1 - \nu} \cdot \frac{c(k_i, \lambda_{n,j}, z_t)}{\eta \cdot z_t \cdot k_i^v \cdot [1 + \phi \cdot (1 - 2Y(e))]^{\lambda_{n,j}}}
\]

Compute \( d_t(k_i, \lambda_{n,j}, z_t), g_t(k_i, \lambda_{n,j}, z_t), \) and \( y_t(k_i, \lambda_{n,j}, z_t) \) so that
\[d_t(k_i, \lambda_{n,j}, z_t) = (1 - \lambda_{n,j}) \frac{\delta}{2} \left( \varphi_{u,t}(k_i, \lambda_{n,j}, z_t)^2 + \varphi_{-u,t}(k_i, \lambda_{n,j}, z_t)^2 \right) + \lambda_{n,j} \delta \varphi_{n,t}(k_i, \lambda_{n,j}, z_t)^2 \]

\[g_t(k_i, \lambda_{n,j}, z_t) = \exp \left( \log(\gamma) \left( 1 - \frac{1 - \lambda_{n,j}}{1 + \kappa \varphi_{u,t}(k_i, \lambda_{n,j}, z_t)} - \frac{\lambda_{n,t}(k_i, \lambda_{n,j}, z_t)}{(1 + \kappa \varphi_{n,t}(k_i, \lambda_{n,j}, z_t))^2} \right) \right) - 1 \]

\[y_t(k_i, \lambda_{n,j}, z_t) = z_t k_i^\gamma h_t(k_i, \lambda_{n,j}, z_t)^{1-\nu} \left[ 1 + \bar{\omega} (1 - 2Y(e))^{\lambda_{n,j}} \right] \left[ 1 + \lambda_{n,j} \bar{\omega} (1 - 2Y(e)) \right] \]

Compute also \(k_{t+1}(k_i, \lambda_{n,j}, z_t)\) and \(\lambda_{n,t+1}(k_i, \lambda_{n,j}, z_t)\) such as

\[k_{t+1}(k_i, \lambda_{n,j}, z_t) = \frac{(1 - \theta)k_i + y_t(k_i, \lambda_{n,j}, z_t) - c_t(k_i, \lambda_{n,j}, z_t) - d_t(k_i, \lambda_{n,j}, z_t)}{1 + g_t(k_i, \lambda_{n,j}, z_t)} \]

\[\lambda_{n,t+1}(k_i, \lambda_{n,j}, z_t) = \lambda_{n,j} \]

\[\ldots + (1 - \lambda_{n,j}) \frac{\kappa \varphi_{u,t}(k_i, \lambda_{n,j}, z_t) + \bar{\alpha}}{(1 + \kappa \varphi_{u,t}(k_i, \lambda_{n,j}, z_t))(1 + \kappa \varphi_{-u,t}(k_i, \lambda_{n,j}, z_t))} \]

\[\ldots - \lambda_{n,j} 2 \frac{\kappa \varphi_{n,t}(k_i, \lambda_{n,j}, z_t)}{(1 + \kappa \varphi_{n,t}(k_i, \lambda_{n,j}, z_t))^2} \]

(4) Then compute the possible levels of future consumption

\[c_{t+1}(k_{t+1}(k_i, \lambda_{n,j}, z_t), \lambda_{n,t+1}(k_i, \lambda_{n,j}, z_t), z_s) = \exp \left( \sum_{j_k=0}^{n_k} \sum_{j_{\lambda}=0}^{n_{\lambda}} \omega_c(j_k, j_{\lambda}|z_s) T_{j_k} \left( \psi_k \left( \log(k_{t+1}(\cdot)) \right) \right) T_{j_{\lambda}} \left( \psi_{\lambda} \left( \lambda_{n,t+1}(\cdot) \right) \right) \right) \]

and

\[h_{t+1}(k_{t+1}(k_i, \lambda_{n,j}, z_t), \lambda_{n,t+1}(k_i, \lambda_{n,j}, z_t), z_s) = \exp \left( \sum_{j_k=0}^{n_k} \sum_{j_{\lambda}=0}^{n_{\lambda}} \omega_h(j_k, j_{\lambda}|z_s) T_{j_k} \left( \psi_k \left( \log(k_{t+1}(\cdot)) \right) \right) T_{j_{\lambda}} \left( \psi_{\lambda} \left( \lambda_{n,t+1}(\cdot) \right) \right) \right) \]

(5) Evaluate the residuals

\[R_c(k_i, \lambda_{n,j}, z_t) = c_t(k_i, \lambda_{n,j}, z_t)^{\eta(1-\sigma)-1} (1 - h_t(k_i, \lambda_{n,j}, z_t))^{(1-\eta)(1-\sigma)} - \beta \sum_{s=0}^{n_s} \pi(z_s|z_t) Z_c(z_s|k_i, \lambda_{n,j}, z_t) \]
with \( \pi(z_s|z_l) \) the transition probability and

\[
\Xi_c(z_s|k_i, \lambda_{n,j}, z_l) = \left( c_{t+1}(k_{t+1}(), \lambda_{n,t+1}(), z_s) \right)^{\eta(1-\sigma)-1} \\
\times \left( 1 - h_{t+1}(k_{t+1}(), \lambda_{n,t+1}(), z_s) \right)^{1-\eta(1-\sigma)} \\
\times \left( v_{z_s} \left( \frac{h_{t+1}(k_{t+1}(), \lambda_{n,t+1}(), z_s)}{k_{t+1}()} \right)^{1-v} \frac{[1 + \varphi(1-2Y(\epsilon))]^{\lambda_{n,t+1}()} + 1 - \theta}{1 + \varphi(1-2Y(\epsilon))} \right)
\]

for all \( i = 1, \ldots, n_k \) and \( j = 1, \ldots, n_\lambda \).

(6) Compute \( y_{t+1}(k_{t+1}(k_i, \lambda_{n,j}, z_l), \lambda_{n,t+1}(k_i, \lambda_{n,j}, z_l), z_s) \) from

\[
y_{t+1}(k_{t+1}(), \lambda_{n,t+1}(), z_s) = \\
z_s k_{t+1}()^v h_{t+1}(k_{t+1}(), \lambda_{n,t+1}(), z_s)^{1-v} \frac{[1 + \varphi(1-2Y(\epsilon))]^{\lambda_{n,t+1}()} + 1 - \theta}{1 + \varphi(1-2Y(\epsilon))}
\]

Compute also

\[
\mathcal{F}_{u,t}(k_i, \lambda_{n,j}, z_l) = \frac{\delta}{\kappa} \varrho_{u,t}(\cdot) \left( 1 + \kappa \varrho_{u,t}(\cdot) \right)^2 \left( 1 + \kappa \varrho_{u,t}(\cdot) \right) \varrho_{u,t}(\cdot) + \bar{\alpha}
\]

and

\[
\mathcal{F}_{n,t}(k_i, \lambda_{n,j}, z_l) = \frac{\delta}{\kappa(1-\bar{\alpha})} \varrho_{u,t}(\cdot) \left( 1 + \kappa \varrho_{u,t}(\cdot) \right)^2 \left( 1 + \kappa \varrho_{u,t}(\cdot) \right)
\]

as well as

\[
\mathcal{F}_{u,t+1}(k_{t+1}(k_i, \lambda_{n,j}, z_l), \lambda_{n,t+1}(k_i, \lambda_{n,j}, z_l), z_s) = \\
\frac{\delta}{\kappa} \varrho_{u,t+1}(k_{t+1}(), \lambda_{n,t+1}(), z_s) \left( 1 + \kappa \varrho_{u,t+1}(k_{t+1}(), \lambda_{n,t+1}(), z_s) \right)^2 \\
\times \frac{\left( 1 + \kappa \varrho_{u,t+1}(k_{t+1}(), \lambda_{n,t+1}(), z_s) \right)^2}{\varrho_{u,t+1}(k_{t+1}(), \lambda_{n,t+1}(), z_s) + \bar{\alpha}}
\]

and

\[
\mathcal{F}_{n,t+1}(k_{t+1}(k_i, \lambda_{n,j}, z_l), \lambda_{n,t+1}(k_i, \lambda_{n,j}, z_l), z_s) = \\
\frac{\delta}{\kappa(1-\bar{\alpha})} \varrho_{u,t+1}(k_{t+1}(), \lambda_{n,t+1}(), z_s) \left( 1 + \kappa \varrho_{u,t+1}(k_{t+1}(), \lambda_{n,t+1}(), z_s) \right)^2 \\
\times \left( 1 + \kappa \varrho_{u,t+1}(k_{t+1}(), \lambda_{n,t+1}(), z_s) \right)
\]

with

\[
\varrho_{u,t+1}(k_{t+1}(k_i, \lambda_{n,j}, z_l), \lambda_{n,t+1}(k_i, \lambda_{n,j}, z_l), z_s) = \\
\exp \left( \sum_{j_k=0}^{n_k} \sum_{j_\lambda=0}^{n_\lambda} \omega_{\varrho_u} (j_k, j_\lambda | z_s) T_{j_k} \left( \psi_k \left( \log(\lambda_{n,t+1}()) \right) \right) T_{j_\lambda} \left( \psi_\lambda \left( \lambda_{n,t+1}() \right) \right) \right)
\]

\[
\varrho_{-u,t+1}(k_{t+1}(k_i, \lambda_{n,j}, z_l), \lambda_{n,t+1}(k_i, \lambda_{n,j}, z_l), z_s) = \\
\exp \left( \sum_{j_k=0}^{n_k} \sum_{j_\lambda=0}^{n_\lambda} \omega_{\varrho_{-u}} (j_k, j_\lambda | z_s) T_{j_k} \left( \psi_k \left( \log(\lambda_{n,t+1}()) \right) \right) T_{j_\lambda} \left( \psi_\lambda \left( \lambda_{n,t+1}() \right) \right) \right)
\]
\[
\varrho_{n,t+1}(k_{t+1}(k_i,\lambda_{n,j},z_l),\lambda_{n,t+1}(k_i,\lambda_{n,j},z_l),z_s) = \\
\exp \left( \sum_{j_k=0}^{n_k} \sum_{j_l=0}^{n_l} \omega_{\varrho_n}(j_k,j_l|z_s) T_{j_k}(\psi_k(\log(k_{t+1}))) T_{j_l}(\psi_\lambda(\lambda_{n,t+1}())) \right)
\]

(7) Evaluate the residuals

\[
R_{\varrho_n}(k_i,\lambda_{n,j},z_l) = c_t(k_i,\lambda_{n,j},z_l)^{(1-\sigma)-1} \left( 1 - h_t(k_i,\lambda_{n,j},z_l) \right)^{(1-\eta)(1-\sigma)} T_{u,t}(k_i,\lambda_{n,j},z_l) \\
- \beta \sum_{s=1}^{n_z} \pi(z_s|z_l) \Xi_{\varrho_n}(z_s|k_i,\lambda_{n,j},z_l)
\]

with

\[
\Xi_{\varrho_n}(z_s|k_i,\lambda_{n,j},z_l) = \left( 1 + g_t(\cdot) \right)^{(1-\sigma)} \left( c_{t+1}(k_{t+1}(\cdot),\lambda_{n,t+1}(\cdot),z_s) \right)^{(1-\sigma)-1} \\
\times \left( 1 - h_{t+1}(k_{t+1}(\cdot),\lambda_{n,t+1}(\cdot),z_s) \right)^{(1-\eta)(1-\sigma)} \\
\times \left[ 1 - \frac{\varrho_n,_{t+1}(k_{t+1}(\cdot),\lambda_{n,t+1}(\cdot),z_s)}{1 + \varrho_n,_{t+1}(k_{t+1}(\cdot),\lambda_{n,t+1}(\cdot),z_s)^2} \right] \\
- \frac{\bar{h}}{2} \left( \varrho_{u,t+1}(k_{t+1}(\cdot),\lambda_{n,t+1}(\cdot),z_s)^2 - \varrho_{n,t+1}(k_{t+1}(\cdot),\lambda_{n,t+1}(\cdot),z_s)^2 \right) \\
+ \frac{\kappa \varrho_{n,t+1}(k_{t+1}(\cdot),\lambda_{n,t+1}(\cdot),z_s)}{1 + \varrho_{n,t+1}(k_{t+1}(\cdot),\lambda_{n,t+1}(\cdot),z_s)^2} \\
\times \left( \varrho_{u,t+1}(k_{t+1}(\cdot),\lambda_{n,t+1}(\cdot),z_s) + \bar{h} \right) \\
= c_t(k_i,\lambda_{n,j},z_l)^{(1-\sigma)-1} \left( 1 - h_t(k_i,\lambda_{n,j},z_l) \right)^{(1-\eta)(1-\sigma)} T_{u,t}(k_i,\lambda_{n,j},z_l) \\
- \beta \sum_{s=1}^{n_z} \pi(z_s|z_l) \Xi_{\varrho_n}(z_s|k_i,\lambda_{n,j},z_l)
\]
with
\[\Xi_{q_n}(z_s|k_i, \lambda_{n,j}, z_l) = (1 + g_t(\cdot))^{\eta(1-\sigma)}(c_{t+1}(k_{t+1}(\cdot), \lambda_{n,t+1}(\cdot), z_s))^{\eta(1-\sigma)-1}\]
\[\times(1 - h_{t+1}(k_{t+1}(\cdot), \lambda_{n,t+1}(\cdot), z_s))^{(1-\eta)(1-\sigma)}\]
\[-\frac{\delta}{2} \left( \varrho_{n,t+1}(k_{t+1}(\cdot), \lambda_{n,t+1}(\cdot), z_s)^2 - \varrho_{u,t+1}(k_{t+1}(\cdot), \lambda_{n,t+1}(\cdot), z_s)^2 \right)\]
\[+ \frac{\kappa \varrho_{n,t+1}(k_{t+1}(\cdot), \lambda_{n,t+1}(\cdot), z_s)}{1 + \kappa \varrho_{n,t+1}(k_{t+1}(\cdot), \lambda_{n,t+1}(\cdot), z_s)}^2 \mathcal{F}_{u,t+1}(k_{t+1}(\cdot), \lambda_{n,t+1}(\cdot), z_s)\]
\[+ \left(1 - \frac{\kappa \varrho_{n,t+1}(k_{t+1}(\cdot), \lambda_{n,t+1}(\cdot), z_s)}{1 + \kappa \varrho_{n,t+1}(k_{t+1}(\cdot), \lambda_{n,t+1}(\cdot), z_s)}^2 \right)\]
\[- \frac{(\kappa \varrho_{u,t+1}(k_{t+1}(\cdot), \lambda_{n,t+1}(\cdot), z_s) + \bar{\alpha})}{(1 + \kappa \varrho_{u,t+1}(k_{t+1}(\cdot), \lambda_{n,t+1}(\cdot), z_s))(1 + \kappa \varrho_{u,t+1}(k_{t+1}(\cdot), \lambda_{n,t+1}(\cdot), z_s))}\]
\[\times \mathcal{F}_{n,t+1}(k_{t+1}(\cdot), \lambda_{n,t+1}(\cdot), z_s)\]
\[= \frac{\mathcal{F}_{u,t}}{1 + \kappa \varrho_{n,t}} + \frac{\kappa \varrho_{n,t} \mathcal{F}_{n,t}}{1 + \kappa \varrho_{n,t}} - \frac{\delta}{\kappa} \varrho_{n,t} (1 + \kappa \varrho_{n,t})^2\]
\[(8) \text{ if all residuals are close enough to zero then stop, otherwise update parameters } \omega_c, \omega_h, \omega_{\varrho_n}, \omega_{\varrho_u}, \text{ and } \omega_{\varrho_{-u}} \text{ and go back to step } (3).\]

**Appendix D. Building a Consistent Dataset**

In this section, I provide the sources and the details of imputations used to construct the database used in the calibration process. The main sources are U.S. statistical agencies (BEA, BLS, SSA) and previous studies. They are publicly available. Adjusted accounts are underlined. Imputations appear with a star. When no mention is given, a variable can be directly found in original sources as given in the Table [7] that gives the source for the data use in this paper.

The purpose is to follow the double-entry methodology of national accounts and to build adjusted gross domestic product ($Y$) which consist in the sum of adjusted consumption ($C$), adjusted investment ($X$), and adjusted R&D expenses ($D$), equals to the gross domestic income which consists in adjusted labor compensation ($W$), plus adjusted capital income ($\Pi$), and adjusted consumption of fixed capital ($CFC$), such as

\[C + X + D = Y = W + \Pi + CFC.\]

The main imputations are related to home production, government capital services, Businesses funding R&D, durable goods and ambiguous capital services. Calculations are detailed in the following sections.
### Table 6: Sources for the Dataset

<table>
<thead>
<tr>
<th>Series</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gross Domestic Product ((GDP)); Consumption of non durable goods ((CND)), durable goods ((CD)), and services ((CS)); Private Investment ((X)); Net Exports ((NX))</td>
<td>NIPA–Table 1.1.5 (BEA)</td>
</tr>
<tr>
<td>Government Expenditure ((CG)); Gov. Investment ((XG))</td>
<td>NIPA–Table 3.9.5 (BEA)</td>
</tr>
<tr>
<td>R&amp;D funding by government ((DG)); R&amp;D funding by Private Universities &amp; Colleges ((DUC)); R&amp;D funding by Businesses ((DB))</td>
<td>National Patterns of R&amp;D resources–Table 5 (NFS)</td>
</tr>
<tr>
<td>General gov. compensation for employees ((WG))</td>
<td>NIPA–Table 3.10.5 (BEA)</td>
</tr>
<tr>
<td>Gov. CFC ((CFCG)); Private CFC ((CFCP)); Private net interests and rental income ((NIP; RIP)); Private corporate profits ((CorpP)); Private Proprietors’ income ((PropP)); current transfers payments ((TransP)), taxes less subsidies plus current Gov.’s surplus ((TaxP)); statistical discrepancy ((Disc)); Total compensation of employees ((W))</td>
<td>NIPA–Table 1.10 (BEA)</td>
</tr>
<tr>
<td>Housing CFC ((CFCH))</td>
<td>Fixed Assets Accounts–Table 5.4 (BEA)</td>
</tr>
<tr>
<td>Depreciation of durable goods ((CFCCD))</td>
<td>Fixed Assets Accounts–Table 8.4 (BEA)</td>
</tr>
<tr>
<td>Private non residential capital ((KP)); Government capital ((KG)); Private residential capital ((KH)); Durable goods capital ((KD))</td>
<td>Fixed Assets Accounts–Table 1.1 (BEA)</td>
</tr>
</tbody>
</table>
Housing net interests and rental income \((NIH; RIH)\); Housing corporate profits \((CorpH)\); Housing Proprietors’ income \((PropH)\); current transfers payments \((TransH)\), taxes less subsidies plus current Gov.’s surplus \((TaxH)\)

Civilian non-institutional population 16 and over \((POP)\)

Average hourly wage for housekeeping \((hWHK)\)

Average Wage Index \((AWI)\)

Non market hours worked \((NMH^*)\)

---

**A. Adjusted Consumption Expenditures**

Adjusted Consumption Expenditures \((C)\) consists in Private Consumption of non durable goods and services, plus government expenditures of services. From the Personal Consumption Expenditures, we subtract the Durable Goods line, and add government expenditures. Since R&D funding by government \((DG)\) and private universities and college expenses \((DH)\) are accounted as Consumption expenses, we remove these amounts from consumption. Finally, we add government capital services \((GKS)\), non market work services \((HHS)\), services from durable goods \((DGS)\). Then, we have

\[
C = CND + CS + CG - DG - DUC + GKS^* + HHS^* + CDS^*.
\]

**B. Adjusted Investment**

Adjusted Investment \((X)\) consists in Private Gross Domestic Investment \((X)\), Government Investment \((XG)\), Durable Goods consumption \((CD)\), and net exports \((NX)\). We have then

\[
X = X + XG + CD + NX.
\]
C. R&D Expenditures

Research & Development expenditures ($D$) are directly given by

$$D = DG + DUC + DB.$$ 

D. Adjusted Compensation of Employees

Adjusted compensation of employees ($W$) consists of government compensation of employees ($WG$), plus labor income from housing sector including housing ambiguous capital income ($WH^*$), plus private compensation of employees including R&D and ambiguous capital labor income share ($WB^*$), plus non market households labor income imputation ($HHS^*$), so that

$$W = WG + WH^* + WB^* + HHS^*.$$ 

E. Adjusted Capital Income

Adjusted Capital Income ($\Pi$) is made of Private excluding Housing Unambiguous Capital Income and excluding corporate profits ($UnKB$), Private excluding Housing Corporate profits ($CorpB$), Private excluding Housing Ambiguous Capital Income excluding Corporate Profits ($AmKB^*$), Private excluding Housing Ambiguous Corporate Profits ($AmCorpB^*$), Business R&D related Capital Income ($\Pi DB^*$), Housing Capital income including ambiguous capital Income ($\Pi H^*$), Gov. capital Income ($GKS^*$), and Durable Goods Capital Income ($\Pi CD^*$). Finally

$$\Pi = UnKB + CorpB + AmKB^* + AmCorpB^* + \Pi DB^* + \Pi H^* + GKS^* + \Pi CD.$$ 

F. Adjusted Consumption of Fixed Capital

Adjusted Consumption of fixed capital ($CFC$) is the sum of consumption of government fixed capital ($CFCG$), Housing fixed capital consumption ($CFCH$), Durable Goods depreciation ($CFCCD$), and private excluding housing consumption of fixed capital ($CFCB^*$) which is computed as the difference between total Private CFC ($CFCP$) less housing CFC

$$CFCB^* = CFCP - CFCH.$$ 

So

$$CFC = CFCG + CFCH + CFCCD + CFCB^*.$$ 

G. Private Capital Income, Excluding Housing

Private, excluding housing, Capital Income is made of unambiguous capital income ($UnKB$), Corporate Profits ($CorpB$), and the capital share of Ambiguous Capital ($AmKB$).
1. Unambiguous capital income, excluding corporate Profits.— Unambiguous capital income excluding corporate profits ($UnKB$) consists in net interests ($NIB^*$) and rental income ($RIB^*$). They are obtained by subtracting housing net interests ($NIH$) and rental income ($RIH$) from private net interests ($NIP$) and rental income ($RIP$).

$$UnKB^* = NIB + RIB = (NIP - NIH) + (RIP - RIH).$$

Similarly

$$CorpB^* = CorpP - CorpH,$$ 
where $CorpP$ and $CorpH$ stand for Private and housing corporate profits respectively.

2. Ambiguous Income.— Private excluding housing ambiguous income ($AmIB^*$) includes Proprietors' income ($PropB$), current transfers payments ($TransB$), taxes less subsidies plus current Gov.'s surplus ($TaxB$), plus statistical discrepancy, i.e., the difference between GDP and GDI ($Disc$). Again, they are obtained by difference between total private and housing. We have

$$AmIB^* = PropB + TransB + TaxB + DiscB = AmIP^* - AmIH^*,$$
where $AmIP^*$ is private ambiguous income and $AmIH^*$ the part related to housing and

$$AmIP^* = PropP + TransP + TaxP + Disc,$$
$$AmIH^* = PropH + TransH + TaxH.$$

3. Compensation for employees.— Compensation in the private sector excluding housing and prior to R&D and ambiguous capital adjustments ($Wb^*$) is obtained by subtracting general government compensation ($WG$) and housing non adjusted compensation ($Wh$) to total compensation ($W$)

$$Wb^* = W - WG - Wh.$$

4. Private excluding housing capital share.— Assuming that the capital share of ambiguous income is equal to the share of capital in total income, $v^*_{KP}$, the Private excluding housing capital share is obtained by solving for $v^*_{KP}$ equation

$$UnKB^* + CorpB^* + CFCB^* + v^*_{KP} AmIB^*$$

$$= v^*_{KP} (UnKB^* + CorpB^* + CFCB^* + AmIB^* + Wb^*).$$

It comes

$$v^*_{KP} = \frac{UnKB^* + CorpB^* + CFCB^*}{UnKB^* + CorpB^* + CFCB^* + Wb^*}.$$

5. Private excl. housing income and profits from ambiguous capital.— Total ambiguous capital income is then given by

$$AmKTB^* = v^*_{KP} AmIB^*.$$
This income consists in capital returns \((AmKB^*)\) and pure rents \((AmCorpB^*)\). We use the corporate profits to unambiguous capital including corporate profits ratio to breakdown ambiguous capital income between capital returns and profits, i.e.,

\[
AmCorpB^* = \frac{CorpB^*}{CorpB^* + UnKB^*} AmKTB^*
\]

and

\[
AmKB^* = AmKTB^* - AmCorpB^*.
\]

6. **Private excluding housing capital income related to R&D.**— Private excluding housing capital income related to private funding R&D, \(\Pi DB^*\), production is computed with the share of capital income \(v_{KP}\) according to

\[
\Pi DB^* = v_{KP} DB.
\]

7. **Housing ambiguous capital income.**— Housing ambiguous capital income \((AmKH^*)\) is computed in the same way:

\[
AmKH^* = v_{KH} AmIH^*
\]

with

\[
v_{KH} = \frac{NIH + RIH + CorpH + CFCH}{NIH + RIH + CorpH + CFCH + Wh}.
\]

8. **Private compensation of employees.**— Having computed the capital share of R&D and ambiguous capital for private excluding housing and housing, we can subtract and compute labor share so that total compensation for private excluding housing is

\[
WB^* = Wb^* + (AmIB^* - AmKTB^*) + (DB - \Pi DB^*)
\]

and

\[
WH^* = Wh^* + (AmIH^* - AmKH^*).
\]

\[
H. Government capital and Durable Goods Services
\]

As in [Cooley and Prescott (1993)] and [Gomme and Rupert (2007)], government capital and durable goods services are computed from the nominal return to private capital \((r_{KP}^*)\) compute as

\[
r_{KP}^* = \frac{UnKB^* + CorpB^* + AmKTB^* + \Pi DB^*}{KP}.
\]

where \(KP\) is the beginning period private capital non residential stock. As in [Gomme and Rupert (2007)], we find a quite high rate of return as regards standard values used in the literature, about 14.5% on average over the 1959-2008 period. See [Gomme and Rupert (2007)] and the references herein for explanations.

Then, Government capital income is given by

\[
GKS^* = r_{KP}^* KG
\]
and durable goods capital income
\[ \Pi CD^* = r^*_{KP} KD. \]
Note that services form durable goods \((DGS)\) is then given by
\[ DGS^* = \Pi CD^* + CFCCD. \]

I. Non market households labor income

Non market households labor income \((HHS^*)\) is the product of the hourly replacement cost of non market work \((RC^*)\), the number of hours of non market work per capita \((NMH^*)\) and the population \((POP)\), so that
\[ HHS^* = RC^* \times NMH^* \times POP. \]

Following Landfield, Fraumeni, and Vojtech (2009), we use the average hourly wage of housekeepers \((hWHK)\) to value the non market hours worked. However, consistent figures are available from 1999 onwards only. I use national average wage index \((AWI)\) from the Social Security Administration (SSA) to backcast the series.

Non market hours worked are taken from Ramey (2009) and casted forward from 2006 to 2008 with American time use survey (ATUS). In order to keep the data preprocessing simple, our definition of non market activities include Households activities, Purchasing goods and services, Caring for and helping household members, and Caring for and helping non-household adults.

Appendix E. Computing the compensation parameter

In section III, we use two specific relations to compute the compensation parameter \(\omega(\epsilon)\). To get these equations, recall the definition of the compensation parameter
\[ \sum_{t=0}^{\infty} \beta^t u((1 + \omega)C_t, h_t) = W^*(s_0) \]
In the general case \((\sigma > 1)\), it comes
\[ \sum_{t=0}^{\infty} \beta^t \left[ \frac{((1 + \omega)C_t)^\eta (1 - h_t)^{1-\eta}](1-\sigma)}{1-\sigma} - 1 \right] = W^*(s_0) \]
\[ \Rightarrow \sum_{t=0}^{\infty} \beta^t \frac{(1 + \omega)^\eta (1 - h_t)^{1-\eta}](1-\sigma)}{1-\sigma} - 1 = W^*(s_0) \]
\[ \Rightarrow \sum_{t=0}^{\infty} \beta^t (1 + \omega)^\eta (1 - h_t)^{1-\eta} \frac{1}{1-\sigma} + \frac{(1 + \omega)^\eta (1-\sigma)}{1-\sigma} \frac{1}{(1-\sigma)(1-\beta)} = W^*(s_0) \]
\[ \Rightarrow (1 + \omega)^\eta (1-\sigma) \left( W(s_0) + \frac{1}{(1-\sigma)(1-\beta)} \right) = W^*(s_0) + \frac{1}{(1-\sigma)(1-\beta)} \]
Finally we find
\[
\omega = \left( \frac{(1 - \sigma)(1 - \beta) W^* (s_0) + 1}{(1 - \sigma)(1 - \beta) W(s_0) + 1} \right)^{\frac{1}{\eta(1-\sigma)}} - 1.
\]

In the logarithmic case, this expression reduces to
\[
\lim_{\sigma \to 1} \omega = \exp \left( \frac{1 - \beta}{\eta} \left( W^* (s_0) - W(s_0) \right) \right) - 1.
\]

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