Forecasting Value-at-Risk (VaR) using Fractionally Integrated Models of Conditional Volatility

Stavros Degiannakis, Christos Floros and Pamela Dent

Department of Economics, University of Portsmouth

Contact Details

Postal Address:

Dr Stavros Degiannakis, Dr Christos Floros and Pamela Dent

Portsmouth Business School

University of Portsmouth

Richmond Building

Portland Street

Portsmouth, UK

PO1 3DE

Telephone: (0044) 023 9284 4244/4023

Emails: Stavros.degiannakis@port.ac.uk; Christos.floros@port.ac.uk and Pamela.dent@port.ac.uk

Forecasting Value-at-Risk (VaR) using Fractionally Integrated Models of Conditional Volatility

Abstract

VaR (Value-at-Risk) modelling is mainly focused on producing one-step-ahead conditional variance forecasts. The present study compares the performance of the long memory FIGARCH model, with that of the short memory GARCH specification, in the forecasting not only of daily but also multi-period VaR across 21 stock indices. The dataset is comprised of daily data covering the period from 1989 to 2009. The research addresses the question of whether or not accounting for long memory in the conditional variance specification improves the accuracy of the VaR forecasts produced, particularly for longer time horizons. Accounting for fractional integration in the conditional variance model does not appear to improve the accuracy of the forecasts for the 1-day-ahead, 10-day-ahead and 20-day-ahead forecasting horizons relative to the short memory GARCH specification. Furthermore, in the case of the 10-step-ahead forecasting horizon its use appears to result in a significant reduction in forecasting accuracy. Therefore, a long memory volatility model compared to a short memory GARCH model does not appear to improve the VaR forecasting accuracy, even for longer forecasting horizons. Additionally, the results suggest that underestimation of the true VaR figure becomes less prevalent as the forecasting horizon increases.

Key words: VaR, FIGARCH, Forecasting, Volatility, stock indices

JEL Classifications: G17; G15; C15; C32; C53

1. Introduction – Motivation and Review of Literature

Value-at-Risk (VaR) is an important tool in risk measurement and the management of the financial assets. Originally used internally by financial institutions to assess risk, VaR assumed greater significance when the Basel Committee encouraged its use through the 1996 Market Risk Amendment to the 1988 Basel Accord. Subsequently, the Basel Committee has refined the regulations relating to the use of VaR, allowing greater flexibility for certain financial institutions to use their own internal VaR models subject to the models being approved by the regulator (Basel, 2006).

VaR quantifies the maximum amount of loss for a portfolio of assets under normal market conditions over a given period of time and at a certain confidence level (95% or 99%). Although financial institutions have flexibility over the model which is used to estimate VaR, the regulations prescribe that they use up to one year of historical data to calculate the daily VaR for their positions, at the 99% confidence level. This daily VaR should be up scaled to a 10-day VaR figure to represent the banks having a 10-day 'holding period' for any given position. The recent financial crash has highlighted the importance and need for reliable models to predict VaR, and has led to further amendments to the regulations, which now require financial institutions to additionally calculate a 'stressed value-at-risk' measure using a one year data period in which the bank incurred significant losses (Basel, 2009).

Within the literature the ability of a variety of increasingly complex models (both parametric and non-parametric) to estimate and forecast VaR has been tested. These models can account variously for certain features of financial asset returns such as heteroskedasticity, asymmetry or leverage effects, leptokurtic distribution and long memory (hyperbolic decline of the conditional variance). See Alexander (2008) for more details. The various competing models have been compared using a range of distributions for the standardised residuals (Normal, Student's t, skewed Student's t, GED), across a number of markets for different levels of statistical significance, for both long and short positions.

At present, the findings in the literature are highly inconsistent as to which is the optimal model for estimating VaR. The best model appears to vary, amongst other factors, with the length of the data series, the market for which VaR is being estimated and the assumptions regarding the distribution of the standardised residuals (Angelidis et al., 2004). Furthermore, a model found to be superior for estimating VaR for long positions may not be optimal for estimating VaR for short positions due to the asymmetric distribution of financial returns (Shao et al., 2009).

A significant number of authors have highlighted the empirical success of the Generalised Autoregressive Conditional Heteroscedasticity (GARCH) framework (Bollerslev, 1986) to model low and high frequency returns and volatility, with many papers focussing on the selection of the optimal GARCH specification in order to calculate and predict VaR. For example, Giot and Laurent (2003a) report that the skewed Student APARCH model is superior for estimating both in-sample and out-of-sample VaR to models based on symmetrical error distributions. Subsequently, Giot and Laurent (2004), using stock index and exchange rate data, demonstrated that a skewed Student APARCH model of daily returns performs at least as well as more complex VaR models based on realised volatility. McMillan and Kambouroudis (2009) find that the APARCH specification is preferred for modelling 99% VaR (for long positions), arguing that the RiskMetrics, FIGARCH and HYGARCH models only perform well in forecasting the volatility of small emerging markets and for broader VaR measures.

The reliability of VaR measures depends on the correct specification of the underlying models (Caporin, 2003). According to Caporin (2003) and Tang and Shieh (2006), the best model for measuring VaR is the fractionally integrated GARCH (or FIGARCH). Furthermore, Caporin (2003) shows how the VaR is affected by model misspecification when the variance follows a long memory conditional heteroskedasticity process. In particular, he shows that the the Kupiec (1995) test[1] leads to the choice of a misspecified model.

Danielsson and Morimoto (2000) argue that conditional VaR models lead to more volatile VaR predictions than unconditional models. However, Kuester et al. (2006), comparing the out-of-sample performance of existing methods for predicting VaR using daily data from Nasdaq Composite index, find that a hybrid GARCH method with an extreme value theory-based approach performs best. Furthermore, they report that only conditional volatility models produce adequate one-step-ahead VaR forecasts, whilst unconditional models have a tendency to produce clustered VaR violations.

The search for effective VaR models has led to increasingly complex specifications being proposed. Practically, financial institutions would have to weigh up the supposed forecasting benefits of a highly parameterised model (compared with the simpler RiskMetrics[2] approach), with the additional cost and complexity of its estimation.

It should be acknowledged that whilst numerous studies demonstrate that financial volatility, especially for daily and higher frequency data, is highly persistent (Baillie et al., (1996), Bollerslev and Mikkelsen (1996), Nagayasu (2008)), there is a growing body of literature which questions whether some of this evidence of long memory within financial volatility may be spurious, and may be being caused, or appear to be being caused, by the existence of structural breaks over time within financial volatility (see for example, Granger and Hyung, 2004). Recently, McMillan and Ruiz (2009) have cautioned that evidence of long memory within volatility weakens if the unconditional variance is allowed to vary over time, rather than being assumed to be constant. Furthermore, Baillie and Morana (2009) have introduced a new Adaptive FIGARCH (A-FIGARCH) specification which accounts for both long memory and structural change within the conditional variance process. They use Monte Carlo simulation to demonstrate that this new specification outperforms the FIGARCH model in the presence of structural breaks, and provides improvements with regard to bias and efficiency associated with the estimation of the model parameters, without any significant computational cost.

---

[1] The Kupiec (1995) test evaluates whether or not the observed proportion of VaR violations resulting from the estimated VaR figure of a particular model is statistically different to the expected proportion of VaR violations determined by the significance level of the test.

[2] The RiskMetrics is equivalent to an IGARCH model with a normal distribution for the standardised residuals. The inferiority of this approach has been highlighted in the literature by, amongst others, McMillan and Kamboroudis (2009) and Giot and Laurent (2003b). The latter suggest that the complexity of the modelling software required to produce highly parameterised models explains why the simpler RiskMetrics technique remains popular in practice.

Grané and Veiga (2008) demonstrate that long memory (FIGARCH and HYGARCH) models outperform short memory GARCH specifications, but like the majority of VaR studies, limit their backtesting to forecasting horizon of just one trading day. By contrast, financial institutions are required by the Basel Committee to calculate the VaR of their positions for at least a 10-day holding period in order to calculate their minimum capital risk requirements (Basel, 2009). Although the Basel Committee suggest that 10-day VaR may be calculated by augmenting 1-day VaR using the square root of time rule[3], Engle (2004) and Danielsson (2002) criticise this technique on the basis that it makes the invalid assumption that the returns are independent and identically distributed and that volatilities over time are constant. Danielsson and Zigrand (2006) show that the square root of time scaling rule can lead to an underestimation of market risk, especially for longer time horizons. Nonetheless, this approach is widely employed, for example by Beltratti and Morana (1999) who find for short horizons of one, five and ten days, that the FIGARCH model produces similar VaR forecasts to the simpler GARCH model for high frequency exchange rate data.

Hartz et al. (2006) employ a resampling technique based on the bootstrap and bias correction step to improve the multi-period VaR forecasts produced by the simple normally distributed AR(1)-GARCH model. They employ another standard multi-period forecasting technique of iterating the conditional mean and conditional variance specifications, using expected values where the returns or innovations are inestimable. Brooks and Persand (2003) use a similar technique to investigate methods of evaluating multi-period volatility forecasts produced by a range of GARCH family and other linear models. Kinateder and Wagner (2010) show that a scaling-based GARCH-LM technique produces superior VaR forecasts to a benchmark fully parametric GARCH model utilising the Drost-Nijman (1993) formula for multiday volatility forecasts, especially for the five and ten day horizons.

Multiperiod VaR may be estimated using a variety of techniques, including parametric or variance-covariance approaches (e.g. GARCH), non-parametric approaches (e.g. historical simulation), semi-parametric approaches (e.g. CAViar, Extreme Value Theory) and Monte Carlo simulation (Dionne et al., 2009). For example, Semenov (2009) proposes a historical simulation technique which allows the accurate estimation of 1-day and 10-day VaR figures conditional on the historical sensitivity of assets returns (within a portfolio) to various macroeconomic factors (risk factor betas) over a period of time. Dionne et al. (2009) use a Monte Carlo approach to estimate intraday VaR (IVaR) using tick-by-tick data. Employing a log ACD-ARMA-EGARCH model[4], they find that the approach produces reliable estimates of intra-day risk. The model benefits from its greater informational content than IVaR estimates based on regularly spaced data, and a greater flexibility with regard to the estimation time horizon.

The key contribution of this research is to introduce a new adaptation to the FIGARCH model of the Monte Carlo simulation technique for estimating multiperiod VaR originally presented by Christoffersen (2003) for the GARCH framework. The paper seeks to establish the impact that accounting for long memory within the conditional variance process has upon the accuracy of the VaR

[3] To account for the non-linear price characteristics of options contracts, financial institutions are expected to move towards calculating a full 10-day VaR for positions involving such contracts (Basel, 2006).
[4] A log autoregressive conditional duration –autoregressive moving average – exponential generalised autoregressive conditional heteroskedastic model

forecasts produced. To the best of the authors' knowledge, this paper extends the existing literature, which is primarily focused on 1-step-ahead VaR forecasts, by comparing the ability of the simple GARCH model, with that of the FIGARCH model, in forecasting i) 1-step-ahead, ii) 10-step-ahead and iii) 20-step-ahead VaR. We compare the VaR forecasting performance of the GARCH and FIGARCH models on daily data across 21 leading stock indices worldwide, at the 95% confidence level.

Our results suggest that accounting for fractional integration within the conditional variance process does not improve the accuracy of the forecasts for the 1-day-ahead, 10-day-ahead and 20-day-ahead forecasting horizons relative to the short memory GARCH specification. Furthermore, in the case of the 10-step-ahead forecasting horizon its use appears to result in a significant reduction in forecasting accuracy. Therefore, the GARCH model is both simpler to estimate, and has a comparable, if not a superior, performance to the FIGARCH specification in terms of VaR forecast accuracy. Our results will provide valuable information to risk analysts and managers.

The remainder of the paper is organised as follows: Section 2 shows the methods for modelling 1-step-ahead and multiple-step-ahead VaR, while Section 3 describes our data. Section 4 presents the empirical analysis of this paper and Section 5 concludes the paper and summarises the main findings.

2. GARCH and VaR Models

2.1 Modelling GARCH and FIGARCH

To successfully capture the characteristics of financial returns data, many papers use GARCH family models under different distributional assumptions. In this paper, the assumption is made that the data generating process for the continuously compounded returns series, $\{y_t\}_{t=0}^{T} = \left\{\log \frac{p_t}{p_{t-1}}\right\}_{t=0}^{T}$, where p_t is the closing price on trading day t, follows an ARCH process (Engle, 1982):

$$y_t = \mu_t + \varepsilon_t$$

$$\varepsilon_t = z_t \sigma_t \text{ where } z_t \sim f(0, 1, \theta) \tag{1.1}$$

$$\sigma_t^2 = g\ (I_{t-1}; w)$$

The conditional mean is modelled using an ARMA (1,0)[5]. The error term ε_t is distributed on z_t which has a density function $f(\cdot)$, where $E(z_t) = 0$, Var $(z_t) = 1$, and θ represents the vector of parameters of f to be estimated. The conditional variance of the error term, σ_t^2, is a time-varying, positive and measureable function $g(\cdot)$ of the information set (I_{t-1}) at time t, with w the vector of parameters to be estimated in the conditional variance equation.

This research compares the performance of the GARCH (1,1) specification, introduced by Bollerslev (1986), with that of the fractionally integrated (FIGARCH) model, which allows for long memory within the conditional volatility of the returns (Baillie et al., 1996). In order to reduce the degree of parameterisation of the models, and to focus the analysis solely on the distinction in the performance between the long memory and short memory specifications, the models will initially be estimated under the standard normal density function for z_t:

$$f(z_t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z_t^2}{2}} \tag{1.2}$$

For the GARCH (*p*, *q*) specification, $g(\cdot)$ takes the functional form:

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{q} \alpha_i\, \varepsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j\, \sigma_{t-j}^2 \tag{1.3}$$

Equation 1.3 can be rewritten with lag operators as follows:

$$\sigma_t^2 = \alpha_0 + \alpha(L)\varepsilon_t^2 + \beta\ (L)\sigma_t^2 \tag{1.4}$$

Where $\alpha(L)$ and $\beta\ (L)$ are lag operator polynomials of order q and p respectively (Harris and Sollis, 2003). The roots of $\alpha(L)$ and $[1 - \alpha(L) - \beta\ (L)]$ must lie outside the unit circle for stability, and all of the parameters should be positive (Baillie, 1996). In this research, it is assumed that $p = q = 1$, since an order of one lag has been shown in the literature to be adequate in modelling conditional volatility (Angelidis and Degiannakis (2007)).

[5] Research suggests that the specification of the conditional mean is not important to the forecasting of the conditional variance. However, the proposed specification allows for discontinuous or non-synchronous trading in the stocks making up an index (Angelidis and Degiannakis, 2007)

Turning to the rate of decay of shocks to the conditional volatility process, Baillie et al. (1996) noted that the distinction between integrated specifications, where shocks affect the optimal volatility forecast indefinitely, as for example in the IGARCH (p, q) specification given by:

$$\emptyset\ (L)\ (1-L)\varepsilon_t^2 = \omega + [1-\ \beta(L)]\ \ (\varepsilon_t^2\ -\sigma_t^2) \tag{1.5}$$

and covariance stationary models, where shocks to the volatility process decay exponentially, such as the GARCH (p, q) specification, was too sharp. To solve this, Baillie et al. (1996) introduced the FIGARCH (p, d, q) process (defined in Equation 1.6), by replacing the first difference operator from Equation 1.5 with the fractional differencing operator$(1-L)^d$:

$$\emptyset\ (L)\ (1-L)^d\varepsilon_t^2 = \omega + [1-\ \beta(L)]\ \ (\varepsilon_t^2\ -\sigma_t^2) \tag{1.6}$$

Where $\emptyset(L) = \{\,1-\ \alpha(L)-\ \beta(L)\}(1-L)^{-1}$ and $\beta\ (L)$ is as in Equation 1.4. The roots of $\emptyset(L)$ and [1 - $\beta(L)$] lie outside of the unit circle.

The fractional differencing operator $(1-L)^d$ is defined as:

$$(1-L)^d = \sum_{j=0}^{\infty}\pi_j\ L^j \text{ where } \pi_j = \frac{\Gamma(j-d)}{\Gamma(j+1)\Gamma(-d)}$$

$$(1-L)^d = 1-\sum_{j=1}^{\infty}\pi_j L^j = 1-dL-\frac{1}{2!}d(1-d)L^2-\frac{1}{3!}d(1-d)(2-d)L^3-\cdots \tag{1.7}$$

In the FIGARCH model, $0 < d < 1$ indicates that shocks to the conditional variance decay at a hyperbolic rate (Baillie et al., 1996). The FIGARCH model nests the IGARCH(p, q) where $d = 1$, as well as the GARCH (p, q), where $d = 0$. FIGARCH processes are strictly stationary and ergodic but are not weakly stationary since the second moment is infinite. Once again, in this study it is assumed that $p = q = 1$.

2.2 Modelling one-step-ahead and multiple-step-ahead VaR

VaR is a single figure which represent a portfolio's worst possible outcome (either a significant loss when a long position is held, or an exceptionally high return if a short position is held), which is likely to occur under normal market conditions over a pre-determined period and for a given confidence level (95% or 99%). Thus it is a simple and easy to understand measure of market risk. However, the use of VaR has a number of limitations. Firstly, there is no indication of the size of the loss when it exceeds the VaR figure. Furthermore, VaR is not sub-additive, which means that the VaR of an overall portfolio may be greater than the sum of the VaRs of its component parts. This latter problem can be overcome by calculating the Expected Shortfall of the portfolio which is a coherent risk measure.[6] Moreover, the majority of VaR models suffer from too many VaR violations, suggesting an underestimation of market risk (Kuester et al., 2006).

Having estimated the parameters of the model, the one-step-ahead VaR is calculated using the following equation:

---

[6] A risk measure ρ is coherent if it is in accordance with the properties of (i) sub-additivity, (ii) homogeneity, (iii) monotonicity and (iv) risk-free condition. These are described in the following equations: (i) $\rho(x) + \rho(y) \leq \rho(x + y)$; (ii) $\rho(tx) = t\rho(x)$; (iii) $\rho(x) \geq \rho(y)\ if\ x \leq y$ and (iv) $\rho(x + n) = \rho(x) - n$. For further details, see Artzner et al. (1999).

$$VaR_{t+1|t} = \mu_{t+1|t} + F(a;\theta^{(t)})\sigma_{t+1|t} \quad (2.1)$$

where $\mu_{t+1|t}$ and $\sigma_{t+1|t}$ are the conditional forecasts of the mean and of the standard deviation at time $t+1$, given the information available at time t, respectively. $F(a;\theta^{(t)})$ is the ath quantile of the assumed distribution, given the estimated parameters θ at time t.

The key innovation of this paper is the estimation of multiple step ahead VaR for the FIGARCH specification. The methodology presented below is based on the numerical technique presented in Xekalaki and Degiannakis (2010) and was originally proposed by Christoffersen (2003). Consider the AR(1)-FIGARCH$(1,d,1)$ model with normally distributed conditional innovations in the framework which follows[7]:

$$\begin{gathered} y_t = c_0(1-c_1) + c_1 y_{t-1} + \varepsilon_t \\ \varepsilon_t = \sigma_t z_t \\ \sigma_t^2 = a_0 + (a_1 - b_1)\varepsilon_{t-1}^2 + \sum_{j=1}^{\infty}\left(\frac{d\Gamma(j-d)}{\Gamma(1-d)\Gamma(j+1)}L^j\left(\varepsilon_t^2 - a_1\varepsilon_{t-1}^2\right)\right) + b_1\sigma_{t-1}^2 \\ z_t \sim N(0,1). \end{gathered} \quad (2.2)$$

To generate the τ step ahead VaR forecast for the FIGARCH framework we employ a Monte Carlo simulation technique. The out-of-sample observations for each index ($\tilde{T}$) are divided into $\frac{\tilde{T}}{\tau}$ non-overlapping intervals of observations, with τ observations in each interval.[8] For each non-overlapping interval, we proceed as follows:

1) **Step 1.1:** Generate 5,000 pseudo-random numbers from a standard normal distribution, which are used to compute the innovations for period $t+1$ onwards.

   Step 1.2: Generate the return at time $t+1$ using the AR(1) process given in Equation 2.3, in which the value of the error term at time $t+1$ (which cannot be estimated), is simulated using the relation $\breve{\varepsilon}_{t+1} = \sigma_{t+1|t}\breve{z}_{i,1}$ from the GARCH framework.

   Step 1.3: Estimate the two-step ahead conditional variance using Equation 2.4. The values of the innovations up to time t are extracted from the Ox estimations, whilst the value of the innovation at time $t+1$ is estimated, as detailed above, using the relation from the GARCH framework, and is treated separately.

2) **Step 2.1:** Generate a further 5,000 pseudo-random numbers from the standard normal distribution, to be used to simulate the innovations for period $t+1$ onwards.

   Step 2.2: Generate the simulated return for $t+2$ using the AR(1) framework in Equation 2.5.

   Step 2.3: Estimate the simulated conditional variance for $t+3$ using Equation 2.6. All innovation terms relating to periods $t+1$ onwards are simulated using the relation from the GARCH framework.

3) Repeat the process for Step 3 through to Step $\tau - 1$

---

[7] Note that AR(1) is presented as $y_t = c_0 + e_t, e_t = c_1 e_{t-1} + \varepsilon_t$, thus $(y_t - c_0) = c_1(y_{t-1} - c_0) + \varepsilon_t$.

[8] The use of non-overlapping intervals is necessary to avoid autocorrelation in the forecast errors.

4) **Step $\boldsymbol{\tau}.\mathbf{1}$**: Generate a further 5,000 pseudo-random numbers from the standard normal distribution, to be used to simulate the innovations for period $t+1$ onwards.

   Step $\boldsymbol{\tau}.\mathbf{2}$: Generate the simulated return for $t+\tau$ using the AR(1) framework in Equation 2.7.

   Step $\boldsymbol{\tau}.\mathbf{3}$: Construct a density function for the returns at time $t+\tau$ using the 5,000 simulated returns. Calculate the 95% VaR figure for the left-hand tail of this distribution.

5) Repeat points 1-4 for each of the non-overlapping intervals of τ observations, such that a total of $\tilde{T}/\tau$ VaR forecasts will be produced.

The algorithm to estimate the τ -day-ahead VaR can be written mathematically as follows:

Step 1.1: Generate random numbers $\{\breve{z}_{i,1}\}_{i=1}^{MC}$ from the standard normal distribution, where MC denotes the number of draws (see Note 1).

Step 1.2: Create the hypothetical returns of time $t+1$, as (see Note 2):

$$\breve{y}_{i,t+1} = \sigma_{t+1|t}\breve{z}_{i,1} + c_0^{(t)}\left(1-c_1^{(t)}\right) + c_1^{(t)}y_t \text{, for } i = 1,\dots,MC. \tag{2.3}$$

Step 1.3: Create the forecast variance for time $t+2$ as (see Note 3):

$$\breve{\sigma}_{i,t+2}^2 = a_0^{(t)} + \left(a_1^{(t)} - b_1^{(t)}\right)\left(\sigma_{t+1|t}\breve{z}_{i,1}\right)^2 + d\left(\left(\sigma_{t+1|t}\breve{z}_{i,1}\right)^2 - a_1\varepsilon_{t/t}^2\right) + \sum_{j=2}^{\infty}\left(\frac{d^{(t)}\Gamma\left(j-d^{(t)}\right)}{\Gamma\left(1-d^{(t)}\right)\Gamma\left(j+1\right)}L^j\left(\varepsilon_{t+2|t+j}^2 - a_1^{(t)}\varepsilon_{t+1|t+j}^2\right)\right) + b_1^{(t)}\sigma_{t+1|t}^2. \tag{2.4}$$

Step 2.1: Generate further random numbers, $\{\breve{z}_{i,2}\}_{i=1}^{MC}$, $\breve{z}_{i,2} \sim N(0,1)$.

Step 2.2: Calculate the hypothetical returns of time $t+2$,

$$\breve{y}_{i,t+2} = \breve{\sigma}_{i,t+2}\breve{z}_{i,2} + c_0^{(t)}\left(1-c_1^{(t)}\right) + c_1^{(t)}\breve{y}_{i,t+1} \text{, for } i = 1,\dots,MC. \tag{2.5}$$

Step 2.3: Create the forecast variance for time $t+3$ as (see Note 4):

$$\breve{\sigma}_{i,t+3}^2 = a_0^{(t)} + \left(a_1^{(t)} - b_1^{(t)}\right)\left(\breve{\sigma}_{i,t+2}\breve{z}_{i,2}\right)^2 + d\left(\left(\breve{\sigma}_{i,t+2}\breve{z}_{i,2}\right)^2 - a_1\left(\breve{\sigma}_{i,t+1}\breve{z}_{i,1}\right)^2\right) + \frac{1}{2!}d\left(1-d\right)\left(\left(\breve{\sigma}_{i,t+1}\breve{z}_{i,1}\right)^2 - a_1\varepsilon_{t/t}^2\right) + \sum_{j=3}^{\infty}\left(\frac{d^{(t)}\Gamma\left(j-d^{(t)}\right)}{\Gamma\left(1-d^{(t)}\right)\Gamma\left(j+1\right)}L^j\left(\varepsilon_{t+3|t+j}^2 - a_1^{(t)}\varepsilon_{t+2|t+j}^2\right)\right) + b_1^{(t)}\breve{\sigma}_{i,t+2}^2. \tag{2.6}$$

…

Step τ .1: Generate further random numbers, $\{\breve{z}_{i,\tau}\}_{i=1}^{MC}$, $\breve{z}_{i,\tau} \sim N(0,1)$.

Step τ .2: Calculate the hypothetical returns of time $t+\tau$,

$$\breve{y}_{i,t+\tau} = \breve{\sigma}_{i,t+\tau}\breve{z}_{i,\tau} + c_0^{(t)}\left(1-c_1^{(t)}\right) + c_1^{(t)}\breve{y}_{i,t+\tau-1}. \tag{2.7}$$

Step τ .3: Calculate the τ -day VaR as $VaR_{t+\tau|t}^{(1-p)} = \mathrm{f}_a\left(\{\breve{y}_{i,t+\tau}\}_{i=1}^{MC}\right)$.

In a similar fashion, τ-step-ahead VaR forecasts are also produced for the GARCH specification using the methodology originally proposed by Christoffersen (2003).

For the purposes of this research, a model is considered to accurately forecast the τ-step-ahead VaR if it cannot be rejected by both the unconditional and independence hypotheses. The unconditional hypothesis is evaluated using a test developed by Kupiec (1995), and examines the null hypothesis that the observed violation rate is statistically equal to the expected violation rate.

In order to test this null hypothesis, the following likelihood ratio statistic is used:

$$LR_{UC} = 2\ln\left((1-\pi_0)^{\tilde{T}-N}(\pi_0)^N\right) - 2\ln\left((1-p)^{\tilde{T}-N}p^N\right) \sim\chi_1^2 \qquad (2.8)$$

where N is the number of days on which a violation occurred across the total VaR estimation period $\tilde{T}$, $\frac{N}{\tilde{T}} = \pi_0$ is the observed violation rate and p is the expected violation rate. The null hypothesis will be rejected wherever the violation rate which a model exhibits is too high or too low relative to p, although no account is taken of any dependence between the instances of VaR violation.

The Kupiec (1995) test cannot distinguish if the distribution of the exceptions is uncorrelated over time. Potential clustering of the VaR violations is an important consideration, and is tested for using the independence hypothesis. This forms part of the test put forward by Christoffersen (1998) which simultaneously examines whether i) the observed VaR failure rate is equal to the expected failure rate, and ii) the instances of VaR failure are independent. The first part of this is tested using the likelihood ratio statistic described by Equation 2.8, whilst the second part is examined using the likelihood ratio statistic given below:

$$LR_{IN} = 2\ln\left((1-\pi_{01})^{n_{00}}\pi_{01}^{n_{01}}(1-\pi_{11})^{n_{10}}\pi_{11}^{n_{11}}\right) - 2\ln\left((1-\pi_0)^{n_{00}+n_{10}}\pi_0^{n_{01}+n_{11}}\right) \sim\chi_1^2 \qquad (2.9)$$

In Equation 2.9, n_{ij} is the number of observations with value i followed by j, for $i,j = 0{,}1$, and $\pi_{ij} = \frac{n_{ij}}{\sum_j n_{ij}}$ are the corresponding probabilities. In this case, $i,j = 1$ indicates that a violation has occurred, whereas $i,j = 0$ indicates the converse. π_{ij} indicates the probability that $j\ (= 0{,}1)$ occurs at time t, given that $i\ (= 0{,}1)$ occurred at time $t-1$. The null hypothesis is $H_0\colon \pi_{01} = \pi_{11}$, which is tested against the alternative hypothesis $H_a\colon \pi_{01} \neq \pi_{11}$. The purpose of the test is to examine the null hypothesis that the VaR failures are independent and are spread over the whole estimation period, against the alternative hypothesis that the failures tend to be clustered.

If the null hypothesis of both the unconditional and independence hypotheses is not rejected for a particular model, then it is possible to conclude that the model produces the expected proportion of VaR violations, and that these violations are not clustered together. However, it does not provide a method for distinguishing between the performance of the various models for which this is the case.

In the event of a VaR violation, the expected shortfall (ES) is defined as the conditional expected loss. The τ -day ahead expected shortfall (ES) forecast for long trading positions is the τ-day-ahead expected value of the loss, given that the return at time $t+\tau$ falls below the corresponding value of the VaR forecast, and is defined by:

$$ES_{t+\tau|t}^{(1-p)} = E\left(y_{t+\tau} \mid \left(y_{t+\tau} \le VaR_{t+\tau|t}^{(1-p)}\right)\right), \qquad (2.10)$$

The value of the τ -day-ahead ES measure is given by:

$$ES_{t+\tau|t}^{(1-p)} = E\left(VaR_{t+\tau|t}^{(1-\tilde{p})}\right),\ \forall 0 < \tilde{p} < p\,. \qquad (2.11)$$

Following Dowd (2002), to calculate the Expected Shortfall we divide the tail of the probability distribution of returns into a large number $\tilde{k}$ of slices each with identical probability mass, estimate the τ-day-ahead VaR attached to each slice and find the mean of these VaRs to estimate the τ-day-ahead ES[9].

$$ES_{t+\tau|t}^{(1-p)} = \tilde{k}^{-1} \sum_{i=1}^{\tilde{k}} \left(VaR_{t+\tau|t}^{(1-p+ip(\tilde{k}+1)^{-1})} \right). \qquad (2.12)$$

Finally, to estimate the model's ability to forecast losses when the VaR is violated, the mean squared error $MSE = \tau\tilde{T}^{-1}\sum_{t=1}^{\tilde{T}} \psi_{t+\tau}$, for $t+\tau$=11, 21, 31, ..., $\tilde{T}$ (for a ten-day-ahead forecasting step) is calculated based on the quadratic loss function in Equation 2.13.

$$\psi_{t+\tau} = \begin{cases} \left(y_{t+\tau} - ES_{t+\tau|t}^{(1-p)} \right)^2 & if\ a\ violation\ occurs \\ 0 & otherwise \end{cases} \qquad (2.13)$$

Where $\tilde{T}$ is the total number of out-of-sample one-step-ahead VaR forecasts for a given index (refer to Section 3 for more details). Thus for each VaR failure we compare the actual return to the forecasted return, given that the VaR is violated. The model will be deemed to perform well if:

i) the observed failure rate equals the expected failure rate (Kupiec (1995) test);

ii) the VaR failures occur independently of each other (Christoffersen (1998) test);

iii) the MSE based on the quadratic loss between the actual and expected returns in the event of a VaR violation is minimised.

3. Data Description

In order to examine the robustness of the VaR and ES forecasting performance of the selected volatility models, the VaR forecasts were generated using daily returns data from 21 developed market stock indices[10]. The data, which was obtained from *Datastream* for the period from 12th January 1989 until 12th February 2009, was conditioned to remove any non-trading days. Thus the total number of log returns for a given index ($\hat{T}$) ranged from 4,924 for the Japanese and Korean indices, to 5,072 for the Dutch index. Based on a rolling sample of T=2,000 observations, a total of $\tilde{T} = \hat{T} - T$ out-of-sample forecasts were produced for each model (with the parameters of the conditional mean, conditional variance and density function re-estimated each trading day).[11]

Descriptive statistics for the daily log returns for the selected indices are given in Table 1. All of the returns distributions are leptokurtic and the majority are negatively skewed. The Jarque-Bera test results indicate that none of the log returns series follow a Gaussian distribution. The absolute value of the log returns are significantly positively autocorrelated for a high number of lags. Examining the

---

[9] In this study we take $\tilde{k}$ to be 5,000

[10] The indices were AEX Index (AMSTEOE), ATHEX Composite (GRAGENL), Austrian Traded Index (ATXINDX), CAC 40 (FRCAC40), Dax 30 Performance (DAXINDX), Dow Jones Industrial (DJINDUS), FTSE 100 (FTSE100), Ireland SE Overall (ISEQUIT), Hang Seng (HNGKNGI), Korea SE Composite (KORCOMP), Madrid SE General (MADRIDI), Mexico IPC (MXIPC35), NASDAQ Composite (NASCOMP), NASDAQ 100 (NASA 100), Nikkei 225 Stock Average (JAPDOWA), NYSE Composite (NYSEALL), OMX Stockholm (SWSEALI), Portugal PSI General (POPSIGN), S&P500 Composite (S&PCOMP), S&P/TSX Composite (TTOCOMP) and Swiss Market (SWISSMI).

[11] The estimations were carried out using the G@RCH 6 (Laurent (2009)) package of Ox (Doornik (2009)).

correlograms for the indices, the decay in the value of the autocorrelation coefficients is initially rapid, before slowing, suggestive of the hyperbolic decay typical for a long memory volatility process.[12]

Table 1: Descriptive Statistics								
INDEX	Observations	Mean	Median	Std. Dev.	Skewness	Kurtosis	Jarque-Bera	Probability[1]
AMSTEOE	5072	0.016836	0.067192	1.395972	-0.16156	9.766062	9696.817	0.000
ATXINDX	4945	0.019052	0.054244	1.408684	-0.28461	10.83047	12700.47	0.000
DAXINDX	5044	0.025316	0.083406	1.48699	-0.12667	8.012514	5293.989	0.000
DJINDUS	5029	0.026568	0.049052	1.125719	-0.11793	11.44804	14966.53	0.000
FRCAC40	5050	0.0135	0.033854	1.412313	-0.0377	7.751845	4752.41	0.000
FTSE100	5051	0.016534	0.040528	1.150512	-0.11488	9.561969	9073.327	0.000
GRAGENL	4936	0.032748	0.018962	1.736119	0.044387	7.844572	4828.592	0.000
HNGKNGI	4944	0.042274	0.062797	1.725812	0.007238	12.05344	16884.78	0.000
ISEQUIT	5017	0.010447	0.049028	1.257524	-0.65183	13.22265	22200.69	0.000
JAPDOWA	4924	-0.02745	-0.01268	1.578394	-0.02021	8.277182	5713.949	0.000
KORCOMP	4924	0.011606	0.041168	1.911157	-0.11475	7.005832	3303.052	0.000
MADRIDI	4992	0.02815	0.077387	1.283485	-0.20082	8.496868	6318.39	0.000
MXIPC35	5003	0.088184	0.103812	1.640208	0.029248	8.16869	5569.752	0.000
NASA100	5031	0.041204	0.121181	1.900873	0.098471	7.94713	5138.512	0.000
NASCOMP	5042	0.031029	0.114602	1.57385	-0.04783	8.913875	7349.36	0.000
NYSEALL	5035	0.025034	0.058148	1.122936	-0.3696	15.23759	31532.73	0.000
POPSIGN	4984	0.021379	0.018262	0.966043	-0.42194	15.9026	34719.61	0.000
S&PCOMP	5039	0.022856	0.048103	1.17312	-0.19843	12.1539	17626.3	0.000
SWISSMI	5023	0.025303	0.067464	1.218007	-0.14656	9.126867	7874.477	0.000
SWSEALI	4999	0.030183	0.066004	1.393286	0.148244	7.555802	4341.474	0.000
TTOCOMP	5029	0.021684	0.061911	1.055662	-0.76711	14.25989	27059.97	0.000

[1] This column displays the p-value for the Jarque-Bera test which has as its null hypothesis that the returns series follow a Gaussian distribution

[12] Correlograms for the absolute log returns of the 21 indices are available from the authors on request.

4. Empirical Analysis

The results for the one-step-ahead VaR forecasting across the 21 indices for both the FIGARCH and GARCH specifications are shown in Table 2. Overall, the fractionally integrated modelling of conditional volatility does not appear to improve the forecasting accuracy of VaR across the 21 stock indices for the one-step-ahead time horizon. Furthermore, the results appear to corroborate findings from the literature that VaR models are not robust across different markets, so that the optimal model varies from one index to the next (Angelidis et al., 2004; McMillan and Kambouroudis, 2009).

According to the results of the Kupiec (1995) test at the 5% significance level, the observed violation rate is not statistically different to the expected violation rate (5%) for the one-step-ahead VaR forecasts produced by both the GARCH and FIGARCH models for the ATXINDX, GRAGENL, HNGKNGI, MXIPC35, POPSIGN and S&PCOMP indices. This is also the case for the one-step-ahead VaR forecasts produced by the FIGARCH specification for the JAPDOWA and MADRIDI indices, and for the one-step-ahead VaR forecasts produced by the GARCH model for the DJINDUS index. In general, the models appear to underestimate the true VaR figure, as the observed proportion of VaR violations exceeds the expected value of 5% in almost all cases, sometimes by a large amount. This is in accordance with the findings of Kuester et al. (2006) who report that the majority of VaR models suffer from excessive VaR violations due to the models underestimating the true VaR figure.

According to the Christoffersen (1998) test, the VaR violations are independently distributed for the majority of the stock indices for both models, with just one exception, that of the ATXINDX for the FIGARCH $(1, d, 1)$ specification. However, although there is limited evidence of clustering of the VaR violations, this is overridden by the results of the Kupiec test suggesting widespread underestimation of the true VaR figure by both models.

Table 3 shows the results for the 10-step-ahead VaR forecasting. For this forecasting horizon, the long memory FIGARCH specification appears to significantly underperform the GARCH model. According to the Kupiec test, the FIGARCH specification produces an observed exception rate which is not statistically different to the anticipated failure rate of 5% for only 4 of the 21 indices. The corresponding figure for the GARCH model is 14 out of 21 indices. Although this represents an improvement over the long memory specification, it once again suggests that the modelling results are not robust across the different indices tested. Whilst the results of the Christoffersen test indicate that the VaR violations are independently distributed throughout the sample, this result is of overshadowed by the earlier results from the Kupiec test.

Table 4 shows the results for the forecasting of 20-step-ahead VaR across the 21 indices for both the FIGARCH and GARCH models. For this longer time horizon the performance of the FIGARCH model improves from the 10-step-ahead forecasting period, although the Kupiec test results suggest that the observed exception rate is not statistically different to the expected failure rate for just 8 of the indices. Furthermore, the Christoffersen test results suggest that for one of these indices, namely the MXIPC35, the VaR violations are not independently distributed. Conversely, the performance of the GARCH model worsens from the 10-day-horizon, and is now only marginally better than that of the FIGARCH model, with the Kupiec test indicating an adequate forecasting performance for 9 of the 21 indices, with none of these 9 indices showing evidence of clustering of VaR violations according to the Christoffersen test.

For all forecasting time horizons and amongst the indices for which the models perform adequately according to the Kupiec and Christoffersen tests, there is a greater range between the maximum and minimum MSE when VaR is modelled using the GARCH framework compared to the FIGARCH model. Another emerging pattern suggests that the longer the VaR forecasting time horizon, the less both models underestimate the true VaR. For the 1-day ahead time horizon, the observed failure rate was more than 5% in all 21 cases for the FIGARCH model and in 20 cases for the GARCH specification. At the 10-day horizon the observed failure rate exceeded 5% in 18 cases (FIGARCH) and 16 cases (GARCH), whilst for the 20-day horizon the observed failure rate exceeded 5% in 14 and 11 cases for the FIGARCH and GARCH models respectively.

Table 2: 1-step-ahead VaR modelling results							
Index	**Number of 1-step-ahead VaR forecasts**	**Average VaR**	**Average ES**	**MSE**	**Observed exception rate**	**Kupiec p-value**	**Christoffersen p-value**
PART A. ARMA (1,0) - FIGARCH (1,d,1)							
AMSTEOE	3072	-2.2792	-2.8731	0.0410	0.0599	0.0145*	0.5369
ATXINDX	2945	-2.0090	-2.5336	0.0420	0.0557	0.1639	0.0278*
DAXINDX	3044	-2.4260	-3.0590	0.0400	0.0647	0.0004**	0.3937
DJINDUS	3029	-1.8316	-2.3105	0.0363	0.0581	0.0458*	0.5662
FRCAC40	3050	-2.2516	-2.8375	0.0381	0.0636	0.0009**	0.4328
FTSE100	3051	-1.8886	-2.3785	0.0271	0.0597	0.0174*	0.3296
GRAGENL	2936	-2.5534	-3.2104	0.0521	0.0548	0.2361	0.1624
HNGKNGI	2944	-2.6380	-3.3248	0.0684	0.0540	0.3243	0.8833
ISEQUIT	3017	-2.0152	-2.2600	0.0605	0.0620	0.0035**	0.6102
JAPDOWA	2924	-2.4826	-3.1168	0.0500	0.0575	0.0705	0.9069
KORCOMP	2924	-3.1120	-3.9140	0.1006	0.0612	0.0071**	0.9902
MADRIDI	2992	-2.0189	-2.5497	0.0353	0.0551	0.2035	0.6936
MXIPC35	3003	-2.4147	-3.0587	0.0471	0.0546	0.2529	0.4732
NASA100	3031	-3.1023	-3.9150	0.0575	0.0591	0.0260*	0.8498
NASCOMP	3042	-2.6646	-3.3614	0.0440	0.0648	0.0003**	0.2375
NYSEALL	3035	-1.8148	-2.2886	0.0293	0.0583	0.0402*	0.1448
POPSIGN	2984	-1.5276	-1.9288	0.0378	0.0576	0.0613	0.7195
S&PCOMP	3039	-1.9304	-2.4338	0.0323	0.0576	0.0608	0.7135
SWISSMI	3023	-1.9190	-2.4226	0.0321	0.0642	0.0006**	0.6541
SWSEALI	2999	-2.2546	-2.8519	0.0386	0.0580	0.0492*	0.7090
TTOCOMP	3029	-1.7537	-2.2138	0.0431	0.0650	0.0003**	0.2309
PART B. ARMA (1,0) - GARCH (1,1)							
AMSTEOE	3072	-2.2717	-2.8630	0.0441	0.0628	0.0017**	0.2553
ATXINDX	2945	-1.9673	-2.4814	0.0457	0.0574	0.0721	0.1665
DAXINDX	3044	-2.3908	-3.0150	0.0422	0.0673	0.0000**	0.0715
DJINDUS	3029	-1.8436	-2.3259	0.0343	0.0555	0.1748	0.8158
FRCAC40	3050	-2.2340	-2.8152	0.0379	0.0643	0.0005**	0.8558
FTSE100	3051	-1.8950	-2.3860	0.0299	0.0633	0.0012**	0.2666
GRAGENL	2936	-2.5960	-3.2636	0.0513	0.0497	0.9459	0.3093
HNGKNGI	2944	-2.6234	-3.3063	0.0649	0.0571	0.0851	0.4251
ISEQUIT	3017	-2.0311	-2.2792	0.0592	0.0626	0.0021**	0.7237
JAPDOWA	2924	-2.4657	-3.0959	0.0451	0.0592	0.0269*	0.8025
KORCOMP	2924	-3.1300	-3.9360	0.1034	0.0612	0.0071	0.7417
MADRIDI	2992	-1.9849	-2.5076	0.0376	0.0615	0.0053**	0.2700
MXIPC35	3003	-2.4233	-3.0696	0.0485	0.0546	0.2529	0.7303
NASA100	3031	-3.1273	-3.9452	0.0619	0.0600	0.0138*	0.9826
NASCOMP	3042	-2.6740	-3.3731	0.0504	0.0667	0.0001**	0.0377*
NYSEALL	3035	-1.8209	-2.2964	0.0331	0.0603	0.0116	0.5397
POPSIGN	2984	-1.5747	-1.9879	0.0350	0.0563	0.1213	0.6044
S&PCOMP	3039	-1.9582	-2.4686	0.0347	0.0559	0.1401	0.1986
SWISSMI	3023	-1.8662	-2.3566	0.0347	0.0708	0.0000**	0.2306
SWSEALI	2999	-2.2357	-2.8287	0.0408	0.0610	0.0073**	0.7047
TTOCOMP	3029	-1.7501	-2.2088	0.0431	0.0660	0.0001**	0.1779

Table 3: 10-step-ahead VaR modelling results							
Index	**Number of 10-step-ahead VaR forecasts**	**Average VaR**	**Average ES**	**MSE**	**Observed exception rate**	**Kupiec p-value**	**Christoffersen p-value**
PART A. ARMA (1,0) - FIGARCH (1,d,1)							
AMSTEOE	307	-2.2143	-2.9164	0.1003	0.0391	0.0039**	0.4799
ATXINDX	294	-1.9578	-2.5870	0.0550	0.0612	0.0074**	0.1032
DAXINDX	504	-2.3837	-3.1187	0.0530	0.0757	0.0000**	0.3089
DJINDUS	502	-1.7669	-2.2889	0.0280	0.0695	0.0000**	0.6636
FRCAC40	305	-2.1960	-2.8345	0.0295	0.0426	0.0554	0.5723
FTSE100	305	-1.8245	-2.3744	0.0742	0.0557	0.1543	0.9576
GRAGENL	293	-2.5373	-3.4059	0.0621	0.0648	0.0005**	0.0284*
HNGKNGI	294	-2.5982	-3.3747	0.0840	0.0612	0.0073**	0.4091
ISEQUIT	301	-1.9125	-2.4978	0.0328	0.0797	0.0000**	0.9502
JAPDOWA	292	-2.4568	-3.1693	0.2446	0.0582	0.0487*	0.3375
KORCOMP	292	-3.0196	-3.9230	0.0876	0.0514	0.7481	0.7938
MADRIDI	299	-1.9659	-2.5761	0.0382	0.0635	0.0011**	0.8331
MXIPC35	300	-2.3522	-3.1283	0.0589	0.0500	0.9900	0.2080
NASA100	303	-2.9765	-3.8647	0.1303	0.0759	0.0000**	0.3462
NASCOMP	304	-2.5644	-3.3831	0.2638	0.0592	0.0239*	0.9426
NYSEALL	303	-1.7591	-2.2930	0.0850	0.0627	0.0021**	0.1291
POPSIGN	298	-1.4947	-2.0610	0.0213	0.0638	0.0010**	0.1169
S&PCOMP	303	-1.8543	-2.4071	0.0504	0.0627	0.0017**	0.0238*
SWISSMI	302	-1.8773	-2.4733	0.0580	0.0629	0.0018**	0.4710
SWSEALI	299	-2.2084	-2.9238	0.0657	0.0736	0.0000**	0.6622
TTOCOMP	302	-1.7096	-2.2458	0.0659	0.0596	0.0167*	0.9108
PART B. ARMA (1,0) - GARCH (1,1)							
AMSTEOE	307	-2.2739	-2.9716	0.1026	0.0423	0.0450*	0.5685
ATXINDX	294	-1.9533	-2.5699	0.0553	0.0680	0.0000**	0.0419*
DAXINDX	504	-2.3863	-3.1146	0.0563	0.0954	0.0000**	0.4000
DJINDUS	502	-1.8516	-2.3906	0.0224	0.0563	0.1100	0.1536
FRCAC40	305	-2.2315	-2.8631	0.0246	0.0426	0.0554	0.5723
FTSE100	305	-1.9029	-2.4583	0.0674	0.0492	0.8318	0.7618
GRAGENL	293	-2.7451	-3.7361	0.0493	0.0580	0.0550	0.0747
HNGKNGI	294	-2.6550	-3.4372	0.0761	0.0578	0.0596	0.3335
ISEQUIT	301	-2.0239	-2.6241	0.0207	0.0764	0.0000**	0.5029
JAPDOWA	292	-2.5059	-3.2176	0.2253	0.0514	0.7481	0.7938
KORCOMP	292	-3.1490	-4.0669	0.0756	0.0514	0.7481	0.7938
MADRIDI	299	-1.9703	-2.5691	0.0270	0.0602	0.0133*	0.9282
MXIPC35	300	-2.4919	-3.3124	0.0509	0.0467	0.3902	0.2408
NASA100	303	-3.1383	-4.0272	0.0886	0.0693	0.0000**	0.0545
NASCOMP	304	-2.6540	-3.4912	0.2334	0.0559	0.1435	0.9603
NYSEALL	303	-1.8349	-2.3786	0.0800	0.0561	0.1356	0.3161
POPSIGN	298	-1.6664	-2.3728	0.0158	0.0570	0.0871	0.1636
S&PCOMP	303	-1.9813	-2.5435	0.0379	0.0462	0.3137	0.2433
SWISSMI	302	-1.8297	-2.4231	0.0538	0.0563	0.1227	0.3179
SWSEALI	299	-2.2493	-2.9797	0.0472	0.0602	0.0116*	0.1281
TTOCOMP	302	-1.7648	-2.3085	0.0650	0.0563	0.1100	0.9658

Table 4: 20-step-ahead VaR modelling results							
Index	**Number of 20-step-ahead VaR forecasts**	**Average VaR**	**Average ES**	**MSE**	**Observed exception rate**	**Kupiec p-value**	**Christoffersen p-value**
PART A. ARMA (1,0) - FIGARCH (1,d,1)							
AMSTEOE	153	-2.1929	-2.9664	0.0837	0.0523	0.5432	0.3457
ATXINDX	147	-1.9353	-2.6041	0.1201	0.0748	0.0000**	0.8431
DAXINDX	152	-2.3759	-3.1815	0.0349	0.0461	0.3040	0.4457
DJINDUS	151	-1.7433	-2.2948	0.1522	0.0728	0.0000**	0.8219
FRCAC40	152	-2.1893	-2.8817	0.0727	0.0461	0.2926	0.4093
FTSE100	152	-1.8114	-2.4085	0.0518	0.0658	0.0001**	0.1414
GRAGENL	146	-2.5173	-3.4558	0.0398	0.0616	0.0051**	0.3042
HNGKNGI	147	-2.5828	-3.4149	0.0349	0.0476	0.5394	0.4010
ISEQUIT	150	-1.8749	-2.4824	0.0341	0.0800	0.0000**	0.3054
JAPDOWA	146	-2.4164	-3.1703	0.0082	0.0411	0.0219*	0.4716
KORCOMP	146	-2.9924	-3.9593	0.1267	0.0479	0.5963	0.3993
MADRIDI	149	-1.9552	-2.6116	0.0367	0.0537	0.3446	0.3388
MXIPC35	150	-2.3390	-3.1549	0.0707	0.0467	0.3902	0.0282*
NASA100	151	-2.9303	-3.8604	0.0610	0.0596	0.0171*	0.5458
NASCOMP	152	-2.5020	-3.3686	0.3173	0.0592	0.0239*	0.2853
NYSEALL	151	-1.7308	-2.2929	0.0356	0.0728	0.0000**	0.1868
POPSIGN	149	-1.4615	-2.0694	0.0094	0.0671	0.0000**	0.2284
S&PCOMP	151	-1.8266	-2.4045	0.0400	0.0728	0.0000**	0.8219
SWISSMI	151	-1.9058	-2.5744	0.0793	0.0464	0.3464	0.3090
SWSEALI	149	-2.1658	-2.9428	0.1061	0.0738	0.0000**	0.0342*
TTOCOMP	151	-1.7006	-2.2734	0.1151	0.0728	0.0000**	0.8219
PART B. ARMA (1,0) - GARCH (1,1)							
AMSTEOE	153	-2.2896	-3.0813	0.0222	0.0392	0.0039**	0.4825
ATXINDX	147	-1.9360	-2.6161	0.1250	0.0680	0.0000**	0.1512
DAXINDX	152	-2.3736	-3.1831	0.0233	0.0395	0.0055**	0.5207
DJINDUS	151	-1.8627	-2.4515	0.1364	0.0662	0.0001**	0.2318
FRCAC40	152	-2.2477	-2.9269	0.0627	0.0461	0.2926	0.4093
FTSE100	152	-1.9136	-2.5249	0.0644	0.0526	0.4865	0.3440
GRAGENL	146	-2.8315	-4.0362	0.0643	0.0548	0.2361	0.3667
HNGKNGI	147	-2.6596	-3.5204	0.0514	0.0408	0.0176*	0.4732
ISEQUIT	150	-2.0047	-2.6492	0.0244	0.0667	0.0001**	0.6854
JAPDOWA	146	-2.4931	-3.2518	0.0043	0.0342	0.0000**	0.5501
KORCOMP	146	-3.1844	-4.2056	0.0768	0.0411	0.0219*	0.4716
MADRIDI	149	-1.9788	-2.6439	0.0217	0.0470	0.4666	0.4044
MXIPC35	150	-2.5278	-3.4393	0.0684	0.0400	0.0090**	0.0130*
NASA100	151	-3.1448	-4.1064	0.0660	0.0464	0.3739	0.3090
NASCOMP	152	-2.6196	-3.5615	0.2221	0.0526	0.5144	0.3440
NYSEALL	151	-1.8336	-2.4266	0.0321	0.0728	0.0000**	0.1868
POPSIGN	149	-1.7368	-2.6582	0.0289	0.0604	0.0121*	0.2802
S&PCOMP	151	-1.9922	-2.6065	0.0111	0.0530	0.4555	0.4207
SWISSMI	151	-1.8528	-2.5202	0.0793	0.0530	0.4642	0.4207
SWSEALI	149	-2.2321	-3.0435	0.0942	0.0470	0.4490	0.3137
TTOCOMP	151	-1.8018	-2.4249	0.0958	0.0596	0.0167*	0.5458

5. Conclusion and Suggestions for Further Research

This research has examined whether or not accounting for fractional integration in the volatility process improves VaR forecasting performance, particularly as the forecasting time horizon lengthens. To this end, the paper proposes a new adaptation and application of the Monte Carlo simulation technique of Christoffersen (2003) to estimating multiple step ahead VaR forecasts using the FIGARCH model. The models were tested across 21 leading stock indices worldwide over the period from 1989-2009, at the 95% confidence level, for 1-step-ahead, 10-steps-ahead and 20-steps-ahead VaR forecasts.

The modelling results suggest that despite evidence of persistence in the volatility process, accounting for long memory in the model did not improve the VaR forecasting accuracy relative to the short memory specification, and in the case of the 10-step-ahead forecasting horizon led to a significant reduction in the forecast accuracy. Kuester et al. (2006) find that the majority of VaR models suffer from excessive VaR violations, implying an underestimation of market risk. Our results suggest that for both modelling specifications underestimation of the true VaR becomes less prevalent as the forecasting time horizon increases. In accordance with the findings of previous research (for example, Beltratti and Morana, 1999), the GARCH specification might be preferred to the FIGARCH model given that it performs at least as well, if not better than, the long memory specification in forecasting VaR across all time horizons, and is a much simpler model to estimate.

Future research might seek to improve the forecasting performance of the models by re-estimating them using the skewed Student's t distribution for the standardised residuals, which in accounting for the skewness and excess kurtosis of financial returns has been widely suggested to improve VaR forecasting accuracy relative to models based on symmetrical distributions (Giot and Laurent 2003a, 2004; Tang and Shieh, 2006; McMillan and Kamboroudis, 2009). Due to the use of non-overlapping intervals, as the forecasting time horizon increases, the number of VaR forecasts produced decreased by a factor equal to the length of the forecast period. As a result, particularly for the 20-day time horizon, the results of the Kupiec and Christoffersen tests were highly sensitive to the number of VaR violations such that a very small number of additional (or fewer) violations can be pivotal in whether or not the forecasting performance of the model is deemed to be adequate. Furthermore, the Kupiec model has been shown to lack power when the number of observations is small (Crouhy et al., 2001). Future research would require the use of a much longer time series of observations for the results to be more robust. Additionally, the emerging observation that the underestimation of the true VaR becomes less prevalent as the forecasting time horizon increases also warrants further investigation.

Acknowledgements

Dr. Christos Floros and Dr. Stavros Degiannakis acknowledge the support from the European Community's Seventh Framework Programme (FP7-PEOPLE-IEF) funded under grant agreement no. PIEF-GA-2009-237022.

References

Alexander, C. (2008). *Value-at-Risk Models*. John Wiley & Sons Ltd.

Angelidis, T., Benos, A. and Degiannakis, S. (2004). The use of GARCH models in VaR estimation. *Statistical Methodology, 1,* 105-128

Angelidis, T. and Degiannakis, S. (2007). Backtesting VaR models: a two-stage procedure. *Journal of Risk Model Validation, 1* (2), 1-22

Artzner, P., Delbaen, F., Eber, J. and Heath, D., (1999). Coherent Measures of Risk. *Mathematical Finance, 9* (3), 203-228.

Baillie, R. (1996). Long memory processes and fractional integration in econometrics. Journal of Econometrics, 73, 5-59.

Baillie, R., Bollerslev, T. and Mikkelsen, H. (1996). Fractionally integrated generalised autoregressive conditional heteroscedasticity. *Journal of Econometrics, 74*, 3-30

Baillie R. and Morana, C. (2009). Modelling long memory and structural breaks in conditional Variances : An adaptive FIGARCH approach. *Journal of Economic Dynamics and Control, 33,* 1577-1592

Basel Committee on Banking Supervision (1988). *International Convergence of Capital Measurement and Capital Standards*. Basel

Basel Committee on Banking Supervision (1996)*. Amendment to the Capital Accord to incorporate Market Risks.* Basel.

Basel Committee on Banking Supervision (2006). *International Convergence of Capital Measurement and Capital Standards – A Revised Framework (Comprehensive Version)*. Basel.

Basel Committee on Banking Supervision (2009). *Revisions to the Basel II Market Risk Framework*. Basel

Beltratti, A. and Morana, C. (1999). Computing value at risk with high frequency data. *Journal of Empirical Finance, 6,* 431-444.

Bollerslev, T. (1986). Generalised autoregressive conditional heteroscedasticity. *Journal of Econometrics, 31*, 307-327

Bollerslev, T. and Mikkelsen, H. (1996). Modeling and pricing long memory in stock market volatility. Journal of Econometrics, 73, 151-184.

Brooks, C. and Persand, G., 2003, Volatility Forecasting for Risk Management. *Journal of Forecasting, 22*, 1-22.

Caporin, M. (2003). Evaluating Value-at-Risk measures in presence of long memory conditional volatility, *GRETA Working paper n.05.03*.

Christoffersen, P. (1998) Evaluating interval forecasts. *International Economic Review, 39,* 841-862.

Christoffersen, P. (2003) *Elements of Financial Risk Management.* CA: Elsevier Science.

Crouhy, M., Galai, D. and Mark, R. (2001). *Risk Management* [Electronic version]. New York: McGraw-Hill.

Danielsson, J. (2002), The emperor has no clothes: Limits to risk modelling. *Journal of Banking and Finance, 26*, 1273-1296.

Daníelsson, J. and Zigrand, J.-P. (2006): On time-scaling of risk and the square root-of-time-rule. *Journal of Banking and Finance, 30*, 2701–2713.

Danielsson, J. and Morimoto, Y. (2000), Forecasting extreme financial risk: A critical analysis of practical methods for the Japanese market. *Monetary and Economic Studies*, 18(2), 25-48.

Dionne, G., Duchesne, P. and Pacurar, M., 2009, Intraday Value at Risk (IVaR) using tick-by-tick data with application to the Toronto Stock Exchange, *Journal of Empirical Finance, 16,* 777-792.

Doornik, J. A., 2009. Ox: Object Oriented Matrix Programming, 6.0. Timberlake Consultants Press, London.

Dowd, K. (2002). *Measuring Market Risk.* New York: Wiley.

Drost, F. C. and T. E. Nijman (1993): Temporal aggregation of GARCH processes. *Econometrica, 61*, 927–909.

Engle, R.F. (1982). Autoregressive conditional heteroskedasticity with estimates of the variance of UK inflation. *Econometrica, 50*, 987-1008

Engle, R. (2004). Risk and Volatility: Econometric Models and Financial Practice. The American Economic Review, 94 (3), 405-420.

Giot, P. and Laurent, S. (2003a). Value-at-Risk for Long and Short Trading Positions. *Journal of Applied Econometrics, 18* (6), 641-664.

Giot, P. and Laurent, S. (2003b). Market risk in commodity markets: a VaR approach. Energy Economics, 25, 435-457.

Giot, P. and Laurent, S. (2004). Modelling daily Value-at-Risk using realised volatility and ARCH type models. *Journal of Empirical Finance, 11*, 379-398.

Grané, A. and H. Veiga (2008): Accurate minimum capital risk requirements: a comparison of several approaches. *Journal of Banking and Finance, 32* (11), 2482–2492.

Granger, C. and Hyung, N., (2004), Occasional structural breaks and long memory with an application to the S&P500 absolute returns, *Journal of Empirical Finance, 11*, 399-421.

Harris, R. and Sollis, R. (2003). *Applied Time Series Modelling and Forecasting.* New York: Wiley.

Hartz, C., Mittnik, S. and Paolella, M. (2006). Accurate value-at-risk forecasting based on the normal-GARCH model. *Computational Statistics and Data Analysis*, 51(4), 2295-2312.

Kinateder, H. And Wagner, N. 2010. Market Risk Prediction under Long Memory: When VaR is higher than expected. *Working Paper Passau University*

Kuester, K. Mittnik, S. and Paolella, M. S. (2006). Value-at-Risk prediction: a comparison of alternative strategies. *Journal of Financial Econometrics*, 4(1), 53-89.

Kupiec, P. (1995) Techniques for verifying the accuracy of risk measurement models. *Journal of Derivatives*, 2, 73-84.

Laurent S. (2009). G@RCH 6, Estimating and Forecasting ARCH Models, London: Timberlake Consultants Press.

McMillan, D. and Kambouroudis, D. (2009). Are Riskmetrics forecasts good enough? Evidence from 31 stock markets. *International Review of Financial Analysis, 18,* 117-124

McMillan, D. and Ruiz, I. (2009). Volatility persistence, long memory and time-varying unconditional mean: Evidence from 10 equity indices. *The Quarterly Review of Economics and Finance, 49*, 578-595.

Nagayasu, J., (2008). *Japanese Stock Movements from 1991-2005: evidence from high- and low-frequency data. Applied Financial Economics, 18* (4), 295-307.

Semenov, A. (2009). Risk factor beta conditional Value-at-Risk. *Journal of Forecasting*. 28(6), 549-558.

Shao,X.-D., Lian Y.-J. and Yin, L.-Q. (2009). Forecasting Value-at-Risk using high frequency data: The realized range model. *Global Finance Journal*. 20(2), 128-136.

Tang, T. and Shieh, S. (2006). Long memory in stock index futures markets: A value-at-risk approach. Physica A, 366, 437-448.

Xekalaki, E.and Degiannakis, S. (2010). ARCH Models for Financial Applications. John Wiley & Sons.

Appendix

Note 1: We adopt Christoffersen's (2003) symbol (MC) for the number of draws.

Note 2: The one-day-ahead variance forecast is computed as:

$$\sigma_{t+1|t}^{2} = a_0^{(t)} + \left(a_1^{(t)} - b_1^{(t)}\right)\varepsilon_{t|t}^{2} + \sum_{j=1}^{\infty}\left(\frac{d^{(t)}\Gamma\left(j-d^{(t)}\right)}{\Gamma\left(1-d^{(t)}\right)\Gamma\left(j+1\right)}L^{j}\left(\varepsilon_{t+1|t+j}^{2} - a_1^{(t)}\varepsilon_{t|t+j}^{2}\right)\right) + b_1^{(t)}\sigma_{t|t}^{2}.$$

Note 3: This is based on the two-steps-ahead volatility forecasting formula: $\breve{\sigma}_{i,t+2}^{2} = a_0^{(t)} + \left(a_1^{(t)} - b_1^{(t)}\right)\left(\sigma_{t+1|t}\breve{z}_{i,1}\right)^{2} + \sum_{j=1}^{\infty}\left(\frac{d^{(t)}\Gamma\left(j-d^{(t)}\right)}{\Gamma\left(1-d^{(t)}\right)\Gamma\left(j+1\right)}L^{j}\left(\varepsilon_{t+2|t+j}^{2} - a_1^{(t)}\varepsilon_{t+1|t+j}^{2}\right)\right) + b_1^{(t)}\sigma_{t+1|t}^{2}$. Note that $\sum_{j=1}^{\infty}\left(\frac{d^{(t)}\Gamma\left(j-d^{(t)}\right)}{\Gamma\left(1-d^{(t)}\right)\Gamma\left(j+1\right)}L^{j}\left(\varepsilon_{t+2|t+j}^{2} - a_1^{(t)}\varepsilon_{t+1|t+j}^{2}\right)\right)$ is decomposed into $dL\left(\varepsilon_{t+2/t+1}^{2} - a_1\varepsilon_{t+1/t+1}^{2}\right) + \frac{1}{2!}d\left(1-d\right)L^{2}\left(\varepsilon_{t+2/t+2}^{2} - a_1\varepsilon_{t+1/t+2}^{2}\right) + \frac{1}{3!}d\left(1-d\right)\left(2-d\right)L^{3}\left(\varepsilon_{t+2/t+3}^{2} - a_1\varepsilon_{t+1/t+3}^{2}\right) + \ldots$, or $d\left(\varepsilon_{t+1/t}^{2} - a_1\varepsilon_{t/t}^{2}\right) + \frac{1}{2!}d\left(1-d\right)\left(\varepsilon_{t/t}^{2} - a_1\varepsilon_{t-1/t}^{2}\right) + \frac{1}{3!}d\left(1-d\right)\left(2-d\right)\left(\varepsilon_{t-1/t}^{2} - a_1\varepsilon_{t-2/t}^{2}\right) + \ldots$. Note also that the term $\varepsilon_{t+1/t}^{2}$, which is not estimable, is replaced by the simulated component $\left(\sigma_{t+1|t}\breve{z}_{i,1}\right)^{2}$. Therefore, the forecast variance for time $t+2$ can be computed as:

$$\breve{\sigma}_{i,t+2}^{2} = a_0^{(t)} + \left(a_1^{(t)} - b_1^{(t)}\right)\left(\sigma_{t+1|t}\breve{z}_{i,1}\right)^{2} + b_1^{(t)}\sigma_{t+1|t}^{2} +$$
$$d\left(\left(\sigma_{t+1|t}\breve{z}_{i,1}\right)^{2} - a_1\varepsilon_{t/t}^{2}\right) + \frac{1}{2!}d\left(1-d\right)\left(\varepsilon_{t/t}^{2} - a_1\varepsilon_{t-1/t}^{2}\right) + \frac{1}{3!}d\left(1-d\right)\left(2-d\right)\left(\varepsilon_{t-1/t}^{2} - a_1\varepsilon_{t-2/t}^{2}\right) + \ldots$$

Note 4: This is based on the three-steps-ahead volatility forecasting formula: $\breve{\sigma}_{i,t+3}^{2} = a_0^{(t)} + \left(a_1^{(t)} - b_1^{(t)}\right)\left(\sigma_{t+2|t}\breve{z}_{i,2}\right)^{2} + \sum_{j=1}^{\infty}\left(\frac{d^{(t)}\Gamma\left(j-d^{(t)}\right)}{\Gamma\left(1-d^{(t)}\right)\Gamma\left(j+1\right)}L^{j}\left(\varepsilon_{t+3|t+j}^{2} - a_1^{(t)}\varepsilon_{t+2|t+j}^{2}\right)\right) + b_1^{(t)}\sigma_{t+2|t}^{2}$. Note that $\sum_{j=1}^{\infty}\left(\frac{d^{(t)}\Gamma\left(j-d^{(t)}\right)}{\Gamma\left(1-d^{(t)}\right)\Gamma\left(j+1\right)}L^{j}\left(\varepsilon_{t+3|t+j}^{2} - a_1^{(t)}\varepsilon_{t+2|t+j}^{2}\right)\right)$ is decomposed into $dL\left(\varepsilon_{t+3/t+1}^{2} - a_1\varepsilon_{t+2/t+1}^{2}\right) + \frac{1}{2!}d\left(1-d\right)L^{2}\left(\varepsilon_{t+3/t+2}^{2} - a_1\varepsilon_{t+2/t+2}^{2}\right) + \frac{1}{3!}d\left(1-d\right)\left(2-d\right)L^{3}\left(\varepsilon_{t+3/t+3}^{2} - a_1\varepsilon_{t+2/t+3}^{2}\right) + \ldots$, or $d\left(\varepsilon_{t+2/t}^{2} - a_1\varepsilon_{t+1/t}^{2}\right) + \frac{1}{2!}d\left(1-d\right)\left(\varepsilon_{t+1/t}^{2} - a_1\varepsilon_{t/t}^{2}\right) + \frac{1}{3!}d\left(1-d\right)\left(2-d\right)\left(\varepsilon_{t/t}^{2} - a_1\varepsilon_{t-1/t}^{2}\right) + \ldots$. Note also that the terms $\varepsilon_{t+2/t}^{2}$ and $\varepsilon_{t+1/t}^{2}$, which are not estimable, are replaced by the simulated components $\left(\sigma_{t+2|t}\breve{z}_{i,2}\right)^{2}$ and $\left(\sigma_{t+1|t}\breve{z}_{i,1}\right)^{2}$, respectively. Therefore, the forecast variance for time $t+3$ can be computed as:

$$\breve{\sigma}_{i,t+3}^{2} = a_0^{(t)} + \left(a_1^{(t)} - b_1^{(t)}\right)\left(\breve{\sigma}_{i,t+2}\breve{z}_{i,2}\right)^{2} + b_1^{(t)}\breve{\sigma}_{i,t+2}^{2} +$$
$$d\left(\left(\breve{\sigma}_{i,t+2}\breve{z}_{i,2}\right)^{2} - a_1\left(\breve{\sigma}_{i,t+1}\breve{z}_{i,1}\right)^{2}\right) + \frac{1}{2!}d\left(1-d\right)\left(\left(\breve{\sigma}_{i,t+1}\breve{z}_{i,1}\right)^{2} - a_1\varepsilon_{t/t}^{2}\right) + \frac{1}{3!}d\left(1-d\right)\left(2-d\right)\left(\varepsilon_{t/t}^{2} - a_1\varepsilon_{t-1/t}^{2}\right) + \ldots$$