Risk of Rare disasters, Euler equation errors and the Performance of the C-CAPM

Olaf Posch\(^{(a,b)}\) and Andreas Schrimpf\(^{(b,c)}\)

\(^{(a)}\) Aarhus University, \(^{(b)}\) CREATES, \(^{(c)}\) Bank for International Settlements

January 2012

Abstract

This paper shows that the consumption-based asset pricing model (C-CAPM) with low probability events accounts for several empirical weaknesses of the canonical model. Rare events rationalize the large pricing errors (Euler equation errors) of the C-CAPM as typically found in the data. This is remarkable, since Lettau and Ludvigson (2009) show that most popular extensions of the C-CAPM cannot rationalize large pricing errors. We illustrate (analytically and in Monte Carlo simulations) that implausible estimates of risk aversion and time preference are not puzzling in this framework and emerge as a result of rational pricing errors. We obtain such pricing errors for both an endowment and – as a novelty – for a production economy.

\textit{JEL classification: }E21, G12, O41

\textit{Keywords: }Euler equation errors, Rare disasters, C-CAPM

\*The authors appreciate financial support from the Center for Research in Econometric Analysis of Time Series, CREATES, funded by The Danish National Research Foundation. The views expressed in this paper are those of the authors and do not necessarily reflect those of the Bank for International Settlements.
1 Introduction

It is a commonly held perception in the asset pricing literature that the workhorse of financial economics – the consumption-based asset pricing model (C-CAPM) – has failed on a grand scale.\(^1\) Most prominent is the failure of the model to account for the US equity premium with any plausible values of risk aversion, which has been labeled as ‘equity premium puzzle’ (Mehra and Prescott, 1985). These limitations of the standard C-CAPM generated a huge literature of extensions in order to achieve better empirical performance.\(^2\) An open question remains, however, why the leading asset-pricing models fail on one particular dimension: these models have severe trouble to explain why the standard C-CAPM generates large pricing errors (or Euler equation errors as in Lettau and Ludvigson, 2009, p.255),

“Unlike the equity premium puzzle, these large Euler equation errors cannot be resolved with high values of risk aversion. To explain why the standard model fails, we need to develop [...] models that can rationalize its large pricing errors.”

This paper makes three contributions. Firstly, we show that the Euler equation puzzle of Lettau and Ludvigson (2009) can be explained by the Barro-Rietz rare disaster hypothesis (Rietz, 1988; Barro, 2006, 2009), i.e., infrequent but sharp contractions (as during historical events such as the Great Depression or World War II). Hence, consumption-based models with low-probability events qualify as a class of models which is able to rationalize the large pricing errors of the canonical model. In fact, by including low-probability rare events the standard C-CAPM not only explains the equity premium (as shown in Barro, 2006), but also can generate large Euler equation errors as found in the empirical data.

Secondly, we derive analytical expressions for asset returns, the stochastic discount factor (SDF), and Euler equation errors, both in an endowment economy and – as a novelty – in a production economy with rare disasters. Our closed-form expressions shed light on the endogenous time-varying behavior of asset returns in a (neoclassical) production economy, and the effects of rare disasters on Euler equation errors in general equilibrium.

Thirdly, we present extensive Monte Carlo evidence to investigate the impact of rare consumption disasters on the plausibility of C-CAPM parameter estimates with particular focus on the parameter of relative risk aversion. We find that implausibly high empirical estimates for the risk aversion and for the subjective time preference parameters – as typically found in the empirical literature – are not puzzling in a world with rare disasters.

\(^1\)The consumption-based asset pricing models has its roots in the seminal articles by Rubinstein (1976), Lucas (1978), and Breeden (1979).

\(^2\)An non-exhaustive list of prominent modifications of the consumption-based model includes habit formation preferences (Campbell and Cochrane, 1999), long-run risk (Bansal and Yaron, 2004), heterogeneous agents and limited stock market participation (e.g. Guvenen, 2009).
Our analysis builds on the continuous-time formulation of dynamic stochastic general equilibrium (DGSE) models. In contrast to Lettau and Ludvigson (2009), this formulation allows us to obtain tractable analytical results and analytical expressions for asset prices and Euler equation errors for both the endowment economy and the neoclassical production economy in general equilibrium, which are particularly useful to obtain our main insights.

The motivation for considering rare events as a solution to asset pricing puzzles is intuitive and goes back at least to Rietz (1988). Recently, the hypothesis has received a lot of attention due to Barro (2006), which backs up the calibration of his model by historical estimates of consumption disasters for a broad set of countries over a very long period. In the Barro-Rietz framework, asset prices reflect risk premia for infrequent and severe disasters where the representative investor’s consumption drops sharply. If such rare consumption disasters are expected by investors ex-ante and thus reflected in their original consumption and investment decisions, but happen not to occur in sample, a high historical equity premium of the magnitude observed for the US data can materialize. Barro shows that a calibrated version of the standard C-CAPM with rare events is able to explain the level of the US equity premium at plausible parameters of risk aversion. In this paper we show that the rare events hypothesis helps explaining other dimensions beyond the equity premium puzzle, namely the pricing errors and implausible estimates for structural parameters.

We first focus on the Euler equation puzzle of Lettau and Ludvigson (2009). The authors convincingly show through extensive simulations that leading extensions of the consumption-based model – such as the long-run risk models (Bansal and Yaron, 2004), habit formation models (Campbell and Cochrane, 1999), and the limiting participation model (Guvenen, 2009) – are not able to explain why the standard model fails so dramatically and delivers large pricing errors.

In this paper we show that one simple extension of the consumption-based models, i.e., allowing for low-probability events, can rationalize large pricing errors of the C-CAPM with power utility. This results is illustrated both analytically and in Monte Carlo simulations. Thus C-CAPM models with low-probability events constitute a class that can rationalize the large pricing errors. Moreover, we also show the effects of rare events on the estimation of structural model parameters in finite samples. As our simulations show, implausible parameter estimates for risk aversion and time preference – as found empirically – are not puzzling in such framework. If asset prices reflect risk adjustments for rare events, but

---

3 This is related to the statement in Cochrane (2005, p.30) that the US economy and other countries with high historical equity premia may simply constitute very lucky cases of history.

4 As documented by a multitude of empirical studies, the standard C-CAPM leaves a substantial fraction of the average return unexplained when the model is asked to account for differences in average returns across different assets (see, e.g., empirical results in Hansen and Singleton, 1982; Lettau and Ludvigson, 2001).
these events do not occur in sample, the biases in estimates of structural parameters of the model can be substantial and emerge as a result of the pricing errors. This bias reflects the ‘small-sample bias’ of the Peso problem hypothesis (cf. Veronesi, 2004). To summarize, rare events not only rationalize the large Euler equation errors but can also explain why estimated parameters of risk aversion and subjective time preference obtained by many empirical studies are so grossly different from what is expected from economic theory.

The remainder of the paper is as follows. Section 2 provides a formal definition of the Euler equation errors, presents some empirical benchmark estimates and gives an intuitive preview of our main analytical and simulation-based results. Section 3 derives asset prices in an endowment economy and in a production economy with rare disasters. Section 4 derives analytical expressions for Euler equation errors in general equilibrium. Section 5 contains Monte-Carlo evidence which shows that rare events model work well in explaining several dimensions of the empirical weaknesses of the standard consumption-based model including large Euler equation errors. Section 6 concludes.

2 Euler equation errors

In this section we provide a brief discussion of the definition of Euler equation errors, the empirical facts typically encountered in the data as well as a brief preview of how rare disasters may help rationalizing the empirical puzzles.

2.1 Euler equation errors and their empirical counterparts

Consider the standard first-order condition implied by the canonical version of the C-CAPM with time-separable utility functions,

\[ u'(C_t) = e^{-\rho} E_t [u'(C_{t+1}) R_{t+1}] , \quad u' > 0, \ u'' < 0. \]  

(1)

The optimality condition (1) is referred to as the Euler equation. It implicitly determines the optimal path of per capita consumption \(C_t\), given gross returns \(R_{t+1}\) on the investor’s savings (or assets), and \(\rho > 0\) is a subjective time-discount rate. We define the stochastic discount factor (SDF) as the process \(m_s/m_t \equiv e^{-\rho(s-t)}u'(C_s)/u'(C_t)\) such that, for any security with price \(P_{i,t}\) and instantaneous payoff \(X_{i,s}\) at some future date \(s \geq t\), we have

\[ m_t P_{i,t} = E_t [m_s X_{i,s}] \implies 1 = E_t [(m_s/m_t) R_{i,s}] , \]  

(2)
where \( R_{i,s} \equiv X_{i,s}/P_{i,t} \) denotes the security’s return.\(^5\) In discrete-time models, the SDF at date \( s = t + 1 \) is usually defined as \( M_{t+1} \equiv m_{t+1}/m_{t} \). Hence, the Euler condition (2) can be used to discount expected payoffs on any asset to find their equilibrium prices: the agent is indifferent between investing into the various assets if (2) is satisfied. In this paper we study how the properties of the SDF explain pricing errors and how the SDF is determined by the general equilibrium of the economy.

Any deviations from (2) represent Euler equation errors,

\[
e_{R}^i \equiv E_t[(m_s/m_t)R_{i,s}] - 1, \quad e_{X}^i \equiv E_t[(m_s/m_t)(R_{i,s} - R_{b,s})],
\]

based on the gross return on any tradable asset, \( R_{i,s} \) or as a function of excess returns over a reference asset, \( R_{i,s} - R_{b,s} \), e.g., the return on a bond (Lettau and Ludvigson, 2009). In what follows, we refer to either \( e_{R}^i \) or \( e_{X}^i \) as the Euler equation error, whereas to their empirical counterparts \( \hat{e}_{R}^i \) and \( \hat{e}_{X}^i \) as the estimated Euler equation error for the \( i \)th asset. The latter are defined for specific utility functions, e.g., for power utility with risk aversion \( \gamma \),

\[
\hat{e}_{R}^i \equiv E_t[e^{-(s-t)\hat{\rho}(C_s/C_t)^{-\hat{\gamma}}}R_{i,s}] - 1, \quad \hat{e}_{X}^i \equiv E_t[e^{-(s-t)\hat{\rho}(C_s/C_t)^{-\hat{\gamma}}}(R_{i,s} - R_{b,s})],
\]

where \( \hat{\rho} \) and \( \hat{\gamma} \) denote the estimated parameters of time-preference and risk aversion. These estimates are usually obtained by the generalized method of moments (GMM) of Hansen (1982) by minimizing a quadratic function of the pricing errors. The fit of the model is often expressed by the root mean squared error (RMSE), which is a summary measure of the magnitude of the fitted Euler equation errors.

### 2.2 Euler equation errors and empirical puzzles

As mentioned in the introduction, it is a well-established fact that the standard C-CAPM with power utility is incapable of explaining cross-sectional variation in average asset returns. In other words, the model produces substantial pricing errors (Euler equation errors) when fitted to the data. In order to obtain some benchmark estimates to be used in our theoretical section, we estimate the parameters of a standard C-CAPM pricing kernel \( \beta(C_{t+1}/C_t)^{-\gamma} \) with US postwar data (1947:Q2-2009:Q3). A large literature has focused on the performance of the model to simultaneously explain the return on a broad stock market portfolio and the return on a riskless asset such as the US Treasury Bill (T-Bill). Moreover, it has also been shown that the C-CAPM fails to explain the return differences among stock portfolios sorted by size and book-to-market ratios (see, e.g. Lettau and Ludvigson, 2001).

\(^5\)An alternative set up is to consider assets which pay a random dividend continuously in time. One could then define the discrete time return to be the return on a portfolio which re-invests the continuously paid dividends (Eraker and Shaliastovich, 2008, p.525).
Following Lettau and Ludvigson (2009), we estimate the model for two sets of test assets: a market portfolio, $R_{m,t}$, and the T-Bill, $R_{b,t}$, and the two assets together with 6 size and book-to-market portfolios ($R_{FF,t}$, obtained from Kenneth French’s website).\footnote{The estimation is based on gross-returns deflated by the PCE deflator. The series of consumption is obtained from the NIPA tables (real consumption of nondurables and services, expressed in per capita terms). The estimation is conducted by standard GMM with the identity matrix as a weighting matrix.} We obtain the non-surprising result that the C-CAPM is flawed: the point estimates of the model’s parameters are implausible: $\hat{\beta} = 1.5$, $\hat{\gamma} = 123.0$ for $R_{m,t}$, $R_{b,t}$ ($\hat{\beta} = 1.4$, $\hat{\gamma} = 101.6$ when we include $R_{FF,t}$). These estimates are grossly inconsistent with economic theory. A subjective time discount factor greater than one implies that people would value future consumption more than current consumption, while risk aversion of the magnitude estimated is far higher than the microeconomic evidence on individuals’ behavior in risky gambles.

In addition, the Euler equation errors are economically large. Here, the RMSE amounts to 2.49% (3.05%) p.a. for the two-asset case (the larger cross-section), leaving a substantial fraction of the cross-sectional variation of average returns unexplained. It is puzzling to the econometrician why individuals seem to accept surprisingly large and persistent pricing errors. Economically, this result implies that consumers seem to accept a 2.5 dollar pricing error for each 100 dollar spent. As Lettau and Ludvigson (2009) further demonstrate, it is not possible to reduce the Euler equation error to smaller magnitudes (or even to zero) by choosing other parameter constellations. Additionally, they convincingly show that all the newly proposed theories of consumption-based asset pricing, as referred to before, are not capable of rationalizing the large pricing errors of the canonical model.

However, as we argue below, the consumption-based models coupled with stochastically occurring rare events of the Barro-Rietz type, which just happen not to occur within the sample, are able to rationalize the pricing errors of the C-CAPM and produce substantial biases in parameter estimates – akin to we observe in the empirical data.

### 2.3 Rare events and Euler equation errors - A preview

While the optimality conditions (1) and (2) are very general pricing formulas which must be fulfilled in most consumption-based models, the continuous-time formulation helps to make the effects on pricing errors more explicit as the distributional assumptions directly appear. Allowing for rare disasters, suppose that a continuous-time formulation of the C-CAPM implies for a dynamic model the following Euler equation (as shown below)

\[
\begin{align*}
    d\pi_t &= -(r^f_t - \rho)u'(C_t)dt - \pi_t u'(C_t)dB_t + (u'(C_t) - u'(C_{t-}))(dN_t - \lambda_t dt),
\end{align*}
\]

\[ (5) \]

where $r^f_t$ is the (shadow) risk-free rate, $\pi_t$ is a measure of risk, $B_t$ is a standard Brownian motion, and $N_t$ is a standard Poisson process capturing rare events occurring at the arrival
rate \( \lambda_t \), and \( C_{t-} \) is the left-limit, \( C_{t-} \equiv \lim_{s\to t} C_s \), for \( s < t \) (cf. Merton, 1971). We obtain the SDF from the Euler equation: use It\( \bar{\text{A}} \)'s formula to rewrite (5) for \( s \geq t \) as

\[
d \ln u'(C_t) = -(r^f_t - \rho + \frac{1}{2} \pi^2_t)dt - \pi_t dB_t + (\ln u'(C_{t-}) - \ln u'(C_t)) (dN_t - \lambda_t dt).
\]

Now integrate and equate discounted marginal utility in \( s \) and \( t \),

\[
e^{-\rho(s-t)} \frac{u'(C_s)}{u'(C_t)} \equiv \exp \left( -\int_t^s (r^f_v + \frac{1}{2} \pi^2_v) dv - \int_t^s \pi_v dB_v + \int_t^s \ln \left( \frac{u'(C_v)}{u'(C_{v-})} \right) (dN_v - \lambda_v dv) \right)
\]

defines the \textit{stochastic discount factor} (also known as pricing kernel or state-price density).

We are now prepared to make our two main points resolving the empirical puzzles. First, the presence of rare events can generate quite persistent pricing errors in finite samples. For illustration, consider a risk-free asset with gross return \( R_{f,s} \equiv \exp(\int_s^t r^f_v dv) \). From (3) and (6), we obtain the Euler equation error, \textit{conditioned} on no disasters as

\[
e^f_{R|N_s-N_t=0} = E_t [ (m_s/m_t) R_{f,s}|N_s - N_t = 0 ] - 1 = \exp \left( -\int_t^s \ln \left( \frac{u'(C_v)}{u'(C_{v-})} \right) \lambda_v dv \right) - 1 < 0 \text{ for } u'(C_v)/u'(C_{v-}) > 1,
\]

which is strictly negative for the case of disasters. Our result shows that the individuals accept persistent pricing errors for the events that happen not to occur in normal times.\(^7\) As shown in Barro (2006), rare disasters have been sufficiently frequent and large enough to explain the equity premium puzzle. As we show below analytically and by simulations, low-probability events are quantitatively important for Euler equation errors as well.

Our second main point is based on \textit{estimated} Euler equation errors and the associated parameter estimates. Provided we have economically substantial Euler equation errors, the standard GMM procedure (also the Empirical Likelihood methods) of obtaining parameter estimates will be extremely biased. As we show below, implausible high estimates of the parameter of relative risk aversion – of similar magnitudes as in empirical studies – will generally appear in samples where the sample frequency of rare disasters differs from the population value.

### 3 Asset pricing models with rare events

This section computes general equilibrium consumption and asset returns in endowment and production economies. These measures are used below to compute Euler equation errors.

\(^7\)This result refers to Hansen and Jagannathan (1991, p.250), who note that the sample volatility may be substantially different than the population volatility if consumers anticipate that extremely bad events can occur with small probability when such events do not occur in the sample.
3.1 Lucas’ endowment economy with rare disasters

Consider a fruit-tree economy and a riskless asset in normal times but with default risk (government bond) similar to Barro (2006) using the formulation as in Posch (2011). Similar papers consider time-varying disaster probabilities (Gabaix, 2008; Wachter, 2009), which will not substantially affect our result and thus is not the focus of our analysis.

3.1.1 Description of the economy

Technology. Consider an endowment economy (Lucas, 1978). Suppose production is entirely exogenous: no resources are utilized, and there is no possibility of affecting the output of any unit at any time, $Y_t = A_t$ where $A_t$ is the stochastic technology. Output is perishable. The law motion of $A_t$ will be taken to follow a Markov process,

$$dA_t = \bar{\mu} A_t dt + \bar{\sigma} A_t dB_t + (\exp(\bar{\nu}) - 1) A_{t-} dN_t,$$

(7)

where $B_t$ is a Brownian motion, and $N_t$ is a Poisson process with arrival rate $\lambda$. The jump size is proportional to its value an instant before the jump, $A_{t-}$, ensuring that $A_t$ does not jump negative. The notation $A_{t-}$ denotes the left-limit, $A_{t-} \equiv \lim_{s \to t} A_s$, for $s < t$.

Suppose ownership of fruit-trees with productivity $A_t$ is determined at each instant in a competitive stock market, and the production unit has outstanding one perfectly divisible equity share. A share entitles its owner to all of the unit’s instantaneous output in $t$. Shares are traded at a competitively determined price, $P_{i,t}$. Suppose that for the risky asset,

$$dP_{i,t} = \mu P_{i,t} dt + \sigma P_{i,t} dB_t + P_{i,t-} J_t dN_t,$$

(8)

and for a government bill with default risk

$$dP_{b,t} = P_{b,t} r dt + P_{b,t-} D_t dN_t, \quad \text{where} \quad D_t = \begin{cases} 0 & \text{with} \quad 1 - q \\ \exp(\kappa) - 1 & \text{with} \quad q \end{cases}$$

(9)

is the default risk in case of a disaster, and $q$ is the probability of default (cf. Barro, 2006).

Preferences. Consider an economy with a single consumer, interpreted as a representative “stand in” for a large number of identical consumers. The consumer maximizes discounted expected life-time utility

$$U_0 \equiv E \int_0^\infty e^{-\rho t} u(C_t) dt, \quad u' > 0, \quad u'' < 0.$$

(10)

Assuming no dividend payments, the budget constraint reads

$$dW_t = ((\mu - r)w_t W_t + rW_t - C_t) dt + w_t \sigma W_t dB_t + ((J_t - D_t)w_{t-} + D_t) W_{t-} dN_t,$$

(11)
where $W_t$ is real financial wealth and $w_t$ denote a consumer’s share holdings.

*Equilibrium properties.* In this economy, it is easy to determine equilibrium quantities of consumption and asset holdings. The economy is closed and all output will be consumed, $C_t = Y_t$, and all shares will be held by capital owners.

### 3.1.2 Obtaining the Euler equation

Suppose that the only asset is the market portfolio,

$$dp_M(t) = \mu_M p_M(t)dt + \sigma_M p_M(t)dB_t - \zeta_M(t) dN_t,$$

where $\zeta_M(t)$ is considered as an exogenous stochastic jump-size, defining

$$\mu_M \equiv (\mu - r)w_t + r, \quad \sigma_M \equiv w_t \sigma, \quad \zeta_M(t) \equiv (D_t - J_t)w_t - D_t.$$

(13)

The consumer obtains income and has to finance its consumption stream from wealth,

$$dW_t = (\mu_M W_t - C_t) dt + \sigma_M W_t dB_t - \zeta_M(t) W_t dN_t.$$

(14)

One can think of the original problem with budget constraint (11) as having been reduced to a simple Ramsey problem, in which we seek an optimal consumption rule given that income is generated by the uncertain yield of a (composite) asset (cf. Merton, 1973).  

Define the value function as

$$V(W_0) \equiv \max_{\{C_t\}_{t=0}^{\infty}} U_0, \quad s.t. \quad (14), \quad W_0 > 0.$$

(15)

Using the Bellman equation (see appendix), we obtain the first-order condition as

$$u'(C_t) = V_W(W_t),$$

(16)

for any $t \in [0, \infty)$, making consumption a function of the state variable $C_t = C(W_t)$.

It can be shown that the Euler equation is (cf. Posch, 2011)

$$du'(C_t) = ((\rho - \mu_M + \lambda)u'(C_t) - \sigma_M^2 W_t u''(C_t) C_W - E^\mathcal{S} [u'(C((1 - \zeta_M(t))W_t))(1 - \zeta_M(t))\lambda]) dt$$

$$- \pi_t u'(C_t) dB_t + (u'(C((1 - \zeta_M(t_-))W_{t_-})) - u'(C(W_{t_-}))) dN_t,$$

(17)

which implicitly determines the optimal consumption path, where the traditional market price of risk can be defined as $\pi_t \equiv -\sigma_M W_t u''(C_t) C_W / u'(C_t)$. We defined $C_W$ as the marginal propensity to consume out of wealth, i.e., the slope of the consumption function.

---

8A more comprehensive approach considers the portfolio problem which is available on request.
3.1.3 General equilibrium prices

This section shows that general equilibrium conditions pin down the prices in the economy. We use the stochastic differential for consumption implied by the Euler equation (17) and the market clearing condition $C_t = A_t$ together with the exogenous dividend process (7).

Proposition 3.1 (Asset pricing) In general equilibrium, market clearing implies

$$\mu_M - r = -\frac{u''(C_t)C_W W_t}{u'(C(W_t))} \sigma_M^2 - \frac{u'(e^\rho C(W_t))}{u'(C(W_t))} \left( (1 - e^\delta)q - \zeta_M \right) \lambda$$

(18)

$$\sigma_M = \bar{\sigma} C_t / (C_W W_t)$$

(19)

$$r = \rho - \frac{u''(C_t)C_t}{u'(C_t)} \bar{\mu} - \frac{1}{2} \frac{u''(C_t)C_t^2}{u'(C_t)} \sigma^2 + \lambda - (1 - (1 - e^\delta)q) \frac{u'(e^\rho C_t)}{u'(C_t)} \lambda.$$  

(20)

as well as implicitly the portfolio jump-size

$$C((1 - \zeta_M(t))W_t) = \exp(\bar{\nu})C(W_t).$$

(21)

Proof. cf. appendix ■

As a result, the higher the subjective rate of time preference, $\rho$, the higher is the general equilibrium interest rate to induce individuals to defer consumption (cf. Breeden, 1986). For convex marginal utility (decreasing absolute risk aversion), $u''(c) > 0$, a lower conditional variance of dividend growth, $\sigma^2$, and a higher conditional mean of dividend growth, $\bar{\mu}$, and a higher default probability, $q$, decrease the bond price and increases the interest rate.

3.1.4 Explicit solutions

As shown in Merton (1971), the standard dynamic consumption and portfolio selection problem has explicit solutions where consumption is a linear function of wealth. For later references, we provide the solution for constant relative risk aversion (CRRA).

Proposition 3.2 (CRRA preferences) If utility exhibits constant relative risk aversion, i.e., $-u''(C_t)C_t / u'(C_t) = \gamma$, then the optimal consumption function is proportional to wealth,

$$C_t = C(W_t) = b W_t, \text{ where } b \equiv (\rho + \lambda - (1 - \gamma)\mu_M - (1 - \zeta_M)^{1-\gamma} \lambda + (1 - \gamma)\gamma \frac{1}{2} \sigma_M^2) / \gamma.$$  

Proof. see Posch (2011) ■

Corollary 3.3 The implicit risk premium is

$$RP = \gamma \sigma^2 + e^{-\gamma \bar{\rho}} (1 - e^\rho) \lambda.$$  

(22)

whereas the disaster risk of the market premium in (18) is $e^{-\gamma \bar{\rho}} (1 - e^\rho - (1 - e^\delta)q) \lambda$.  

9
### 3.1.5 Stochastic discount factor

This section computes the stochastic discount factor (SDF). We obtain the SDF along the lines of (5) to (6) from the Euler equation (17), which in general equilibrium is

\[
\begin{align*}
du'(C_t) &= (\rho - r)u'(C_t)dt + (1 - e^\kappa)u'(e^\kappa C_t)q\lambda dt - (u'(e^\kappa C_t) - u'(C_t))\lambda dt \\
&= -\pi_t u'(C_t)dB_t + (u'(e^\kappa C_t) - u'(C_t))dN_t, 
\end{align*}
\]

where the deterministic term consists firstly of the difference between the subjective rate of time preference and the riskless rate, secondly a term which transforms this rate into the certainty equivalent rate of return (shadow risk-free rate), and thirdly the compensation of time preference and the riskless rate, secondly a term which transforms this rate into a martingale. For \( s \geq t \), we obtain

\[
m_s/m_t \equiv \exp \left( -\int_t^s \left( \rho - \frac{u''(C_v)C_v}{u'(C_v)} \mu - \frac{1}{2} \frac{u''''(C_v)C_v^2}{u'(C_v)^2} \sigma^2 + \frac{1}{2} \pi^2 \right) dv \right) \\
\times \exp \left( -\int_t^s \pi_t dB_v + \int_t^s (\ln u'(e^\kappa C_t) - \ln u'(C_t)) dN_v \right),
\]

as the stochastic discount factor, which can be used to price any asset in this economy. For the case of CRRA preferences, (23) simplifies to

\[
m_s/m_t = \exp \left( -(r - e^{-\gamma \rho})(1 - e^\kappa)q\lambda + \frac{1}{2}(\gamma \sigma)^2 + (e^{-\gamma \sigma} - 1)\lambda(s - t) \right) \\
\times \exp \left( -\gamma \hat{\sigma}(B_s - B_t) - \gamma \hat{\nu}(N_s - N_t) \right),
\]

where

\[
r = \rho + \gamma \hat{\mu} - \frac{\gamma}{2} (1 + \gamma) \sigma^2 + \lambda - (1 - (1 - e^\kappa)q) e^{-\gamma \sigma} \lambda
\]

is the (shadow) risk-free rate, or the rate of return of any zero-supply riskless security. For \( \gamma < 0 \) and \( \kappa < 0 \), the presence of rare events increases the risk-free rate of return.

### 3.1.6 General equilibrium consumption growth rates and asset returns

This section derives consumption growth rates and equilibrium asset returns for various financial claims. These measures are important for computing Euler equation errors.

**Consumption.** Consumption growth rates are exogenous in the endowment economy. Thus, consumption growth rates can be obtained from the dividend process (7),

\[
\ln(C_s/C_t) = \ln(A_s/A_t) = (\hat{\mu} - \frac{1}{2} \hat{\sigma}^2)(s - t) + \hat{\sigma}(B_s - B_t) + \hat{\nu}(N_s - N_t).
\]

**Risky asset.** Consider a claim which pays a dividend \( X_{i,t+1} = A_{t+1} \), i.e., an instantaneous return in period \( s = t + 1 \). Using the pricing kernel (24) together with (2) implies

\[
R_{c,t+1} = \exp \left( \rho + \gamma \hat{\mu} - \frac{\gamma}{2} (1 + \gamma) \sigma^2 - \frac{1}{2} (1 - \gamma)^2 \sigma^2 - (e^{(1-\gamma)\rho} - 1)\lambda \right) \\
\times \exp \left( \hat{\sigma}(B_{t+1} - B_t) + \hat{\nu}(N_{t+1} - N_t) \right).
\]

10
Riskless asset. From (24) and (2), the equilibrium asset return of any riskless security is

\[ R_{f,t+1} = \exp \left( r - e^{-\gamma} (1 - e^\gamma) q \lambda \right). \]  

(28)

whereas \( R_{b,t+1} = e^{r+\lambda_t dN_t} \) is the equilibrium return for any riskless asset which is subject to default risk, e.g., issued exogenously by the government.

3.2 A production economy with rare events

This section obtains the pricing kernel for an economy where Euler equation errors arise as a result of rare technological improvements in a production economy (cf. Wälde, 2005). As before, an extensive discussion of the model and its solution is in Posch (2011).

3.2.1 Description of the economy

Technology. At any time, the economy has some amounts of capital, labor, and knowledge, and these are combined to produce output. The production function is a constant return to scale technology \( Y_t = A_t F(K_t, L) \), where \( K_t \) is the aggregate capital stock, \( L \) is the constant population size, and \( A_t \) is the stock of knowledge or total factor productivity (TFP), which is driven by a standard Brownian motion \( B_t \) and a Poisson process \( \tilde{N}_t \) with arrival rate \( \tilde{\lambda} \),

\[ dA_t = \tilde{\mu} A_t dt + \tilde{\sigma} A_t dB_t + (\exp(\tilde{\nu}) - 1) A_t - d\tilde{N}_t. \]  

(29)

We introduce jumps in TFP as there is empirical evidence of Poisson jumps in output growth rates which, however, may not necessarily reflect consumption disasters (Posch, 2009).

The capital stock increases if gross investment exceeds stochastic capital depreciation,

\[ dK_t = (I_t - \delta K_t) dt + \sigma K_t dZ_t + (\exp(\nu) - 1) K_t - dN_t, \]  

(30)

in which \( Z_t \) is a standard Brownian motion (uncorrelated with \( B_t \)), and \( N_t \) is a Poisson process with arrival rate \( \lambda \). The jump size in the capital stock is proportional and has a degenerated distribution.\(^9\) Note that only for \( \sigma = \nu = 0 \), the capital stock (physical asset) is instantaneously riskless (cf. Merton, 1975).

Preferences. Consider an economy with a single consumer, interpreted as a representative “stand in” for a large number of identical consumers. The consumer maximizes expected life-time utility

\[ U_0 \equiv E_0 \int_0^\infty e^{-\rho t} u(C_t) dt, \quad u' > 0, \quad u'' < 0 \]  

(31)

\(^9\)As in Cox, Ingersoll and Ross (1985, p.366), individuals can invest in physical production indirectly through firms or directly, in effect creating their own firms. There is a market for instantaneous borrowing and lending at the interest rate \( r_t = Y_t, \) which is determined as part of the competitive equilibrium of the economy. There are markets for contingent claims which are all zero-supply assets in equilibrium.
subject to
\[ dW_t = ((r_t - \delta)W_t + w_t W_t - C_t)dt + \sigma W_t dZ_t + J_t W_t dN_t. \]  
(32)

\[ W_t \equiv K_t / L \] denotes individual wealth, \( r_t \) is the rental rate of capital, and \( w_t W_t \) is labor income.

Equilibrium properties. In equilibrium, factors of production are rewarded with value marginal products, \( r_t = Y_K \) and \( w_t W_t = Y_L \). The goods market clearing condition demands
\[ Y_t = C_t + I_t. \]  
(33)

Solving the model requires the aggregate capital accumulation constraint (30), the goods market equilibrium (33), equilibrium factor rewards of perfectly competitive firms, and the first-order condition for consumption. It is a system of stochastic differential equations determining, given initial conditions, the paths of \( K_t, Y_t, r_t, w_t W_t \) and \( C_t \), respectively.

3.2.2 Obtaining the Euler equation

Define the value function as
\[ V(W_0, A_0) = \max_{\{C_t\}_{t=0}^{\infty}} U_0 \text{ s.t. } (32) \text{ and } (29), \]  
(34)
denoting the present value of expected utility along the optimal program. Using the Bellman equation similar to the endowment economy, we obtain the first-order condition as
\[ u'(C_t) = V_W(W_t, A_t), \]  
(35)
for any \( t \in [0, \infty) \), making consumption a function of the state variables \( C_t = C(W_t, A_t) \).

It can be shown that the Euler equation is (cf. appendix)
\[ du'(C_t) = (\rho - (r_t - \delta) + \lambda + \bar{\lambda})u'(C_t)dt - u'(C(e^{r_t} W_t, A_t))e^{r_t} \lambda dt - u'(C(W_t, e^{r_t} A_t))\bar{\lambda} \]
\[ -\sigma^2 u''(C_t)C_t W_t dt + u''(C_t)(C_t A_t \tilde{\sigma} dB_t + C_t W_t \tilde{\sigma} dZ_t) \]
\[ + [u'(C(W_t, e^{r_t} A_t)) - u'(C(W_t, A_t))]d\bar{N}_t \]
\[ + [u'(C(e^{r_t} W_t, A_t)) - u'(C(W_t, A_t))]dN_t, \]  
(36)
which implicitly determines the optimal consumption path. Comparing to the Euler equation in the endowment economy (17), the stochastic discount factor implied by (36) now has richer dynamics with time-varying interest rates and two sources of low-probability events.

3.2.3 General equilibrium prices

Note that the physical asset is the only asset that is held in equilibrium, henceforth the market portfolio. Since all other assets are zero-supply assets, we can price any financial claim as if they were traded assets using the stochastic discount factor.
3.2.4 Explicit solutions

A convenient way to describe the behavior of the economy is in terms of the evolution of $C_t$, $A_t$, and $W_t$. Similar to the endowment economy there are explicit solutions available, due to the non-linearities only for specific parameter restrictions. Below we use two known restrictions where the policy function $C_t = C(A_t, W_t)$ (or consumption function) is available, and many economic variables can be solved in closed form.

**Proposition 3.4 (linear-policy-function)** If the production function is Cobb-Douglas, $Y_t = A_t K_t^\alpha L^{1-\alpha}$, utility exhibits constant relative risk aversion, i.e., $-u''(C_t)C_t/u'(C_t) = \gamma$, and $\alpha = \gamma$, then optimal consumption is linear in wealth.

$$\alpha = \gamma \Rightarrow C_t = C(W_t) = \phi W_t$$

where

$$\phi \equiv (\rho - (e^{(1-\gamma)\nu} - 1)\lambda + (1 - \gamma)\delta)/\gamma + 1/2(1 - \gamma)\sigma^2 \tag{37}$$

**Proof.** see appendix

**Corollary 3.5** The implicit risk premium is

$$RP|_{\alpha = \gamma} = \gamma \sigma^2 + e^{-\gamma \nu}(1 - e^\nu)\lambda. \tag{38}$$

**Proposition 3.6 (constant-saving-function)** If the production function is Cobb-Douglas, $Y_t = A_t K_t^\alpha L^{1-\alpha}$, utility exhibits constant relative risk aversion, i.e., $-u''(C_t)C_t/u'(C_t) = \gamma$, and the subjective discount factor is

$$\bar{\rho} \equiv (e^{-\theta \nu} - 1)\tilde{\lambda} + (e^{(1-\alpha \gamma)\nu} - 1)\lambda - \gamma \tilde{\mu} + 1/2(\gamma(1 + \gamma)\tilde{\sigma}^2 - \alpha \gamma(1 - \alpha \gamma)\sigma^2) - (1 - \alpha \gamma)\delta,$$

then optimal consumption is proportional to current income (i.e., non-linear in wealth).

$$\rho = \bar{\rho} \Rightarrow C_t = C(W_t, A_t) = (1 - s)A_t W_t^\alpha, \quad \gamma > 1, \quad \text{where} \quad s \equiv 1/\gamma \tag{39}$$

**Proof.** see appendix

**Corollary 3.7** The implicit risk premium is

$$RP|_{\rho = \bar{\rho}} = \alpha \gamma \sigma^2 + e^{-\alpha \gamma \nu}(1 - e^\nu)\lambda. \tag{40}$$

It is interesting to note that the market premium (or implicit risk premium) does not reward the risk associated with a stochastic TFP process. The intuitive reason is that at the aggregate level all contingent claims are in zero supply. Hence, the only asset that affects the intertemporal investment opportunities of the market is the physical asset.
3.2.5 Stochastic discount factor

Similar to the endowment economy, the SDF is obtained along the lines of (5) to (6) from the Euler equation (36). For \(s \geq t\), we obtain

\[
m_s/m_t = \exp \left( - \int_t^s \left( r_l - \delta - \lambda - \tilde{\lambda} + \frac{u'(C(e^\rho W_t, A_t))}{u'(C(W_t, A_t))} \rho + \frac{u'(C(W_t, e^\rho A_t))}{u'(C(W_t, A_t))} \hat{\rho} \right) dt \right. \\
+ \frac{u''(C_l)C_W W_t}{u'(C_l)} \sigma^2 dt - \frac{1}{2} \int_t^s \frac{(u''(C_l))^2}{(u'(C_l))^2} ((C_A A_t \sigma)^2 + (C_W W_t \sigma)^2) dt \\
+ \int_t^s \frac{u''(C_l)}{u'(C_l)} (C_A A_t \sigma dB_t + C_W W_t \sigma dZ_t) \\
+ \int_t^s \ln \left( \frac{u'(C(e^\rho W_{t-}, A_{t-}))}{u'(C(W_{t-}, A_{t-}))} \right) dN_t + \int_t^s \ln \left( \frac{u'(C(W_{t-}, e^\rho A_{t-}))}{u'(C(W_{t-}, A_{t-}))} \right) d\bar{N}_t.
\]

as the stochastic discount factor, which can be used to price any asset in this economy. For the case of CRRA preferences we obtain for our closed-form solutions,

\[
m_s/m_t \big|_{\alpha=\gamma} = \exp \left( - \int_t^s (r_l - \delta) dt + [\lambda - e^{(1-\gamma)\nu} \lambda + \gamma \sigma^2 - \frac{1}{2} (\gamma \sigma)^2] (s - t) \right) \\
\times \exp(-\gamma \sigma (Z_s - Z_t) - \gamma \nu (N_s - N_t)), \tag{41}
\]

\[
m_s/m_t \big|_{\rho=\bar{\rho}} = \exp \left( - \int_t^s (r_l - \delta) dt + [(1 - e^{(1-\sigma)\nu}) \lambda + (1 - e^{-\gamma \bar{\rho}}) \bar{\lambda}] (s - t) \right) \\
\times \exp \left( (\gamma \alpha \sigma^2 - \frac{1}{2} (\gamma \sigma)^2 - \frac{1}{2} (\alpha \gamma \sigma)^2) (s - t) - \gamma \bar{\sigma} (B_s - B_t) \right) \\
\times \exp \left( -\alpha \gamma \sigma (Z_s - Z_t) - \alpha \gamma \nu (N_s - N_t) - \gamma \bar{\nu} (\bar{N}_s - \bar{N}_t) \right). \tag{42}
\]

In the general case, the implicit risk premium will be time-varying and asymmetric over the business cycle (Posch, 2011). This also implies that the SDF is no longer available in closed form, which complicates any Monte Carlo study without generating new insights.

3.2.6 General equilibrium consumption growth rates and asset returns

This section derives consumption growth rates and equilibrium asset returns for various financial claims. We focus on two parametric restrictions under which consumption, the pricing kernel, as well as asset returns on various claims are available in closed form. This strategy greatly simplifies our effort later to compute Euler equation errors.

Consumption. In contrast to the endowment economy, the (neoclassical) production economy introduces transitional dynamics, which imply that consumption growth rates, at least transititionally, are endogenous and will depend on the specific solution.

Given the closed-form solutions as of Propositions 3.4 and 3.6, it is straightforward to
obtain consumption growth rates (cf. appendix),
\[
\ln(C_s/C_t)|_{\alpha=\gamma} = \frac{1}{\alpha} \int_t^s r_v dv - (\phi + \delta + \frac{1}{2} \sigma^2)(s - t) + \sigma(Z_s - Z_t) + \nu(N_s - N_t). \tag{43}
\]
\[
\ln(C_s/C_t)|_{\rho=\bar{\rho}} = \frac{1}{\gamma} \int_t^s r_v dv + (\bar{\mu} - \frac{1}{2} \sigma^2 - \alpha \delta - \frac{1}{2} \alpha \sigma^2)(s - t) + \bar{\sigma}(B_s - B_t) + \alpha \sigma(Z_s - Z_t) + \alpha \nu(N_s - N_t) + \bar{\nu}(\bar{N}_s - \bar{N}_t). \tag{44}
\]

*Risky assets.* Asset prices in this economy are driven by the rental rate of physical capital. For our parametric restriction, we obtain these capital rewards in closed-form. As shown in the appendix, the dynamics for the rental rate of capital are given by
\[
dr_t = c_1(c_2 - r_t)rt dt + (\alpha - 1)\sigma r_t dZ_t + \bar{\sigma} r_t dB_t + (\exp((\alpha - 1)\nu) - 1)r_t dN_t + \exp(\nu) - 1)1_{t-} d\bar{N}_t. \tag{45}
\]
This result is remarkable as it implies a specific structure for a tendency of this rate towards some equilibrium value $c_2$ at the speed of reversion of $c_1$ (cf. Posch, 2009),
\[
c_1|_{\alpha=\gamma} \equiv \frac{1 - \alpha}{\alpha}, \quad c_2|_{\alpha=\gamma} \equiv \alpha \phi + \alpha \delta - \frac{1}{2} \alpha(\alpha - 2)\sigma^2 - \frac{\alpha}{\alpha - 1} \bar{\mu},
\]
\[
c_1|_{\rho=\bar{\rho}} \equiv \frac{1 - \alpha}{\alpha}, \quad c_2|_{\rho=\bar{\rho}} \equiv \alpha \gamma \delta - \frac{1}{2} \alpha \gamma(\alpha - 2)\sigma^2 - \frac{\alpha}{\alpha - 1} \bar{\mu}.
\]

Consider a risky bond that pays continuously at the rate, $r_t$. Investing into this asset gives the random dividend process $X_{b,t+1} = e^{\int_t^{t+1} r_v ds}$. Using the pricing kernels (41) or (42) together with (2) implies
\[
R_{b,t+1}|_{\alpha=\gamma} = \exp \left( \int_t^{t+1} (r_s - \delta) ds - (\gamma \sigma^2 + e^{-\gamma \nu}(1 - e^{\nu}) \lambda) \right), \tag{46}
\]
\[
R_{b,t+1}|_{\rho=\bar{\rho}} = \exp \left( \int_t^{t+1} (r_s - \delta) ds - (\gamma \sigma^2 + e^{-\alpha \gamma \nu}(1 - e^{\nu}) \lambda) \right). \tag{47}
\]

Consider a claim on output which pays $X_{c,t+1} = A_{t+1}K_{t+1}^\alpha$, i.e., an instantaneous return in period $s = t + 1$. As shown in the appendix, for the case of $\alpha = \gamma$, we obtain a closed-form expression for the asset’s return,
\[
R_{c,t+1}|_{\alpha=\gamma} = \exp \left( \int_t^{t+1} (r_s - \delta) ds - \frac{1}{2} \bar{\sigma}^2 - \lambda + e^{(1-\gamma)\nu} \lambda - \gamma \sigma^2 + \frac{1}{2} \gamma(\gamma \sigma^2 - (e^\delta - 1)\lambda) \right) \times \exp \left( \bar{\sigma}(B_{t+1} - B_t) + \alpha \sigma(Z_{t+1} - Z_t) + \alpha \nu(N_{t+1} - N_t) + \bar{\nu}(\bar{N}_{t+1} - \bar{N}_t) \right). \tag{48}
\]

Similarly, consider a claim on capital, which pays $X_{c,t+1} = K_{t+1}^{\alpha \gamma}$ at date $s = t + 1$. This particular function has been chosen in order to get a closed-form expression in the case where $\rho = \bar{\rho}$, which turns out to be
\[
R_{c,t+1}|_{\rho=\bar{\rho}} = \exp \left( \int_t^{t+1} (r_s - \delta) ds - \lambda + e^{(1-\alpha \gamma)\nu} \lambda - \gamma \sigma^2 + \frac{1}{2} (\alpha \gamma \sigma^2) \right) \times \exp \left( \alpha \gamma \sigma(Z_{t+1} - Z_t) + \alpha \gamma \nu(N_{t+1} - N_t) \right). \tag{49}
\]
In what follows, the equilibrium consumption growth rates and asset returns are employed to compute Euler equation errors in the production economy.

4 Euler equation errors in general equilibrium

This section computes the Euler equation errors in endowment and production economies. It shows that the Barro-Rietz ‘rare disaster hypothesis’ generates large pricing errors. It also shows that the standard approach of estimating the parameters of relative risk aversion and time preference is severely affected in samples where the rare events anticipated by consumers do not occur (cf. Hansen and Jagannathan, 1991, p.250).

4.1 Euler equation errors in finite samples

We illustrate the approach of computing Euler equation errors using the endowment economy. A similar approach is applicable to production economies using our closed-form solutions.

Consider two assets, i.e., the government bill, \( R_{b,t+1} \), and the claim on dividends, \( R_{c,t+1} \).

From the definition of Euler equation errors (3), for any asset \( i \) and CRRA preferences

\[
e^i_C = E_t \left[ e^{-\gamma \tilde{\rho} + \frac{1}{2} \gamma \sigma^2 - \gamma \tilde{\sigma} (B_{t+1} - B_t) - \gamma \tilde{\nu} (N_{t+1} - N_t)} R_{i,t+1} \right] - 1,
\]

where we inserted the SDF from (24) and the (shadow) risk-free rate (25). Note that Euler equation errors based on excess returns can be obtained from \( e^i_X = e^i_R - e^i_B \) for any asset \( i \).

Risky asset. Inserting the one-period equilibrium return on the risky asset gives

\[
e^r_R = E_t \left[ e^{-\frac{1}{2} (1-\gamma)^2 \tilde{\sigma}^2 - (e^{(1-\gamma)\tilde{\nu}} - 1) \lambda + (1-\gamma) \tilde{\sigma} (B_{t+1} - B_t) + (1-\gamma) \tilde{\nu} (N_{t+1} - N_t)} - 1 \right].
\]

Conditional on no disasters, on average we can rationalize Euler equation errors

\[
e^r_{R|N_{t+1}-N_t=0} = E_t \left[ e^{-\frac{1}{2} (1-\gamma)^2 \tilde{\sigma}^2 - (e^{(1-\gamma)\tilde{\nu}} - 1) \lambda + (1-\gamma) \tilde{\sigma} (B_{t+1} - B_t)} - 1 \right] = \exp \left( -e^{(1-\gamma)\tilde{\nu}} - 1 \right) - 1.
\]

For Barro’s calibration of \( \lambda = 0.017, \tilde{\nu} = -0.4 \), the absolute Euler equation error is about 3.9% for \( \gamma = 4 \) and further increases with risk aversion. Hence, we argue that the Euler equation error can be large in finite samples. We cannot rule out that we simply measure the disaster risk, as the probability of no disaster occurring in a randomly selected sample of \( T = 50 \) years is \( p(N_{t+T} - N_t = 0) = e^{-\lambda T} = 43\% \).

Riskless asset. Inserting the one-period equilibrium returns on the government bill and the truly riskless asset (\( q = 0 \)), we obtain Euler equation errors

\[
e^b_R = E_t \left[ e^{-\tilde{\nu} (1-e^q) \tilde{\sigma} - (e^{(1-\gamma)\tilde{\nu}} - 1) \lambda - \frac{1}{2} (\gamma \sigma^2 - \gamma \tilde{\sigma} (B_{t+1} - B_t) - \gamma \tilde{\nu} (N_{t+1} - N_t)) + \int_0^{1+1} \ln(1+D_s) dN_s} - 1, \right.
\]

\[
e^f_R = E_t \left[ e^{-\frac{1}{2} (\gamma \sigma^2 + (e^{(1-\gamma)\tilde{\nu}} - 1) \lambda - \gamma \tilde{\sigma} (B_{t+1} - B_t) - \gamma \tilde{\nu} (N_{t+1} - N_t))} - 1. \right.
\]
Conditional on no disasters, on average we can rationalize Euler equation errors

\[ e^b_{R|N_{t+1}-N_t=0} = \exp\left(-e^{-\rho} - 1\right) + e^{-\gamma}(1-e^{\kappa})q\lambda - 1, \]
\[ e^f_{R|N_{t+1}-N_t=0} = \exp\left(-e^{-\rho} - 1\right) - 1. \]

(52)

(53)

Obviously, the presence of default risk reduces the Euler equation error in quiet times for that particular asset. Neither the disaster nor the default occurred in the sample.

For the production economy, we would obtain for the risky bond

\[ e^b_{R|N_{t+1}-N_t=0|\alpha=\gamma} = \exp\left((1-e^{-\rho})\lambda\right) - 1, \]
\[ e^b_{R|N_{t+1}-N_t=0|\bar{\rho}=\rho} = \exp\left((1-e^{-\alpha\gamma})\lambda\right) - 1. \]

(54)

(55)

The claims on capital and output do not generate persistent pricing errors as long as there are no rare events in total factor productivity (as in Wälde, 2005). In such cases,

\[ e^c_{R|\bar{N}_{t+1}-\bar{N}_t=0|\alpha=\gamma} = \exp\left((1-e^{\gamma})\bar{\lambda}\right) - 1, \]
\[ e^c_{R|\bar{N}_{t+1}-\bar{N}_t=0|\bar{\rho}=\bar{\rho}} = \exp\left((1-e^{-\gamma\bar{\rho}})\bar{\lambda}\right) - 1, \]

Two remarks are noteworthy. First, the claims were chosen in order to obtain analytical expressions for Euler equation errors. In general – conditional on no disasters – claims on assets or technology may also produce substantial pricing errors. Second, as we use excess returns below we can rationalize Euler equation errors for the claims, \( e^X - e^b_R \).

The root mean square error (RMSE) is defined as the average Euler equation errors across both assets – in our case the excess return and the bill return – and observation periods,

\[ RMSE = \left(\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{2}(e^c_{X,t})^2 + \frac{1}{2}(e^b_{R,t})^2\right]\right)^{\frac{1}{2}}. \]

(56)

4.2 Estimated Euler equation errors

We illustrate the implications for the estimated Euler equation errors for the endowment economy. A similar derivation for the production economy is straightforward. Consider the government bill, \( R_{b,t+1} \), and the claim on dividends, \( R_{c,t+1} \).

Using estimated Euler equation errors in (4), for any asset \( i \) and CRRA preferences

\[ \hat{e}^i_R = E_t\left[e^{-\bar{\rho}} - \gamma\hat{\rho} + \frac{1}{2}\bar{\gamma}\hat{\sigma}^2 - \gamma\hat{\rho}(\hat{B}_{t+1} - \hat{B}_t) - \gamma\hat{\rho}(N_{t+1} - N_t)R_{b,t+1}\right] - 1, \]

where we inserted the equilibrium consumption growth rate from (26). The estimated Euler equation errors for excess returns can be obtained from \( \hat{e}^X = \hat{e}^i_R - \hat{e}^b_R \) for any asset \( i \).
Risky asset. Conditional on no disasters, the estimated Euler equation errors for the one-period equilibrium return on the risky claim are

\[
\epsilon_{R[t]}^{c|\bar{N}_{t+1}-N_t=0} = E_t \left[ e^{\rho \tau + (\gamma - \gamma)(\bar{\mu} - \frac{1}{2} \bar{\sigma}^2) - \frac{1}{2}((1 - \gamma)^2 - (1 - \bar{\gamma})^2) \bar{\sigma}^2} \right] - 1 \\
= \exp \left( \rho - \hat{\rho} + (\gamma - \hat{\gamma})(\bar{\mu} - \frac{1}{2} \bar{\sigma}^2) - \frac{1}{2}((1 - \gamma)^2 - (1 - \bar{\gamma})^2) \bar{\sigma}^2 \right) \\
\times \exp \left( -(e^{(1 - \gamma)\rho} - 1)\lambda \right) - 1. \tag{57}
\]

This result clearly shows that in order to minimize the estimated Euler equation errors, the parameter estimates are biased as long as \((e^{(1 - \gamma)\rho} - 1)\lambda \neq 0.\)

Riskless asset. Conditional on no disasters, the estimated Euler equation errors for the one-period equilibrium return the government bill are

\[
\epsilon_{R[t]}^{b|\bar{N}_{t+1}-N_t=0} = E_t \left[ e^{\rho \tau - \frac{1}{2}((\gamma - \bar{\gamma})^2 + \gamma\sigma(B_{t+1} - B_t))} \right] - 1 \\
= \exp \left( \rho - \hat{\rho} + (\gamma - \bar{\gamma})(\bar{\mu} - \frac{1}{2} \bar{\sigma}^2) - (\gamma^2 - \bar{\gamma}^2) \frac{1}{2} \bar{\sigma}^2 \right) \\
\times \exp \left( -(1 - (1 - e^\kappa)q) e^{-\gamma^b} - 1)\lambda \right) - 1, \tag{58}
\]

where \(\hat{\gamma} \equiv \hat{\rho} + \frac{1}{2} \bar{\gamma}(1 + \hat{\gamma}) \bar{\sigma}^2\) defines the risk-free rate in a world without rare disasters. Hence, the effect of rare disasters on the equilibrium risk-free rate, \(\lambda - (1 - (1 - e^\kappa)q) e^{-\gamma^b} \lambda\), can only be captured by \(r - \hat{r}\) through biased estimates of \(\rho\) and \(\gamma\).

In the standard approach, the GMM procedure chooses \(\hat{\rho}\) and \(\hat{\gamma}\) such as to minimize the Euler equation errors across assets (cf. Lettau and Ludvigson, 2009). In particular, we would encounter the square root of the average Euler equation errors for the \(t\)th observation,

\[
\text{RMSE}_t = \sqrt{\frac{1}{2} \left( \epsilon_{R,t}^{c} - \epsilon_{R,t}^{b} \right)^2 + \frac{1}{2} \left( \epsilon_{R,t}^{b} \right)^2}.
\]

Consider the case of rare disasters, i.e., \(\hat{\nu} < 0\) and \(\kappa < 0\). In this case, the Euler equation error for the risky claim (conditional on no disasters) in (51) on average is positive for \(\gamma < 1\), whereas negative for \(\gamma > 1\). Further, for the government bill (conditional on no disasters) in (52) on average is unambiguously negative as the risk free rate presumably is biased downwards. Therefore, the procedure chooses parameters \(\hat{\gamma}\) and \(\hat{\rho}\) such that they increase \(\hat{\gamma}\) in (58), taking account of the effects on the estimated Euler equation errors in (57).

\[\text{In order to minimize (57) and (58), both equations should hold simultaneously,}\]

\[
\rho - \hat{\rho} + (\gamma - \bar{\gamma})(\bar{\mu} - \frac{1}{2} \bar{\sigma}^2) - \frac{1}{2}((1 - \gamma)^2 - (1 - \bar{\gamma})^2) \bar{\sigma}^2 \approx (e^{(1 - \gamma)\rho} - 1)\lambda, \\
\rho - \hat{\rho} + (\gamma - \bar{\gamma})(\bar{\mu} - \frac{1}{2} \bar{\sigma}^2) - (\gamma^2 - \bar{\gamma}^2) \frac{1}{2} \bar{\sigma}^2 \approx ((1 - (1 - e^\kappa)q) e^{-\gamma^b} - 1)\lambda.
\]

Hence, in order to minimize Euler equation errors, the GMM procedure tends to bias both \(\hat{\rho}\) and \(\hat{\gamma}\).
5 Monte Carlo experiments

In this section we provide Monte Carlo evidence on the impact of low-probability events on the performance of the consumption-based asset pricing model. In particular, we seek a better understanding of how the estimated parameters and pricing errors are affected by the presence of rare events. The general setup of this analysis is as follows.

We first simulate equilibrium asset prices and consumption paths from a reasonably parameterized consumption-based model with rare events.\(^{11}\) We consider both the endowment economy outlined above and the production economy for which explicit solutions for asset prices can be derived. Consistent with the sample lengths of typical empirical studies of the C-CAPM (e.g. Lettau and Ludvigson, 2001), the sample paths have a length of 50 years and we use 5,000 Monte Carlo draws.\(^{12}\) We then study the performance of a standard C-CAPM with power utility that is fitted to these data by GMM, similar to Lettau and Ludvigson (2009). We are mainly interested in this context if the estimated C-CAPM generates Euler equation errors using the simulated series. This is of particular interest since Lettau and Ludvigson have shown that other prominent models in the recent asset pricing literature fail to produce substantial pricing errors. Moreover, we are interested in the biases in parameter estimation induced by the presence of rare events in the data.

[DETAILED RESULTS AND DISCUSSION TO BE ADDED]

6 Conclusion

In this paper we study the impact of rare disasters (such as wars or natural catastrophes) on Euler equation errors and the empirical performance of the consumption-based asset pricing model in general. For this purpose, we derive closed-form asset pricing implications and Euler equation errors both in an endowment as well as a production economy with stochastically occurring disasters. We also investigate in extensive simulations the impact of rare disasters on estimates of structural parameters of the consumption-based model and the empirical performance of the model. Thus, our paper seeks to provide a better understanding of why the standard model fails so dramatically when fitted to the data.

Allowing for rare events in an otherwise standard C-CAPM explains why the canonical model generates large and persistent Euler equation errors when confronted by the data. Hence, consumption-based models with rare events qualify as a class of models which can rationalize the Euler equation puzzle of Lettau and Ludvigson (2009). We also show an-

\(^{11}\)The parameterization is fairly standard and similar with those elsewhere in the literature on rare events (see for instance, Barro, 2009 or Wachter, 2009).

\(^{12}\)The simulated data are sampled at a quarterly frequency.
alytically and through simulations that the dismal empirical performance and implausible estimates of risk aversion and time preference are not puzzling in a world with rare disasters - if these disasters happen not to occur in sample. In our simulations based on realistic calibration for the disaster distributions (cf. Barro, 2006), we are able to reproduce very closely the failure of the standard consumption-based model that is observed in the data. Overall, the results presented in this paper suggest that rare events not only help explaining the equity premium puzzle but also account for other important dimensions of the empirical failure of the standard consumption-based model.

References


A Appendix

A.1 Computing moments

A.1.1 A lemma for $E(c^kN_s)$

The following lemmas are required to compute the stochastic discount factor in the text.\footnote{We are indebted to Ken Sennewald and Klaus Wälde for discussions.}

\textbf{Lemma A.1} The conditional mean of $c^kN_s$ conditioned on the information set at time $t$ is

$$E_t[c^kN_s] = c^{kN_t}e^{(c^k-1)\lambda(s-t)}, \quad s > t, \quad c, k = \text{const.}$$

Note that for integer $k$, these are the raw moments of $c^N_s$.

\textbf{Proof.} We can trivially rewrite $c^kN_s = c^{kN_t}c^{(N_s-N_t)k}$. Thus, $E_t[c^kN_s] = E_t[c^{(N_s-N_t)k}]$. Computing this expectation requires the probability that a Poisson process jumps $n$ times between $t$ and $s$. Formally,

$$E_t[c^{(N_s-N_t)k}] = \sum_{n=0}^{\infty} e^{-(s-t)\lambda}\frac{(s-t)\lambda n}{n!} = e^{-(s-t)\lambda+\lambda(s-t)\lambda} = e^{(s-t)\lambda},$$

where $\frac{(s-t)\lambda n}{n!}$ is the probability of $N_s = n$, and $\sum_{n=0}^{\infty} \frac{(s-t)\lambda n}{n!} = 1$ is the probability function over the whole support of the Poisson distribution used in the last step.

\textbf{Lemma A.2} The unconditional mean of $c^kN_s$ is

$$E[c^{(N_s-N_t)k}] = e^{(c^k-1)\lambda(s-t)}, \quad s > t, \quad c, k = \text{const.}$$

\textbf{Proof.} This proof simply applies lemma A.1. 

A.2 Lucas’ endowment economy with rare disasters

A.2.1 Bellman equation

The Bellman equation becomes when choosing the control $C_s \in \mathbb{R}_+$ at time $s$

$$\rho V(W_s) = \max_{C_s} \left\{ u(C_s) + (\mu MW_s - C_s) V_W + \frac{1}{2} \sigma^2 MW^2_s V_{WW} + (E^C [V((1 - \zeta_M(s))W_s)] - V(W_s)) \lambda \right\}.$$
A.2.2 General equilibrium prices

Using the inverse function, we are able to determine the path for consumption \((u'' \neq 0)\). From the Euler equation (17), we obtain

\[
dC_t = ((\rho - \mu_M + \lambda)u'(C_t)/u''(C_t) - \sigma_M^2 W_tC_W - \frac{1}{2} u'''(C_t) \sigma_M^2 W_t^2 C_W^2
- \lambda E^C [u'(C((1 - \zeta_M(t))W_t))(1 - \zeta_M(t))] \lambda/u''(C_t)) dt
+ \sigma_M W_t C_W dB_t + (C((1 - \zeta_M(t))W_t - C(W_t -)) dN_t,
\]

(59)

where we employed the inverse function \(c = g(u'(c))\) which has

\[
g'(u'(c)) = 1/u''(c), \quad g''(u'(c)) = -u'''(c)/(u''(c))^3.
\]

Economically, concave utility \((u'(c) > 0, u''(c) < 0)\) implies risk aversion, whereas convex marginal utility, \(u''(c) > 0\), implies a positive precautionary saving motive. Accordingly, \(-u''(c)/u'(c)\) measures absolute risk aversion, whereas \(-u'''(c)/u''(c)\) measures the degree of absolute prudence, i.e., the intensity of the precautionary saving motive (Kimball, 1990).

Because output is perishable, using the market clearing condition \(Y_t = C_t = A_t\), and

\[
dC_t = \bar{\mu}C_t dt + \sigma_C dB_t + (\exp(\bar{\nu}) - 1)C_t - dN_t,
\]

(60)

the parameters of price dynamics are pinned down in general equilibrium. In particular, we obtain \(J_t\) implicitly as function of \(\bar{\nu}, D_t\) (stochastic investment opportunities), and the curvature of the consumption function, where \(\tilde{C}(W_t) \equiv C((1 - \zeta_M(t))W_t)/C(W_t)\) defines optimal consumption jumps. In equilibrium, market clearing requires the percentage jump in aggregate consumption to match the size of the disaster, \(\exp(\bar{\nu}) = \tilde{C}(W_t)\), and thus \(\exp(\bar{\nu}) = C((1 + (J_t - D_t)W_t + D_t)W_t)/C(W_t)\) implies a constant jump size. For example, if consumption is linear homogeneous in wealth, the jump size of the market portfolio is

\[
\zeta_M = \frac{\zeta_M}{e^\bar{\nu} - 1}.
\]

Similarly, the market clearing condition pins down \(\sigma_M W_t C_W = \tilde{\sigma}C_t\), and

\[
\mu_M - r = -\frac{u''(C_t) W_t C_W}{u'(C(W_t))} \sigma_M^2 - \frac{u'(e^\bar{\nu} C(W_t))}{u'(C(W_t))} ((1 - e^\kappa)q - \zeta_M) \lambda.
\]

(62)

Inserting our results back into (59), we obtain that consumption follows,

\[
dC_t = (\rho - r + \lambda) \frac{u'(C_t)}{u'(C_t)} dt - \frac{1}{2} \frac{u''(C_t)}{u'(C_t)} \sigma_M^2 W_t^2 C_W^2 dt - (1 - (1 - e^\kappa)q) \frac{u'(e^\bar{\nu} C_t)}{u'(C_t)} \lambda dt
+ \sigma_M W_t C_W dB_t + (C((1 - \zeta_M(t))W_t - C(W_t -)) dN_t.
\]

This in turn determines the return on the government bill

\[
r = \rho - \frac{u''(C_t) C_t}{u'(C_t)} \bar{\mu} - \frac{1}{2} \frac{u'''(C_t) C_t^2}{u'(C_t)} \sigma^2 + \lambda - (1 - (1 - e^\kappa)q) \frac{u'(e^\bar{\nu} C_t)}{u'(C_t)} \lambda.
\]

(63)
A.2.3 General equilibrium consumption growth rates and asset returns

Consumption. Consumption growth rates are exogenous in the endowment economy. Thus, consumption growth rates can be obtained from the dividend process (7),

\[ A_s = A_t e^{(\bar{\mu} - \frac{1}{2}\bar{\sigma}^2)(s-t) + \bar{\sigma}(B_s - B_t) + \bar{\nu}(N_s - N_t)} \] (64)

\[ \Leftrightarrow \ln(C_s/C_t) = \ln(A_s/A_t) = (\bar{\mu} - \frac{1}{2}\bar{\sigma}^2)(s-t) + \bar{\sigma}(B_s - B_t) + \bar{\nu}(N_s - N_t). \] (65)

Risky asset. Consider a claim which pays a dividend \( X_{c,t+1} = A_{t+1} \), i.e., an instantaneous return in period \( s = t + 1 \),

\[ R_{c,t+1} = \frac{A_{t+1}}{P_{c,t}}. \] (66)

From (2) we obtain the price of this asset in terms of the consumption good as

\[ P_{c,t} = E_t \left[ \frac{m_{t+1}}{m_t} A_{t+1} \right] 
\]

\[ = e^{-(\rho+(\gamma-1)\bar{\mu} + (\gamma-1)\bar{\sigma}^2)E_t \left[ e^{(1-\gamma)\bar{\sigma}(B_{t+1} - B_t)} \right] E_t \left[ e^{(1-\gamma)\bar{\nu}(N_{t+1} - N_t)} \right] A_t}
\]

\[ = e^{-\rho+(\gamma-1)\bar{\mu} + (\gamma-1)\bar{\sigma}^2 + \frac{1}{2}(1-\gamma)^2\bar{\sigma}^2 + (e^{(1-\gamma)\bar{\nu}}-1)\lambda A_t.}
\]

Inserting this result together with (64) into (66) finally gives (27).

Riskless asset. Consider a riskless asset which is subject to default risk. Use the random payoff \( X_{b,t+1} = A_t e^{r + f_{t+1}^1 \ln(1+D_s)dN_s} \) and (2) which gives the price of the government bill as

\[ P_{b,t} = E_t \left[ \frac{m_{t+1}}{m_t} A_t e^{r + f_{t+1}^1 \ln(1+D_s)dN_s} \right] 
\]

\[ = e^{-(\rho-r+\gamma\bar{\mu} - \frac{1}{2}\gamma\bar{\sigma}^2)E_t \left[ e^{-\gamma\bar{\sigma}(B_{t+1} - B_t)} \right] E_t \left[ e^{(e^{\ln(1+D_s)}-\gamma\bar{\nu}-1)\lambda} \right] A_t}
\]

\[ = e^{-(\rho-r+\gamma\bar{\mu} - \frac{1}{2}\gamma\bar{\sigma}^2) + \frac{1}{2}(\gamma\bar{\sigma})^2 + q(e^{\kappa}\gamma\bar{\nu}-1)\lambda + (1-q)(e^{-\gamma\bar{\nu}-1})\lambda A_t} = A_t.
\]

This in turn gives the return of the government bill with default risk.

A.3 A production economy with rare events

A.3.1 The Bellman equation and the Euler equation

As a necessary condition for optimality the Bellman’s principle gives at time \( s \)

\[ \rho V(W_s, A_s) = \max_{C_s} \left\{ u(C_s) + \frac{1}{dt} E_s dV(W_s, A_s) \right\}. \]
Using Itô’s formula yields
\[
dV(W_s, A_s) = V_W(dW_s - J_sW_s - dN_t) + V_A(dA_s - ((\exp(\nu) - 1)A_t - d\tilde{N}_t)) \\
+ \frac{1}{2} \left( V_{AA} \bar{s}_t^2 A_s^2 + V_{WW} \bar{\sigma}_s^2 W_s^2 \right) dt \\
+ [V(W_s, A_s) - V(W_{s-}, A_{s-})](d\tilde{N}_t + dN_t)
\]
\[
= ((r_s - \delta)W_s + w_s^L - C_s)V_W dt + V_W \sigma_s W_s dZ_s + V_A \bar{A}_s dt + V_A \bar{\sigma}_s dB_s \\
+ \frac{1}{2} \left( V_{AA} \bar{s}_t^2 A_s^2 + V_{WW} \bar{\sigma}_s^2 W_s^2 \right) dt + [V(e^\nu W_{s-}, A_{s-}) - V(W_{s-}, A_{s-})]d\tilde{N}_t \\
+ [V(W_{s-}, e^\nu A_{s-}) - V(W_{s-}, A_{s-})]d\tilde{N}_t.
\]
Using the property of stochastic integrals, we may write
\[
\rho V(W_s, A_s) = \max_{C_s} \left\{ u(c_s) + ((r_s - \delta)W_s + w_s^L - C_s)V_W + \frac{1}{2} \left( V_{AA} \bar{s}_t^2 A_s^2 + V_{WW} \bar{\sigma}_s^2 W_s^2 \right) \\
+ V_A \bar{A}_s + [V(e^\nu W_s, A_s) - V(W_s, A_s)]\lambda + [V(W_s, e^\nu A_s) - V(W_s, A_s)]\bar{\lambda} \right\}
\]
for any \( s \in [0, \infty) \). Because it is a necessary condition for optimality, we obtain the first-order condition (35) which makes optimal consumption a function of the state variables.

For the evolution of the costate we use the maximized Bellman equation
\[
\rho V(W_t, A_t) = u(C(W_t, A_t)) + ((r_t - \delta)W_t + w_t^L - C_t)V_W + V_A \bar{A}_t \\
+ \frac{1}{2} \left( V_{AA} \bar{s}_t^2 A_t^2 + V_{WW} \bar{\sigma}_t^2 W_t^2 \right) + [V(e^\nu W_t, A_t) - V(W_t, A_t)]\lambda \\
+ [V(W_t, e^\nu A_t) - V(W_t, A_t)]\bar{\lambda},
\]
where \( r_t = r(W_t, A_t) \) and \( w_t^L = w(W_t, A_t) \) follow from the firm’s optimization problem, and the envelope theorem (also for the factor rewards) to compute the costate,
\[
\rho V_W = \bar{\mu}_t A_t V_{AW} + ((r_t - \delta)W_t + w_t^L - C_t)V_{WW} + (r_t - \delta)W_t + \frac{1}{2} \left( V_{WWAA} \bar{s}_t^2 A_t^2 + V_{WWWW} \bar{\sigma}_t^2 W_t^2 \right) \\
+ V_{WWW} \bar{\sigma}_t^2 W_t + [V(e^\nu W_t, A_t)e^\nu - V(W_t, A_t)]\lambda + [V(W_t, e^\nu A_t) - V(W_t, A_t)]\bar{\lambda}.
\]
Collecting terms we obtain
\[
(r_t - \delta + \lambda + \bar{\lambda})V_W = V_{AW} \bar{\mu}_t A_t + ((r_t - \delta)W_t + w_t^L - C_t)V_{WW} \\
+ \frac{1}{2} \left( V_{WWAA} \bar{s}_t^2 A_t^2 + V_{WWWW} \bar{\sigma}_t^2 W_t^2 \right) \\
+ \bar{\sigma}_t^2 V_{WWW} W_t + V_{WW} \bar{\sigma}_t^2 W_t + V_{WW} \bar{\sigma}_t^2 W_t + V_{WW} \bar{\sigma}_t^2 W_t + V_{WW} \bar{\sigma}_t^2 W_t
\]
Using Itô’s formula, the costate obeys
\[
dV_W = V_{AW} \bar{\mu}_t A_t dt + V_{AW} \bar{\sigma}_t A_t dB_t + \frac{1}{2} \left( V_{WWAA} \bar{s}_t^2 A_t^2 + V_{WWWW} \bar{\sigma}_t^2 W_t^2 \right) dt + V_{WWW} \sigma_t W_t dZ_t \\
+ ((r_t - \delta)W_t + w_t^L - C_t)V_{WWW} dt + [V(W_t, A_t) - V(W_{t-}, A_{t-})](d\tilde{N}_t + dN_t)
\]
where inserting yields

\[
dV_W = (\rho - (r_t - \delta) + \lambda + \bar{\lambda})V_W dt - V_W(e^{\nu}W_t, A_t)e^{\nu}\lambda - V_W(W_t, e^\delta A_t)\bar{\lambda} - \sigma^2 V_W W_t dt + V_{AW} A_t \sigma dB_t + V_{Wt} W_t \sigma dZ_t + [V_W(e^{\nu}W_{t-}, A_{t-}) - V_W(W_{t-}, A_{t-})]dN_t + [V_W(W_{t-}, e^\delta A_{t-}) - V_W(W_{t-}, A_{t-})]d\bar{N}_t,
\]

which describes the evolution of the costate variable. As a final step, we insert the first-order condition (35) to obtain the Euler equation (36).

**A.3.2 Proof of Proposition 3.4**

The idea of this proof is to show that using an educated guess of the value function, the maximized Bellman equation (67) and the first-order condition (35) are both fulfilled. We guess that the value function reads

\[
V(W_t, A_t) = \frac{C_t W_t^{1-\theta}}{1-\theta} + f(A_t).
\]

From (35), optimal consumption is a constant fraction of wealth,

\[
C_t^{1-\theta} = C_t W_t^{-\theta} \iff C_t = C_t^{-1/\theta} W_t.
\]

Now use the maximized Bellman equation (67), the property of the Cobb-Douglas technology, \(F_K = \alpha A_t K_t^{\alpha - 1} L^{1-\alpha} \) and \(F_L = (1-\alpha)A_t K_t \theta L_{t-}^{-\alpha} \), together with the transformation \(K_t \equiv LW_t\), and insert the solution candidate,

\[
\rho \frac{C_t W_t^{1-\theta}}{1-\theta} = \frac{C_t^{1-\frac{1-\theta}{\theta}} W_t^{1-\theta}}{1-\theta} + (\alpha A_t W_t^{\alpha - 1} W_t - \delta W_t + (1-\alpha)A_t W_t^{\alpha} - C_t^{-1/\theta} W_t)C_t W_t^{1-\theta} - \frac{1}{2} \theta C_t W_t^{1-\theta} \sigma^2 - g(A_t) + (e^{(1-\theta)\nu} - 1) \frac{C_t W_t^{1-\theta}}{1-\theta} \lambda.
\]

where we defined \(g(A_t) \equiv \rho f(A_t) - f(A_t) - \frac{1}{2} \int f(A_t) \int d\tilde{A}_t^2 - [f(e^\delta A_t) - f(A_t)] \bar{\lambda}. \) When imposing the condition \(\alpha = \theta \) and \(g(A_t) = C_t A_t \) it can be simplified to

\[
(\rho - (e^{(1-\theta)\nu} - 1)\lambda) \frac{C_t W_t^{1-\theta}}{1-\theta} + g(A_t) = \frac{C_t^{-1/\theta} W_t^{1-\theta}}{1-\theta} + (A_t W_t^{\alpha-\theta} - \delta W_t^{1-\theta} - C_t^{-1/\theta} W_t^{1-\theta})C_t W_t^{1-\theta} - \frac{1}{2} \theta C_t W_t^{1-\theta} \sigma^2
\]

\[
\iff (\rho - (e^{(1-\theta)\nu} - 1)\lambda) W_t^{1-\theta} = -\theta C_t^{-1/\theta} W_t^{1-\theta} - (1-\theta) \delta W_t^{1-\theta} - \frac{1}{2} \theta (1-\theta) W_t^{1-\theta} \sigma^2,
\]

which implies that \( C_t^{-1/\theta} = (\rho - (e^{(1-\theta)\nu} - 1)\lambda + (1-\theta) \delta + \frac{1}{2} \theta (1-\theta) \sigma^2)/\theta. \) This proves that the guess (68) indeed is a solution, and by inserting the guess together with the constant, we obtain the optimal policy function for consumption.
A.3.3 Proof of Proposition 3.6

The idea of this proof follows Section A.3.2. An educated guess of the value function is

$$V(W_t, A_t) = \frac{C_1 W_t^{1-\alpha}}{1 - \alpha \theta} A_t^{-\theta}. \quad (69)$$

From (35), optimal consumption is a constant fraction of income, \(C_t^{-\theta} = C_1 W_t^{-\alpha} A_t^{-\theta}\) or equivalently \(C_t = C_1^{1/\theta} W_t^\alpha A_t\). Now use the maximized Bellman equation (67), the property of the Cobb-Douglas technology, \(F_K = \alpha A_t K_t^{\alpha - 1} L^{1-\alpha}\) and \(F_L = (1 - \alpha) A_t K_t^{\alpha} L^{-\alpha}\), together with the transformation \(K_t = LW_t\), and insert the solution candidate,

$$\rho V(W_t, A_t) = \frac{C_1^{1/\theta} W_t^{\alpha - \theta} A_t^{1-\theta}}{1 - \theta} + [(r_t - \delta)W_t + W_t^L - C(W_t, A_t)]V_W + V_A \tilde{\mu} A_t$$

$$+ \frac{1}{2} (V_A \tilde{\sigma}^2 A_t^2 + V_W \tilde{\sigma}^2 W_t^2) + [V(e^\nu W_t, A_t) - V(W_t, A_t)]\lambda$$

$$+ [V(W_t, e^\nu A_t) - V(W_t, A_t)]\bar{\lambda}.$$

Inserting the guess and collecting terms which is equivalent to

$$(\rho - (e^{(1-\alpha)\nu} - 1)\lambda - (e^{-\theta\nu} - 1)\bar{\lambda}) \frac{C_1 W_t^{1-\alpha}}{1 - \alpha \theta} A_t^{-\theta} =$$

$$\frac{C_1^{1/\theta} W_t^{\alpha - \theta} A_t^{1-\theta}}{1 - \theta} - \frac{C_1 W_t^{1-\alpha}}{1 - \alpha \theta} \tilde{\mu} A_t^{-\theta}$$

$$+ (\alpha A_t W_t^\alpha - \delta W_t + (1 - \alpha) A_t W_t^\alpha - C_1^{1/\theta} W_t^\alpha A_t) \frac{C_1 W_t^{1-\alpha}}{1 - \alpha \theta} A_t^{-\theta}$$

$$+ \frac{1}{2} (\theta(1 + \theta)\tilde{\sigma}^2 - \alpha \theta(1 - \alpha \theta)\tilde{\sigma}^2) \frac{C_1 W_t^{1-\alpha}}{1 - \alpha \theta} A_t^{-\theta}.$$

Collecting terms gives

$$\rho + \theta \tilde{\mu} - \frac{1}{2} (\theta(1 + \theta)\tilde{\sigma}^2 - \alpha \theta(1 - \alpha \theta)\tilde{\sigma}^2) + (1 - \alpha \theta)\delta - (e^{(1-\alpha)\nu} - 1)\lambda - (e^{-\theta\nu} - 1)\bar{\lambda} =$$

$$\left(\frac{\theta}{1 - \theta} C_1^{1/\theta} + 1\right) (1 - \alpha \theta) A_t W_t^{\alpha - 1}.$$

which has a solution for \(C_1^{-1/\theta} = (\theta - 1)/\theta\) and

$$\rho = (e^{-\theta\nu} - 1)\bar{\lambda} + (e^{(1-\alpha)\nu} - 1)\lambda - \theta \tilde{\mu} + \frac{1}{2} (\theta(1 + \theta)\tilde{\sigma}^2 - \alpha \theta(1 - \alpha \theta)\tilde{\sigma}^2) - (1 - \alpha \theta)\delta.$$

This proves that the guess (69) indeed is a solution, and by inserting the guess together with the constant, we obtain the optimal policy function for consumption.
A.3.4 Obtaining the rental rate of capital

The rental rate of capital, \( r_t = Y_t \), in a neoclassical Cobb-Douglas economy where output is defined as \( Y_t \equiv A_t Y(K_t, L) = A_t K_t^\alpha L^{1-\alpha} \) follows from the stochastic differential

\[
dA_t K_t^{\alpha-1} = (\alpha - 1) A_t K_t^{\alpha-2} (Y_t - C_t - \delta K_t) dt + (\alpha - 1) \sigma A_t K_t^{\alpha-2} dZ_t + (\alpha - 1) \sigma A_t K_t^{\alpha-2} dN_t + \frac{1}{2} (\alpha - 1) (\alpha - 2) K_t^{\alpha-3} \sigma^2 A_t dt + K_t^{\alpha-1} (dA_t - (\exp(\bar{\nu}) - 1) A_t d\bar{N}_t)
\]

which implies

\[
dr_t = \frac{1 - \alpha}{\alpha} \left( \alpha C_t/K_t + \alpha \delta - \frac{1}{2} \alpha (\alpha - 2) \sigma^2 - \frac{\alpha}{\alpha - 1} \bar{\mu} - r_t \right) r_t dt + (\alpha - 1) \sigma r_t dZ_t + \bar{\sigma} r_t dB_t + (\alpha - 1) \sigma r_t dN_t + (\exp(\bar{\nu}) - 1) r_t d\bar{N}_t.
\]

For \( \alpha = \gamma \) we obtain

\[
dr_t = \frac{1 - \alpha}{\alpha} \left( \alpha \phi + \alpha \delta - \frac{1}{2} \alpha (\alpha - 2) \sigma^2 - \frac{\alpha}{\alpha - 1} \bar{\mu} - r_t \right) r_t dt + (\alpha - 1) \sigma r_t dZ_t + \bar{\sigma} r_t dB_t + (\alpha - 1) \sigma r_t dN_t + (\exp(\bar{\nu}) - 1) r_t d\bar{N}_t,
\]

where we defined \( c_1 \equiv \frac{1 - \alpha}{\alpha} \) and \( c_2 \equiv \alpha \phi + \alpha \delta - \frac{1}{2} \alpha (\alpha - 2) \sigma^2 - \frac{\alpha}{\alpha - 1} \bar{\mu} \).

For \( \rho = \bar{\rho} \) we obtain

\[
dr_t = \frac{1 - \alpha}{\alpha} \left( \rho \phi - \frac{1}{2} \rho (\rho - 2) \sigma^2 - \frac{\rho}{\rho - 1} \bar{\mu} - sr_t \right) r_t dt + (\alpha - 1) \sigma r_t dZ_t + \bar{\sigma} r_t dB_t + (\alpha - 1) \sigma r_t dN_t + (\exp(\bar{\nu}) - 1) r_t d\bar{N}_t,
\]

where we defined \( c_1 \equiv \frac{1 - \alpha}{\alpha \gamma} \) and \( c_2 \equiv \alpha \gamma \delta - \frac{1}{2} \alpha \gamma (\alpha - 2) \sigma^2 - \frac{\alpha \gamma}{\alpha - 1} \bar{\mu} \).

Because the stochastic differential equation for \( r_t \) is reducible, it has the solution

\[
r_s = \Theta_{s,t} r_t^{-1} + c_1 \int_t^s \Theta_{v,t} dv
\]

where \( \Theta_{s,t} \equiv e^{(c_1) c_2 - \frac{1}{2} (\alpha - 1) \sigma^2 \gamma (s - t) + (2s - Z_t)(\alpha - 1) \sigma (B_s - B_t) \bar{\sigma} + (\alpha - 1) \nu (N_s - N_t) + \bar{\nu} (\bar{N}_s - \bar{N}_t)} \). Observe that the closed-form solution simplifies the problem of simulating Euler equation errors.
A.3.5 General equilibrium consumption growth rates and asset returns

Consumption. Observe that the solution to (29) is for \( s \geq t \)

\[
A_s = A_t e^{\left( \mu - \frac{1}{2} \sigma^2 \right) (s-t) + \sigma (B_s - B_t) + \nu (N_s - \bar{N}_t)}
\]

\[
\leftrightarrow \ln(A_s/A_t) = (\mu - \frac{1}{2} \sigma^2) (s-t) + \sigma (B_s - B_t) + \nu (N_s - \bar{N}_t).
\] (70)

Similarly, we obtain growth rates of the capital stock from (30)

\[
\ln(K_t/K_s) = \int_s^t \left( r_v/\alpha - C_v/K_v - \delta - \frac{1}{2} \sigma^2 \right) dv + \sigma (Z_t - Z_s) + \nu (N_t - N_s).
\] (71)

For the case of \( \alpha = \gamma \), as from Proposition 3.4, consumption is a linear function in the capital stock \( C_t = \phi K_t \). Hence, the consumption growth rate is \( \ln(C_s/C_t)|_{\alpha=\gamma} = \ln(K_t/K_s) \) which gives (43). For the case of \( \rho = \bar{\rho} \), as from Proposition 3.6, consumption is a constant fraction of output, \( C_t = (1-s)Y_t \), and thus we obtain the consumption growth rate as \( \ln(C_s/C_t)|_{\rho=\bar{\rho}} = \ln(Y_s/Y_t) = \ln(A_s/A_t) + \alpha \ln(K_s/K_t) \), which finally gives (44).

Risky assets. Consider a risky bond that pays continuously at the rate, \( r_t \). Investing into this asset gives the random payoff \( X_{b,t+1} = e^{\int_t^{t+1} r_t \, dt} \). From (2) we obtain the price

\[
P_{b,t} = E_t \left[ \frac{m_{t+1}}{m_t} e^{\int_t^{t+1} r_s \, ds} \right]
\]

\[
\Rightarrow P_{b,t}|_{\alpha=\gamma} = e^{\delta + \gamma \sigma^2 + e^{-\gamma} \lambda - e^{(1-\gamma) \nu} \lambda},
\]

\[
P_{b,t}|_{\rho=\bar{\rho}} = e^{\delta + \gamma \sigma^2 + e^{-\alpha} \lambda - e^{(1-\alpha) \nu} \lambda}
\]

Hence, we obtain the returns for the bond as in (46) and (47).

Consider a claim on output which pays \( X_{c,t+1} = A_{t+1} K_{t+1}^\alpha \), i.e., an instantaneous return in period \( s = t + 1 \),

\[
R_{c,t+1} = \frac{A_{t+1} K_{t+1}^\alpha}{P_{c,t}}.
\] (72)

where from (70) and (71), we obtain output at date \( s \geq t \) from

\[
A_s K_s^\alpha = A_t K_t^\alpha e^{\left( \mu - \frac{1}{2} \sigma^2 \right) (s-t) + f_s^t (r_v - \alpha C_v/K_v - \delta - \frac{1}{2} \sigma^2) dv + \sigma (B_s - B_t) + \sigma (Z_s - Z_t) + \nu (N_s - N_t) + \nu (N_t - \bar{N}_t)}
\]

From (2) we obtain the price of this asset in terms of the consumption good as

\[
P_{c,t} = E_t \left[ \frac{m_{t+1}}{m_t} A_{t+1} K_{t+1}^\alpha \right]
\]

\[
\Rightarrow P_{c,t}|_{\alpha=\gamma} = A_t K_t^\alpha E_t \left[ e^{\mu - \frac{1}{2} \sigma^2 - \alpha \gamma - \delta - \frac{1}{2} \sigma^2 + \delta + \lambda - e^{(1-\gamma) \nu} \lambda + \gamma \sigma^2 - \frac{1}{2} \lambda (\gamma \sigma^2 + \sigma (B_{t+1} - B_t) + \nu (\bar{N}_{t+1} - \bar{N}_t))} \right]
\]\n
\[
= A_t K_t^\alpha e^{-\left( \alpha \gamma + \delta + \frac{1}{2} \sigma^2 - \delta - \lambda + e^{(1-\gamma) \nu} \lambda - \gamma \sigma^2 - \frac{1}{2} \lambda (\gamma \sigma^2 + \sigma (B_{t+1} - B_t) + \nu (\bar{N}_{t+1} - \bar{N}_t)) \right)},
\]

\[
P_{c,t}|_{\rho=\bar{\rho}} = A_t K_t^\alpha e^{\left( \alpha \gamma + \delta + \frac{1}{2} \sigma^2 - (1-e^{-\gamma}) \lambda - (1-e^{-\gamma}) \lambda \gamma \sigma^2 + \frac{1}{2} \lambda (\gamma \sigma^2 + \sigma (B_{t+1} - B_t) + \nu (\bar{N}_{t+1} - \bar{N}_t)) \right)}
\]

\[
\times E_t \left[ e^{-\int_t^{t+1} (x-1 \sigma^2 + \delta) ds + (1-\gamma) \left( \delta (B_{t+1} - B_t) + \nu (Z_{t+1} - Z_t) + \nu (N_{t+1} - N_t) + \nu (\bar{N}_{t+1} - \bar{N}_t)) \right)} \right],
\]
where the latter needs to be determined numerically. Observe that for the case of $\alpha = \gamma$, we obtain the claim of this asset in closed form, with the return (48).

Consider a claim on capital

$$ R_{c,t+1} = \frac{K_{t+1}^{\alpha\gamma}}{P_{c,t}}, \tag{73} $$

where from (71), we obtain the function of capital as

$$ K_s^{\alpha\gamma} = K_t^{\alpha\gamma} e^{\int_t^s (\gamma r - \alpha C_v/K_v - \alpha \gamma - \frac{1}{2} \alpha \gamma^2) ds + \alpha \gamma \sigma (Z_s - Z_t) + \alpha \gamma \nu (N_s - N_t)}. $$

From (2) we obtain the price of this asset in terms of the consumption good as

$$ P_{c,t} = E_t \left[ \frac{m_{t+1}}{m_t} K_{t+1}^{\alpha\gamma} \right] $$

$$ \Rightarrow P_{c,t} |_{\alpha = \gamma} = K_t^{\alpha\gamma} e^{\delta + \lambda - e^{(1-\gamma)\nu} \lambda + \gamma \sigma^2 - \frac{\lambda}{2} (\gamma \sigma)^2 - \alpha \gamma \phi - \alpha \gamma \delta - \frac{1}{2} \alpha \gamma^2} \times E_t \left[ e^{-\int_t^{t+1} (1-\gamma) r_s ds + (\alpha - 1) \sigma (Z_s - Z_t) + (\alpha - 1) \gamma \nu (N_s - N_t)} \right], $$

$$ P_{c,t} |_{\rho = \tilde{\rho}} = K_t^{\alpha\gamma} e^{-\alpha \gamma \phi - \delta - (1-e^{(1-\gamma)\nu}) \lambda - (1-e^{-\gamma}) \lambda + \gamma \sigma^2 + \frac{1}{2} (\gamma \sigma)^2} \times E_t \left[ e^{-\gamma \sigma (Z_{t+1} - Z_t) - \gamma \nu (N_{t+1} - N_t)} \right] = K_t^{\alpha\gamma} e^{-\alpha \gamma \phi - \delta - (1-e^{(1-\gamma)\nu}) \lambda - \gamma \sigma^2 + \frac{1}{2} (\alpha \gamma) - \frac{1}{2} (\alpha \gamma^2)}, $$

with the return to the equity claim for the latter yields (49).

Riskless asset. From (41) or (42) and (2), we obtain for any riskless security

$$ R_{f,t+1} |_{\alpha = \gamma} = \left( E_t \left[ e^{-\int_t^{t+1} (r_s - \delta) ds + \lambda - e^{(1-\gamma)\nu} \lambda + \gamma \sigma^2 - \frac{\lambda}{2} (\gamma \sigma)^2 - \gamma \sigma (Z_{t+1} - Z_t) - \gamma \nu (N_{t+1} - N_t)} \right] \right)^{-1}, $$

$$ R_{f,t+1} |_{\rho = \tilde{\rho}} = \left( E_t \left[ e^{-\int_t^{t+1} (r_s - \delta) ds + (1-e^{(1-\gamma)\nu}) \lambda + (1-e^{-\gamma}) \lambda + \gamma \sigma^2 - \frac{1}{2} (\gamma \sigma)^2 - \frac{1}{2} (\alpha \gamma)^2} \times e^{-\gamma \sigma (B_{t+1} - B_t) - \alpha \gamma \sigma (Z_{t+1} - Z_t) - \alpha \gamma \nu (N_{t+1} - N_t) - \gamma \nu (N_{t+1} - N_t)} \right] \right)^{-1}, $$

which is not available in closed-form, but will be time-varying.

A.3.6 Euler equation errors

Consider two assets, i.e., the risky bond, $R_{b,t+1}$, and the claim on capital or output, $R_{c,t+1}$.

From the definition of Euler equation errors (3), for any asset $i$ and CRRA preferences

$$ e^i_{R} |_{\alpha = \gamma} = E_t \left[ e^{-\int_t^{t+1} (r_s - \delta) ds + \lambda - e^{(1-\gamma)\nu} \lambda + \gamma \sigma^2 - \frac{1}{2} (\gamma \sigma)^2 - \gamma \sigma (Z_{t+1} - Z_t) - \gamma \nu (N_{t+1} - N_t) R_{i,t+1}} \right] - 1, $$

$$ e^i_{R} |_{\rho = \tilde{\rho}} = E_t \left[ e^{-\int_t^{t+1} (r_s - \delta) ds + (1-e^{(1-\gamma)\nu}) \lambda + (1-e^{-\gamma}) \lambda + \gamma \sigma^2 - \frac{1}{2} (\gamma \sigma)^2 - \frac{1}{2} (\alpha \gamma)^2} \times e^{-\gamma \sigma (B_{t+1} - B_t) - \alpha \gamma \sigma (Z_{t+1} - Z_t) - \alpha \gamma \nu (N_{t+1} - N_t) - \gamma \nu (N_{t+1} - N_t)} R_{i,t+1} \right] - 1, $$

where we inserted the SDFs from (41) and (42). Note that Euler equation errors based on excess returns can be obtained from $e^X_i = e^i_R - e^i_R$ for any asset $i$. 
Conditional on no disasters, on average we can rationalize Euler equation errors

\[
e^{-\gamma_0} - e^{-\gamma_0} = \exp \left( (1 - e^{-\gamma_0}) \lambda \right) - 1,
\]

or, conditional on no rare events, on average we can rationalize Euler equation errors

\[
e^{-\gamma_0} - e^{-\gamma_0} = \exp \left( (1 - e^{-\gamma_0}) \lambda \right) - 1.
\]

Similarly, inserting the return on the claims on output (48) and capital (49) yields

\[
e^{-\gamma_0} - e^{-\gamma_0} = \exp \left( (1 - e^{-\gamma_0}) \lambda \right) - 1,
\]

or, conditional on no rare events, on average we can rationalize Euler equation errors

\[
e^{-\gamma_0} - e^{-\gamma_0} = \exp \left( (1 - e^{-\gamma_0}) \lambda \right) - 1.
\]

A.3.7 Estimated Euler equation errors

Consider two assets, i.e., the risky bond, \( R_{b,t+1} \), and the claim on capital or output, \( R_{c,t+1} \).

Using estimated Euler equation errors in (4), for any asset \( i \) and CRRA preferences

\[
\hat{e}_R^i|_{\alpha=\gamma} = E_t \left[ e^{(1-e^{-\gamma})\lambda - \frac{1}{2}(\gamma\sigma)^2} - e^{(1-e^{-\gamma})\lambda - \frac{1}{2}(\gamma\sigma)^2} \right] - 1,
\]

\[
\hat{e}_R^i|_{\rho=\tilde{\rho}} = E_t \left[ e^{(1-e^{-\gamma})\lambda - \frac{1}{2}(\gamma\sigma)^2} - e^{(1-e^{-\gamma})\lambda - \frac{1}{2}(\gamma\sigma)^2} \right] - 1,
\]

where we used the equilibrium consumption growth rates from (43) and (44). The estimated Euler equation errors for excess returns can be obtained from \( \hat{e}_X = \hat{e}_R - \hat{e}_F \) for any asset \( i \).

Risky assets. Inserting the one-period equilibrium returns for the bond into

\[
\hat{e}_R^b|_{\alpha=\gamma} = E_t \left[ e^{(1-e^{-\gamma})\lambda - \frac{1}{2}(\gamma\sigma)^2} - e^{(1-e^{-\gamma})\lambda - \frac{1}{2}(\gamma\sigma)^2} \right] - 1,
\]

\[
\hat{e}_R^b|_{\rho=\tilde{\rho}} = E_t \left[ e^{(1-e^{-\gamma})\lambda - \frac{1}{2}(\gamma\sigma)^2} - e^{(1-e^{-\gamma})\lambda - \frac{1}{2}(\gamma\sigma)^2} \right] - 1.
\]