Learning-To-Forecast with Genetic Algorithms

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Mikhail Anufriev†, Cars Hommes‡ and Tomasz Makarewicz‡

†University of Technology, Sydney
‡Center for Nonlinear Dynamics in Economics and Finance at University of Amsterdam, Tinbergen Institute

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Abstract

In this paper we show that Genetic Algorithms (GA) offer a simple explanation for human behavior in the Learning-to-Forecast experiment (Heemeijer, Hommes, Sonnemans, and Tuinstra, 2009; Bao, Heemeijer, Hommes, and Sonnemans, 2012). In the first experiment two treatments are studied: under positive (negative) feedback realized price depends positively (negatively) upon individual forecasts. A common finding is that people converge quickly to the fundamental price under negative feedback, but the price oscillates under positive feedback.

We propose to model agents as boundedly rational optimizers who learn how to maximize the forecasting precision of their prediction heuristics through GA. They use a simple linear (first-order) prediction rule which depends on past prices and predictions, the average price so far and the observed trend. The coefficients of the rule are not fixed; instead the agents are endowed with a small number of different coefficient sets and update their specifications with Genetic Algorithms optimization procedure. In each period they choose one set of coefficients based on its hypothetical performance from the previous period. As in the experiments, the agents are independent and cannot share information.

In line with the Heemeijer, Hommes, Sonnemans, and Tuinstra (2009) experiment, we study simulations in which the initial predictions are sampled from the empirical distribution and hence the GA agents update their rules for 49 periods. Our results closely resemble the experimental data both at the individual and the aggregate level for both types of feedback. GA agents converge to the fundamental price under the negative feedback, but exhibit oscillating predictions in the case of positive feedback. We later show that the same model explains well the experiment by Bao, Heemeijer, Hommes, and Sonnemans (2012), which replicates the Heemeijer, Hommes, Sonnemans, and Tuinstra (2009) experiment design for a changing fundamental price.

Our model is also in line with previous analytical models and offers a unified explanation for their partial insights. Regardless of the feedback, heterogeneity in terms of forecasting rules persists. Trend extrapolation becomes important in the positive feedback type of markets, which amplifies the oscillating behavior of the realized prices.

1 Introduction

In this paper we discuss a model in which the agents optimize a simple linear rule with the use of Genetic Algorithms (GA) in order to form price expectations. We will show that our model explains various experimental observations (or stylized facts) and has a strong empirical interpretation. Moreover it generalizes and motivates the existing theoretical models, in specific the Heuristic Switching Model (HSM) by Anufriev, Hommes, and Philipse (2012). This model, despite its good fit to the experimental data, offers only a stylized explanation of human behavior.

Our contribution is twofolds. We motivate the HSM model and show that its result hold under plausible, realistic psychological foundations. To be specific, our GA model allows agents
to learn individually about different prediction heuristics; hence they are flexible to choose one such heuristic depending on the market conditions; and finally they remain heterogeneous in terms of considered heuristics. We argue that these three features are critical for any model, which is supposed to be a realistic description of human learning to forecast. Second we investigate our GA model and argue that people are likely to learn to forecast with simple heuristics, which are yet cleverly chosen for particular market conditions.

The price expectations play a formidable role in many economic environments. Firms often need to organize their production long before the product finally reaches the market. Consumers have to smooth their spendings on durable goods, which they cannot afford all at the same time. Financial institutions invest in shares and derivatives, hoping for a future profit from a price change. Central banks try to maintain inflation at a certain level.

In all these cases, economic agents have to optimize their behavior conditional on what they think will be the future price. Therefore, price expectations are often a key element in economic models. For instance, the majority of the business cycle literature relies on dynamic stochastic general equilibrium (DSGE) models, in which such expectations explicitly shape the saving and investment decisions of companies and individual consumers. Another example is the financial literature, in which the asset pricing question has become one of the most important issues.

The two aspects of the literature are: the empirical question what prediction rules are actually used by the empirical economic agents; and what prediction rules they should use. In our paper, we focus on the former question.

1.1 The failure of rational expectations

In the traditional literature, the expectations are taken to be perfectly rational (rational expectations, RE) (Muth, 1961). The fundamental assumption of the classical economic framework is that the learning is unimportant and the agents have perfect information about the markets. Based on that, the agents will seek to optimize their behavior in a self-consistent fashion in the sense of (1) a mutual and complementary best-response and (2) consistency of the expected economic conditions with the realized market outcomes.

In practice the real agents do not have the perfect information about the market immediately. The traditional framework defends RE in the following way. The agents want to avoid inconsistent behavior, especially they want to avoid systematically wrong beliefs (Muth, 1961). Human beings are capable of uncanny rationality and thus can easily deal with difficult problems, especially if motivated by high stakes (such as profit). As a result, despite some initial errors, they are likely to shrewdly adapt to the underlying market conditions. If we leave them for a while, they converge to behavior which is as if they were perfectly rational, as if they knew the actual fundamental equilibrium of the economy.

Notice that the above-mentioned defense is stylized and does not come with an explicit proof. Furthermore the traditional models are not in line with this defense and take the agents as directly, not as ‘as-if’, perfectly rational: the learning is never included even in the short-
run, initial dynamics. For instance, the DSGE models usually take the market fluctuations as caused by random productivity shocks, for which the optimal response of prices, capital endowment or employment is sluggish due to market frictions (like matching frictions in the labor market). On the other hands, the agents are assumed to know what is the new optimal behavior and try to adapt to it as soon as possible.

This optimism towards learning follows the philosophical roots of the modern economics in the Vienna Circle program. The logical positivism took the 19th century physics as a splendid example of the possibilities of human learning, but also as the ultimate paragon of learning. Philosophers like Rudolph Carnap first explicated the scientific method as a careful combination of empirical observations with formal (mathematical or logical) and verifiable models, in which there should be no metaphysical prejudice. Second they claimed that this epistemology is both simple and efficient, and furthermore that it is the unique way to the objective truth.

This program has been attacked and finally rejected by the majority of philosophers of science. Nevertheless its main intuition – that there exists a unique and successful path to the objective truth – has prevailed as a popular concept and has also pervaded the ‘folk’ scientific methodology. Given this and the influence of neo-positivism and its behaviorism on the early modern economics, it is no surprise that the Vienna approach towards learning lies as a foundation to the psychology which serves as the workhorse in most of the economic research.

What many economists have missed was the collapse of this intuition (and of the whole Vienna program) after the advances in logics and philosophy of science. The former comes mostly in the works of Kurt Gödel, who showed that no axiomatic system can prove the consistency of its axioms by itself. This has severe consequences towards the Vienna stance that logic is ‘immediately true’, since the choice of the formal language is arbitrary and not ‘obvious’. The latter is seen e.g. in the works of philosophers like Wittgenstein, Khun or Quine who, much in the spirit of David Hume, emphasized the necessarily arbitrary link between formal models and the empirical data.

The major result of this discussion is that, in the best case scenario, a priori we can propose only a very general model of learning, which for all empirical applications is far to abstract. It follows that we cannot propose one and unique, specific rational cognitive structure which is immediately obvious as the proper one for an unprejudiced mind. For us economists this has an important consequence: we cannot simply assume that people will efficiently learn their way through the economic environment. Instead, these are empirical questions: what the economic agents take as obvious, both in terms of what models can they use and what they think of how other agents think of the economy; how (and to what extent) they use their experience to update their beliefs about the markets and other agents; whether these methods have anything to do with what we would call rational reasoning; and whether they enable convergence towards any type of equilibrium.
1.2 Alternative explanations

This approach is already visible in the economic literature (see for instance Colander, Howitt, Kirman, Leijonhufvud, and Mehrling (2008) on DSGE models). Experimental economists have repeatedly demonstrated that the RE fail to explain many empirical studies (for example Smith, Suchanek, and Williams (1988) were able to generate a market bubble in a simple asset pricing environment). Furthermore there exists a vast literature on the so called bounded rationality (Smith, 2010), which is also often data-driven (Nunes, 2010).

This literature emphasizes two important stylized facts about the empirical predictions. First, the agents quite often rely on simple ‘rules of thumb’. Second, the resulting dynamics can be quite complex even if the agents exhibit very simple behavior.

In respect to the first observation, the empirical agents are often untrained in statistics, cannot access abundant data sets, or simply face overly complex environment. If the structure of the economy changes constantly, it may be impossible to identify its key parameters, due to lack of the data, or even due to some fundamental econometric issues. In any case, the agents cannot rely on sophisticated economic concepts such as ‘fundamental price’ or ‘equilibrium’. Instead, they can only use simple heuristics, which may have a surprisingly successful performance.

For the example of the price expectations, many laboratory experiments suggest that people use simple rules: naive, adaptive or trend extrapolating (see Hommes (2011) for a discussion). Also, analysis of larger polls leads to the same conclusion (Nunes, 2010).

A more surprising, scientific example is the Sims critique (Sims, Goldfeld, and Sachs, 1982). Christopher Sims wrote that in practice large structural models are outperformed by the basic linear auto-regression. The reason is that the former are often stylized, but still cannot be properly identified. On the other hand, Wold decomposition theorem implies that any stationary time series can be represented as an autoregressive moving average (ARMA) process, which depends on a handful of parameters and so can be reliably estimated for most empirical cases. The result is that since 1980ties, vast part of the empirical work on time series relies on the reduced form linear estimators.

The ARMA models are simple heuristics in the sense in which this term is often used by the behavioral economists. ARMA representation forfeits the ‘full picture’ of the structural models, where each parameter or equation has a straightforward economic interpretation. Instead, reduced form estimations are never interpreted and serve only as a forecasting tool. They are re-estimated routinely and used to predict the distribution of the time series up to a handful of future periods. See for instance discussion by Ghent (2009) that it is possible for more involved structural models to outperform the pure vector autoregressive approach.

Even if the economic agents use simple heuristics, the resulting dynamics (especially in the short-run) can become complicated. First of all, people learn and adapt. Experiments suggest that people may be limited to a small number of simple rules (like trend extrapolation versus adaptive expectations), but will adhere to one or other based on their current performance (Hommes, 2011; Anufriev, Hommes, and Philipse, 2012). Furthermore their rules are
commonly heterogeneous: not all extrapolate the trend to the same degree, just as not all rely in the same way on the past prices (Heemeijer, Hommes, Sonnemans, and Tuinstra, 2009).

This means that people indeed learn to forecast. The dynamics of price expectations are ‘as if’ perfectly rational only in some special environments. Else they can exhibit complicated and unpredictable patterns. For the economists this results in a twofolds problem. First, it is not clear what mechanism generates these specific heuristics; most theoretical models are stylized and cannot be motivated in a systematic way with proper psychological foundations. Second, for the case of most data sets (both experimental and field), the agents are awarded also for other tasks than just forecasting. As a result, it may be difficult to distinguish, for instance, the strategic behavior from the pure prediction process, and how these two co-evolve during an experiment.

1.3 The research agenda

In our paper we focus on the learning to forecast. We present a model in which the agents use Genetic Algorithms to optimize a simple linear forecasting rule. We will show that this model has both theoretical and empirical virtues. It is efficient in explaining the experimental predictions. It also generalizes and motivates the existing stylized models (like the heuristic switching model, Anufriev, Hommes, and Philipse (2012)). Finally it is by far the most realistic model we have seen in the literature: it allows agents to learn and efficiently update their heuristics; yet it does not limit them to any small set of these heuristics, but instead preserves the expected degree of individual heterogeneity.

We use two experiments (Heemeijer, Hommes, Sonnemans, and Tuinstra, 2009; Bao, Heemeijer, Hommes, and Sonnemans, 2012), in which the agents are asked specifically to predict the next period price and hence are rewarded only for that. The first experiment is based on a simple and stable feedback between the expectations and the realized prices. We will use this experiment to fine-tune our model. It is difficult to properly test a model which is on such a level of complexity as our. Therefore, we will focus on showing that the model (1) generates time paths which have a distribution that looks reasonably close to the experimental ones and (2) replicates the stylized facts which are found in the experiments.

The second experiment is more complicated, since here the fundamental feedback structure changes twice throughout the time. We will show that our model is robust against this non-stationary environment.

2 Learning-to-Forecast experiment

In this section we describe the Heemeijer, Hommes, Sonnemans, and Tuinstra (2009) experiment. We will also discuss the Heuristic Switching Model (HSM) and how it serves as a good stylized description of the experimental behavior. We will define the three important concepts: flexibility, heterogeneity and individual learning, which we argue are the key notions for any
model that explains the way in which people learn to forecast.

2.1 Experiment setup

In the Learning-to-Forecast experiment reported by Heemeijer, Hommes, Sonnemans, and Tuinstra (2009), subjects were asked to predict one period ahead price of a good. In the two treatments with positive and negative feedback, the realized price depended on the forecasts in positive or negative manner respectively. There were in total seven groups with the positive and six groups with the negative feedback, each consisting of six subjects who participated in the experimental economy for 50 periods. The law of motion for the negative feedback economy was

\[ p_t = \frac{20}{21} (123 - \bar{p}_e^t) + \eta_t \]

and for the positive feedback economy

\[ p_t = \frac{20}{21} (3 + \bar{p}_e^t) + \eta_t \]

where \( \bar{p}_e^t = \sum_{i=1}^{6} p_{i,t} \) is the average prediction from that period and \( \eta_t \sim NID(0,0.25) \) is a small normal shock.

Regardless of the feedback, the self-consistent prediction based on perfectly rational expectations is \( p^f = 60 \) (which we call the fundamental price). Also both markets are ‘stable’ in the sense that the system converges to the fundamental under pure naive expectations such that \( p_{e,t} = p_{t-1} \).

Subjects were given neither the law of motion of their economies nor a sample time-path of the prices, but were informed about the feedback between the predictions and prices in qualitative terms. The experiment results were different for the two feedbacks. In the negative feedback, predictions, which are far from the fundamental, are likely to be ‘punished’. For instance if on average the agents predict a low price, the realized price will be high. As a result, subjects in all six groups were quickly tamed to expect prices close to the fundamental. This followed a brief span of chaotic, uncorrelated (both between the agents and for each agent) predictions.

The positive feedback groups behaved differently. In this market structure, if all the subjects predict price significantly lower than the fundamental, the realized price will be in between the predictions and the fundamental, but also very close to the average prediction. This encourages the subject to remain conservative with their forecasts (which also means that they are able to coordinate faster than under the negative feedback treatment) and follow the price growth by a little. Then, since they would undershoot the realized price for some number of periods, they seem to loose patience and start to extrapolate the trend, which results in an increasing velocity of their predictions and the realized prices. As a result, instead of converging to the fundamental, subjects overshoot it. Reverse story thus continues: subjects
follow decreasing prices and undershoot the fundamental.

These oscillations remain significant far until the 50th period in five out of seven groups. Only in one group there is a clear convergence, while the remaining group cannot be clearly evaluated. Despite the oscillations, subjects in all seven groups managed to coordinate their predictions in a handful of periods: individual forecasts were quite similar even if far from the fundamental.

2.2 Stylized facts

To sum up, the experiment resulted in the following stylized facts. For the negative feedback:

1. Predictions from initial periods are volatile for each subject and not correlated between the subjects;
2. Afterwards all the predictions and the prices converge to the fundamental price \( p^f = 60 \).

For the positive feedback:

1. Subjects quickly (quicker than in the case of the negative feedback) coordinate and hence their predictions are highly correlated;
2. Prices are likely to oscillate throughout the whole experimental time of 50 periods.

The coordination can be expressed as the standard deviation of the agents’ predictions from a given period. The distance from the fundamental price \( p^f = 60 \) measures the degree of convergence. This corresponds to Figure (4) in the Heemeijer, Hommes, Sonnemans, and Tuinstra (2009) paper. Notice that the authors actually report the absolute distance from the fundamental. Their point is that the positive feedback dynamics do not converge and in most periods the prices are far from the fundamental value. In the following analysis we will use signed (not absolute) distance from the fundamental instead. In this way we will be able to control for the proper shape of the oscillation, which should alternate in over- and undershooting the fundamental.

2.3 Heterogeneity

After reporting the experimental results, Heemeijer, Hommes, Sonnemans, and Tuinstra (2009) focus on the first-order rule (FOR)

\[
p_{e,t} = \alpha_1 p_{t-1} + \alpha_2 p_{e,t-1} + \alpha_3 60 + \beta (p_{t-1} - p_{t-2}) + \eta_t
\]

for \( \alpha_1, \alpha_2, \alpha_3 > 0, \alpha_1 + \alpha_2 + \alpha_3 = 1, \beta \in [-1, 1] \) and \( \eta_t \) being some well-behaved IID random variable. The authors estimated this rule separately for each subject, based on their predictions from the last 40 periods. Notice that the third term, the fundamental which is associated with the \( \alpha_3 \) coefficient, was used for the sake of stationarity of the estimation.
The original estimation allows the agents to be heterogeneous in terms of prediction heuristics. It seems natural that for some people the trend is more important than for the others, who are more likely to focus for example on an anchor. The former are associated with (3) with $\beta$ close to unity, while the latter would take $\beta$ closer to zero. In fact we would expect the empirical trend extrapolation to have a continuous distribution. The same holds for the adaptive heuristic. Figure 7 from Heemeijer, Hommes, Sonnemans, and Tuinstra (2009) shows the empirical extent of heterogeneity between the subjects. Even for a small sample of 78 agents, their rules are clearly varied. We will refer to this phenomenon of the between-agent heuristic differentiation as ‘heterogeneity’.

2.4 Flexibility

Anufriev, Hommes, and Philipse (2012) have shown that a Heuristic Switching Model (HSM) can replicate a good share of the experimental stylized facts, including the oscillations under the positive feedback treatment and the convergence to the fundamental under the negative feedback treatment. That model postulated the subjects to be endowed with a small number of prediction heuristics, from which they choose one forecasting strategy based on their relative past performance. The specific heuristics used by authors were:

- **Adaptive** $p_{i,t} = \alpha p_{i,t-1} + (1 - \alpha)p_{i,t-1}$ for $w \in [0, 1]$,
- **Trend extrapolation** $p_{i,t} = p_{i,t-1} + \beta(p_{t-1} - p_{t-2})$ for $\beta \in [-1, 1]$.

Notice that both heuristics are a special case of the first-order rule. The adaptive heuristic is the FOR with $\alpha_3 = \beta = 0$, while the trend extrapolation is the FOR with $\alpha_2 = \alpha_3 = 0$.

The idea of the model is the following. Each subject is endowed with these two heuristics. At the beginning of each period, she evaluates the hypothetical realized performance (the mean squared error, $MSE_t(h)$) of both heuristics in the previous period. In the next step, she computes the hypothetical estimated performance as a variable with memory of the realized performances. To be specific, $MSE_t(h)$ is defined as $\mu MSE_{t-1}(h) + (1 - \mu)MSE_{t-1}(h)$, where $\mu \in (0, 1)$ and $MSE_t(h)$ is the hypothetical estimated performance of the heuristic $h$. It means that the full hypothetical estimated performance is a moving average of the hypothetical realized performances.

For the current period $t$, the agent draws one of the two heuristics, where the probabilities come from the logit transformation of heuristics’ hypothetical estimated performance:

$$
Prob[\text{agent } i \text{ using heuristic } h_1] = \frac{\exp(\gamma MSE_{t-1}(h_1))}{\exp(\gamma MSE_{t-1}(h_1)) + \exp(\gamma MSE_{t-1}(h_2))},
$$

where $h_1, h_2 \in \{\text{trend extrapolation}, \text{adaptive expectations}\}$ and $\gamma$ is a sensitivity parameter.

Anufriev, Hommes, and Philipse (2012) work on a plausible intuition: people are flexible in the sense that they can adapt their behavior to the current circumstances. For instance, if there is a persistent trend within the data, people might abandon a notion of a static anchor,
and then use anchor-based heuristics again if the time series stabilizes. We will refer to this within-agent heterogeneity as ‘flexibility’.

Notice that this concept is different than heterogeneity. The flexibility, as is the case in the Anufriev, Hommes, and Philipse (2012) paper, is possible if the agents have the same heuristic sets. The only restriction is that this set – possibly homogeneous between the agents – contains more than one heuristic.

2.5 Individual learning and the place for Genetic Algorithms

Both papers offer a partial insight into the experimental data. Heemeijer, Hommes, Sonnenmanns, and Tuinstra (2009) show the importance of the heterogeneity, but they still impose a single rule on each agent. Agents are different, but each individual must stick to one particular forecasting heuristic regardless of market conditions. The reverse holds for Anufriev, Hommes, and Philipse (2012). The authors do not restrict individual agents to one rule, but they impose all the agents to use the same set of rules.

Another restriction of both papers is that the agents do not learn in the proper sense. In the HSM model by Anufriev, Hommes, and Philipse (2012), the agents are flexible but they cannot adjust the set of strategies. Imagine that there is an agent who can switch between the two described heuristics: adaptive expectations and trend extrapolation, where the latter takes $\beta = 1$ as the trend coefficient. It can happen that the agent faces an environment where there is a persistent trend, yet its effect is not particularly strong. In this case a trend extrapolation heuristic with $\beta = 0.5$ would outperform the one which is used by our agent. Yet she could not take that into consideration – she is bound to use either the inefficiently strong trend extrapolation, or the stationary adaptive heuristic. In practice this is unrealistic, since people are likely to optimize their behavior to the specific circumstances of the current market situation.

Altogether one could generalize the HSM to include a wider set of strategies, that could also differ between the agents. If these sets were dense enough, that would approximate the individual learning. As a result, this model would incorporate all the three features – the heterogeneity, the flexibility and the individual learning – that, we argue, are necessary for any realistic model of learning to forecast.

In the next section we will discuss Genetic Algorithms. We will argue that a model, which incorporates Genetic Algorithms, is essentially such a general version of the HSM. Furthermore it is more plausible and elegant, as well as offers a better psychological interpretation.

3 Genetic Algorithms

In this section we explain the optimization procedure known as Genetic Algorithms. We define its general structure and explain how it can be plausibly interpreted as a feature of human brains; and how it can play a major role in the forecasting process.
3.1 Genetic Algorithms as optimization procedure

Consider a maximization problem where the target function $F$ of $N$ arguments $\theta_n$ is such that a straightforward analytical solution is unavailable. Instead, we need to use a numerical optimization procedure.

Traditional maximization algorithms, like the Broyden-Fletcher-Goldfarb-Shanno (BFGS) algorithm, iterate a candidate argument for the optimum of the target function $F$ by (1) estimating the curvature around the candidate and (2) using this curvature to find the optimal direction and length of the change to the candidate solution. This so called ‘hill-climbing’ algorithm is very efficient in its use of the shape of the target function. On the other hand, it will fail if the target function is ‘ill-behaved’: non-continuous or almost flat around the optima, has kinks or breaks. Here the curvature cannot be reliably estimated. Another problem is that the BFGS may perform poorly for a problem of large dimensionality.

The Genetic Algorithms are based on fundamentally different approach and therefore can be used for a wider class of problems. The basic idea is that we should focus on a population of arguments which compete only in terms of their respective function value. This competition is modeled in an evolutionary fashion: mutation operators allow for a blind-search experimentation, but the probability that a particular candidate will survive over time is relative to its functional value. As a result, the target function may be as general as necessary, while the arguments can be of any kind, including real numbers, integers, probabilities or binary variables. The only constraint is that each argument must fall into a predefined bounded interval $a_n, b_n$.

3.2 Binary strings

Genetic algorithm (GA) uses $H$ chromosomes $g_{h,t} \in H$ which are binary strings divided into $N$ genes $g_{n,h,t}$, each encoding one candidate parameter $\theta_{n,h,t}$ for the argument $\theta^n$. A chromosome $h \in \{1, \ldots, H\}$ at time $t \in \{1, \ldots, T\}$ is specified as

$$g_{h,t} = \{g_{h,t,1}, \ldots, g_{h,t,N}\},$$

such that each gene $n \in \{1, \ldots, N\}$ has its length equal to an integer $L_n$ and is a string of binary entries (bites)

$$g_{n,h,t} = \{g_{n,1,h,t}, \ldots, g_{n,L_n,h,t}\}, \quad g_{n,l,h,t} \in \{0, 1\} \text{ for each } j \in \{1, \ldots, L_n\}.$$

The interpretation of (5) and (6) is straightforward. An integer $\theta^n$ is simply encoded by (6) with its binary notation. Consider now an argument $\theta^n$ which is a probability. Notice that $\sum_{l=0}^{L_n-1} 2^j = 2^{L_n} - 1$. It follows that a particular gene $g_{n,h,t}$ can be decoded as a normalized sum

$$\theta_{n,h,t} = \sum_{l=1}^{L_n} \frac{g_{n,l,h,t} \cdot 2^{l-1}}{2^{L_n} - 1}.$$
A gene of zeros only is therefore associated with \( \theta_n = 0 \), a gene of ones only – with \( \theta_n = 1 \), while other possible binary strings cover the \([0, 1]\) interval with an \( \frac{1}{2^{L_n - 1}} \) increment. Any desired precision can be achieved with this representation. Since \( 2^{-10} \approx 10^{-3} \), the precision close to one over trillion \( (10^{-12}) \) is obtained by a mere of 40 bites.

A real variable \( \theta^n \) from an \([a_n, b_n]\) interval can be encoded in a similar fashion, by a linear transformation of a probability:

\[
\theta^n_{h,t} = a_n + (b_n - a_n) \sum_{l=1}^{L_n} g^n_{h,t} \frac{l-1}{2^{L_n} - 1}
\]

where the precision of this representation is given by \( \frac{b_n - a_n}{2^{L_n - 1}} \). Notice that one can approximate an unbounded real number by reasonably large (in absolute value) \( a_n \) or \( b_n \), since the loss of precision is easily undone by a longer string.

Binary representation has two advantages. First, as will be discussed later, it allows for an efficient search through the parameter space. Second, any type of a well-defined argument can be translated into a string of logical values.

### 3.3 Evolutionary operators

The core of GA are evolutionary operators. GA iterates the population of chromosomes for \( T \) periods, where \( T \) is either large and predefined, or depends on some convergence criterion. First, at each period \( t \in \{1, \ldots, T\} \) each chromosome has its fitness equal to a nondecreasing transformation of the function value \( \mathcal{F} \). This transformation is defined as \( V(\mathcal{F}(\theta_{h,t})) \equiv V(h_{k,t}) \rightarrow \mathbb{R}^+ \cap \{0\} \). For example, a non-negative function can be used directly as the fitness. If the problem is to minimize a function, a popular choice is the exponential transformation of the function values, similar to the one used in the logit specification in (4).

Chromosomes at each period can undergo the following evolutionary operators: procreation, mutation, crossover and election. These operators first generate an offspring population of chromosomes from the parent population \( t \) and therefore transform both populations into a new generation of chromosomes \( t + 1 \) (notice the division of the process). In the following analysis we will use all four of them.

#### 3.3.1 Procreation

For the population at time \( t \), GA picks \( \Pi \subseteq \mathbb{H} \) subset of \( \pi \) chromosomes and picks \( \kappa < \pi \) of them into a set \( \mathbb{K} \). The probability that the chromosome \( h \in \Pi \) will be picked into the \( \mathbb{K} \) as its \( z \)th element (where \( z \in \{1, \ldots, \kappa\} \)) is usually defined by the power function:

\[
\text{Prob}(g_z = g_{h,t}) = \frac{V(g_{h,t})}{\sum_{j \in \mathbb{K}} V(g_{j,t})}.
\]

\(^1\)Nevertheless there are GA extensions with real-valued genes. See Haupt and Haupt (2004) for an introductory discussion.
This procedure is repeated with differently chosen \( \Pi \)'s until the number of chromosomes in all such sets \( \mathcal{K} \)'s is equal to \( H \). For instance, the roulette is procreation with \( \iota = H \) and \( \kappa = 1 \): GA picks randomly one chromosome from the whole population, where each chromosome has probability of being picked equal to its function value relative to the function value of all other chromosomes. This is repeated exactly \( H \) times.

So called tournaments are often used for the sake of computational efficiency. Here, \( \iota << H \). For instance, GA could divide the chromosomes into pairs and sample two offspring from each pair. We will use the full roulette operator.

Procreation is modeled on the basic natural selection mechanism. We consider subsets of the original population (or maybe the whole population at once). Out of each such a subset, we pick a small number of chromosomes, giving advantage to these which perform better. We repeat this procedure until the offspring generation is as large as the old one. Thus the new generation is likely to be ‘better’ than the old one.

3.3.2 Mutation

For each generation \( t \in \{1, \ldots, T\} \), after the procreation has taken the place, each binary entry in each new chromosome has a predefined \( \delta_m \) probability to be swapped: ones turned into zeros and vice versa. In this way the chromosomes represent now different numbers and may therefore attain better fit. Moreover, the mutation operator is where the binary representation becomes most useful.

If the bites, which are close to the end of the gene, mutate, the new argument will be substantially different from the original one. On the other hand, small changes can be obtained by mutating bites from the beginning of the gene. Both changes are equally likely! In this way, GA can easily evaluate arguments which are both far away from and close to what the chromosomes are currently encoding. As a result, GA efficiently converges to the maximum, but also are likely not to fixate on a local maximum. This requires no additional investigation of the initial conditions, as is the case of BFGS.

3.3.3 Crossover

Let \( 0 \leq C_L, C_H \leq \sum_{n=1}^{N} L_n \) be two predefined integers. Crossover operator divides the population of chromosomes into pairs and hence exchanges the first \( C_L \) and the last \( C_H \) bites between chromosomes in each pair with a predefined probability \( \delta_c \). This operator facilitates experimentation in a different way than the mutation operator: here the chromosomes experiment with different compositions of the individual arguments, which on their own are already successful.

3.3.4 Election

The experimentation done by the mutation and crossover operators does not need to lead to efficient binary sequences. For instance, a chromosome which actually decodes the optimal
argument should not mutate at all. To counter this effect, it is customary to divide the creation of a new generation into two stages. First, the chromosomes procreate and undergo mutation and crossover in some predefined order. Next, the resulting set of chromosomes is compared in terms of fitness with the parent population. Thus, offspring will be passed to the new generation only if it (weakly) outperforms the parent chromosome. In this way each generation will be at least as good as the previous one, what in many cases facilitates convergence.

3.4 Behavioral interpretation

Brains can have a good use of numerical procedures, since these can have straightforward biological applications. The obvious example is the LtF experiment, in which the subjects observed numbers – prices – and sought to process them somehow into another numbers – predictions. However, the nature gives us very similar examples in much less sophisticated situations. Consider a problem where an animal seeks to intercept a moving object – be that a human trying to catch a ball (Chardenon, Montagne, Buekers, and Laurent, 2002) or a dragonfly following its prey (Olberg, Worthington, and Venator, 2000). In both cases, the brain of such an animal needs to translate one number – the number of retina photoreceptor cells which were stimulated in a particular fashion – into another number, which is the number of particular nervous signals sent to particular muscles, what thus determines the exact muscle contraction.

The efficiency of a particular numerical response depends on the surrounding environment: for the interception problem this may depend on factors such as the wind strength or the type of the ground. Presumably a brain does not perform these activities blindly. Of course it will not even try to solve a fundamental problem: given the current distribution of wind strength and direction, where the ball is likely to fall (see the paper by Chardenon, Montagne, Buekers, and Laurent (2002) for a good discussion). Instead, it will rather try to adjust the precision of its numerical reaction to changes in stimuli. For instance, if the wind is strong, one may be forced to run faster, and thus a small change in the retina stimuli will lead to a substantially stronger muscle contraction.

We propose to model this optimization of human brain by GA. As mentioned, GA can be used for very different problems and allow for any well defined argument. On the other hand, it is surprisingly simple and efficient, even if the target function is ill-behaved and allows for no proper analytical solution. By the simplicity we mean both the conceptual elegance and the actual programming effort. Therefore it is likely that our brains are endowed with a computational routine which can be represented by the GA, since evolution promotes elegant and versatile solutions.

Notice that the brain should be able to access this numerical tool directly, not only through the conscious control. First, if our evolutionary reasoning is true, this device must be far older than the developed self-awareness. Second, many of its uses – like the muscle contraction – should be instant, instinctive, whereas conscious control is reserved for tasks in which the
benefits of a more sophisticated planning outweigh the computational (eo ipso time) costs. In practice we should expect that even if the human subjects decide to take a direct control over that module, majority of the exercise still remains outside their conscious reach.

This is in line with our experience. Quite often we have to quickly come up with some number (such as a rough estimate of a travel time, or a more precise expectations of shop prices) and we are able to do so even though we do not perform any explicit, conscious computations. This is even so for the case where we try to catch a ball.

3.5 Experimental validity

Even if people exploit GA for some specific practical problems (like intercepting a ball), is it still reasonable to think that they have been using them in the LtF experiment as well? The ideal econometric approach to forecasting is to estimate an ARMA or GARCH representation of the time series and use it as a prediction heuristic. If these forecasts are basis for the subject’s policy function, which may then influence the time series itself, she could test the significance of her impact and thus figure out the optimal and (as closely as possible) self-consistent prediction given the past data. The agent could also work on a fine structural model – if she can access and estimate one.

In practice the econometric approach may be infeasible. The agent must be able to use these techniques in the first place, which requires both skills and software. This amplifies in the case of the structural estimations: if the agent seeks to learn the fundamental price of an asset, she needs to know this concept and know how to use it for this particular market and the data set. Second problem is that the time series need to be long, otherwise identification and precision issues arise. Imagine that you want to estimate a fundamental price by taking the average realized price – if you have a sample of 10 volatile observations, the confidence intervals are unlikely to be precise enough for any practical purposes or testing.

Therefore we argue that the subjects of the Heemeljer, Hommes, Sonnemans, and Tuinstra (2009) experiment did not use any proper econometric technique, but rather relied on informal heuristics and rules of thumb. Furthermore we argue that they did not use any economic model to solve the problem. Any economic intuition – like a fundamental price – becomes useless as the econometrics is infeasible.

Notice that the agents were actually never asked directly for such equilibrium concepts like the fundamental price the Nash Equilibrium of the model. Upfront they could not even say whether the short-run dynamics of the price will have anything to do with its long-run attraction; or if such actually exists. Instead they were simply asked to predict the price, regardless of whether it was behaving odd or converging to an equilibrium – and this is the task that the subjects focused on. Not to predict the long-run behavior of the prices, but simply to ‘catch that number’ in the next period.

\(^2\)Notice that not all the experiment participants are likely to have underwent a proper statistical training. Neither were they given laptops with Stata.
Recall the example of the dragonfly. This crude creature at every second will try to maintain a secure angle between its direction and the prey’s direction, so that in a due time its advantage in velocity will secure an interception (Olberg, Worthington, and Venator, 2000). Change the labels and variable interpretation, and the experimental subjects and the dragonflies deal with a problem of the same fundamental structure. Since human cannot access sophisticated ‘reasoned’ means to solve it, it is natural to expect them to adhere to the instincts they use for similar problems outside economics – like interception exercises.

3.6 GA and other learning models

GA can be interpreted as a generalized version of the above mentioned forecasting heuristics, the first-order rule and the heuristic switching model. Given its psychological plausibility, it is therefore an ideal candidate to motivate these models. Furthermore, it is also a generalized version of many other learning models, such the famous experience-weighted attraction model (Camerer and Hua Ho (1999), EWA).

HSM model has a very similar structure to GA since they both repeatedly evaluate performance of a small set of heuristics/chromosomes against the target function. The main difference is that in HSM the target function (moving average of the prediction error) changes over time, while the set of heuristics is static. In the traditional use of GA the reverse holds, a constantly mutating set of chromosomes maximizes a static, predefined function. Nevertheless there is no conceptual restriction that the GA target fitness should be static. We will show in the following section how to condition the performance of the heuristics, which are generated by the GA-based learning, on the evolving time path of the realized prices. In this way we will preserve the flexibility of the HSM model, while at the same time allowing for the heterogeneity and the individual learning through the use of evolutionary operators.

A similar interpretation holds for the EWA learning (and its special cases, reinforcement and fictitious learning). Here the chromosomes would encode the number of a strategy or, alternatively, probabilities of a mixed strategy. Hence they would be evaluated against some predefined transformation of the past experience: stream of hypothetical discounted payoffs or an estimated ‘representative behavior’ of other players, or a mixture of both. GA representation becomes especially interesting for games which are finite, but nevertheless have an enormous action space. In the classical EWA model the agent evaluates each strategy, which may be computationally involving for such cases of large games. In practice it may be easier for the subject to focus on a small number of mixed or pure strategies and allow her to experiment over time.

4 The model

In this section we present our Genetic Algorithm based model and argue that it generates time paths which greatly resemble the experimental results. Second we discuss (and interpret) the
specification of the model and comment on the behavior, which is predicted by the model.

4.1 Setup of the model

We consider the Genetic Algorithm (GA) optimization procedure as specified in the previous section. The idea of the model is the following. Each agent needs to predict price in the current period. To do so, she cannot use concepts like e.g. fundamental price. She relies on a simple linear rule. Nevertheless, she does not use one blindly. Instead, she works on a small set of such specific rules with different coefficients. At each period, she updates them using GA and their hypothetical past performance (in terms of mean squared error, MSE). As a result, she learns how to forecast the prices in the optimal fashion.

To be specific, each agent \( i \in \{1, \ldots, 6\} \) focuses on the following linear prediction rule:

\[
 p_{e,i,t} = \alpha p_{t-1} + (1 - \alpha)p_{e,i,t-1} + \beta(p_{t-1} - p_{t-2}).
\]

Notice that this is essentially a deterministic version of (3), with the additional restriction \( \alpha_3 = 0 \). We will discuss this restriction in the following subsection.

This general rule needs actual parameters to work. For instance, naive agents would pick \( \alpha = 1 \) and \( \beta = 0 \). In general, the \( \alpha \) parameter is the weight of the previous realized price, in contrast to the previous prediction: the higher it is, the less conservative is the agent in her predictions. The trend extrapolation parameter \( \beta \) shows the extent to which the agent believes that the past price changes are meaningful. We assume that each agent is endowed with \( H = 20 \) such rules with some specific coefficients

\[
 p_{e,h,i,t}(s) = \alpha_{h,t} p_{s-1} + (1 - \alpha_{h,t})p_{e,i,s-1} + \beta_{h,t}(p_{s-1} - p_{s-2}), \quad h \in \{1, \ldots, H\}.
\]

Notice that this rule \( h \) depends on the timing in two ways. First, it has some specific coefficients at time \( t \). Second, it can be used to determine a hypothetical prediction based on the data from period \( s \). For instance, \( p_{h,i,t}(t) \) is a prediction of the current price based on the most updated version of a heuristic \( h \), and \( p_{h,i,t}(t-1) \) is the same rule (with the same coefficients) as if it was used in the previous period.

By the definition, the price weight must be \( \alpha \in [0, 1] \). For the trend extrapolation, the possible interval is less obvious. Intuition suggests that \( \beta \in [-1, 1] \) is natural, since it is (speaking loosely) ‘stationary’. One can easily show that as long as the price generating process is of the linear form

\[
 p_t = A + B p_t^e,
\]

basic adaptive rule (rule (10) with \( \beta = 0 \)) has a unique steady state at the fundamental price \( p^f = \frac{A}{1 - B} \) as long as \(|B| < 1 \) (which means that the prediction errors are not amplified). Notice that (1) and (2) are special cases of (12).

Once we allow for the trend extrapolation, the fundamental price is still the unique steady
state. Nevertheless its stability is now much more complicated and one can show that for $B$
close to $-1$ and large enough $\beta$, the system is not stable even if both coefficients are in the
unit circle, $\beta, B \in (-1,1)$.

Furthermore it is not clear whether the subjects have to behave in a ‘stationary’ fashion. This is an interesting
intellectual concept that helps to solve some theoretical models. Yet in reality people may be forced to deal with
problems which in the practical time perspective are not stationary. This means that they may even be interested in
very strong trend extrapolation with $\beta > 1$, for instance if they face exponentially growing prices. We will later show that in
fact $\beta \in [-1,1]$ is outperformed by the slightly wider interval $\beta \in [-1.1,1.1]$ for the case of

Another issue is whether this interval should be symmetric. One can imagine that people
may be strong trend followers, but avoid even moderate contrarian heuristics, or vice versa.
Therefore for now we assume that $\beta \in [\beta_L, \beta_H]$ and the exact allowed degree of the trend extrapolation
is an empirical issue.

Given the set of rules (11), define the fitness of each rule $h$ for the agent $i$ at moment $t$
against the price $s$ as

$$ F_{i,t}(h, s) = \exp(-\gamma(p_{h,i,t}^e(s) - p_s)^2), $$

where $\gamma$ is the sensitivity parameter. For the actual fitness measure in the following analysis
we will use $\gamma = 1$. Notice that this is a simple exponent of the MSE of the prediction, which
would be generated by the rule $h$ (in its form the period $t$) based on data from period $s$, against
the price from that moment $s$. We will be interested in $s = t - 1$ and $s = t$, as explained
later. We use the logit transformation, since the agents want to minimize prediction error.
If a heuristic is perfectly successful, its fitness is equal to unity; and as the MSE diverges to
infinity, the fitness converges to zero.

The timing of the model for each period $t$ is the following:

1. Agents pick particular heuristics and generate their predictions, and the probability that
   a heuristic is chosen decreases with its past hypothetical performance;

2. The market price is realized according to (12);

3. Agents update their heuristics using one GA iteration.

In the first step, the agents use the roulette operator, with which they randomly pick
exactly one heuristic from the whole set of heuristics. Recall that the roulette distribution is
defined by the equation (9), for which the agents use $F_{i,t}(h, t-1)$ (the hypothetical performance
of the heuristics in the previous period) as the weight $V$ for the probabilities of picking each
heuristic.

After the agents choose their heuristics, they generate their predictions $p_{h,i,t}^e(t)$ and thus the
market price is realized. In our case, this means either the negative or the positive feedback
market with $|B| = \frac{20}{21}$ and the fundamental price $p^f = 60$. Afterwards the agents learn:
they observe the hypothetical performance of their heuristic sets and update them with one iteration of GA. The important assumption is that each agents does that independently – just like as in the experimental setting (and most real scenarios), they do not observe other agents’ heuristics. Therefore they cannot use the latter in any way (for any evolutionary operator).

To be specific, each heuristic corresponds to its respective chromosome. We decided to encode each parameter with 20 bites (corresponding to the precision of $10^{-6}$), which gives a chromosome of length 40. The model is not sensitive against this choice. Agents use all the four evolutionary operators, as described in the previous section: procreation, mutation, crossover and election. Each operator has straightforward interpretation.

Procreation means that the agents focus on those heuristics, which performed well in the previous period. The procreation uses the roulette operator again, with fitness given as $F_{i,t}(h,t)$ and $\gamma = 1$ – the hypothetical performance of a heuristic in comparison with the current price.

Mutation and crossover are the out-of-the-box experimentation. Agents do not wish to be restrained to the initial set of heuristics, but neither they want to evaluate all the possible heuristics. Instead, they work on a small set of heuristics and experiment with their specification and mixture. We decided to use $\delta_m = 0.01$ and $\delta_c = 0.9$. The model is not sensitive for a reasonable choice of these parameters. The crossover exchanges the $\alpha$ entry between two chromosomes.

Finally the election operator ensures that only successful experimentation is transmitted into the next period. For instance, if an agent considers for a change a highly extrapolative heuristic, she will preserve it only if there is an actual trend in the data, which was not properly picked up by her other heuristics.

4.2 Initial period

The first period (or the initialization of the model) is another issue and is specific to the actual experimental setting. Both Heemeejer, Hommes, Sonnemans, and Tuinstra (2009) and Bao, Heemeejer, Hommes, and Sonnemans (2012) ask the subjects to give an initial prediction in the first period without supporting them with any sample of the prices. This means that the very first prediction is purely uninformed and cannot come from any learning. We therefore take the initial predictions (predictions in the period 1) as exogenous.

This is not very important for the negative feedback. However, under the positive feedback the results depend on the initial price. For example, HSM predicts the empirical oscillations only if the initial price substantially differs from the fundamental (Anufriev, Hommes, and Philipse, 2012). In the Heemeejer, Hommes, Sonnemans, and Tuinstra (2009) experiment, the fundamental price is $p^f = 60$, whereas the the subjects were asked to give their initial

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3One could imagine that the agents focus on strategies that performed better in a series of periods. We will show that the assumption of long memory is not necessary to explain the data. It also has a good interpretation: agents do not know how meaningful are the previous periods for the current conditions and hence may be likely to disregard past or discount it heavily.
prediction in the $[0, 100]$ interval. The midpoint (or the focal point) of this interval is lower than the fundamental and moreover the subjects were unevenly likely to predict less than the focal point: almost half of the initial predictions were less than 50, while one-third was exactly 50.

Diks and Makarewicz (2012) investigate the distribution of these initial conditions in a systematic fashion. They argue that they were generated by a non-continuous distribution, which they called Winged Focal Point (WFP). The initial prediction is taken as the focal point 50 with some positive probability, otherwise it falls into one of two ‘wings’ – uniform distributions that span from 50 towards 0 and 100. The authors estimate this composite distribution as

$$
p_{i,1}^e = \begin{cases} 
\epsilon_i^1 \sim U(9.546, 50) & \text{with probability } 0.45739, \\
50 & \text{with probability } 0.30379, \\
\epsilon_i^2 \sim U(50, 62.793) & \text{with probability } 0.23882.
\end{cases}
$$

For all the following Monte Carlo (MC) experiments which are set in the Heemeijer, Hommes, Sonnemans, and Tuinstra (2009), we sample the initial predictions from the (14) distribution. Therefore, the actual GA learning starts at the end of the second period. At the beginning of the second period, the agents randomly generate their sets of heuristics and pick one at random (with equal probabilities).

Notice that at the period 2, the rule (11) requires $p_1$ as the past price, but also $p_1 - p_0$ for the trend extrapolation. It is plausible that after the first period (with only one realized price!), the agents do not think that there is any actual trend in the data – yet. Moreover Diks and Makarewicz (2012) notice that the initial predictions are close neither to the fundamental price, nor to the focal point. Therefore, it seems that these two points were not used by the agents as any sort of a natural reference point. Since we do not see any other possible reference point (a natural estimator of $p_0$ given the information, which was provided to the subjects), we argue that the agents disregarded the trend in the second period and set $\beta = 0$. This is effectively the same as behaving as if $p_0 = p_1$.

### 4.3 Identification and testing

There are 13 groups in the original Heemeijer, Hommes, Sonnemans, and Tuinstra (2009) experiment, each with 6 subjects that interact for 50 periods. We decided to leave one of the positive feedback groups, in which the results were heavily influenced by one participant with ‘off’ behavior. This participant twice predicted suddenly very large prices and thus the whole group exhibited volatile behavior. This leaves us with six positive and six negative feedback groups. We used the full data from that sample, which leaves us with 3600 data points.

Still we observe 3 cases (in the positive feedback treatment groups), where a participant in some later period would suddenly predict a very different price than the current realizations. These three cases seemed to be an attempt to outsmart other participants: like the subjects
wanted to push the prices far from the current consensus, be the only one to predict that and hence collect a premium. These attempts where punished by the experiment design, since each subject has only a 1/6 power over the realized price. Subjects would never try that again. For the sake of presentation, we replace these unimportant outlying predictions with an average of the subject’s predictions from the period before and after.

The amount of data seems impressing, nevertheless it is to small for any meaningful estimation. First, the behavior within each group (apart from a very small number of initial predictions) was highly correlated. In a sense, we have only 12 multidimensional observations. Even under a more liberal approach, the data is in fact a set of 72 individual time paths. Furthermore, the time series in the case of the negative feedback are always very close to the fundamental price and offer little variability.

On the other hand, our model is highly non-linear. The issue is not in the number of parameters per se. The actual problem is that the distribution of heuristics is enormous. Moreover, the distribution of heuristics conditional on the heuristics in the previous period is non-trivial, non-smooth and cannot be described by a handful of parameters (like the distribution moments).

Therefore we do not attempt to estimate the parameters of our model, which includes the allowed interval for the trend extrapolation, [\(\beta_L, \beta_H\)], the number of chromosomes (independent heuristics) per agent, the length of each gene (or the bites assigned for each coefficient), the mutation and crossover probabilities and finally the \(\gamma\) sensitivity parameter. For all these parameters apart from the trend extrapolation interval, we use the specification described in the previous subsection. We did perform robustness checks and the model seems to work fine regardless of the specific choices, as long as they are ‘reasonable’ (for example, very high sensitivity parameter seemed to generate overly large oscillations for the positive feedback treatment).

Moreover testing of the model is tricky. We want to show that our model can replicate the stylized fact which were reported by Heemeijer, Hommes, Sonnemans, and Tuinstra (2009). In this working version of the paper we focus on graphical analysis. Even though the model is complex, one can easily sample its time paths. We compute 1000 such time paths and display the 95% confidence intervals (CI) for (1) the difference of the realized price from the fundamental price \(p_f = 60\) and (2) the standard deviation of the individual predictions at a given period. Recall that these two variables represent (1) the convergence or oscillations of the prices under the negative and positive feedback treatment respectively and (2) coordination between the agents. The simulations were computed in Ox (Doornik, 2007) and the code is available at request.

Each simulation from the MC sample is based on a different set of random numbers. This means that in each simulation the initial heuristics are different (also between the agents!) and mutate differently. On the other hand, recall that the price equations (1) for the negative feedback and (2) for the positive feedback are – conditional on the predictions – stochastic as well, since they are normally distributed. For the sake of stability of the results, in each
simulation we use the same errors as in the experiment.

In the following investigation we will first comment on the importance of an anchor. Hence we will discuss the possible specification for the allowed trend extrapolation $[\beta_L, \beta_H]$. In the next section we will also argue that the model performs well for the 2012 experiment reported in Bao, Heemeijer, Hommes, and Sonnemans (2012). At all these stages our argument is somewhat informal. We claim that a specification has a good fit if for both test variables, the simulated CI include the majority of the ones from the experiment and if the CI are not too wide (to exclude models with overwhelming variance).

4.4 The anchor

Recall that Heemeijer, Hommes, Sonnemans, and Tuinstra (2009) estimate for each subject the FOR with the anchor, given in (3). In fact their specification is not fully realistic, since the agents do not have to know the fundamental price. A more plausible version of the FOR would be therefore

$$p_{e,t}^f = \alpha_1 p_{t-1} + \alpha_2 p_{e,t-1}^f + \alpha_3 p_t + \beta(p_{t-1} - p_{t-2}) + \eta_t,$$

where $p_t^f = \frac{1}{t} \sum_{s=1}^t p_t$ is the average price so far. Indeed that (less the random term) was our initial choice for (10).

It turns out that restricting the weight of the anchor to zero does not change much the distribution of the model’s time paths. We report the MC studies for the model with and without the anchor, that is for the rules (15) and (10), on Figure 1 and Figure 2 respectively.

There is almost no difference between the two figures. In fact, we investigated the GA model based on the FOR with the anchor (15) and found the following. For the negative feedback, all the three variables: realized price, past expectation and anchor quickly converge to the fundamental price $p_f = 60$ and so the choice of the three corresponding coefficients is arbitrary: anchor neither helps nor disturbs. In the case of the positive feedback, the average price so far, despite the oscillations, is quite stable and slowly moves towards the fundamental. Given the presence of oscillations, this quickly discourages the agents to rely on the anchor.

This means that we do not need to rely on the notion of an anchor in order to explain the experiment. Notice that this result is intuitive, as discussed earlier. Since the subjects do not know upfront anything about the stability of the realized market variables, why should they even consider any type of an anchor?

4.5 Trend extrapolation

We will now focus on the degree to which the agents extrapolate the trend. As seen on the Figure 2 the trend extrapolation coefficient $\beta \in [-1, 1]$ (the baseline model) gives already
Negative feedback

(a) Distance from the fundamental

Positive feedback

(b) Distance from the fundamental

(c) Predictions standard deviation

(d) Predictions standard deviation

Figure 1: Monte Carlo for the GA model with the full First-Order-Rule (15) with $\beta \in [-1, 1]$: distance from the fundamental and coordination (standard deviation of the individual predictions) over time. Green line is the experimental median, black pluses are real observations; blue lines are the 95% confidence interval and red line is the median for the GA model. Left column displays the negative feedback, right the positive feedback.

Quite successful model. Under the negative feedback, the agents, after some initial time of chaos, converge to the fundamental price. Then the move of the prices is caused mostly by the random errors in the price equation. On the other hand, the positive feedback results in highly coordinated predictions, which still fluctuate around the fundamental price. Also the coordination seems to emerge faster for this case.

In both cases, the 95% CI of the model’s simulated time paths contains the median of the experimental price distance from the fundamental, as well as majority of the observations. Still the model has some problems in explaining the degree to which the real subjects have been overshooting the fundamental. To further investigate this issue, we investigate the model with $\beta \in [-0.5, 0.5]$ (conservative model) and $\beta \in [-1.5, 1.5]$ (explosive model). The results are presented on Figure 3 and Figure 4 respectively.

Figure 3 shows the importance of the trend extrapolation. In line with Anufriev, Hommes, and Philipse (2012), without this heuristic, the agents do not oscillate, but rather slowly converge towards the fundamental. This implies that initially the positive feedback encourages conservative behavior. Since the realized prices are close to the predicted ones, at first the agents are unwilling to change their forecasts much. What nevertheless happens is that they systematically undershoot the price, so after a small number of periods they lose patience and focus on extrapolating the trend.

This resembles a sad story where a donkey tries to catch a carrot. Since the carrot is tied
Negative feedback  
Positive feedback

(a) Distance from the fundamental  
(b) Distance from the fundamental

(c) Predictions standard deviation  
(d) Predictions standard deviation

Figure 2: Monte Carlo for the GA model with the First-Order-Rule (10) with $\beta \in [-1, 1]$: distance from the fundamental and coordination (standard deviation of the individual predictions) over time. Green line is the experimental median, black pluses are real observations; blue lines are the 95% confidence interval and red line is the median for the GA model. Left column displays the negative feedback, right the positive feedback.

to the wagon, the poor critter cannot grab it and, frustrated, runs faster and faster. In the experiment, similar frustration results in the agents overshooting the fundamental. On the other hand, the design of the experiment means that this cannot go forever and once the price is far above the fundamental, it looses its momentum. Agents pause once they notice that the price stops growing (or even decreases). This on its own causes the price to start falling. And now the agents systematically overshoot a constantly falling price, which completes one cycle of the oscillation.

Figure 4 shows that once we allow the agents in the GA model to use highly unstable heuristics (with the trend extrapolation in the $[-1.5, 1.5]$ interval), the resulting oscillations are enormous. The ‘loss of patience’ effect is so strong, that the agents can easily overshoot the fundamental price by more than 20% of the allowed maximum prediction (which were constrained to the $[0, 100]$ interval). This model contains almost all the experimental observations, but this comes at the price of an excess volatility of the results. The distribution of the simulated time paths of the model does not have to be so wide to explain the experiment.

This suggests that the best model would have the trend extrapolation in between the baseline and the explosive model. After some trial and error search, we conclude that a reasonable trade-off between a proper degree of oscillations and the variance of the result is given by $\beta \in [-1.1, 1.1]$. This means that the agents are allowed to extrapolate the trend in a non-stationary way, but only by a little. The MC results are presented on Figure 5. This
Negative feedback  Positive feedback

(a) Distance from the fundamental  (b) Distance from the fundamental

(c) Predictions standard deviation  (d) Predictions standard deviation

Figure 3: Monte Carlo for the GA model with the First-Order-Rule (10) with $\beta \in [-0.5, 0.5]$: distance from the fundamental and coordination (standard deviation of the individual predictions) over time. Green line is the experimental median, black pluses are real observations; blue lines are the 95% confidence interval and red line is the median for the GA model. Left column displays the negative feedback, right the positive feedback.

specification (which we call the extrapolative) has a fine fit to the experimental stylized facts.

The final issue is whether the trend extrapolation coefficient has to be taken from a symmetric interval. If that is indeed the case, people distinguish between contrarian and positive trend extrapolations. We check three possible cases: asymmetric baseline, asymmetric extrapolative and asymmetric explosive, which corresponds to the intervals $\beta \in [-0.5, 1]$ (Figure 6), $\beta \in [-0.5, 1.1]$ (Figure 7) and $\beta \in [-0.5, 1.5]$ (Figure 8) respectively. Notice that the simulated time paths are almost the same as in the case of the symmetric intervals. In the same spirit as with the anchor, we therefore claim that the assymetry of the trend extrapolation is not necessary to explain the experiment.

4.6 Expectation heuristics

In the previous subsection we have shown that our model with the trend extrapolation in either $[-1, 1]$ or $[-1.1, 1.1]$ closely resembles the experimental stylized facts (and the latter performs better). The next logical question is: what heuristics were used by the GA agents?

We report the distribution of the chosen coefficients on Figure 9 for $\beta \in [-1, 1]$ and Figure 10 for $\beta \in [-1.1, 1.1]$. Notice that at each period, each agent is endowed with 20 (possibly different) heuristics. We report only those which were used by the agents to generate their predictions.
The first striking result is the negative correlation between the price weight and the trend extrapolation. Even though the 95% CI’s contain the zero axis, the median is clearly negative regardless of the feedback.

In the positive feedback, the price weight $\alpha$ is quite high (with the median close to one), just as the trend extrapolation (the 95% CI does contain the zero axis, but the median is significantly positive). It follows that the agents tend to use the naive expectations, and more equally weighted adapted expectations require substantially stronger trend extrapolation. In either case the agents are not conservative about their expectations, but rather try to follow the prices as much as possible.

In the negative feedback the trend extrapolation is unimportant. Not only the 95% CI contain the zero axis, but also the median is very close to zero. This result is intuitive: in practice, under the negative feedback treatment, predicting anything but the fundamental is quickly punished, while the price process itself has low memory. The agents are thus uninterested in extrapolating small price changes and treat them as idiosyncratic random shocks around a persistently stable steady state. Since both the prices and the predictions quickly converge to this fundamental, the specific price weight can be anything.

Notice that these results are in line with the theoretical discussion by Anufriev, Hommes, and Philipse (2012). They also seem to be reasonable: persistent steady state (such as in the negative feedback) tames the agents, but once we allow the price process to have a long
memory (as is the case of the positive feedback), the agents will look closely at the trends. This policy means that they are unable to predict the switches of a cycle, but most of the time they are able to follow it closely. As a side note, we think that this resembles (in a very stylized way the general attitude of an (especially small) investor in the financial markets, which are a good example of a positive feedback between the expectations and the prices.

5 2012 experiment

Bao, Heemeijer, Hommes, and Sonnemans (2012) have repeated the Heemeijer, Hommes, Sonnemans, and Tuinstra (2009) experiment with a changing fundamental price. In this section we use their to show that our GA model is robust. We first discuss the experiment in detail, and hence report the results for the baseline and extrapolative models. We also argue that this experiment gives additional insight into how the GA agents – and so the actual experimental subjects – learn to forecast.

5.1 Experiment setup and results

In the experiment reported by Bao, Heemeijer, Hommes, and Sonnemans (2012), the subjects have to predict the price one period ahead (for 65 periods). The experiment has essentially the same structure as the Heemeijer, Hommes, Sonnemans, and Tuinstra (2009) experiment.
Negative feedback

Positive feedback

(a) Distance from the fundamental

(b) Distance from the fundamental

(c) Predictions standard deviation

(d) Predictions standard deviation

Figure 6: Monte Carlo for the GA model with the First-Order-Rule (10) with $\beta \in [-0.5, 1]$: distance from the fundamental and coordination (standard deviation of the individual predictions) over time. Green line is the experimental median, black pluses are real observations; blue lines are the 95% confidence interval and red line is the median for the GA model. Left column displays the negative feedback, right the positive feedback.

The two treatments are the negative and the positive feedback between the prices and the expectations. Either feedback is based on the (12) rule, where the prices are a linear function of the predictions plus a constant. In both cases, the linear coefficient corresponding to the average prediction is equal in absolute terms to $B = 20/21$, as was the case in the Heemeijer, Hommes, Sonnemans, and Tuinstra (2009) experiment.

The only difference is that, regardless of the feedback, the constant $A$ changes twice during the experiment. The result is that (both under the negative and the positive feedback treatment) the fundamental price changes from $p_f = 56$ to $p_f = 41$ starting from period $t = 21$ and then to $p_f = 62$ starting from period $t = 44$ until the last period $t = 65$.

The experimental dynamics are similar to those observed in the original 2009 experiment. In the case of the negative feedback, at the beginning of the experiment and then twice when the fundamental changes, there is a brief time of chaotic behavior, but the subjects quickly converge to the (new) fundamental. In the case of the positive feedback, the subjects quickly coordinate in terms of their predictions, but fail to converge to the fundamental. Instead, we observe a time path that would result in oscillations. Since there are two breaks in the fundamental price, the price series smoothly moves towards each fundamental and over- or undershoots it just before the next break.
5.2 Model fit

We investigate the same GA based model as in the previous section. The only difference is that we re-estimate the initial conditions for the new experiment sample, using exactly the same procedure as Diks and Makarewicz (2012) have for the 2009 experiment. Hence we simulate the GA model for 65 periods with the errors to the pricing equation from the 2012 experiment for each simulation. Even though the model was not adjusted in any other way, the simulations are in line with the experimental results. The initial predictions are sampled from a Winged Focal Point distribution

\[
p_{i,1}^{e} = \begin{cases} 
\varepsilon_{i}^{1} \sim U(16.406, 50) & \text{with probability 0.32296}, \\
50 & \text{with probability 0.35159}, \\
\varepsilon_{i}^{2} \sim U(50, 70.312) & \text{with probability 0.32296}.
\end{cases}
\]  

We report the same set of graphs as for the 2009 experiment, see Figure 11 for $\beta \in [-1, 1]$ stationary trend extrapolation and Figure 12 for $\beta \in [-1.1, 1.1]$ with higher possible trend extrapolation. The latter specification again performs slightly better, but the differences are not large. In either case, the negative feedback results in (1) the agents being quickly tamed to the fundamental and (2) brief times of chaos at the beginning of the experiment and then
Negative feedback

Positive feedback

(a) Distance from the fundamental

(b) Distance from the fundamental

(c) Predictions standard deviation

(d) Predictions standard deviation

Figure 8: Monte Carlo for the GA model with the First-Order-Rule (10) with $\beta \in [-0.5, 1.5]$: distance from the fundamental and coordination (standard deviation of the individual predictions) over time. Green line is the experimental median, black pluses are real observations; blue lines are the 95% confidence interval and red line is the median for the GA model. Left column displays the negative feedback, right the positive feedback.

during the fundamental price breaks. Under the positive feedback, (1) the GA agents are much more coordinated and the breaks have almost no effect on that, while (2) the prices tend to oscillate and smoothly, slowly move towards the new fundamentals.

We therefore conclude that our model has a satisfactory resemblance to the experimental data. It replicates all the major stylized facts that were observed in the two treatments, positive and negative feedback.

5.3 Expectation heuristics

In the previous section we have discussed the actual heuristics used by the GA agents under the 2009 experiment setup. We argued that under the negative feedback, trend extrapolation loses on importance and the price weight can be anything, since both the prices and the predictions converge to the fundamental price. On the other hand, under the positive feedback treatment the agents put a significant emphasis on both the price weight and the (positive) trend extrapolation.

Similar pattern is observed in the 2012 experiment. First, under the negative feedback, the trend is unimportant. The price weight is now higher (and almost never close to zero). This is intuitive: if there are price breaks, then there appears a difference between focusing on the past price and on the past prediction. The former policy allows for a faster response to the break in the fundamental, while relying more on the past prediction implies a more
Negative feedback

(a) $\alpha$ price weight
(b) $\alpha$ price weight
(c) $\beta$ trend extrapolation
(d) $\beta$ trend extrapolation
(e) Correlation of $\alpha$ and $\beta$
(f) Correlation of $\alpha$ and $\beta$

Figure 9: Monte Carlo for First-Order-Rule (15) with $\beta \in [-1,1]$: Chosen coefficients of the price weight and the trend extrapolation. Blue lines are 95% confidence interval, purple dotted are 90% CI and red line is the median for the GA model. Left column displays the negative feedback, right the positive feedback. Left column displays the negative feedback, right the positive feedback.

Conservative stance. Since the off-equilibrium behavior is quickly punished, this means that the price gains a higher weight, in particular early after a break in the fundamental (this is seen in the small oscillations in the lower bounds of the CI on the Figure 13a).

Under the positive feedback, the trend becomes much more important. This is clearer than in the simulations for the 2009 experiment, since now the trend extrapolation eventually has strictly positive 95% CI. Again, the price weight becomes very important in this market structure.

These results are in line with the simulations for the 2009 experiment. They actually emphasize the importance of the following the prices heuristic: either through the price weight or through the trend extrapolation. In reality, fundamental price breaks are very common: the economic environment evolves constantly, just as the firms themselves. On the other hand,
these changes may be difficult to detect instantly be the economic agents. Our model suggests that people will learn different forecasting heuristics depending on the type of feedback between their decisions (or predictions) and the aggregate market outcomes. Conservative heuristics are outperformed by rules which put greater emphasis on the past prices.

In this sense the economic agents always tend to be naive, regardless of the market structure. The difference is that if a market structure allows for a strong and positive relation between individual predictions and the realized conditions (aka positive feedback), then the agents adapt by learning to be naive in terms of trends. The result is that the agents endogenously adapt to the problem of stationarity. If the short-run dynamics are non-stationary (for instance, if a price can systematically increase/decrease for some time before it is pushed back to the equilibrium value), then the agents look not on the levels, but rather at the first differences. It means that they can exhibit quite a simple (naive) behavior while at the same
Negative feedback

Positive feedback

(a) Distance from the fundamental
(b) Distance from the fundamental
(c) Predictions standard deviation
(d) Predictions standard deviation

Figure 11: Monte Carlo for the GA model with the First-Order-Rule (10) with $\beta \in [-1, 1]$, 2012 experiment: distance from the fundamental and coordination (standard deviation of the individual predictions) over time. Green line is the experimental median, black pluses are real observations; blue lines are the 95% confidence interval and red line is the median for the GA model. Left column displays the negative feedback, right the positive feedback.

time being able to use it in a surprisingly sophisticated manner.

6 Conclusion

In this paper we discuss a model in which the agents use Genetic Algorithms to optimize a simple linear prediction rule, which is based on past price, individual past prediction and the observed trend. We argue that this model generalizes the Heuristic Switching Model (Anufriev, Hommes, and Philipse, 2012), allowing for individual learning and heterogeneity between agents. We show that the generated time paths of the model replicate the stylized facts, which were observed in the two important experiments conducted by Heemeijer, Hommes, Sonnemans, and Tuinstra (2009) and Bao, Heemeijer, Hommes, and Sonnemans (2012).

In both experiments, subjects were asked to predict prices one period ahead, and the realized price increased or decreased with the predictions under respectively positive and negative feedback treatments. The second experiment modified this setting by including two breaks in the fundamental price. The observed behavior was very similar in the two experiments. Under the negative feedback treatment, agents initially are uncoordinated and report predictions which greatly vary from period to period. After the initial periods of this chaotic behavior, the agents are tamed to converge to the fundamental price. If there is a
Negative feedback

(a) Distance from the fundamental

(b) Distance from the fundamental

(c) Predictions standard deviation

(d) Predictions standard deviation

Figure 12: Monte Carlo for the GA model with the First-Order-Rule (10) with $\beta \in [-1.1, 1.1]$, 2012 experiment: distance from the fundamental and coordination (standard deviation of the individual predictions) over time. Green line is the experimental median, black pluses are real observations; blue lines are the 95% confidence interval and red line is the median for the GA model. Left column displays the negative feedback, right the positive feedback.

break in the fundamental, similar chaotic transition paths are observed before the predictions are pushed to the new fundamental.

Under the positive feedback treatment, a reverse story holds. Subjects quickly coordinate between each other and this is not disrupted by the breaks in the fundamental price. On the other hands, their predictions oscillate around the fundamental price, thus the breaks in the fundamental result in smooth transition paths.

We use the first Heemeijer, Hommes, Sonnemans, and Tuinstra (2009) experiment and its simpler design to fine tune our GA based model. We later show that this model replicates the time paths of both that experiment and the experiment by Bao, Heemeijer, Hommes, and Sonnemans (2012). Regardless of the feedback and the setting, our model generates the same pattern of coordination and transition paths as in the two experiments.

The first contribution of our model is that it reconciles theoretical literature with a realistic psychological framework. It is therefore a successful explanation of how people learn to forecast.

We show that the HSM model (Anufriev, Hommes, and Philipse, 2012), despite its stylized form, can be motivated by a model which is realistic and has a proper psychological foundations. We replicate the major insights of the HSM model – that the trend extrapolation is important only for the positive feedback, in which it generates the oscillating behavior. Our results therefore validate the stylized investigation by Anufriev, Hommes, and Philipse (2012).
Figure 13: Monte Carlo for First-Order-Rule (15) with $\beta \in [-1,1]$ for the 2012 experiment: Chosen coefficients of the price weight and the trend extrapolation. Blue lines are 95% confidence interval, purple dotted are 90% CI and red line is the median for the GA model. Left column displays the negative feedback, right the positive feedback.

On the other hand, we show how the agents’ flexibility can be enhanced by individual learning and agents heterogeneity. In the HSM model, the agents were flexible to choose a particular heuristic that performed better given particular circumstances. Our GA based models retains this structure, while allowing agents to experiment with their heuristic sets: changing the coefficients and focusing on the successful ones. Therefore the heterogeneity of heuristics between the agents emerges endogenously.

The second contribution lies in the results of our model. We argue that – like the agents in the simulations – human subjects are likely to focus on naive expectations. If the market structure enforces a stable time path of the prices, people will look at the past price and disregard the trend. On the other hand, it may happen that the short-run dynamics of the market are more dynamic, in the sense that the prices can grow for some number of periods, before being pushed back to the fundamental. In this case, subjects focus on naive expectations.
Figure 14: Monte Carlo for First-Order-Rule (15) with $\beta \in [-1,1]$ for the 2012 experiment: Chosen coefficients of the price weight and the trend extrapolation. Blue lines are 95% confidence interval, purple dotted are 90% CI and red line is the median for the GA model. Left column displays the negative feedback, right the positive feedback.

in respect to the price changes, which is equivalent to a strong trend extrapolation. This is amplified if the fundamental price is changing over time, a realistic assumption about real markets.

This contrast the dominating framework of the perfectly rational expectations. Traditionally, the economists assumed that people use sophisticated concepts such as fundamental price or long run equilibrium. They disregarded the fact that using such complex notions may be unwise in market practice. The reasons include lack of data or time to use them properly on one hand, lack of proper education or analytical tools on the other. As we know from biology, the evolution selects inverted solutions. It prefers to rely on simple tools, which then can be used in a sophisticated manner if necessary. It is not surprising that we have found a similar pattern in the human learning to forecast.
References


