Investment timing and capital structure under hard and soft budget constraints

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Purpose:
When the firm’s performance is going to be bad, debtholder chooses one of the following two strategies:

- liquidation strategy (hard budget constraint)
- strategic debt service strategy (soft budget constraint)

Then, we consider how two strategies lead to

- investment timing,
- capital structure,
- credit spread.
1. Model setup
2. Investment strategy with all-equity financing
   - standard real options model
     (McDonald and Siegel, 1986, *Q.J.E.*)
3. Investment strategy with debt financing
   - debt financing under hard budget constraint
     (*liquidation strategy*)
   - debt financing under soft budget constraint
     (*strategic debt service strategy*)
4. Model implication
   - comparison between two strategies
5. Concluding remarks
Agents: owner-managed firm and debtholder
Firm has an investment option to output the product
\((X(t))_{t \geq 0}\): price of the product

\[
dX(t) = \mu X(t)dt + \sigma X(t)dz(t), \quad X(0) = x > 0, \quad (1)
\]

where \(\mu > 0\), \(\sigma > 0\), and \((z(t))_{t \geq 0}\): standard Brownian motion.

\(r > 0\): risk-neutral discount factor

\(I > 0\): one-time cost expenditure at investment
\[ \Pi(x) : \text{EBIT (earning before interests and tax)} \]

\[ \Pi(x) := \mathbb{E} \left[ \int_0^{+\infty} e^{-rt} (1 - \tau) QX(t) \, dt \middle| X(0) = x \right] \]

\[ = \frac{1 - \tau}{r - \mu} Qx \quad (2) \]

where \( X(0) = x \)

- \( \tau > 0 \): tax rate,
- \( Q > 0 \): quantity of the product

Investment expenditure is covered by either of as follows:

- all-equity financing
- debt financing under hard budget constraint
- debt financing under soft budget constraint
Value function under all-equity financing

- **Investment timing:**

\[ T_U^i := \inf\{ t \geq 0; X(t) \geq x_U^i \}, \]

where \( x_U^i \): investment trigger.

- superscript “\( i \)” stands for investment strategy.
- subscript “\( U \)” stands for unlevered firm (all-equity financing).

- **Value function:**

\[
E_U(x) := \sup_{T_U^i} \mathbb{E} \left[ e^{-rT_U^i}(\Pi(x_U^i) - 1) \right], \quad (3)
\]

\[
= \sup_{x_U^i} \left( \frac{X}{x_U^i} \right)^\beta (\Pi(x_U^i) - 1), \quad (4)
\]

where \( x < x_U^i \) and \( \beta := \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} > 1. \)
Investment with all-equity financing (benchmark case)

- Firm's investment optimization problem with all-equity financing:

\[
\max_{x_U^i} \left( \frac{x}{x_U^i} \right)^\beta (\Pi(x_U^i) - I) \quad (5)
\]

- Solution:

\[
x_U^{i*} = \frac{\beta}{\beta - 1} \frac{1}{\Pi(1)} I. \quad (6)
\]

- Equity value:

\[
E_U^*(x) = \left( \frac{x}{x_U^{i*}} \right)^\beta \frac{1}{\beta - 1} I \quad (7)
\]

where \(x < x_U^{i*}\). This is because \(\Pi(x_U^{i*}) - I = \frac{1}{\beta - 1} I\).
Debt financing

- Unlevered firm (all-equity financing):
  \[ T_U^i = \inf\{ t \geq 0; X(t) \geq x_U^i \} \]

- Levered firm (debt financing):
  \[ T_j^i = \inf\{ t \geq 0; X(t) \geq x_j^i \}, \quad T_j^d = \inf\{ t \geq T_j^i; X(t) \geq x_j^d \} \]
  where \( j \in \{H, S\} \)
Debt financing under hard budget constraint
Default timing

\[ T^d_H := \inf\{ t \geq T^i_H; X(t) \leq x^d_H \} \]  \hspace{1cm} (8)

where \( x^d_H \): default trigger.

- superscript “\( d \)” stands for default strategy.
- subscript “\( H \)” stands for debt financing under hard budget constraint.

Equity value after investment: For \( t \in [T^i_H, T^d_k] \)

\[
E_H(X(t)) = \mathbb{E}_t \left[ \int_t^{T^d_H} e^{-r(u-t)} (1 - \tau)(QX(u) - c_H)du \right]
\]

\[
= \Pi(X(t)) - (1 - \tau) \frac{c_H}{r} - \left\{ \Pi(x^d_H) - (1 - \tau) \frac{c_H}{r} \right\} \left( \frac{X(t)}{x^d_H} \right)^{\gamma}, \hspace{1cm} (9)
\]

where \( c_H \): coupon for H strategy and

\[
\gamma := \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{\left( \frac{\mu}{\sigma^2} - \frac{1}{2} \right)^2 + \frac{2r}{\sigma^2}} < 0.
\]
• **Debt value after investment:** For \( t \in [T_H^i, T_H^d] \),

\[
D_H(X(t)) = \mathbb{E}_t \left[ \int_t^{T_H^d} e^{-r(u-t)} c_H du + e^{-r(T_H^d-t)} (1 - \alpha) \Pi(x_H^d) \right] \\
= \frac{c_H}{r} - \left\{ \frac{c_H}{r} - (1 - \alpha) \Pi(x_H^d) \right\} \left( \frac{X(t)}{x_H^d} \right)^\gamma,
\]

(10)

where \( \alpha \in (0, 1) \): bankruptcy cost.

• **Total firm value after investment:** For \( t \in [T_H^i, T_H^d] \),

\[
V_H(X(t)) = E_H(X(t)) + D_H(X(t))
\]

\[
= \Pi(X(t)) + \tau \frac{c_H}{r} (1 - \left( \frac{X(t)}{x_H^d} \right)^\gamma) - \alpha \Pi(x_k^d) \left( \frac{X(t)}{x_H^d} \right)^\gamma
\]

(11)

where \( x < x_H^i \).
Optimal default trigger and coupon after investment

\[ x_H^d(c_H) = \arg\max_{x_H^d} E_H(X(t)) = \frac{\gamma}{\gamma - 1} \frac{r - \mu}{Q} \frac{c_H}{r}, \quad (12) \]

\[ c_H(X(t)) = \arg\max_{c_H} V_H(X(t)) = \frac{r}{r - \mu} \frac{\gamma - 1}{\gamma} \frac{1}{h} QX(t), \quad (13) \]

where

\[ h = (1 - \gamma (1 - \alpha + \frac{\alpha}{T}))^{-1/\gamma} \geq 1. \quad (14) \]

That is, these results given in (12) and (13) are exactly the same as in Leland (1994, J.F.).
Investment with debt financing under hard budget const.

- Firm’s investment optimization problem with debt financing under hard budget constraint:

\[
\max_{x_H^i} \left( \frac{x}{x_H^i} \right)^\beta \{ V_H(x_H^i) - I \}, \tag{15}
\]

where \(X(0) = x\). This is because \(E_H(x_H^i) - (I - D_H(x_H^i)) = V_H(x_H^i) - I\).

- Solution:

\[(x_H^i, x_H^d, c_H^i) = (\psi_H x_U, \frac{\psi_H}{h} x_U, \zeta_H I), \tag{16}\]

where

\[
\psi_H = \left(1 + \frac{1}{h} \frac{\tau}{1 - \tau} \right)^{-1} \leq 1, \tag{17}\]

\[
\zeta_H = \frac{\gamma - 1}{\gamma} \frac{\beta}{\beta - 1} \frac{r}{1 - \tau} (h + \frac{\tau}{1 - \tau})^{-1}. \tag{18}\]
• equity value (before investment):

\[ E_H^*(x) = \left( \frac{x}{\psi_H x_{i*}^U} \right)^\beta \frac{1}{\beta - 1} l = \psi_H^{-\beta} E_U^*(x), \quad (19) \]

where \( x < x_{i*}^U \). This is because \( V_H(x_{i*}^U) - l = \frac{1}{\beta - 1} l \).

• credit spread and leverage:

\[ cs_H(x_{i*}^U) = \frac{c_H^*}{D_H(x_{i*}^U)} - r = r \frac{\xi_H}{1 - \xi_H}, \quad (20) \]

\[ LV_H(x_{i*}^U) = \frac{D_H(x_{i*}^U)}{V_H(x_{i*}^U)} = \frac{\gamma - 1}{\gamma} \frac{1}{1 - \tau} \frac{\psi_H}{h} (1 - \xi_H), \quad (21) \]

where

\[ \xi_H = (1 - (1 - \alpha)(1 - \tau)\frac{\gamma}{\gamma - 1}) h^\gamma. \quad (22) \]
Debt financing under soft budget constraint

region a: normal region (normal coupon)
region b: bankruptcy region (reduced coupon)
Equityholder’s and debtholder’s surplus in region b
\( \{X(t) \leq x^d_S\} \)

\[ E^b_S(x) = \eta \{ V^b_S(x) - (1 - \alpha)\Pi(x) \}, \quad (23) \]
\[ D^b_S(x) = (1 - \alpha)\Pi(x) + (1 - \eta) \{ V^b_S(x) - (1 - \alpha)\Pi(x) \}, \quad (24) \]

where
- \( \eta \): equityholder’s bargaining power
- \( 1 - \eta \): debtholder’s bargaining power

Notation
- subscript “S” stands for debt financing under soft budget constraint.
- superscript “b” stands for region b.
Equity value after investment in regions a and b

- region a ($\{ X(t) \geq x_S^d \}$):

$$\begin{align*}
E^a_S(X(t)) &= \mathbb{E}\left[ \int_t^{T^d_S} e^{-r(u-t)} (1 - \tau)(QX(u) - c^a_S)du + e^{-r(T^d_S-t)}E^b_S(x_S^d) \right] \\
&= \Pi(X(t)) - (1 - \tau) \frac{c^a_S}{r} - (1 - \eta \alpha) \Pi(x_S^d) \left( \frac{X(t)}{x_S^d} \right) \gamma \\
&\quad + \frac{c^a_S}{r} \left\{ 1 - \tau + \tau \frac{\eta \gamma}{\beta - \gamma} \right\} \left( \frac{X(t)}{x_S^d} \right) \gamma,
\end{align*}$$

(25)

- superscript “a” stands for region a.

- region b ($\{ X(t) \leq x_S^d \}$):

$$\begin{align*}
E^b_S(X(t)) &= \mathbb{E}\left[ \int_t^{T^d_S} e^{-r(u-t)} (1 - \tau)(QX(u) - c^b_S)du + e^{-r(T^d_S-t)}E^a_S(x_S^d) \right] \\
&= \eta \alpha \Pi(X(t)) - \frac{\tau c^a_S}{r} \frac{\gamma}{\beta - \gamma} \left( \frac{X(t)}{x_S^d} \right)^\beta,
\end{align*}$$

(26)
Debt value after investment in regions a and b

- region a ($\{X(t) \geq x_S^d\}$):

$$D_a^S(X(t)) = \mathbb{E}\left[ \int_t^{T^d_S} e^{-r(u-t)} c_s^a du + e^{-r(T^d_S-t)} D_b^S(x_S^d) \right]$$

$$= \frac{c_s^a}{r} + (1 - \eta \alpha) \Pi(x_S^d) \left( \frac{X(t)}{x_S^d} \right)^\gamma$$

$$- \frac{c_s^a}{r} \left\{ (1 - \tau + \tau \frac{\beta}{\beta - \gamma} + \tau \frac{\eta \gamma}{\beta - \gamma}) \left( \frac{X(t)}{x_S^d} \right)^\gamma \right\}.$$  \hspace{1cm} (27)

- region b ($\{X(t) \leq x_S^d\}$):

$$D_b^S(X(t)) = \mathbb{E}\left[ \int_t^{T^d_S} e^{-r(u-t)} c_s^b du + e^{-r(T^d_S-t)} D_a^S(x_S^d) \right],$$

$$= \left[ (1 - \eta \alpha) \Pi(X(t)) - (1 - \eta) \frac{\tau c_s^a}{r} \frac{\gamma}{\beta - \gamma} \left( \frac{X(t)}{x_S^d} \right)^\beta \right].$$  \hspace{1cm} (29)
Corporate value after investment in regions a and b

region a (\( \{X(t) \geq x_S^d\} \)):

\[
V_S^a(X(t)) = \Pi(X(t)) + \frac{\tau c_S^a}{r} - \frac{\tau c_S^a}{r} \frac{\beta}{\beta - \gamma} \left( \frac{X(t)}{x_S^d} \right)^\gamma,
\]  
(30)

region b (\( \{X(t) \leq x_S^d\} \)):

\[
V_S^b(X(t)) = \Pi(X(t)) - \frac{\tau c_S^a}{r} \frac{\gamma}{\beta - \gamma} \left( \frac{X(t)}{x_S^d} \right)^\beta,
\]  
(31)
Optimization problem is solved from backward.

1. region b
   - coupon payment in region b

2. region a
   - default trigger
   - coupon payment in region a

3. investment trigger
   - coupon payment in region b

$$c_s^b(X(t)) = (1 - \eta\alpha)(1 - \tau)X(t),$$ \hspace{1cm} (32)

This is obtained from ODE of equity value.
Optimal default trigger and coupon payment:

\[ x_S^d(c_S) = \arg\max_{x_S^d} E_S^a(X(t)) \]
\[ = \frac{\gamma}{\gamma - 1} \frac{r - \mu}{Q} \frac{1 - \tau(1 - \eta)}{(1 - \tau)(1 - \eta \alpha)} \frac{c_S^a}{r}, \quad (33) \]

\[ c_S^a(X(t)) = \arg\max_{c_S^a} V_S^a(X(t)) \]
\[ = \frac{r}{r - \mu} \frac{\gamma - 1}{\gamma} \frac{1}{g} \frac{(1 - \tau)(1 - \eta \alpha)}{1 - \tau(1 - \eta)} Q X(t), \quad (34) \]

where

\[ g = \left[ \frac{\beta}{\beta - \gamma} (1 - \gamma) \right]^{-1/\gamma} > 1, \quad (35) \]
Investment with debt financing under soft budget const.

- Firm’s investment optimization problem with debt financing under soft budget constraint:

$$\max_{x_S^i} \left( \frac{x}{x_S^i} \right)^{\beta} \{ V_S^a(x_S^i) - I \}, \quad (36)$$

where $X(0) = x$.

- solution:

$$(x_S^{i*}, x_S^{d*}, c_S^{a*}, c_S^{b*})$$

$$= (\psi_S x_U^{i*}, \frac{\psi_S}{g} x_U^{i*}, \zeta_S I, (1 - \eta \alpha)(1 - \tau)x), \quad (37)$$

where

$$\psi_S = \left( 1 + \tau \frac{1 - \eta \alpha}{1 - \tau(1 - \tau)g} \right)^{-1} \leq 1, \quad (38)$$

$$\zeta_S = \frac{\gamma - 1}{\gamma} \frac{\beta}{\beta - 1} r \left( g \frac{1 - \tau(1 - \eta)}{1 - \eta \alpha} + \tau \right)^{-1}. \quad (39)$$
• Equity value

\[ E_S^*(x) = \left( \frac{x}{\psi_S x^*_U} \right)^\beta \frac{1}{\beta - 1} l = \psi_S^{-\beta} E_U^*(x), \quad (40) \]

where \( x < x_S^* \). This is because \( V_S^a(x_S^*) - l = \frac{1}{\beta - 1} l \).

• credit spread and leverage:

\[ cs_S(x_S^*) = \frac{c_S^*}{D_S(x_S^*)} - r = r \frac{\xi_S}{1 - \xi_S}, \quad (41) \]

\[ LV_S(x_S^*) = \frac{D_S^a(x_S^*)}{V_S^a(x_S^*)} = \frac{\gamma - 1}{\gamma} \frac{1 - \eta \alpha}{1 - \tau(1 - \eta)} \frac{\psi_S}{g} (1 - \xi_S), \quad (42) \]

where

\[ \xi_S = \left( \frac{1}{1 - \gamma} + \tau (\beta - 1)(1 - \eta) \frac{\gamma}{\gamma - 1} \frac{1}{\beta - \gamma} \right) g^\gamma. \quad (43) \]
Comparisons

- **Investment triggers:**
  \[ x_{H}^{i^*} < x_{U}^{i^*}, \quad x_{S}^{i^*} < x_{U}^{i^*}. \]  
  (44)
  
  This is because \( x_{H}^{i^*} = \psi_{H} x_{U}^{i^*} \) and \( x_{S}^{i^*} = \psi_{S} x_{U}^{i^*} \) where \( \psi_{H} \leq 1 \) and \( \psi_{S} \leq 1 \).

- **Debt financing decreases in investment trigger.**
- **It is ambiguous whether \( x_{S}^{i^*} \) is smaller than \( x_{H}^{i^*} \).**

- **Equity values:**
  \[ E_{H}^{*}(x) > E_{U}^{*}(x), \quad E_{S}^{a*}(x) > E_{U}^{*}(x). \]  
  (45)
  
  This is due to \( E_{H}^{*}(x) = \psi_{H}^{-\beta} E_{U}^{*}(x) \) and \( E_{S}^{a*}(x) = \psi_{S}^{-\beta} E_{U}^{*}(x) \) where \( \beta > 1 \).

- **Debt financing increases in equity value.**
- **It is ambiguous whether \( E_{S}^{a*}(x) \) is larger than \( E_{H}^{*}(x) \).**
Investment strategy under $\sigma = 0.1$

$$x_S^{i*}(\eta = 0) = 1.45 < x_H^{i*} = 1.71 < x_S^{i*}(\eta = 1) = 1.90 < x_U^{i*} = 2.31.$$  

($r = 0.07; \alpha = 0.4; \mu = 0.03; l = 50; \tau = 0.4; \sigma 0.1; \eta \in \{0, 1\}$).
Investment strategy under $\sigma = 0.3$

\[ x_S^i(\eta = 0) = 3.75 < x_S^i(\eta = 1) = 4.94 < x_H^i = 5.00 < x_U^i = 6.01. \]

\((r = 0.07; \alpha = 0.4; \mu = 0.03; l = 50; \tau = 0.4; \sigma 0.3; \eta \in \{0, 1\}).\)
Investment strategy with $\sigma$

$$(r = 0.07; \alpha = 0.4; \mu = 0.03; l = 50; \tau = 0.4; \eta \in \{0, 1\}).$$
Results:

<table>
<thead>
<tr>
<th>$\sigma = 0.1$</th>
<th>$\sigma = 0.3$</th>
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<tbody>
<tr>
<td>$\eta = 0$</td>
<td>$\eta = 1$</td>
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<tr>
<td>$x_S &lt; x_H$, $E_S(x) &gt; E_H(x)$</td>
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- When $\sigma = 0.1$, the firm may prefer the hard budget constraint.
- When $\sigma = 0.3$, the firm always prefer the soft budget constraint.
Credit spread with $\sigma$

$$\sigma$$

$$(r = 0.07; \alpha = 0.4; \mu = 0.03; l = 50; \tau = 0.4; \eta \in \{0, 1\}).$$
Leverage with $\sigma$

\[ (r = 0.07; \alpha = 0.4; \mu = 0.03; I = 50; \tau = 0.4; \eta \in \{0, 1\}) \]
We consider the investment timing and capital structure with debt financing with hard and soft budget constraints.

Under hard and soft budget constraints, debt financing always increases the investment and increases equity value.

The firm may prefer the hard budget constraint, depending the combination of parameters such as $\eta$, $\sigma$, $\alpha$, and $\tau$.

- When $\sigma$ is sufficiently small, the firm with larger $\eta$ prefers the hard budget constraint.
- When $\sigma$ is sufficiently large, the firm always prefers the soft budget constraint.