Mean reversion in the real interest rate and the effects of calculating expected inflation

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The real interest rate

• One of the most widely studied variables in empirical finance/macroeconomics, whose stochastic properties have implications for the conduct of monetary policy, consumption based asset pricing models, and real interest rate parity.

• Based on the Fisher hypothesis, for an asset maturing in $k$-periods:

\[ i_t^k = r_t^k + E(\pi_{t+k} | \Omega_t), \]

\[ \Rightarrow r_t^k = i_t^k - E(\pi_{t+k} | \Omega_t). \]

• A critical question is, how do we measure expected inflation?
Measuring expected inflation

• Most studies invoke rational expectations:

\[ E(\pi_{t+k} \mid \Omega_t) = \pi_{t+k} + \nu_{t+k}, \]

where \( \nu_{t+k} \) is a mean zero, presumably serially uncorrelated prediction error. This yields,

\[ r_{t}^{k} = i_{t}^{k} - \pi_{t+k} - \nu_{t+k}. \]

• Based on this representation, researchers use the ex-post real rate \( (r_{t}^{p,k}) \):

\[ r_{t}^{p,k} = i_{t}^{k} - \pi_{t+k} = r_{t}^{k} + \nu_{t+k}. \]
An innocuous choice?

• As argued by Sun and Phillips (2004), finite sample measures of persistence based on the ex post real rate can be biased, with the degree of the bias depending on the inverse of the signal to noise ratio. The bias is increasing in the variance of the inflation forecast errors.
  – They advocate using alternative proxies for expected inflation.

• Evans and Lewis (1995) argue that even under rational expectations, forecast errors may be serially correlated if agents anticipate infrequent shifts in inflation that are not accommodated by policy makers.
  – They use Markov regime switching models to calculate expected inflation.
Measuring expected inflation, $E(\pi_{t+k})$

- We consider the consequences of using different methods to calculate $E(\pi_{t+k})$. We also use two annualization methods for inflation:
  - “Monthly”: $\ln\left(\frac{P_t}{P_{t-1}}\right)^{1200}$, “Year to Year”: $\ln\left(\frac{P_t}{P_{t-12}}\right)^{100}$.

- Data: Eurodollar interest rates (Sveriges Riksbank) and seasonally adjusted CPI (IFS). We consider the following measures for $E(\pi_{t+k})$.
  - Actual inflation (associated with the ex post rate).
  - Forecasted inflation using in sample information.
    - Univariate forecasts based on AR($p$) models.
    - Forecasts using models that add relevant macro-variables, including money growth.
  - Inflation forecasted out of sample, using a rolling regression based on selected ARMA models (5 year window).
  - Inflation forecasts based on Markov regime switching methods (allowing for 3 regimes).
Sample autocorrelations

lags

ACF ex ante rate using Markov switching
ACF ex post rate
Stochastic properties of the real rate.

• Linear unit root tests/cointegration methods:
  – Evidence of unit root.
  – Evidence of an I(0) real rate.

• Some researchers have advocated deviations from standard unit root and cointegration methodology.
  – Long memory
  – Structural breaks
  – Nonlinear modelling with threshold effects
    • Kapetanios, Shin, and Snell (2003), and Christopoulos and Leon Ledesma (2007).
Our contribution

• We consider how calculating and annualizing expected inflation influences measured persistence.
• We document the lack of congruence in the literature on questions of mean reversion when using standard unit root methods.
• We consider how these results are affected by deviations from linear unit root testing, using several recent econometric modeling advances, including long memory, nonlinear, and structural break models.
• We perform Monte Carlo analysis, which provides evidence of why unit root tests have difficulty in detecting mean reversion for precisely the types of processes estimated for the various real rates.
Conventional Unit Root Tests

- **ADF tests**
  \[
  \Delta r_t = c + \alpha r_{t-1} + \sum_{j=1}^{p-1} \zeta_j \Delta r_{t-j} + \varepsilon_t, \text{ test stat: } t_\mu = \frac{\hat{\alpha}}{\hat{\sigma}_\alpha}
  \]
  - Results: Reject \( H_0 \): 3/5 (monthly), 5/5 (year to year).

- **PP test**
  \[
  \Delta r_t = a + \alpha r_{t-1} + u_t, \text{ test stat, } Z_t = \frac{\hat{\gamma}_0^{1/2}}{\hat{\lambda}} t_\hat{\alpha} - \frac{T}{2} \frac{(\hat{\lambda}^2 - \hat{\gamma}_0) \hat{\sigma}_\alpha}{\hat{\lambda}_S}
  \]
  - \( \gamma_0 / \hat{\lambda}^2 \) are the variance/LR variance of \( u_t \) (Newey-West, with a Bartlett kernel).
  - Results: Reject \( H_0 \): 5/5 (monthly), 1/5 (year to year).

- **DF-GLS with lag selection using MAIC:**
  - Test stat: t-stat in an ADF equation based on GLS de-trended data.
  - Results: Reject \( H_0 \): 1/5 (monthly), 1/5 (year to year).

- **Ng-Perron with lag selection using MAIC**
  - Uses modified Z-statistics applied to GLS de-trended data where \( \lambda^2 \) is estimated using an AR spectral based estimator.
  - Results: Reject \( H_0 \): 0/5 (monthly), 1/5 (year to year).
  - Overall: Reject \( H_0 \) 17/40 times.
  - More evidence against \( H_0 \) using ADF test.
Measuring persistence with long memory

• Here, 

\[ r_t - \mu = (1 - L)^{-d} u_t, \]

\[ \Rightarrow r_t = \mu + \sum_{j=0}^{t-1} \frac{\Gamma(j + d)}{\Gamma(d)\Gamma(j + 1)} u_{t-j}. \]

• where \( u_t \) is a general short memory process. Above \( r_t \) is defined as a type-II fractional process (Marinucci and Robinson, 1999).

• As illustrated by Granger and Joyeux (1980) the process is stationary provided \( d < 1/2 \) and is mean reverting for all \( d < 1 \).
  
  – An I(0) process results when \( d = 0 \) and a unit root results when \( d = 1 \).
Estimation of $d$

- Semi-parametric estimators in the frequency domain have proven especially useful.
  
  - It is not necessary to provide a parametric specification for $u_t$.
  - Mitigates confusion on interpreting measured persistence.

- We use the exact Whittle estimator of Shimotsu and Phillips (2005). Established asymptotic theory is applicable for all $d$.

$$
\arg \min_{\{G,d\}} Q_m(G,d), Q_m(G,d) = \frac{1}{m} \sum_{j=1}^m \left( \log \left( \frac{G}{\omega_j} \right)^{-2d} + G^{-1} I_{(1-L)^d r_i}(\omega_j) \right),
$$

where $I_{(1-L)^d r_i}$ is the periodogram of the fractional difference of $r_t$, $m$ is the bandwidth satisfying $(m/T)+(1/m)$ goes to 0 as $T$ tends to infinity, and where $\omega_j$ are the set of Fourier frequencies, $\omega_j=(2\pi j/T)$, $j=1,\ldots,m$. Shimotsu and Phillips show,

$$
\sqrt{m(\hat{d}_{EW} - d)} \xrightarrow{d} N(0,1/4).
$$

where $\hat{d}_{EW}$ denotes the exact Whittle estimator of $d$. 
### Estimation results for long memory when inflation is measured by: $\ln[(P_t/P_{t-1})^{1200}]$

<table>
<thead>
<tr>
<th>Bandwidth</th>
<th>Ex Post</th>
<th>Forecast-AR</th>
<th>Forecast-ARX</th>
<th>Rolling</th>
<th>Switching</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m=T^{0.60}$</td>
<td>0.760</td>
<td>0.766</td>
<td>0.790</td>
<td>0.861</td>
<td>0.757</td>
</tr>
<tr>
<td></td>
<td>[0.938]</td>
<td>[0.937]</td>
<td>[0.961]</td>
<td>[1.031]</td>
<td>[0.928]</td>
</tr>
<tr>
<td>$m=T^{0.65}$</td>
<td>0.539</td>
<td>0.654</td>
<td>0.671</td>
<td>0.644</td>
<td>0.563</td>
</tr>
<tr>
<td></td>
<td>[0.687]</td>
<td>[0.801]</td>
<td>[0.819]</td>
<td>[0.792]</td>
<td>[0.711]</td>
</tr>
<tr>
<td>$m=T^{0.70}$</td>
<td>0.538</td>
<td>0.709</td>
<td>0.742</td>
<td>0.677</td>
<td>0.566</td>
</tr>
<tr>
<td></td>
<td>[0.666]</td>
<td>[0.837]</td>
<td>[0.869]</td>
<td>[0.800]</td>
<td>[0.694]</td>
</tr>
<tr>
<td>$m=T^{0.75}$</td>
<td>0.478</td>
<td>0.706</td>
<td>0.792</td>
<td>0.581</td>
<td>0.529</td>
</tr>
<tr>
<td></td>
<td>[0.589]</td>
<td>[0.817]</td>
<td>[0.903]</td>
<td>[0.691]</td>
<td>[0.639]</td>
</tr>
<tr>
<td>$m=T^{0.80}$</td>
<td>0.480</td>
<td>0.728</td>
<td>0.842</td>
<td>0.594</td>
<td>0.569</td>
</tr>
<tr>
<td></td>
<td>[0.575]</td>
<td>[0.824]</td>
<td>[0.938]</td>
<td>[0.689]</td>
<td>[0.665]</td>
</tr>
<tr>
<td>Mean Est.</td>
<td>0.561</td>
<td>0.713</td>
<td>0.768</td>
<td>0.671</td>
<td>0.597</td>
</tr>
<tr>
<td>AMLE Estimate</td>
<td>0.4656</td>
<td>0.6206</td>
<td>0.5637</td>
<td>0.5311</td>
<td>0.6725</td>
</tr>
<tr>
<td></td>
<td>[0.6566]</td>
<td>[0.8530]</td>
<td>[0.8608]</td>
<td>[0.7403]</td>
<td>[0.8565]</td>
</tr>
</tbody>
</table>

Numbers in brackets denote upper values of 95% confidence intervals about the estimated values of $d$. “AMLE estimate” refers to the estimate of $d$ from the parametric, constrained sum of squares estimator of Beran (1995). For the AMLE estimator, a parametric model must be specified. Here, we use an ARFIMA(1,$d$,0).
<table>
<thead>
<tr>
<th>Test</th>
<th>$d=0.50$</th>
<th>$d=0.60$</th>
<th>$d=0.70$</th>
<th>$d=0.80$</th>
<th>$d=0.90$</th>
<th>$d=1.00$</th>
</tr>
</thead>
<tbody>
<tr>
<td>ADF</td>
<td>0.9890</td>
<td><strong>0.8450</strong></td>
<td><strong>0.4990</strong></td>
<td>0.2910</td>
<td>0.1510</td>
<td>0.0650</td>
</tr>
<tr>
<td>PP</td>
<td>0.9850</td>
<td><strong>0.8040</strong></td>
<td><strong>0.4390</strong></td>
<td>0.2000</td>
<td>0.0820</td>
<td>0.0800</td>
</tr>
<tr>
<td>DF-GLS</td>
<td>0.5350</td>
<td><strong>0.4570</strong></td>
<td><strong>0.3650</strong></td>
<td>0.2310</td>
<td>0.1510</td>
<td>0.0450</td>
</tr>
<tr>
<td>Ng-Perron</td>
<td>0.5070</td>
<td><strong>0.4480</strong></td>
<td><strong>0.3600</strong></td>
<td>0.2280</td>
<td>0.1440</td>
<td>0.0450</td>
</tr>
<tr>
<td>EW</td>
<td>0.9940 [0.5580]</td>
<td><strong>0.9270 [0.6606]</strong></td>
<td><strong>0.6900 [0.7656]</strong></td>
<td>0.3230 [0.8667]</td>
<td>0.1140 [0.9597]</td>
<td>0.0720 [0.9931]</td>
</tr>
</tbody>
</table>

The table above reports the proportion of rejections of the unit root hypothesis. We ran 1000 simulations with 300 observations. The residual variance is set equal to 5. Except when $d=1.00$, the true model is an ARFIMA(1,$d$,0) with the autoregressive parameter equal to 0.40. Value in brackets for “EW” is the mean estimated value for $d$. 
Alternate models capturing persistence

- Nonlinear models and models with structural breaks can also accommodate persistence. A growing literature has demonstrated the potential for spurious long memory (Diebold and Inoue, 2001, Granger and Hyung, 2004).
  - We consider two methods that have recently been proposed for the real interest rate.
    - Smooth transition autoregressive models.
    - Models that accommodate multiple structural breaks in the mean of a process.
STAR Models

• Consider the following zero mean ESTAR model,

\[ r_t = \rho r_{t-1} + \gamma r_{t-1} \{1 - \exp[-\theta r_{t-1}^2]\} + u_t. \]

Imposing \( \rho = 1 \),

\[ \Rightarrow \Delta r_t = \gamma r_{t-1} \{1 - \exp[-\theta r_{t-1}^2]\} + u_t. \]

• Hypothesis testing is complicated by the presence of nuisance parameters, which can be overcome by taking the appropriate Taylor series expansion of the transition function. The result,

\[ \Delta r_t = \delta r_{t-1}^3 + e_t, \text{ where } e_t = u_t \text{ under } H_0. \]

• The unit root null is given by \( \delta = 0 \). To account for serial correlation in \( e_t \), we can add lags of \( \Delta y_t \) (Kapetanios et. al, 2003) or make a non-parametric correction (Rothe and Sibbertsen, 2006), resulting in the following test-statistic (where \( s_e^2 \) and \( \lambda^2 \) are the variance and LR variance of \( e_t \)):

\[ Z_{NL}(t) = \frac{s_e}{\hat{\lambda}} t \delta - \frac{3}{2} \sum_{t=2}^{T} y_{t-1}^2 \left( \hat{\lambda}^2 - \hat{s}_e^2 \right) \frac{1}{\left( \hat{\lambda}^2 \sum_{t=2}^{T} y_{t-1}^6 \right)^{1/2}}. \]
Structural breaks

- We consider here the tests for a unit root against the null of m-unknown number of structural breaks in the intercept.
- Suppose the real rate is governed by:

\[ r_t = c + \rho r_{t-1} + \delta_1 DU_{1,t} + \delta_2 DU_{2,t} + \ldots + \delta_m DU_{m,t} + \sum_{j=1}^{p-1} \zeta_j \Delta r_{t-j} + \varepsilon_t, \]

where, \( \begin{cases} DU_{j,t} = 0, & \text{if } [\lambda_j T] \leq t \\ = 1, & \text{o/w.} \end{cases} \), \( \lambda_j \) is the break fraction for break \( j \).

- As described in Kapetanios (2005), and building on Bai and Perron (1998), we use an iterative algorithm to determine the break dates.
- The null hypothesis for a unit root is: \( \rho=1 \). Let \( \tau_i \) denote the set of t-statistics for \( \rho=1 \) across all possible partitions associated with the j-th break point. Then the t-statistic is given by minimum of the set, \( \tau^1 U \tau^2 U \ldots U \tau^m \).
Structural breaks

- We use Monte Carlo analysis to calculate 90%, 95%, and 99% critical values for at most 1, 2, 3, 4, and 5 structural breaks.
- We simulate 10000 random walks and calculate the associated test-statistic for 325 observations (the size of our common sample) to yield finite sample critical values. Unlike Kapetanios (2005), we do not include a linear trend in our specification.
- For the sample period in common, Rapach and Wohar (2005) find evidence of 3 structural breaks in US real rates. We adopt this choice for the maximum number of breaks, and allow for up to 8 lags in our test equation.
- The finite sample critical values are given by:
  - 90%: -5.8706
  - 95%: -6.1449
  - 99%: -6.8679
The test results for a unit root against the ESTAR alternative are based on the Z-test of Rothe and Sibbertsen (2006). The tests of Kapetanios et al., based on an ADF equation, yielded a rejection of the unit root null in 7/10 cases. For the rolling regression above, using a year to year annualization, given strong evidence for additional breaks because of a rolling sample, we allowed for up to 5 breaks in the intercept.

<table>
<thead>
<tr>
<th></th>
<th>Annualization: $\ln(P_t/P_{t-1})_{1200}^1$</th>
<th>Annualization: $\ln(P_t/P_{t-12})_{100}^1$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Test Stat:</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>ESTAR Alternative</td>
<td>-5.333*</td>
<td>-8.186*</td>
</tr>
<tr>
<td>3-breaks Alternative</td>
<td>-3.741*</td>
<td>-4.987</td>
</tr>
<tr>
<td>Ex Post</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Forecast – AR</td>
<td>-3.202*</td>
<td>-5.400</td>
</tr>
<tr>
<td>Forecast-ARX</td>
<td>-6.210*</td>
<td>-6.817*</td>
</tr>
<tr>
<td>Rolling$^a$</td>
<td>-3.745*</td>
<td>-7.877*</td>
</tr>
<tr>
<td>Regime-Switch</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

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Real rates (monthly annualization) using a rolling regression to forecast inflation along with associated break dates.
Conclusions

• We consider the time series properties of the US real interest rate using multiple measures of expected inflation.
• Standard unit root tests, taken as a whole, provide little guidance as to whether or not the real interest rate is mean reverting. The strongest evidence against a unit root emerges from the use of the ADF test.
• Implementation of long memory methods provides highly robust evidence against the unit root null.
• Monte Carlo experiments whose parametric design is based on the properties of the real rates in this paper suggest that a lack of congruence amongst the unit root tests may be due to strong persistence in the data. For the parameterizations of interest, the Exact Whittle estimator of Shimotsu and Phillips (2005) has significantly more power to reject the unit-root null.
Conclusions

• Nonlinearity and structural breaks can also generate observed persistence. We consider both ESTAR nonlinearity and a model that allows for an arbitrary number of breaks in the intercept as alternatives.
  – The nonlinear ADF and PP versions of the test result in a rejection in 17/20 cases (compared with 14/20 for the standard versions).
  – The test allowing for structural breaks under the alternative, which is based on an ADF specification, resulted in a rejection of the unit root null 8/10 times. The break model generates very sensible break dates consistent with economic theory.
• The tests can be sensitive to a number of factors. Since the test is based on an ADF specification, there is approximately the same amount of evidence against a unit root (compared to the linear ADF w/o breaks), even when allowing for structural breaks.
• We would like to consider tests that allow for structural breaks both under the null and alternative (as in Carrion-i-Silvestre, Kim, and Perron, 2009), and that have good size and power properties should there be structural breaks in the conditional variance (given the Great Moderation).