Homotopy Methods for Reliable Projection Method Computations with Dynamic Stochastic Models

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Paper Outline

Projection Methods and Collocation

System Solution Strategies

- Traditional
- Informal Homotopies
- Formal Homotopies

Lessons Learned

Future Directions
Projection Methods and Collocation

- Typical DSGE models have no analytic closed form solutions

\[ E_t[F(x_{t-1}, x_t, x_{t+1}, \epsilon_t)] \]

- Projection methods provide one method to approximate solution

\[ \hat{x}_t = \sum_{j=0}^{M} w_{ij} \phi_j(s_t) \]

- Often \( \phi_j \) taken to be set of orthogonal polynomials (Outer product of Chebyshev polynomials for example) Collocation Method determines \( w_{ij} \) by

\[ E_t[F(s^j_{t-1}, \hat{x}_t(s^j_{t-1}), \hat{x}_{t+1}(s^j_{t-1}))] = 0 \]

for all \( s^j \) collocation points
Difficult Nonlinear equation system

\[ E_t[F(s_{t-1}^j, \hat{x}_t(s_{t-1}^j), \hat{x}_{t+1}(s_{t-1}^j))] = 0 \]

• Often difficult to solve

• Newton’s method requires an initial guess

• Appropriate levels for selecting polynomial weights not always apparent

• other approximation strategies also have similar numerical problems
The Collection of Models

<table>
<thead>
<tr>
<th>Name</th>
<th>$N$</th>
<th>$s$</th>
<th>$k$</th>
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<td>2</td>
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<td>6</td>
<td>5</td>
<td>2</td>
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<td>Anderson Kim Yun</td>
<td>15</td>
<td>3</td>
<td>2</td>
</tr>
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<td>EDO Model</td>
<td>58</td>
<td>32</td>
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- Focus on feasibility as prerequisite to speed and accuracy
- Convergence of Newton’s method requires a good initial guess
Traditional Strategies

Zeroth Order

- For Dynamic Models zeroth order solution is steady state
- In this case, given the steady state solution, Easy to infer Chebyshev polynomial Weights
- Use Analytic Results for steady state when available

Low Order, Low State Space Dimension

- Guessing Chebyshev polynomial basis weights problematic above zero order
- Authors have used perturbation solution to compute initial guess of polynomial basis weights
- Dimension of state space complicates this but still doable
Perturbation Solution Starting Point

Simplest Uni-variate Case

\[ a_n = \frac{\int_{-1}^{1} \frac{T_n(x)p(x)}{\sqrt{1-x^2}} \, dx}{\int_{-1}^{1} \frac{T_n(x)T_n(x)}{\sqrt{1-x^2}} \, dx} \]

| Coefficients for \(a + bx + cx^2 + dx^3\) |
|---|---|---|---|
| 1 | \(x\) | \(2x^2 - 1\) | \(4x^3 - 3x\) |
| \(\frac{2a+c}{2}\) | \(\frac{4b+3d}{4}\) | \(\frac{c}{2}\) | \(\frac{d}{4}\) |

More Complicated Cases

- Transform to variable ranging from (-1 to 1) when \(x\) varies from \((x_{min}, x_{max})\)
- Computing weight for a given outer product of polynomials straightforward but tedious
- Mapping these coefficients to collocation basis representation even more tedious
The Zeroth Order Approximation

- Newton’s method has finite basin of convergence
- Simply guessing can be tedious or impractical
- The zeroth order perturbation solution is exactly the same problem
- Different parameter setting for a given model can affect the size of the basin of attraction
- Fortunately, basins overlap – Solutions to one model in basin of convergence for other models
- Parameter homotopy useful
Judd’s JET Example

\[ k_t - (f(k_{t-1}) - c_t) \]
\[ u'(c_t) - \beta * u'(c_{t+1}) * f'(k_t) \]

with

\[ u(c) = \frac{1}{(1 + \gamma)c^{(1+\gamma)}} \]
\[ f(k) = Ak^\alpha; \]

and

\[ \beta = .95, \alpha = .25, \gamma = -.9, A = \frac{1}{\alpha\beta} \]
Judd’s model parameter Specific finite basin of convergence

• **JET parameter values (51% of values converge)**

• **lower discount rate (54% of values converge )**
Informal Homotopies

- Solve an easier more tractable variant
- Use interim solution to bootstrap toward desired problem
- Consider a sequence of problems with overlapping basins of convergence terminating in the problem of interest
Move Lower End to Upper

\[ N(\hat{\Lambda}(s_t, \cdot, \mathcal{I}, \mathcal{M}), \mathcal{W}), \gamma^*, \{\gamma_0, \mathcal{W}\} \]

1: \textbf{procedure} MoveLowerEndToUpper

2: \quad Q := \text{true}

3: \quad \gamma := \gamma_0

4: \quad \mathcal{W} = \mathcal{W}_0

5: \quad \textbf{while} \ Q \ \textbf{do}

6: \quad \quad \{\gamma, \mathcal{W}\} := \text{FindBetterParams}

7: \quad \quad \textbf{if} \ \gamma = \gamma^* \ \textbf{then}

8: \quad \quad \quad \quad Q := \text{false}

9: \quad \quad \textbf{end if}

10: \quad \textbf{end while}

11: \textbf{end procedure}
Find Better Params

\[ \mathcal{N}(\hat{\Lambda}(s_t, \cdot, \mathcal{I}, \mathcal{M}), \cdot), \gamma^*, \{\gamma_0, \mathcal{W}_0\} \]

1: procedure FINDBETTERPARAMS
2: \( \nu \in (0, 1) \)
3: \( \gamma := 1 \)
4: \( Q := \text{true} \)
5: while \( Q \) do
6: \( \mathcal{W}^* := \mathcal{N}(\hat{\Lambda}(s_t, \eta(\gamma, \gamma_0, \gamma^*), \mathcal{I}, \mathcal{M}), \mathcal{W}) \)
7: \( \text{if} \ \mathcal{W}^* = \emptyset \text{ then} \)
8: \( \gamma := \nu \times \gamma; \)
9: \( \text{else} \)
10: \( Q := \text{false} \)
11: \( \text{end if} \)
12: \( \text{end while} \)
13: return(\{\eta(\gamma, \gamma_0, \gamma^*), \mathcal{W}^*\})
14: end procedure
Higher Order Approximations

• Newton’s method has finite basin of convergence

• Different parameter setting for a given model can affect the size of the basin of attraction

• Perturbation solution typically lies in basin of convergence.

• Simply guessing can be tedious or impractical

• Basins overlap – Solutions to one model in basin of convergence for other models

• Parameter homotopy still useful

• In addition range homotopy useful
Judd Higher Order Solutions

Using $C = 3.21053 + 0.895794(K - 1)$,

- JET values

- lower discount rate
Widen Range To Full

\[ \mathcal{N}(\hat{\Lambda}(s_t, \Upsilon, \mathcal{I}(\cdot), \mathcal{M}), \mathcal{W}_0) \]

1: \textbf{procedure} \textsc{WidenRangeToFull} \\
2: \hspace{1em} Q := \text{true} \\
3: \hspace{1em} \mathcal{W} = \mathcal{W}_0 \\
4: \hspace{1em} \textbf{while} Q \textbf{ do} \\
5: \hspace{2em} \{\gamma, \mathcal{W}\} := \text{FindWiderRange} \\
6: \hspace{2em} \textbf{if} \gamma := 1 \textbf{ then} \\
7: \hspace{3em} Q := \text{false} \\
8: \hspace{2em} \textbf{end if} \\
9: \hspace{1em} \textbf{end while} \\
10: \textbf{end procedure}
Widen Range

1: procedure FindWiderRange
2: \( \nu \in (0, 1) \)
3: \( \gamma := 1 \)
4: \( Q := \text{true} \)
5: while \( Q \) do
6: \( \mathcal{W}^* := \mathcal{N}(\hat{\Lambda}(s_t, \Upsilon, \mathcal{I}(\gamma), \mathcal{M}), \mathcal{W}_0)) \)
7: if \( \mathcal{W}^* = \emptyset \) then
8: \( \gamma := \nu \times \gamma \)
9: else
10: \( Q := \text{false} \)
11: end if
12: end while
13: return(\( \gamma, \mathcal{W}^* \))
14: end procedure
Shrink To Middle Provides Some Improvement in Robustness

Using the "ShrinkToMiddleThenExpandRangeFind-ZeroStrategy modest improves robustness.
Formal Homotopy Methods

- Using Watson's freely available Curve Tracking Routines: HOMPACK

- "default" homotopy map

\[ \rho_a(\lambda, x) = \lambda F(x) + (1 - \lambda)(x - a) \]

- Find zero of F by following the zero curve of \( \rho_a \) from 0 to 1.

- Code designed to follow more general \( \rho_a \)

- Implements several strategies for following the zero curve
Lessons Learned

- Zeroth Order and first order hardest
- Large range for variables harder than small range
- Should first compute to high order over small range
- Perturbation solutions can provide good initial guess of weights
- Curve tracking promising but no panacea. Mathematica’s FindRoot performed better in some cases than default homotopy
- Increasing polynomial orders in round robin fashion seems to work more reliably than simultaneous increments
Future Directions

• Automate Using perturbation solution as initialization point high order, high dimension problems

• Investigate using perturbation solutions in homotopy

• Investigate linearizing about arbitrary points and for providing a homotopy for global projection method solutions