Robust Optimization and its Guarantees

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Consider the general optimization problem:
\[
\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad f_i(x) \leq 0, \quad i = 1, \ldots, m \\
x & \in \mathbb{R}^n
\end{align*}
\]

where:
- \( f_0(x) \) is the objective function
- \( f_i(x) \leq 0 \) are the constraints
- \( x \in \mathbb{R}^n \) are the decision variables

If \( f_0, f_1, \ldots, f_m \) are convex in \( x \) then:
- (P) is polynomial time solvable (tractable)

If not then:
- (P) is a global optimization problem \( \rightarrow \) NP-hard (intractable)
Convex Conic Optimization

- The class of Conic Programs is comprised of:
  - **Linear Programs (LP):**
    \[
    \text{minimize} \quad c^T x \\
    \text{subject to} \quad Ax \leq b
    \]
  - **Second-Order Cone Programs (SOCP):**
    \[
    \text{minimize} \quad c^T x \\
    \text{subject to} \quad \|A_i x + b_i\|_2 \leq f_i^T x + d_i, \quad i = 1, \ldots, m
    \]
  - **Semidefinite Programs (SDP):**
    \[
    \text{minimize} \quad c^T x \\
    \text{subject to} \quad F_0 + \sum_{i=1}^n F_i x_i \succeq 0
    \]
- Can all be solved using **very efficient software packages**
Second-Order Cone or Ice Cream Cone

\[ \{(x, y, z) : \sqrt{x^2 + y^2} \leq z\} \]
Positive Semidefinite Cone

\[ x \geq 0, xz - y^2 \geq 0: \]

\[ \left\{ \begin{bmatrix} x & y \\ y & z \end{bmatrix} \in \mathbb{S}^2 : \begin{bmatrix} x & y \\ y & z \end{bmatrix} \succeq 0 \right\} \]
For every **Primal** Problem a **Dual** Problem can be constructed:

**Primal Problem:**

\[
\begin{aligned}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax \leq b, \quad x \geq 0
\end{aligned}
\]  

**Dual Problem:**

\[
\begin{aligned}
\text{maximize} & \quad b^T y \\
\text{subject to} & \quad A^T y \geq c, \quad y \geq 0
\end{aligned}
\]

If (P) is feasible, i.e.:

\[
\exists \hat{x} \in \mathbb{R}^n \text{ such that } A\hat{x} \leq b, \quad \hat{x} \geq 0
\]

then

\[
\text{Opt}(P) = \text{Opt}(D) \quad (\text{Strong Duality})
\]
Robust Portfolios with Derivative Insurance Guarantees

Berc Rustem, D. Kuhn, S. Zymler
Mean-Variance Portfolio Optimization

**Optimal Asset Allocation Problem**

Choose \( w \in \mathbb{R}^n \) for high portfolio return, keeping associated risk \( \rho(w) \) low.

- Mean-Variance Portfolio Optimization:

  \[
  \begin{align*}
  \text{maximize} & \quad w^T \mu - \lambda w^T \Sigma w \\
  \text{subject to} & \quad w \in \mathcal{W}.
  \end{align*}
  \]

  Tradeoff between expected return \( w^T \mu \) and risk \( \rho(w) \equiv w^T \Sigma w \), where \( \lambda \) : risk aversion parameter.
Let $\tilde{r}$ denote the asset returns. Portfolio return is $w^T\tilde{r}$.

Ben-Tal and Nemirovski [1], Rustem and Howe [3], suggest investor wants to maximize portfolio return:

$$\max_{w \in \mathcal{W}} w^T\tilde{r}$$

$\mathbf{r} \in \mathcal{U}$,

$$\mathcal{U}_r = \{ \mathbf{r} \mid (\mathbf{r} - \mu)^T \Sigma^{-1} (\mathbf{r} - \mu) \leq \delta^2 \}$$

Robust optimization: worst-case approach

$$\max_{w \in \mathcal{W}} \min_{\mathbf{r} \in \mathcal{U}_r} w^T \mathbf{r} \equiv \max_{w \in \mathcal{W}} w^T \mu - \delta \| \Sigma^{1/2} w \|_2.$$
Probabilistic Guarantees

- Return means $\mu$ and covariance $\Sigma \succ 0$ known $\tilde{r}$, **not** entire distribution.

- Set $\mathcal{P}$ contains all distributions with mean $\mu$ and covariance $\Sigma$.

- El Ghaoui et al. [2] show for any $w \in \mathcal{W}, p$

  $$\delta = \sqrt{p/(1 - p)} \implies \inf_{P \in \mathcal{P}} \mathbb{P}\{w^T\tilde{r} \geq \min_{r \in \mathcal{U}_r} w^T r\} = p$$

  with probability $p$ return better than worst-case return over all distributions in $\mathcal{P}$.

- Let $\theta^* = \max_{w \in \mathcal{W}} \min_{r \in \mathcal{U}_r} w^T r$, then

  $$w^{* T} r \geq \theta^* \quad \forall r \in \mathcal{U}_r.$$  

  **Non-inferiority** of robust portfolios is a form of weak insurance.
Support Information and Coherency

- Support information about $\tilde{r}$:
  \[ B = \{ r : l \leq r \leq u \} \quad (\text{Always true: } B = \{ r : r \geq 0 \}) \]
  - $r$: a realization of $\tilde{r}$.
- Add support information to $U_r$:
  \[ U_r = \{ r \in B \mid (r - \mu)^T \Sigma^{-1} (r - \mu) \leq \delta^2 \} \]
  - Strong convex duality:
    \[
    \max \min_{w \in \mathcal{W}} w^T r \equiv \max_{w \in \mathcal{W}, s \geq 0} \mu^T (w - s) - \delta \| \Sigma^{1/2} (w - s) \|_2.
    \]
    - $s$: dual variable.
- Consider $\rho$:
  \[
  \rho(w) = \min_{s \geq 0} - \mu^T (w - s) + \delta \| \Sigma^{1/2} (w - s) \|_2.
  \]
  - $\rho$ is a coherent risk-measure.
- maximize worst-case return $\iff$ minimize coherent risk risk!
Uncertainty Set: Illustration

\[ \begin{array}{c}
\text{r}_1 \\
\text{r}_2 \\
\end{array} \]
Parameter Uncertainty

- Estimate true mean $\mu$ and covariance $\Sigma \rightarrow$ considerable uncertainty.

- Optimal portfolio sensitive to errors in $\mu$ estimate: $\hat{\mu} \rightarrow$ error-maximization effect.

- i.i.d. $\tilde{r}$ for $M$ samples $\hat{\mu}$ approx:

$$\hat{\mu} \sim \mathcal{N}(\mu, \Lambda), \quad \Lambda = (1/M)\Sigma$$

- Uncertainty set for $\hat{\mu}$

$$\mathcal{U}_\mu = \{\mu \mid (\mu - \hat{\mu})^T \Lambda^{-1} (\mu - \hat{\mu}) \leq \kappa^2, \ e^T (\mu - \hat{\mu}) = 0\}.$$  

- Uncertainty set for $\tilde{r}$ taking account of $\mathcal{U}_\mu$:

$$\mathcal{U}_r = \{r \in B \mid \exists \mu \in \mathcal{U}_\mu, \ (r - \mu)^T \Sigma^{-1} (r - \mu) \leq \delta^2\}$$
Incorporating Options within Robust Framework

- Portfolio return \( \tilde{r}_p = w^T \tilde{r} + (w^d)^T \tilde{r}^d \).
- Set \( w^d \geq 0 \), and \( 1^T w + 1^T w^d = 1 \). (else problem becomes nonconvex - too risky to sell)
- Robust max-min problem:
  \[
  \max_{(w, w^d) \in \mathcal{W}} \min_{r \in \mathcal{U}_r, \ r^d = f(r)} w^T r + (w^d)^T r^d
  \]
- Equivalent to semi-infinite problem:
  \[
  \maximize_{w, w^d, \phi} \phi \\
  \text{subject to} \quad w^T r + (w^d)^T r^d \geq \phi \quad \forall r \in \mathcal{U}_r, \ r^d = f(r) \\
  (w, w^d) \in \mathcal{W}
  \]
- \( \phi^* \) worst-case portfolio return for \( r \in \mathcal{U}_r \).
Incorporating Options within Robust Framework

- Portfolio return \( \tilde{r}_p = w^T \tilde{r} + (w^d)^T \tilde{r}^d \).

- Set \( w^d \geq 0 \), and \( 1^T w + 1^T w^d = 1 \).

- Robust max-min problem:

\[
\max_{(w, w^d) \in \mathcal{W}} \min_{r \in \mathcal{U}_r, \text{ subject to } r^d = f(r)} w^T r + (w^d)^T r^d
\]

- Equivalent semi-infinite problem:

\[
\begin{align*}
\text{maximize} & \quad \phi \\
\text{subject to} & \quad \mu^T (w + B^T y - s) - \delta \left\| \Sigma^{1/2} (w + B^T y - s) \right\|_2 + a^T y \geq \phi \\
& \quad (w, w^d) \in \mathcal{W}, \quad 0 \leq y \leq w^d, \quad s \geq 0
\end{align*}
\]

- At optimality \( \phi^* \) is the worst-case portfolio return when \( r \in \mathcal{U}_r \).
At optimality we obtain the non-inferiority guarantee:

$$w^*_T r + (w^*_d)^T r^d \geq \phi^* \quad \forall r \in U_r, \ r^d = f(r)$$

Extreme events can cause $\tilde{r}$ to be realised outside $U_r \rightarrow$ no more guarantees!

Control deterioration of portfolio return below $\phi$ for any realisation of $\tilde{r}$:

$$w^T r + (w^d)^T r^d \geq \theta \phi \quad \forall r \in B, \ r^d = f(r),$$

where $\theta \in [0, 1]$.

Insurance guarantee expressed as fraction of $\phi$:

- Only hedge against extreme scenarios not covered by non-inferiority guarantee.
- Prevents insurance from being overly expensive.
Guarantee Tradeoff

- The insured robust portfolio optimization model:

  \[
  \text{maximize} \quad \phi \\
  \text{subject to} \quad w^T r + (w^d)^T r^d \geq \phi \quad \forall r \in U_r, \quad r^d = f(r) \\
  w^T r + (w^d)^T r^d \geq \theta \phi \quad \forall r \in B, \quad r^d = f(r) \\
  (w, w^d) \in \mathcal{W}.
  \]

- Has a SOCP reformulation → tractable.

- Model exposes tradeoff between non-inferiority and insurance guarantees:
  - As \( U_r \) increases, \( \phi^* \) decreases.
  - When \( \phi^* \) decreases, so does insurance level \( \theta \phi^* \) and associated insurance costs (premium).
Robust Currency Portfolios

Berc Rustem, R. Fonseca, S. Zymler, W. Wiesemann
Aims

- Robust optimization approach for currency-only portfolio
  - Formulation of triangulation property to avoid non-convexity
- Hedging using currency options and robust optimization
- Explore relationship between robust optimization, size of uncertainty sets, and hedge ratio:
  
  
  \[
  \text{foreign currency options} \\
  \text{foreign currency holdings}
  \]
Notation

- Portfolio with \( n \) different currencies
- \( E_i^0 \) and \( E_i \): today and future spot rates \( i \)th currency: units of domestic currency per unit foreign currency
- \( e_i = E_i / E_i^0 \): return on currency \( i \)

Deterministic Model

\[
\begin{align*}
\max_{w \in \mathbb{R}^n} & \quad w' e \\
\text{s.t.} & \quad w' 1 = 1 \\
& \quad w \geq 0
\end{align*}
\]

- Lacks robustness: small deviations in realised return values from estimates may yield infeasible solution.
Uncertainty Sets

- Currency returns: random parameters within interval or uncertainty set, $\Theta_e$:
  \[
  \Theta_e = \{e \geq 0 : (e - \bar{e})'\Sigma^{-1}(e - \bar{e}) \leq \delta^2\}
  \]

- Joint confidence region as deviations from means are weighted by covariance.
Defining EUR/USD and GBP/USD implies determining cross rate EUR/GBP – *Triangulation Property*

If this relationship does not hold, risk free profit possible.

New constraint needed to exclude arbitrage:

\[
E_i \cdot \frac{1}{E_j} \cdot CE_{ij} = 1, \quad \forall i, j = 1, \ldots, n \quad \text{and} \quad (ij) = 1, \ldots, \frac{n(n - 1)}{2}
\]

\[
\Leftrightarrow e_i \cdot \frac{1}{e_j} \cdot ce_{ij} = 1
\]

Non convex model.
Cross rates do not impact portfolio return: only further restrict uncertainty set of rates

If uncertainty of cross rates is modelled as interval centered at estimate, then:

\[
\begin{align*}
&\underline{ce} \leq ce_{ij} \leq \bar{ce} \\
\iff &\underline{ce} \leq \frac{e_j}{e_i} \leq \bar{ce} \\
\iff &\underline{ce} \cdot e_i \leq e_j \leq \bar{ce} \cdot e_i, \quad \forall e_i \neq 0
\end{align*}
\]

Convexity preserved with \(n(n-1)\) linear inequalities.
Uncertainty Sets II

- Currency returns random expected to be realised within an interval or uncertainty set, $\Theta_e$:

$$\Theta_e = \{ e \geq 0 : (e - \bar{e})'\Sigma^{-1}(e - \bar{e}) \leq \delta^2 \land Ae \geq 0 \}$$
Robust Counterpart

- Portfolio maximization for worst case currency returns within uncertainty set
- $A$: matrix reflecting triangular relationships among rates

Robust Formulation

$$\max_{w \in \mathbb{R}^n} \min_{e \in \Theta_e} w^	op e$$

s.t. $A e \geq 0$, $\forall e \in \Theta_e$

$w^	op 1 = 1$

$w, e \geq 0$
Problem not ready to pass to a solver.

\[
\begin{align*}
\max_{w \in \mathbb{R}^n} & \quad \phi \\
\text{s.t.} & \quad w'e \geq \phi, \quad \forall e \in \Theta_e \\
& \quad Ae \geq 0, \quad \forall e \in \Theta_e \\
& \quad w'1 = 1 \\
& \quad w \geq 0
\end{align*}
\]
Problem not ready to pass to a solver.

Start solving inner min problem wrt exchange rate returns

\[
\begin{align*}
\max_{\mathbf{w} \in \mathbb{R}^n} & \quad \phi \\
\text{s.t.} \quad & \mathbf{w}' \mathbf{e} \geq \phi, \quad \forall \mathbf{e} \in \Theta_e \\
& \mathbf{Ae} \geq 0, \quad \forall \mathbf{e} \in \Theta_e \\
& \mathbf{w}' \mathbf{1} = 1 \\
& \mathbf{w} \geq 0
\end{align*}
\]

\[
\begin{align*}
\min_{\mathbf{e} \in \mathbb{R}^n} & \quad \mathbf{w}' \mathbf{e} \\
\text{s.t.} \quad & \left\| \Sigma^{-1/2} (\mathbf{e} - \bar{\mathbf{e}}) \right\| \leq \delta \\
& \mathbf{Ae} \geq 0 \\
& \mathbf{e} \geq 0,
\end{align*}
\]
The Dual Problem

- Second order cone program: primal and dual have same objective value.

- Dual:

$$\begin{align*}
\max_{\nu, s} \quad & \bar{e}'(w - s) - \delta \nu \\
\text{s.t.} \quad & \|\Sigma^{1/2}(w - s)\| = \nu \\
& s \leq w \\
& s, \nu \geq 0
\end{align*}$$

with $s = A'k + y$
The Dual Problem

- Second order cone program: primal and dual have same objective value.

**Dual:**

\[
\max_{\nu, s} \; \bar{e}'(w - s) - \delta \nu
\]

s.t. \(\|\Sigma^{1/2}(w - s)\| = \nu\)

\[
\begin{align*}
& s \leq w \\
& s, \nu \geq 0
\end{align*}
\]

with \(s = A'k + y\)

**Replace in the original problem:**

\[
\max_{w, s} \; \phi
\]

s.t. \(\bar{e}'(w - s) - \delta \|\Sigma^{1/2}(w - s)\| \geq \phi\)

\[
\begin{align*}
& s \leq w \\
& w'1 = 1 \\
& w, s \geq 0
\end{align*}
\]
Robust Counterpart

- Uncertainty in return estimates incorporated directly
- Asset and currency returns are random
- Maximize portfolio return for worst case return

Robust Formulation

\[
\max_w \min_{r_a, r_e \in \Xi} [\text{diag}(r_a)O r_e]'w
\]

s.t. \(1'w = 1\)

\(w \geq 0\)

where

\(\Xi = \left\{ r_a, r_e \geq 0 : A r_e \geq 0 \land \left( \begin{bmatrix} r_a \\ r_e \end{bmatrix} - \begin{bmatrix} \bar{r}_a \\ \bar{r}_e \end{bmatrix} \right)' \Sigma^{-1} \left( \begin{bmatrix} r_a \\ r_e \end{bmatrix} - \begin{bmatrix} \bar{r}_a \\ \bar{r}_e \end{bmatrix} \right) \leq \delta^2 \right\} \)
Linear Decision Rules for Dynamic Problems

Berc Rustem, R. Fonseca, D. Kuhn, W. Wiesemann, S. Zymler
Maximize portfolio wealth at final stage, \( t = T \), for the worst-case of the stock prices, \( \xi \).

Parameter \( \xi \) is uncertain, defined as \( \xi = [1 \ \xi_1 \ \xi_2 \ \ldots \ \xi_n]^T \), and within an uncertainty set \( \Xi \).

Variable \( w \) is the asset units, while \( v \) is the amount bought/sold.

**Robust Formulation**

\[
\begin{align*}
\max_w \quad & \min_{\xi \in \Xi} (w^T)^T \xi^T \\
\text{s. t.} \quad & E[(w^T)^T \xi^T] \geq \rho \\
& w_0 + \sum_{i=1}^t v^i = w^t \\
& v^T \xi^t = 0 \\
& w^t \geq 0
\end{align*}
\]

\( \forall \ t = 1, \ldots, T \)
“Wait-and-see” variables are affinely dependent on the realizations of the uncertain parameters.

Affinely Adjustable Formulation

\[
\forall t = 1, \ldots, T :\\
\quad w^t = W^t \xi^t, \quad v^t = V^t \xi^t
\]

Final portfolio wealth is:

\[
(\xi^T)^T [C^T W^T] \xi^T
\]

- Matrices \( W^t \) and \( V^t \) are now our decision variables, which map the available observations to decisions.
- Matrices \( C^T \) and \( W_0 \), the initial portfolio units, are constant.
Replacing in our original formulation:

**Affinely Adjustable Formulation**

\[ \text{max} \quad \tau \]

\[ \text{s.t.} \quad (\xi^T)^T [C^T W^T] \xi^T \geq \tau \]

\[ W_0 \xi^t + \sum_{i=1}^{t} V^i \xi^i = W^t \xi^t \]

\[ (\xi^t)^T [1 V^t] \xi^t = 0 \]

\[ W^t \xi^t \geq 0 \]

\[ \forall \ t = 1, \ldots, T \]

Model may be simplified via dual optimization as linear/SOCP/SDP, depending on the uncertainty set formulation of \( \xi \).
A. Ben-Tal and A. Nemirovski. 
Robust solutions of uncertain linear programs. 

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