Computation in an Asymptotic Expansion Method *

Akihiko Takahashi, Kohta Takehara† and Masashi Toda
Graduate School of Economics, the University of Tokyo
7-3-1, Hongo, Bunkyo-ku, Tokyo, 113-0033, Japan

June 9, 2009

Abstract

An asymptotic expansion scheme in finance initiated by Kunitomo and Takahashi [15] and Yoshida[68] is a widely applicable methodology for analytic approximation of the expectation of a certain functional of diffusion processes. [46], [47] and [53] provide explicit formulas of conditional expectations necessary for the asymptotic expansion up to the third order. In general, the crucial step in practical applications of the expansion is calculation of conditional expectations for a certain kind of Wiener functionals. This paper presents two methods for computing the conditional expectations that are powerful especially for high order expansions: The first one, an extension of the method introduced by the preceding papers presents a general scheme for computation of the conditional expectations and show the formulas useful for expansions up to the fourth order explicitly. The second one develops a new calculation algorithm for computing the coefficients of the expansion through solving a system of ordinary differential equations that is equivalent to computing the conditional expectations. To demonstrate their effectiveness, the paper gives numerical examples of the approximation for Λ-SABR model up to the fifth order and a cross-currency Libor market model with a general stochastic volatility model of the spot foreign exchange rate up to the fourth order.

1 Introduction

This paper presents two alternative schemes for computation in an asymptotic expansion approach based on Watanabe theory(Watanabe [66]) in Malliavin

---

*This research is supported by the global COE program “The research and training center for new development in mathematics.”
†Research Fellow of the Japan Society for the Promotion of Science
calculus by extending the preceding papers and also by developing a new calculation algorithm.

To our best knowledge, the asymptotic expansion is first applied to finance for evaluation of an average option that is a popular derivative in commodity markets. [15] and [46] derive the approximation formulas for an average option by an asymptotic method based on log-normal approximations of an average price distribution when the underlying asset price follows a geometric Brownian motion. [68] applies a formula derived more generally by the asymptotic expansion of small diffusion processes. Thereafter, the asymptotic expansion have been applied to a broad class of problems in finance: See [47], [48], [49], [50], Kunitomo and Takahashi [16], [17], [18], [19], Kawai [11], Matsuoka, Takahashi and Uchida [31], Takahashi and Matsushima [51], Takahashi and Saito [52], Takahashi and Yoshida [57], [58], Kobayashi, Takahashi and Tokioka [13], Muroi [33], Osajima [38], Takahashi and Uchida [56], Kunitomo and Kim [14], Kawai and Jäckel [12], and [53], [54], [55].

For other asymptotic methods in finance which do not depend on Watanabe theory, see also Fouque, Papanicolaou and Sircar [5], [6], Henry-Labordere [24], [25], [26], Kusuoka and Osajima [20], Osajima [39] and Siopacha and Teichmann [45].

In the application of the asymptotic expansion based on Watanabe theory, they calculated certain conditional expectations which appear in their expansions and play a key role in computation, by the formulas up to the third order given explicitly in [46], [47] and [53]. In many applications, these formulas give sufficiently accurate approximation, but in some cases, for example in the cases with long maturities or/and with highly volatile underlying variables, the approximation up to the third order may not provide satisfactory accuracies. Thus, the formulas for the higher order computation are desirable. But to our knowledge, asymptotic expansion formulas higher than the third order have not been given yet. This paper provides the general procedures for the explicit computation of conditional expectations in the asymptotic expansion and show the formulas for the approximation up to the fourth order. Moreover, we develop another calculation algorithm which enables us to derive high order approximation formulas in an automatic manner. As a consequence, our approximation generally shows sufficient accuracy with computation of high order expansions, which is confirmed by numerical experiments.

In the following sections, after a brief explanation of the asymptotic expansion in Section 2, Section 3 will provide a computation procedure explicitly for conditional expectations appearing in the expansion and show the formulas for expansions up to the fourth order. Moreover, Section 4 will introduce our new alternative computation algorithm for the asymptotic expansion and derive the fourth order asymptotic expansion formula. Finally, Section 5 will apply our algorithms described in the previous sections to the concrete financial models, and confirm the effectiveness of the higher order expansions by numerical examples in $\lambda$-SABR model and a cross-currency Libor market model with a general stochastic volatility model of the spot foreign exchange rate.

2 Asymptotic Expansion

We consider a $d$-dimensional diffusion process $X^{(\epsilon)}_t = (X^{(\epsilon),1}_t, \cdots, X^{(\epsilon),d}_t)$ which is the solution to the following stochastic differential equation:

$$
\begin{align*}
&dX^{(\epsilon),i}_t = V^i_0(X^{(\epsilon)}_t, \epsilon)dt + \epsilon V^i(X^{(\epsilon)}_t)dW_t \quad (i = 1, \cdots, d) \\
&X^{(\epsilon)}_0 = x_0 \in \mathbb{R}^d
\end{align*}
$$
where $W = (W^1, \ldots, W^d)$ is a $d'$-dimensional standard Wiener process, and $\epsilon \in (0,1]$ is a known parameter. Suppose $V = (V^1, \ldots, V^d): \mathbb{R}^d \mapsto \mathbb{R}^d \otimes \mathbb{R}^{d'}$ satisfies some regularity conditions.

Next, suppose that a function $g : \mathbb{R}^d \mapsto \mathbb{R}$ to be smooth and all derivatives have polynomial growth orders. Then, $g(X_T^{(c)})$ has its asymptotic expansion;

$$g(X_T^{(c)}) = \sum_{n=0}^{\infty} \epsilon^n g_{nT}$$

in $L^p$ for every $p > 1$ as $\epsilon \downarrow 0$. $g_{nT}$, the coefficients in the expansion, can be obtained by Taylor’s formula and represented based on multiple Wiener–Ito integrals.

Let $A_{kt} = \frac{\partial^k X_T^{(c)}}{\partial \epsilon^k} |_{\epsilon=0}$ and $A_{kt}^i$, $i = 1, \ldots, d$ denote the $i$-th elements of $A_{kt}$. In particular, $A_{1t}$ is represented by

$$A_{1t} = \int_0^t Y_t V_{u-1} \left( \frac{\partial}{\partial \epsilon} V_0(X_u^{(0)}, 0) du + V(X_u^{(0)}) dW_u \right)$$

where $Y$ denotes the solution to the differential equation;

$$dY_t = \frac{\partial}{\partial \epsilon} V_0(X_t^{(0)}, 0) Y_t dt; \quad Y_0 = I_d.$$

Here, $\partial V_0$ denotes the $d \times d$ matrix whose $(j, k)$-element is $\partial_k V_0^j = \frac{\partial V_0^j(x, \cdot)}{\partial x_k}$, $V_0^j$ is the $j$-th element of $V_0$, and $I_d$ denotes the $d \times d$ identity matrix.

For $k \geq 2$, $A_{kt}^i$, $i = 1, \ldots, d$ is recursively determined by the following:

$$A_{kt}^i = \int_0^t \partial^k V_0(X_s^{(0)}, 0) ds + \sum_{l=1}^{k} \frac{k!}{l(k-l)!} \sum_{\beta=1}^{l} \sum_{l' \in L_{\beta,k}} \int_0^t \frac{1}{\beta!} \sum_{d_1, \ldots, d_\beta = 1}^{d} \partial^\beta d_1, \ldots, d_\beta \partial_k^{l} V_0^{\beta}(X_s^{(0)}, 0) \prod_{j=1}^{\beta} A_{jt}^{d_j} ds$$

$$+ \sum_{\beta=1}^{k} \sum_{l' \in L_{\beta,k-1}} \int_0^t \frac{1}{\beta!} \sum_{d_1, \ldots, d_\beta = 1}^{d} \partial^\beta d_1, \ldots, d_\beta \partial_k^{l_1} V_0^{\beta}(X_s^{(0)}) \prod_{j=1}^{\beta} A_{jt}^{d_j} dW_s^{\alpha},$$

where $\partial^\beta = \frac{\partial^\beta}{\partial x_1 \cdots \partial x_\beta}$ and

$$L_{\beta,k} = \left\{ l_{\beta} = (l_1, \ldots, l_\beta); l_j \geq 0 (j = 1, \ldots, \beta), \sum_{j=1}^{\beta} l_j = k \right\}.$$

Then, $g_{0T}$ and $g_{1T}$ can be written as

$$g_{0T} = g(X_T^{(0)}),$$

$$g_{1T} = \sum_{i=1}^{d} \partial_i g(X_T^{(0)}) A_{1T}^i.$$

For $n \geq 2$, $g_{nT}$ is expressed as follows:

$$g_{nT} = \sum_{s \in S_n} \left( \frac{n!}{s_1! \cdots s_{n'}!} \right) \prod_{i=1}^{n} \left( \frac{1}{\epsilon} \right)^{s_i} \sum_{\rho^i \in P_{s_i}} \left( \frac{s_i!}{p_1^{s_1} \cdots p_{n'}^{s_{n'}}} \right) \partial_1^{s_1} \cdots \partial_{n'}^{s_{n'}} g(X_T^{(0)}) \prod_{i=1}^{n} (A_{iT})^{\rho_i^i}$$

(4)
where
\[
S_n := \left\{ \vec{s} = (s_1, \cdots, s_n); \ s_l \geq 0 (l = 1, \cdots, n), \ \sum_{l=1}^{n} s_l = n \right\},
\]
\[
P_s := \left\{ \vec{p}^s = (p_1^s, \cdots, p_d^s); \ p_i^s \geq 0 (i = 1, \cdots, d), \ \sum_{i=1}^{d} p_i^s = s \right\}.
\]

Next, normalize \(g(X_T^{(c)})\) to
\[
G^{(c)} = \frac{g(X_T^{(c)}) - g_{1T}}{\epsilon}
\]
for \(\epsilon \in (0, 1]\). Then,
\[
G^{(c)} = g_{1T} + \sum_{n=1}^{\infty} \epsilon^n g_{(n+1)T}
\]
in \(L^p\) for every \(p > 1\) as \(\epsilon \downarrow 0\).

Moreover, let
\[
a_t = (\partial g(X_T^{(0)}))[Y_T Y_t^{-1} V(X_t^{(0)})]
\]
and make the following assumption:

\((\text{Assumption 1})\) \[\Sigma_T = \int_0^T a_t a_t' dt > 0.\]

Note that \(g_{1T}\) follows a normal distribution with variance \(\Sigma_T\) and hence Assumption 1 means that the distribution of \(g_{1T}\) does not degenerate. In application, it is easy to check this condition in most cases.

Next, let \(\Phi\) be a generalized function that belongs to the dual space of the real Schwartz space of rapidly decreasing \(C^\infty\)-functions. Then, the expectation of \(\Phi(G^{(c)})\) is expanded around \(\epsilon = 0\) in the sense of Watanabe\([66]\), Yoshida\([67]\) as follows:

\[
E[\Phi(G^{(c)})] = \sum_{m=0}^{\infty} \frac{1}{m!} \left[ \Phi^{(m)}(g_{1T}) \left( \sum_{n=1}^{\infty} \epsilon^n g_{(n+1)T} \right)^m \right]
\]
\[
= \sum_{m=0}^{\infty} \frac{1}{m!} \left[ \Phi^{(m)}(g_{1T}) E \left( \left( \sum_{n=1}^{\infty} \epsilon^n g_{(n+1)T} \right)^m \epsilon^{1T} \right) \right]
\]
\[
= \sum_{j=0}^{N} \sum_{m=0}^{j} \frac{1}{m!} \left[ \Phi^{(m)}(g_{1T}) \sum_{k \in K_{j,m}} C_{j,m,k} E \left[ X_{j,m,k} g_{1T} \right] \right] + o(\epsilon^N)
\]

where
\[
K_{j,m} = \left\{ (k_1, \cdots, k_{j-m+1}); k_n \geq 0, \ \sum_{n=1}^{j-m+1} k_n = m, \ \sum_{n=1}^{j-m+1} nk_n = j \right\}
\]
and
\[
X_{j,m,k} = \prod_{n=1}^{j-m+1} g_{(n+1)T}^{k_n},
\]
\[
C_{j,m,k} = \prod_{n=1}^{j-m+1} \frac{m!}{k_1! \cdots k_{j-m+1}!}.
\]
3 Computation of Conditional Expectations

3.1 Procedures of Computations

In the previous section, we have

$$E[\Phi(G^{(c)})] = \sum_{j=0}^{N} \sum_{m=0}^{j} \frac{1}{m!} E \left[ \Phi^{(m)}(g_{1T}) \sum_{k \in K_{j,m}} \mathcal{C}_{j,m,k} E\left[ X_{j,m,k} g_{1T} \right] \right] + o(\epsilon^N).$$  

(8)

Then, if we obtain conditional expectations appearing in this expression explicitly, it can be easily calculated since $g_{1T}$ follows a normal distribution.

In particular, letting $\Phi$ be $\delta_x$, the delta function at $x \in \mathbb{R}$, the asymptotic expansion of the density function of $G^{(c)}$ can be obtained as in (28) in the next section.

Here we describe the procedures of evaluating these conditional expectations.

At the beginning of this subsection, we state the following proposition playing an important role in the evaluation.

Proposition 1 Let $J_n(f_n)$ denote the $n$-times iterated Itô integral of $L^2(T^n)$-function $f_n$:

$$J_n(f_n) := \int_0^T \int_0^{t_1} \cdots \int_0^{t_{n-1}} f_n(t_1, \ldots, t_n) dW_{t_2} \cdots dW_{t_1}$$

for $n \geq 1$ and $J_0(f_0) := f_0(\text{constant})$.

Then, its expectation conditional on $J_1(q) = x$ is given by

$$E[J_n(f_n) | J_1(q) = x] = \left( \int_0^T \int_0^{t_1} \cdots \int_0^{t_{n-1}} f_n(t_1, \ldots, t_n) q(t_1) \cdots q(t_n) dt_n \cdots dt_1 \right) \frac{H_n(x; \|q\|^2_{L^2(T^n)})}{(\|q\|^2_{L^2(T^n)})^n}$$

(9)

where $T = [0, T]$, $t_i \in T$ ($i = 1, 2, \ldots, n$) and $H_n(x; \Sigma)$ is the Hermite polynomial of degree $n$ which is defined as

$$H_n(x; \Sigma) := (-\Sigma)^n e^{x^2/2\Sigma} \frac{d^n}{dx^n} e^{-x^2/2\Sigma}.$$

(proof) See Section 3.2. □

Next, we show how to compute the conditional expectations in (8). In the rest of this subsection, we assume $\partial V_0(X_t^{(0)}, 0) \equiv (0, \ldots, 0)$ with no loss of generality, and set $q(t) = a_t = (\partial g(X_t^{(0)}))^t Y_T^{-1} V(X_t^{(0)})$ (then $J_1(q) = g_{1T}$ and $\|q\|^2_{L^2(T^n)} = \Sigma_T$). If this assumption is not satisfied, we can obtain almost the same result by taking conditional expectations with respect to

$$\hat{g}_{1T} := g_{1T} - C$$

instead of $g_{1T}$, where

$$C := (\partial g(X_t^{(0)}))^t \int_0^T Y_T^{-1} \partial V_0(X_t^{(0)}, 0) dt.$$

The procedures consist of three steps.
1. The way to derive an expansion of $A^l_T = \frac{\partial^l X^{(s)}(t)}{\partial \epsilon^l}|_{\epsilon=0}$ is explained. In this stage, there are two alternative ways.

- In one way, as in Lemma 2 in Section 3.2, $A^l_T$ can be expanded as a summation of at most $l$ iterated Itô integrals whose integrands are a family of symmetric $L^2(\mathbb{T})$-functions $\{f^{i,l}_T\}_{i=1}^l$:

$$A^l_T = \sum_{l'=0}^l J_l(f^{i,l}_T)$$

The integrand of $l'$-times iterated Itô integral in this expansion is given by the expectation of the $l'$-th Malliavin derivative of $A^l_T$:

$$\hat{f}^{i,l}_{l'}(t_1, \cdots, t_{l'}) = E[D_{t_1, \cdots, t_{l'}} A^l_T].$$

- In fact, as we can see in (3) every $A^l_T$ is given by finite operations of multiplication, (Lebesgue) integration with respect to time parameters and stochastic integrations. Then, the alternative expansion of $A^l_T$ can be directly calculated via iterated use of Itô’s formula:

$$A^l_T = \sum_{l'=0}^l J_l(f^{i,l}_T)$$

We here briefly advert to the relationship between $\hat{f}^{i,l}_{l'}$ and $f^{i,l}_T$. Note that $A^l_T$ has its Wiener-Chaos expansion as in the proof of Lemma 2 which is described in Section 3.2

$$A^l_T = \sum_{l'=1}^l I_l(\hat{f}^{i,l}_{l'})$$

and that the integrand of $l'$-th order multiple Wiener-Itô integral is given by

$$\hat{f}^{i,l}_{l'}(t_1, \cdots, t_{l'}) = \frac{1}{l!} E[D_{t_1, \cdots, t_{l'}} A^l_T].$$

Then, due to the relationship between an iterated Itô integral and a multiple Wiener-Itô integral of the same order shown in Lemma 1 in Section 3.2, $\hat{f}^{i,l}_{l'} = l'!f^{i,l}_T$ actually coincides with a symmetrization (unnormalized with respect to its norm) of $f^{i,l}_T$:

$$\hat{f}^{i,l}_{l'}(t_1, \cdots, t_{l'}) = \sum_{\sigma} 1_{(t_{(1)} \geq \cdots \geq t_{(l')}}) f^{i,l}_{l'}(t_{(1)}, \cdots, t_{(l')}).$$

2. From the expansion of $A^l_T$, we derive that of $g_{nT}$. Recall that for $n \geq 1$

$$g_{nT} = \sum_{s \in S_n} \left( \frac{n!}{s_1! \cdots s_n!} \right) \prod_{i=1}^n \left( \frac{1}{n!} \right)^{s_i} \sum_{p^{(s_i)}} \left( \frac{s_i!}{p^{(s_i)}_1 \cdots p^{(s_i)}_d} \right) \partial_{p^{(s_i)}_1} \cdots \partial_{p^{(s_i)}_d} g(X_T^{(0)}) \prod_{i=1}^d \left( A^{i,l}_T \right)^{p^{(s_i)}_i}$$

where

$$S_n = \left\{ \tilde{s} = (s_1, \cdots, s_n): s_i \geq 0, \sum_{i=1}^n s_i = n \right\}$$

and

$$P_s = \left\{ p^{(s_i)} = (p^{(s_i)}_1, \cdots, p^{(s_i)}_d): p^{(s_i)}_i \geq 0, \sum_{i=1}^d p^{(s_i)}_i = s \right\}.$$
Then, the expansions of $g_{nT}$ are obtained by applying Itô’s formula iteratively. Moreover, noting that the highest order of the expansion of $g_{nT}$ is $n$, $g_{nT}$ is expressed as

$$g_{nT} = \sum_{i=0}^{n} J_i(f_i^{g_{nT}})$$

with integrands $\{f_i^{g_{nT}}\}_{i=1}^{n}$ obtained via Itô’s formula.

3. Again by iterative applications of Itô’s formula to

$$X^{j,m,k} = \prod_{n=1}^{j-m+1} g_{(n+1)T}^{k_n}$$

where $K_{j,m} = \{(k_1, \ldots, k_{j-m+1}); k_n \geq 0, \sum_{n=1}^{j-m+1} k_n = m, \sum_{n=1}^{j-m+1} nk_n = j\}$, we now have the expansion of $X^{j,m,k}$ in (8) with a finite number of terms as

$$X^{j,m,k} = \sum_{n'=0}^{j+m} J_{n'}(f_{n'}^{j,m,k})$$

with integrands $\{f_{n'}^{j,m,k}\}_{n'=1}^{j+m}$ which can be deduced from $\{\hat{f}_{n'}^{i,l}l_{n'}\}_{l=1}^{i}$ in (10) or $\{f_{n'}^{i,l}\}_{l=1}^{i}$ in (12).

4. From Proposition 1, we conclude that the conditional expectations in (8) are given by

$$E[X^{j,m,k}|g_{1T}] = \sum_{n'=0}^{j+m} \left( \int_{0}^{T} \int_{0}^{t_1} \cdots \int_{0}^{t_{n'-1}} f_{n'}^{j,m,k}(t_1, \ldots, t_{n'})q(t_1) \cdots q(t_{n'})dt_{n'} \cdots dt_1 \right) H_{n'}(x; \Sigma_T) \Sigma_{n'}^{-1}$$

(15)

**Example 1**

At the end of this subsection we show a simple example evaluating $X^{1,1,1}(t) = g_{2T}$ in order to make these procedures clear. Let consider the case when $d = d' = 1$ and $V_0(x,\varepsilon) \equiv 0$. In this case $g_{2T}$ is given by

$$g_{2T} = \frac{1}{2} \partial^2 g(X_T^{(0)}) (A_{1T})^2 + \frac{1}{2} \partial g(X_T^{(0)}) A_{2T}$$

where

$$A_{1T} = \int_{0}^{T} V(X_u^{(0)})dW_u,$$

$$A_{2T} = 2 \int_{0}^{T} \partial V(X_u^{(0)}) A_{1u}dW_u.$$ 

Then, it can be decomposed as the sum of $J_n(\cdot)$ by Itô’s formula;

$$g_{2T} = J_0(f_0^{g_{2T}}) + J_2(f_2^{g_{2T}})$$

where

$$f_0^{g_{2T}} = \frac{1}{2} \partial^2 g(X_T^{(0)}) \int_{0}^{T} V(X_u^{(0)})^2 du,$$

$$f_2^{g_{2T}}(t_1, t_2) = \partial^2 g(X_T^{(0)}) V(X_{t_2}^{(0)}) V(X_{t_1}^{(0)}) + \partial g(X_T^{(0)}) V(X_{t_2}^{(0)}) \partial V(X_{t_1}^{(0)}).$$
From Proposition 1, it follows that

\[
E \left[ X^{1,1,(1)}_{gT} \right] = \left( \partial^2 g(X^{(0)}_T) \right) \sum_{i=0}^{T} V(X^{(0)}_t)^2 dt_2 V(X^{(0)}_t)^2 dt_1 \\
+ \partial g(X^{(0)}_T)^3 \int_0^T \int_0^t V(X^{(0)}_s)^2 dt_2 \partial V(X^{(0)}_s) dt_1 \\
+ \frac{1}{2} \partial^2 g(X^{(0)}_T) \int_0^T V(X^{(0)}_s)^2 du.
\]

3.2 Proof of Lemmas and Proposition in Section 3.1

In this subsection we introduce and prove the important proposition and lemmas used in Section 3.1.

**Proposition 1** The expectation of \( n \)-times iterated Itô integral \( J_n(f_n) \) conditional on \( J_1(q) = x \) is given by

\[
E[J_n(f_n)|J_1(q) = x] = \left( \int_0^T \int_0^t \cdots \int_0^{t_{n-1}} f_n(t_1, \ldots, t_n)q(t_1) \cdots q(t_n)dt_n \cdots dt_1 \right) \frac{H_n(x; \|q\|_{L^2(T)})}{(\|q\|_{L^2(T)})^n}.
\]

(proof) This can be considered as a version of Proposition 3 of Nualart, Üstünel and Zakai [36].

Let \( I_n(\hat{f}) \) denote the multiple Wiener-Itô integral of \( n \)-th order of its integrand \( \hat{f} \in L^2(T^n) \). Then, from Proposition 3 of [36], we know

\[
E[I_n(\hat{f})|I_1(q) = x] = \left( \int_0^T \int_0^t \cdots \int_0^{t_{n-1}} \hat{f}(t_1, \ldots, t_n)q(t_1) \cdots q(t_n)dt_n \cdots dt_1 \right) \frac{H_n(x; \|q\|_{L^2(T)})}{(\|q\|_{L^2(T)})^n}.
\]

Substituting a symmetrization of \( f_n \) defined as in (17) in Lemma 1 below, we obtain the result:

\[
E[J_n(f_n)|J_1(q) = x] = \left( \int_0^T \int_0^t \cdots \int_0^{t_{n-1}} f_n(t_1, \ldots, t_n)q(t_1) \cdots q(t_n)dt_n \cdots dt_1 \right) \frac{H_n(x; \|q\|_{L^2(T)})}{(\|q\|_{L^2(T)})^n}
\]

The following lemma gives us the relationship between the iterated Itô integral and the multiple Wiener-Itô integral of the same order.

**Lemma 1** For any \( L^2(T^n) \)-function \( f_n \) which is not necessarily symmetric it holds that

\[
J_n(f_n) = I_n(\hat{f}_n)
\]

(16)
where \( \hat{f}_n \) is a symmetrization of \( f_n \) defined by
\[
\hat{f}_n(t_1, \cdots, t_n) := \frac{1}{n!} \sum_{\sigma} \mathbb{1}_{(t_{\sigma(1)} \leq \cdots \leq t_{\sigma(n)})} f_n(t_{\sigma(1)}, \cdots, t_{\sigma(n)})
\]  
with taking summation over all permutations \( \sigma \).

(proof) The assertion can be easily shown:
\[
I_n(f) = n! J_n(\hat{f})
\]
\[
= n! J_n \left( \frac{1}{n!} \sum_{\sigma} \mathbb{1}_{(t_{\sigma(1)} \leq \cdots \leq t_{\sigma(n)})} f(t_{\sigma(1)}, \cdots, t_{\sigma(n)}) \right)
\]
\[
= \sum_{\sigma} \int_0^T \int_0^{t_1} \cdots \int_0^{t_{n-1}} \mathbb{1}_{(t_{\sigma(1)} \leq \cdots \leq t_{\sigma(n)})} f(t_{\sigma(1)}, \cdots, t_{\sigma(n)}) dW_{t_1} \cdots dW_{t_n} dW_{t_1}
\]
\[
= \int_0^T \int_0^{t_1} \cdots \int_0^{t_{n-1}} f(t_1, \cdots, t_n) dW_{t_1} \cdots dW_{t_n} dW_{t_1} = J_n(f). \]

Finally we introduce and prove the following lemma.

**Lemma 2** Let \( A_t^{i} := \frac{\partial \mathbb{E} X^{(i)}_t}{\partial \omega_{|t=0}} \) as in Section 2. Then, it has an expansion with a finite number of iterated Itô integrals as
\[
A_t = \sum_{l'=1}^{l} J_{l'}(f_{l'}^{i})
\]  
with \( L^2(T') \)-function \( \{f_{l'}^{i}\} \) whose derivation was explained in Section 3.1.

Before the proof of Lemma 2 we state the following lemma.

**Lemma 3** Assume \( d' = 1 \) for simplicity. Then, the \( n \)-th Malliavin derivative of \( X_t^{(c)} \) are given by
\[
D_{t_1, \cdots, t_n} X_t^{(c),i} = \epsilon \sum_{r=1}^{n} \alpha_n^{i}(t_r; t_1, \cdots, t_{r-1}, t_{r+1}, \cdots, t_n; \epsilon)
\]
\[
+ \epsilon \int_{t_1 \vee \cdots \vee t_n}^{t} \alpha_n^{i}(s, t_1, \cdots, t_n; \epsilon) dW_s + \int_{t_1 \vee \cdots \vee t_n}^{t} \beta_n^{i}(s, t_1, \cdots, t_n; \epsilon) ds
\]  
for \( t \geq t_1 \vee \cdots \vee t_n \) and zero for \( t < t_1 \vee \cdots \vee t_n \), where
\[
\alpha_n^{i}(s, t_1, \cdots, t_n; \epsilon) := \sum_{k_1} \partial_{k_1} \cdots \partial_{k_1} V^{(i)}(X_s^{(c)}) \prod_{p=1}^{\nu} D_{t(M_p)} X_{s(k)}^{(c),k_p},
\]
\[
\beta_n^{i}(s, t_1, \cdots, t_n; \epsilon) := \sum_{k_1} \partial_{k_1} \cdots \partial_{k_1} V_0^{(i)}(X_s^{(c)}) \epsilon \prod_{p=1}^{\nu} D_{t(M_p)} X_{s(k)}^{(c),k_p}
\]  
where \( t(M_p) = t_{r_1}, \cdots, t_{r_p} \) for \( M_p = \{r_1, \cdots, r_p; r_1 < \cdots < r_p\} \subset \{1, \cdots, n\} \), and the sums are taken under the set of all partitions \( M_p \) such that \( M_1 \cup \cdots \cup M_\nu = \{1, \cdots, n\} \).
\[ A_{ll} = \sum_{l' = 1}^{l} I_{l'}(\tilde{f}^{l,l}_{l'}) \]  

(22)

with \( L^2_{\text{sym}}(T') \)-functions \( \{ f^{l,l}_{l'} \} \). Since \( \tilde{f}^{l}_{l'} \) is given by

\[
\tilde{f}^{l,l}_{l'}(t_1, \ldots, t_{l'}) = \frac{1}{l!} \mathbb{E} \left[ D_{t_1, \ldots, t_{l'}} A_{ll}^{\epsilon} \right] = \frac{1}{l!} \mathbb{E} \left[ \frac{\partial^{l}}{\partial \epsilon^{l}} \left. X_{T}^{(i),l} \right|_{\epsilon = 0} \right]
\]

where the last equality holds due to uniqueness of the asymptotic expansion of \( X_{T}^{(i),l} \), in order to prove the expansion (22) it is sufficient to see that for any \( l' > l \),

\[
Y_{l,t}^{i,l} := \left. \frac{\partial^{l}}{\partial \epsilon^{l}} \right|_{\epsilon = 0} D_{t_1, \ldots, t_{l'}} X_{T}^{(i),l}
\]

is equal to zero, which will be proved by induction.

First, it is obvious that this statement holds with \( l = 0 \), because \( X_{T}^{(i)} \) becomes deterministic as \( \epsilon \downarrow 0 \).

Second, from Lemma 3, for \( l \geq 1 \) we have

\[
Y_{l,t}^{i,l} = \sum_{r = 1}^{l' - 1} \left. \frac{\partial^{r - 1}}{\partial \epsilon^{r - 1}} \right|_{\epsilon = 0} \left( \alpha_{l'-1}^{(i)}(t_{r}; t_1, \ldots, t_{r-1}, t_{r+1}, \ldots, t_{l'}; \epsilon) \right) + l \int_{t_1 \vee \cdots \vee t_{l'}} \left. \frac{\partial^{l - 1}}{\partial \epsilon^{l - 1}} \right|_{\epsilon = 0} \left( \alpha_{l}^{(i)}(s, t_1, \ldots, t_{l'}; \epsilon) \right) dW_s + \int_{t_1 \vee \cdots \vee t_{l'}} \left. \frac{\partial^{l}}{\partial \epsilon^{l}} \right|_{\epsilon = 0} \left( \beta_{l}^{(i)}(s, t_1, \ldots, t_{l'}; \epsilon) \right) ds.
\]

for \( t \geq t_1 \vee \cdots \vee t_n \) and \( Y_{l,t}^{i,l} = 0 \) for \( t < t_1 \vee \cdots \vee t_n \).

First and second terms of the right hand side of (23) is summation of the terms (for the second term whose integrands are)

\[
\sum_{p = 1}^{\nu} \frac{\partial^{l_p}}{\partial \epsilon^{l_p}} \left( \partial_{k_1} \cdots \partial_{k_{\nu}} V^{(i)}(X^{(i)}_{s}) \right) \prod_{p = 1}^{\nu} \left. \frac{\partial^{l_p}}{\partial \epsilon^{l_p}} \right|_{\epsilon = 0} D_{t(M_p)} X^{(i),k_p}_{s}
\]

with the conditions \( \sum_{p = 0}^{\nu} l_p = l - 1 \) and \( \sum_{p = 1}^{\nu} \#(M_p) = k, k = l' - 1 \) for the first term and \( k = l' - 1 \) for the second term. Thus, since \( k > l - 1 \) in both cases, for at least one \( p \) we have \( l_p < \#(M_p) \). Then, all of these terms will vanish as \( \epsilon \downarrow 0 \) by the assumption of induction (note that \( l_p \leq \#(M_p) \)).

For the third terms, almost the same result is obtained except that \( \int \partial_{j} V^{(i)}_{0}(X^{(i)}_{s}, \epsilon) Y_{l,t}^{j,l} ds \) for \( j = 1, \ldots, d \) remains.

As a consequence, all terms except for

\[
\sum_{j = 1}^{d} \int_{t_1 \vee \cdots \vee t_{l'}} \partial_{j} V^{(i)}_{0}(X^{(i)}_{s}, 0) Y_{l,t}^{j,l} ds
\]

are equal to zero. Thus we have a trivial linear equation

\[
Y_{l,t}^{i,l} = \int_{t_1 \vee \cdots \vee t_{l'}} \partial V_{0}(X^{(i)}_{s}, 0) Y_{l,t}^{i,l} ds
\]

(24)

whose solution is given by \( Y_{l,t}^{i,l} \equiv (0, \ldots, 0)^{T} \) where \( Y_{l,t}^{i,l} = (Y_{l,t}^{1,l}, \ldots, Y_{l,t}^{d,l})^{T} \). □
3.3 Useful formulas

Finally, we here list up some formulas of conditional expectations often used in asymptotic expansions. Let \( q_i : [0, T] \to \mathbb{R}^m, i = 1, 2, 3, 4, 5, 6, 7 \) are non-random functions and we define \( \Sigma \) as

\[
\Sigma = \int_0^T q_{1v} q_{1v} dv,
\]

where \( z' \) is the transpose of \( z \). We assume that \( 0 < \Sigma < \infty \) and integrability in the following formulas.

Before the list of formulas, we define a notation of iterated integrations for convenience;

\[
F_n(f_1, \ldots, f_n) := \int_0^T \int_0^{t_1} \cdots \int_0^{t_n-1} f_1(t_1) \cdots f_n(t_n) dt_n \cdots dt_1, \ n \geq 1. \quad (25)
\]

1. 
\[
\mathbb{E} \left[ \int_0^T q'_{2v} dW_t \int_0^T q'_{1v} dW_v = x \right] = F_1(q'_2 q_1) \frac{H_1(x; \Sigma)}{\Sigma}
\]

2. 
\[
\mathbb{E} \left[ \int_0^T \int_0^t q'_{2u} dW_u \int_0^T q'_{3s} dW_s \int_0^T q'_{1v} dW_v = x \right] = F_2(q'_2 q_1, q'_3 q_1) \frac{H_2(x; \Sigma)}{\Sigma^2}
\]

3. 
\[
\mathbb{E} \left[ \left( \int_0^T q'_{2u} dW_u \right) \left( \int_0^T q'_{3s} dW_s \right) \int_0^T q'_{1v} dW_v = x \right] = \left( F_1(q'_2 q_1) \times F_1(q'_3 q_1) \right) \frac{H_2(x; \Sigma)}{\Sigma^2} + F_1(q'_2 q_3)
\]

4. 
\[
\mathbb{E} \left[ \int_0^T \int_0^t \int_0^s q'_{2u} dW_u q'_{3s} dW_s q'_{4t} dW_t \int_0^T q'_{1v} dW_v = x \right] = F_3(q'_2 q_1, q'_3 q_1, q'_4 q_1) \frac{H_3(x; \Sigma)}{\Sigma^3}
\]

5. 
\[
\mathbb{E} \left[ \int_0^T \left( \int_0^T q'_{2u} dW_u \right) \left( \int_0^T q'_{3s} dW_s \right) q'_{4t} dW_t \int_0^T q'_{1v} dW_v = x \right] = \left( F_3(q'_2 q_1, q'_3 q_1, q'_4 q_1; T) + F_3(q'_3 q_1, q'_2 q_1, q'_4 q_1) \right) \frac{H_3(x; \Sigma)}{\Sigma^3} + F_2(q'_2 q_1, q'_4 q_1) \frac{H_1(x; \Sigma)}{\Sigma}
\]
6. 
\[
\mathbb{E} \left[ \left( \int_0^T \int_0^t q_{2u} dW_u q_{3t} dW_t \right) \left( \int_0^T q_{4u} dW_u \right) \left| \int_0^T q_{1v} dW_v = x \right. \right] \\
= \left( F_2(q'_2 q_1, q'_3 q_1) \times F_1(q'_4 q_1) \right) \frac{H_3(x; \Sigma)}{\Sigma^4} + \left( F_2(q'_2 q_4, q'_3 q_1) + F_2(q'_2 q_1, q'_3 q_4) \right) \frac{H_4(x; \Sigma)}{\Sigma^4}
\]

7. 
\[
\mathbb{E} \left[ \left( \int_0^T \int_0^t q_{2z} dW_z q_{3t} dW_t \right) \left( \int_0^T \int_0^r q_{4u} dW_u q_{5r} dW_r \right) \left| \int_0^T q_{1v} dW_v = x \right. \right] \\
= \left( F_2(q'_2 q_1, q'_3 q_1) \times F_2(q'_4 q_1, q'_5 q_1) \right) \frac{H_4(x; \Sigma)}{\Sigma^4} + \left\{ \left( F_3(q'_2 q_4, q'_3 q_1, q'_5 q_1) + F_3(q'_2 q_1, q'_3 q_4, q'_5 q_1) \right) \frac{H_3(x; \Sigma)}{\Sigma^4} \\
+ \left( F_3(q'_2 q_4, q'_3 q_1, q'_3 q_1) + F_3(q'_4 q_1, q'_2 q_3, q'_5 q_1) + F_3(q'_4 q_1, q'_2 q_3, q'_3 q_1) \right) \frac{H_4(x; \Sigma)}{\Sigma^4} \right\} \\
+ F_2(q'_2 q_4, q'_3 q_5)
\]

8. 
\[
\mathbb{E} \left[ \left( \int_0^T q_{2t} dW_t \right) \left( \int_0^T q_{3t} dW_t \right) \left( \int_0^T \int_0^r q_{4u} dW_u q_{5r} dW_r \right) \left| \int_0^T q_{1v} dW_v = x \right. \right] \\
= \left( F_2(q'_1 q_2, q'_3 q_4, q'_5 q_5) + F_2(q'_1 q_3, q'_2 q_4, q'_5 q_5) \right) \frac{H_3(x; \Sigma)}{\Sigma^4} + \left( F_2(q'_1 q_2, q'_3 q_4, q'_5 q_5) + F_2(q'_1 q_3, q'_2 q_4, q'_5 q_5) \right) \frac{H_4(x; \Sigma)}{\Sigma^4} \\
+ \left( F_2(q'_1 q_2, q'_3 q_4, q'_5 q_5) + F_2(q'_1 q_3, q'_2 q_4, q'_5 q_5) \right)
\]

9. 
\[
\mathbb{E} \left[ \left( \int_0^T \int_0^t \int_0^u q_{2t} dW_t q_{3u} dW_u q_{4t} dW_t q_{5u} dW_u \right) \left| \int_0^T q_{1u} dW_u = x \right. \right] \\
= F_4(q'_2 q_1, q'_3 q_1, q'_4 q_1, q'_5 q_1) \frac{H_4(x; \Sigma)}{\Sigma^4}
\]

10. 
\[
\mathbb{E} \left[ \left( \int_0^T \int_0^t \int_0^u q_{2t} dW_t \right) \left( \int_0^u q_{3u} dW_u \right) \left( \int_0^T q_{4u} dW_u q_{5t} dW_t \right) \left| \int_0^T q_{1v} dW_v = x \right. \right] \\
= \left( F_4(q'_2 q_1, q'_3 q_1, q'_4 q_1, q'_5 q_1) + F_4(q'_3 q_1, q'_2 q_1, q'_4 q_1, q'_5 q_1) \right) \frac{H_4(x; \Sigma)}{\Sigma^4} \\
+ \left( F_3(q'_2 q_3, q'_4 q_1, q'_5 q_1) \right) \frac{H_2(x; \Sigma)}{\Sigma^4}
\]

11. 
\[
\mathbb{E} \left[ \left( \int_0^T \int_0^t \int_0^u q_{2t} dW_t q_{3u} dW_u \right) \left( \int_0^t q_{4u} dW_u \right) q_{5t} dW_t \left| \int_0^T q_{1v} dW_v = x \right. \right] \\
= \left( F_4(q'_2 q_1, q'_3 q_1, q'_4 q_1, q'_5 q_1) + F_4(q'_3 q_1, q'_2 q_1, q'_4 q_1, q'_5 q_1) \right) \frac{H_4(x; \Sigma)}{\Sigma^4} \\
+ \left( F_3(q'_2 q_3, q'_4 q_1, q'_5 q_1) \right) \frac{H_2(x; \Sigma)}{\Sigma^4}
\]
\[ = \left( F_4(q'_2q_1, q'_2q_1, q'_3q_1, q'_5q_1) + F_4(q'_2q_1, q'_4q_1, q'_3q_1, q'_5q_1) + F_4(q'_4q_1, q'_2q_1, q'_3q_1, q'_5q_1) \right) \frac{H_4(x; \Sigma)}{\Sigma^4} \\
+ \left( F_3(q'_2q_1, q'_3q_1, q'_5q_1) + F_3(q'_2q_1, q'_4q_1, q'_5q_1) \right) \frac{H_2(x; \Sigma)}{\Sigma^2} \]

\[ =: \dot{F}'_{11}^{11}(q_1, q_2, q_3, q_4, q_5) \frac{H_4(x; \Sigma)}{\Sigma^4} + \dot{F}'_{22}^{11}(q_1, q_2, q_3, q_4, q_5) \frac{H_2(x; \Sigma)}{\Sigma^2} \]

12.

\[ E \left[ \int_0^T \left( \int_0^t q'_{2s} dW_s \right) \left( \int_0^t q'_{5u} dW_u \right) \left( \int_0^t q'_{4u} dW_u \right) q'_{4u} dW_t \mid \int_0^T q'_{1v} dW_v = x \right] = \]

\[ \left( \dot{F}'_{11}^{11}(q_1, q_2, q_3, q_4, q_5) + \dot{F}'_{22}^{11}(q_1, q_2, q_3, q_4, q_5) \right) \frac{H_4(x; \Sigma)}{\Sigma^4} \]

\[ + \left\{ \left( F_3(q'_2q_1, q'_3q_1, q'_5q_1) + F_3(q'_2q_1, q'_4q_1, q'_5q_1) \right) \frac{H_2(x; \Sigma)}{\Sigma^2} \right\} \]

\[ =: \dot{F}'_{10}^{13}(q_1, q_2, q_3, q_4, q_5, q_6) \frac{H_5(x; \Sigma)}{\Sigma^5} + \dot{F}'_{13}^{13}(q_1, q_2, q_3, q_4, q_5, q_6) \frac{H_3(x; \Sigma)}{\Sigma^3} + \dot{F}'_{11}^{11}(q_1, q_2, q_3, q_4, q_5, q_6) \frac{H_4(x; \Sigma)}{\Sigma} \]

13.

\[ E \left[ \left( \int_0^T \int_0^t q'_{2s} dW_s q'_{4u} dW_u \right) \mid \int_0^T q'_{1v} dW_v = x \right] = \]

\[ \left( F_2(q'_2q_1, q'_3q_1) \times F_3(q'_4q_1, q'_5q_1, q'_6q_1) \right) \frac{H_5(x; \Sigma)}{\Sigma^5} \]

\[ + \left\{ \left( F_4(q'_4q_1, q'_5q_1, q'_2q_1, q'_3q_1) + F_4(q'_3q_1, q'_2q_1, q'_5q_1, q'_6q_1) + F_4(q'_2q_1, q'_3q_1, q'_5q_1, q'_6q_1) \right) \frac{H_3(x; \Sigma)}{\Sigma^3} \right\} \]

\[ + \left( F_4(q'_4q_1, q'_5q_1, q'_2q_1, q'_3q_1) + F_4(q'_3q_1, q'_2q_1, q'_5q_1, q'_6q_1) + F_4(q'_2q_1, q'_3q_1, q'_5q_1, q'_6q_1) \right) \frac{H_4(x; \Sigma)}{\Sigma} \]

\[ =: \dot{F}'_{10}^{13}(q_1, q_2, q_3, q_4, q_5, q_6) \frac{H_5(x; \Sigma)}{\Sigma^5} + \dot{F}'_{13}^{13}(q_1, q_2, q_3, q_4, q_5, q_6) \frac{H_3(x; \Sigma)}{\Sigma^3} + \dot{F}'_{11}^{11}(q_1, q_2, q_3, q_4, q_5, q_6) \frac{H_4(x; \Sigma)}{\Sigma} \]
In this section we propose a new computational scheme in asymptotic expansion

\[ E \left[ \left( \int_0^T \int_0^t q_{2w} dW_w q_{3t} dW_t \right) \left( \int_0^T \int_0^t q_{4w} dW_w q_{5t} dW_t \right) \left( \int_0^T \int_0^t q_{6w} dW_w q_{7t} dW_t \right) \right] \mid \int_0^T q_{4w} dW_w = x \] 

\[
= \left( F_2(q_2 q_1, q_3 q_1) \times F_2(q_4 q_1, q_5 q_1) \times F_2(q_6 q_1, q_7 q_1) \right) \frac{H_0(x; \Sigma)}{\Sigma^6} \\
+ \left( \tilde{F}_1^{14}(q_1, q_2, q_3, q_4, q_5, q_6, q_7) + \tilde{F}_1^{14}(q_1, q_4, q_5, q_6, q_7, q_2, q_3) + \tilde{F}_1^{14}(q_1, q_6, q_7, q_2, q_3, q_4, q_5) \right) \frac{H_4(x; \Sigma)}{\Sigma^4} \\
+ \left( \tilde{F}_2^{14}(q_1, q_2, q_3, q_4, q_5, q_6, q_7) + \tilde{F}_2^{14}(q_1, q_4, q_5, q_6, q_7, q_2, q_3) + \tilde{F}_2^{14}(q_1, q_6, q_7, q_2, q_3, q_4, q_5) \right) \frac{H_2(x; \Sigma)}{\Sigma^2} \\
+ \left( F_3(q_3 q_6, q_4 q_7, q_5 q_5) + F_3(q_3 q_6, q_5 q_7, q_4 q_5) + F_3(q_4 q_6, q_5 q_7, q_4 q_5) + F_3(q_4 q_6, q_5 q_7, q_5 q_5) \right) \\
+ F_3(q_5 q_6, q_5 q_7, q_4 q_5) + F_3(q_5 q_6, q_5 q_7, q_4 q_5) + F_3(q_5 q_6, q_5 q_7, q_5 q_5) + F_3(q_5 q_6, q_5 q_7, q_5 q_5) \\
\right) 
\]

where

\[
\tilde{F}_1^{14}(q_1, q_2, q_3, q_4, q_5, q_6, q_7) := \left( F_3(q_4 q_6, q_5 q_1, q_6 q_1) + F_3(q_4 q_6, q_6 q_1, q_5 q_1) + F_3(q_6 q_1, q_4 q_5, q_6 q_5) \right) \times F_2(q_4 q_1, q_5 q_1). \\
\tilde{F}_2^{14}(q_1, q_2, q_3, q_4, q_5, q_6, q_7) := F_2(q_2 q_1, q_3 q_1) \times F_2(q_4 q_6, q_5 q_7) \\
+ \left( F_4(q_2 q_4, q_5 q_1, q_3 q_6, q_5 q_1) + F_4(q_2 q_4, q_3 q_6, q_5 q_1, q_5 q_1) + F_4(q_2 q_4, q_3 q_6, q_5 q_1, q_5 q_1) \right) \\
+ \left( F_4(q_2 q_4, q_5 q_1, q_5 q_1, q_6 q_5, q_5 q_1) + F_4(q_2 q_4, q_5 q_1, q_5 q_1, q_6 q_5, q_5 q_1) \right) \\
+ \left( F_4(q_2 q_4, q_5 q_1, q_5 q_1, q_6 q_5, q_5 q_1) + F_4(q_2 q_4, q_5 q_1, q_5 q_1, q_6 q_5, q_5 q_1) \right) \\
+ \left( F_4(q_2 q_4, q_5 q_1, q_5 q_1, q_6 q_5, q_5 q_1) + F_4(q_2 q_4, q_5 q_1, q_5 q_1, q_6 q_5, q_5 q_1) \right) \\
+ \left( F_4(q_2 q_4, q_5 q_1, q_5 q_1, q_6 q_5, q_5 q_1) + F_4(q_2 q_4, q_5 q_1, q_5 q_1, q_6 q_5, q_5 q_1) \right) \\
+ \left( F_4(q_2 q_4, q_5 q_1, q_5 q_1, q_6 q_5, q_5 q_1) + F_4(q_2 q_4, q_5 q_1, q_5 q_1, q_6 q_5, q_5 q_1) \right) \\
+ \left( F_4(q_2 q_4, q_5 q_1, q_5 q_1, q_6 q_5, q_5 q_1) + F_4(q_2 q_4, q_5 q_1, q_5 q_1, q_6 q_5, q_5 q_1) \right) \\
+ \left( F_4(q_2 q_4, q_5 q_1, q_5 q_1, q_6 q_5, q_5 q_1) + F_4(q_2 q_4, q_5 q_1, q_5 q_1, q_6 q_5, q_5 q_1) \right) \\
+ \left( F_4(q_2 q_4, q_5 q_1, q_5 q_1, q_6 q_5, q_5 q_1) + F_4(q_2 q_4, q_5 q_1, q_5 q_1, q_6 q_5, q_5 q_1) \right) \\
+ \left( F_4(q_2 q_4, q_5 q_1, q_5 q_1, q_6 q_5, q_5 q_1) + F_4(q_2 q_4, q_5 q_1, q_5 q_1, q_6 q_5, q_5 q_1) \right) \\
+ \left( F_4(q_2 q_4, q_5 q_1, q_5 q_1, q_6 q_5, q_5 q_1) + F_4(q_2 q_4, q_5 q_1, q_5 q_1, q_6 q_5, q_5 q_1) \right) \\
+ \left( F_4(q_2 q_4, q_5 q_1, q_5 q_1, q_6 q_5, q_5 q_1) + F_4(q_2 q_4, q_5 q_1, q_5 q_1, q_6 q_5, q_5 q_1) \right) \\
\right) 
\]

4 New Computational Scheme

In this section we propose a new computational scheme in asymptotic expansion which is alternative to the method described in the previous section. To compute conditional expectations in the right hand side of (7), we use the following lemma which can be derived from the property of Hermite polynomials.
Lemma 4 Let $X \in L^2(\Omega)$ and $Y$ be a random variable with Gaussian distribution with mean 0 and variance $\Sigma$. Then, the conditional expectation $E[X|Y]$ has following expansion in $L^2(\Omega)$:

$$E[X|Y] = \sum_{n=0}^{\infty} a_n H_n(Y; \Sigma)$$  \hspace{1cm} (26)

where $H_n(x; \Sigma)$ is the Hermite polynomial of degree $n$ which is defined as

$$H_n(x; \Sigma) = (-\Sigma)^n e^{x^2/2\Sigma} \frac{d^n}{dx^n} e^{-x^2/2\Sigma}$$

and coefficients $a_n$ are given by

$$a_n = \frac{1}{(i\Sigma)^n} \left. \frac{d^n}{d\xi^n} \right|_{\xi=0} \left\{ e^{\frac{\xi^2}{2\Sigma}} E[e^{i\xi Y} X] \right\}.$$  \hspace{1cm} (27)

(proof) Since the Hermite polynomials $\{H_n(x; \Sigma)\}$ is the orthogonal basis of $L^2(\mathbb{R}, \mu)$ where $\mu$ is the Gaussian measure on $\mathbb{R}$ with mean 0 and variance $\Sigma$, and $E[X|Y = y] \in L^2(\mathbb{R}, \mu)$, we have the following unique expansion of $E[X|Y = y]$ in $L^2(\mathbb{R}, \mu)$:

$$E[X|Y = y] = \sum_{n=0}^{\infty} a_n H_n(y; \Sigma)$$

and also we have

$$E[X|Y] = \sum_{n=0}^{\infty} a_n H_n(Y; \Sigma)$$

in $L^2(\Omega)$. And note that

$$e^{\xi Y} = e^{-\frac{\xi^2}{2\Sigma}} \sum_{n=0}^{\infty} \frac{H_n(Y; \Sigma)}{n!} (i\xi)^n.$$  \hspace{1cm} (27)

Then,

$$e^{\frac{\xi^2}{2\Sigma} E[e^{i\xi Y} X]} = e^{\frac{\xi^2}{2\Sigma} E[e^{i\xi Y} E[X|Y]]}$$

$$= E\left[ \sum_{m=0}^{\infty} \frac{H_m(Y; \Sigma)}{m!} (i\xi)^m \sum_{n=0}^{\infty} a_n H_n(Y; \Sigma) \right]$$

$$= \sum_{n=0}^{\infty} a_n (i\Sigma)^n \xi^n.$$  \hspace{1cm} (27)

Comparing to the coefficients of the Taylor series of $e^{\frac{\xi^2}{2\Sigma} E[e^{i\xi Y} X]}$ around 0 with respect to $\xi$, we see that $a_n$ can be written as (27). \hfill \square

Recall $\hat{g}_{1T}$ is defined as

$$\hat{g}_{1T} = (\partial g(X_t^{(0)})) \cdot \int_0^T [Y_T Y_t^{-1} V(X_t^{(0)})] dW_t = g_{1T} - C$$

where

$$C = (\partial g(X_t^{(0)})) \cdot \int_0^T Y_T Y_t^{-1} \partial_t V_0(X_t^{(0)}, 0) dt.$$
and define

\[ Z^{(\xi)}_T = \exp \{ i \xi \hat{g}_{1T} + \frac{\xi^2}{2} \Sigma_T \}. \]

Then, from Lemma 4, we have the following expression of \( E[\Phi(G^{(c)})] \):

\[
E[\Phi(G^{(c)})] = \sum_{j=0}^{N} e^j \sum_{m=0}^{j} \sum_{k,K_{j,m}} C^{j,m,k}_{m!} E \left[ \Phi^{(m)}(\hat{g}_{1T} + C) X^{i,m,k} \hat{g}_{1T} \right] + o(e^N)
\]

\[
= \sum_{j=0}^{N} e^j \sum_{m=0}^{j} \sum_{k,K_{j,m}} \frac{C^{j,m,k}_{m!}}{m!} E \left[ \Phi^{(m)}(\hat{g}_{1T} + C) \sum_{l=0}^{\infty} a^{j,m,k}_l H_l(\hat{g}_{1T}; \Sigma_T) \right] + o(e^N)
\]

\[
= \sum_{j=0}^{N} e^j \sum_{m=0}^{j} \sum_{k,K_{j,m}} \sum_{l=0}^{\infty} \frac{a^{j,m,k}_l C^{j,m,k}_{m!}}{m!} E \left[ \Phi^{(m)}(\hat{g}_{1T} + C) H_l(\hat{g}_{1T}; \Sigma_T) \right] + o(e^N)
\]

where

\[ a^{j,m,k}_l = \frac{1}{(i \Sigma_T)^l} \left. \frac{d^l}{d\xi^l} \right|_{\xi=0} \left\{ E[X^{i,m,k}_l Z^{(\xi)}_T] \right\}. \]

In particular, let \( \Phi \) be the delta function at \( x \in \mathbb{R}, \delta_x \), we obtain the asymptotic expansion of density of \( G^{(c)} \):

\[
f_{G^{(c)}}(x) = E[\delta_x(G^{(c)})]
\]

\[
= \sum_{j=0}^{N} e^j \sum_{m=0}^{j} \sum_{k,K_{j,m}} \frac{a^{j,m,k}_l C^{j,m,k}_{m!}}{m!} E \left[ \delta_x^{(m)}(\hat{g}_{1T} + C) H_l(\hat{g}_{1T}; \Sigma_T) \right] + o(e^N)
\]

\[
= \sum_{j=0}^{N} e^j \sum_{m=0}^{j} \sum_{k,K_{j,m}} \sum_{l=0}^{\infty} \frac{a^{j,m,k}_l C^{j,m,k}_{m!}}{m!} (-1)^m \partial_x^m \left\{ H_l(x - C; \Sigma_T) f_{g_{1T}}(x) \right\} + o(e^N)
\]

(28)

where

\[ f_{g_{1T}}(x) := \frac{1}{\sqrt{2\pi \Sigma_T}} \exp \left( -\frac{x^2}{2\Sigma_T} \right). \]

### 4.1 Asymptotic Expansion of Density Function

In this subsection, we propose a new computational method for the asymptotic expansion of the density function (28). In particular, we show that coefficients in the expansion is obtained through a system of ordinary differential equations that is solved easily, and derive a concrete expression of the expansion up to \( e^3 \)-order.

First, we write down the equation (28) more explicitly up to \( e^3 \)-order:

\[
f_{G^{(c)}}(x) = \sum_{l=0}^{\infty} a^{0,0,(0)}_l H_l(x - C; \Sigma_T) f_{g_{1T}}(x)
\]

\[
+ \epsilon \left\{ \sum_{l=0}^{\infty} a^{1,1,(1)}_l \langle -\partial_x \rangle \left\{ H_l(x - C; \Sigma_T) f_{g_{1T}}(x) \right\} \right\}
\]

\[
+ \epsilon^2 \left\{ \sum_{l=0}^{\infty} a^{2,1,(0)}_l \langle -\partial_x \rangle \left\{ H_l(x - C; \Sigma_T) f_{g_{1T}}(x) \right\} \right\}
\]

\[
+ \frac{1}{2} \sum_{l=0}^{\infty} a^{2,2,(0)}_l \partial_x^2 \left\{ H_l(x - C; \Sigma_T) f_{g_{1T}}(x) \right\}
\]

16
where coefficients $a_i^{j,m,k}$ are given by

$$
\begin{align*}
    a_i^{0,0,(0)} &= \frac{1}{(i\Sigma_T)^l} \frac{d^l}{d\xi^l} \left\{ \mathbb{E}[Z_T^{(l)}] \right\}, \\
    a_i^{1,1,(1)} &= \frac{1}{(i\Sigma_T)^l} \frac{d^l}{d\xi^l} \left\{ \mathbb{E}[g_{2T}Z_T^{(l)}] \right\}, \\
    a_i^{2,1,(0,1)} &= \frac{1}{(i\Sigma_T)^l} \frac{d^l}{d\xi^l} \left\{ \mathbb{E}[g_{3T}Z_T^{(l)}] \right\}, \\
    a_i^{2,2,(2,0)} &= \frac{1}{(i\Sigma_T)^l} \frac{d^l}{d\xi^l} \left\{ \mathbb{E}[g_{2T}^2Z_T^{(l)}] \right\}, \\
    a_i^{3,1,(0,0,1)} &= \frac{1}{(i\Sigma_T)^l} \frac{d^l}{d\xi^l} \left\{ \mathbb{E}[g_{4T}Z_T^{(l)}] \right\}, \\
    a_i^{3,2,(1,1,0)} &= \frac{1}{(i\Sigma_T)^l} \frac{d^l}{d\xi^l} \left\{ \mathbb{E}[g_{3T}g_{3T}Z_T^{(l)}] \right\}, \\
    a_i^{3,3,(3,0,0)} &= \frac{1}{(i\Sigma_T)^l} \frac{d^l}{d\xi^l} \left\{ \mathbb{E}[g_{2T}^3Z_T^{(l)}] \right\}.
\end{align*}
$$

(29)

Since $\mathbb{E}[Z_T^{(l)}] = 1$, we have $a_i^{0,0,(0)} = 1$ and $a_i^{0,0,(0)} = 0$ for $l \geq 1$. The other expectations above are expressed in terms of $A_{nT}$ and $Z_T^{(l)}$ as follows:

$$
\begin{align*}
    \mathbb{E}[g_{2T}Z_T^{(l)}] &= \frac{1}{2} \sum_{i,j=1}^d \partial_i \partial_j g(X_T^{(0)}) \mathbb{E}[A_{1T}^i A_{1T}^j Z_T^{(l)}] + \frac{1}{2} \sum_{i=1}^d \partial_i g(X_T^{(0)}) \mathbb{E}[A_{2T}^i Z_T^{(l)}], \\
    \mathbb{E}[g_{3T}Z_T^{(l)}] &= \frac{1}{6} \sum_{i,j,k=1}^d \partial_i \partial_j \partial_k g(X_T^{(0)}) \mathbb{E}[A_{1T}^i A_{1T}^j A_{1T}^k Z_T^{(l)}] + \frac{1}{2} \sum_{i,j=1}^d \partial_i \partial_j g(X_T^{(0)}) \mathbb{E}[A_{2T}^i A_{1T}^j Z_T^{(l)}]
    + \frac{1}{6} \sum_{i=1}^d \partial_i g(X_T^{(0)}) \mathbb{E}[A_{3T}^i Z_T^{(l)}], \\
    \mathbb{E}[g_{2T}^2Z_T^{(l)}] &= \frac{1}{4} \sum_{i,j,k,l=1}^d \partial_i \partial_j \partial_k \partial_l g(X_T^{(0)}) \partial_i \partial_k \partial_l g(X_T^{(0)}) \mathbb{E}[A_{1T}^i A_{1T}^j A_{1T}^k A_{1T}^l Z_T^{(l)}] \\
    &\quad + \frac{1}{2} \sum_{i,j,k=1}^d \partial_i \partial_j g(X_T^{(0)}) \partial_k g(X_T^{(0)}) \mathbb{E}[A_{1T}^i A_{1T}^j A_{2T}^k Z_T^{(l)}] \\
    &\quad + \frac{1}{4} \sum_{i,j=1}^d \partial_i g(X_T^{(0)}) \partial_j g(X_T^{(0)}) \mathbb{E}[A_{2T}^i A_{2T}^j Z_T^{(l)}], \\
    \mathbb{E}[g_{4T}Z_T^{(l)}] &= \frac{1}{24} \sum_{i,j,k,l=1}^d \partial_i \partial_j \partial_k \partial_l g(X_T^{(0)}) \mathbb{E}[A_{1T}^i A_{1T}^j A_{2T}^k A_{1T}^l Z_T^{(l)}].
\end{align*}
$$

17
\[
+ \frac{1}{4} \sum_{i,j,k=1}^{d} \partial_{i} \partial_{j} \partial_{k} g(X_{T}^{(0)}) E[A_{2T} A_{1T}^{k} A_{1T}^{j} Z_{T}^{(0)}] \\
+ \frac{1}{2} \sum_{i,j=1}^{d} \partial_{i} \partial_{j} g(X_{T}^{(0)}) E[A_{2T} A_{1T}^{j} Z_{T}^{(0)}] + \frac{1}{2} \sum_{i,j=1}^{d} \partial_{i} \partial_{j} g(X_{T}^{(0)}) E[A_{3T} A_{1T}^{j} Z_{T}^{(0)}] \\
+ \frac{1}{6} \sum_{i=1}^{d} \partial_{i} g(X_{T}^{(0)}) E[A_{1T} Z_{T}^{(0)}], \]

\[
E[g_{2T} g_{3T} Z_{T}^{(0)}] = \frac{1}{12} \sum_{i,j,k,l,m=1}^{d} \partial_{i} \partial_{j} g(X_{T}^{(0)}) \partial_{k} \partial_{l} g(X_{T}^{(0)}) E[A_{1T}^{k} A_{1T}^{j} A_{1T}^{l} A_{1T}^{m} Z_{T}^{(0)}] \\
+ \frac{1}{12} \sum_{i,j,k,l=1}^{d} \left\{ \partial_{i} g(X_{T}^{(0)}) \partial_{j} \partial_{k} g(X_{T}^{(0)}) + 3 \partial_{i} \partial_{j} g(X_{T}^{(0)}) \partial_{k} \partial_{l} g(X_{T}^{(0)}) \right\} E[A_{1T}^{k} A_{1T}^{j} A_{2T}^{l} Z_{T}^{(0)}] \\
+ \frac{1}{12} \sum_{i,j,k=1}^{d} \partial_{i} \partial_{j} g(X_{T}^{(0)}) \partial_{k} g(X_{T}^{(0)}) E[A_{1T}^{k} A_{1T}^{j} A_{3T}^{l} Z_{T}^{(0)}] \\
+ \frac{1}{12} \sum_{i,j=1}^{d} \partial_{i} g(X_{T}^{(0)}) \partial_{j} g(X_{T}^{(0)}) E[A_{2T}^{1} A_{1T}^{j} Z_{T}^{(0)}], \]

\[
E[g_{2T}^{3} Z_{T}^{(0)}] = \frac{1}{8} \sum_{i,j,k,l,m,n=1}^{d} \partial_{i} \partial_{j} g(X_{T}^{(0)}) \partial_{k} \partial_{l} g(X_{T}^{(0)}) \partial_{m} \partial_{n} g(X_{T}^{(0)}) E[A_{1T}^{k} A_{1T}^{j} A_{1T}^{l} A_{1T}^{m} A_{1T}^{n} Z_{T}^{(0)}] \\
+ \frac{3}{8} \sum_{i,j,k,l,m=1}^{d} \partial_{i} \partial_{j} g(X_{T}^{(0)}) \partial_{k} \partial_{l} g(X_{T}^{(0)}) \partial_{m} g(X_{T}^{(0)}) E[A_{1T}^{k} A_{1T}^{j} A_{1T}^{l} A_{2T}^{m} Z_{T}^{(0)}] \\
+ \frac{3}{8} \sum_{i,j,k,l,m=1}^{d} \partial_{i} \partial_{j} g(X_{T}^{(0)}) \partial_{k} g(X_{T}^{(0)}) \partial_{l} g(X_{T}^{(0)}) E[A_{1T}^{k} A_{1T}^{j} A_{2T}^{l} A_{2T}^{m} Z_{T}^{(0)}] \\
+ \frac{1}{8} \sum_{i,j,k,l,m=1}^{d} \partial_{i} g(X_{T}^{(0)}) \partial_{j} g(X_{T}^{(0)}) \partial_{k} g(X_{T}^{(0)}) \partial_{l} g(X_{T}^{(0)}) E[A_{2T}^{1} A_{2T}^{1} A_{2T}^{l} Z_{T}^{(0)}] \]

where \(A_{1T}\) is given by (2), and \(A_{2T}\), \(A_{3T}\) and \(A_{4T}\) are expressed as

\[
A_{2T} = \int_{0}^{t} Y_{1} Y_{u}^{-1} \left( \sum_{j,k=1}^{d} \partial_{j} \partial_{k} V_{0}(X_{u}^{(0)}, 0) A_{1u}^{k} A_{1u}^{j} du + 2 \sum_{j=1}^{d} \partial_{j} V_{0}(X_{u}^{(0)}, 0) A_{1u}^{j} du \right) \\
+ \partial_{2} V_{0}(X_{u}^{(0)}, 0) du + 2 \sum_{j=1}^{d} \partial_{j} V(X_{u}^{(0)}, A_{1u}^{j}) dW_{u}, \]

\[
A_{3T} = \int_{0}^{t} Y_{1} Y_{u}^{-1} \left( \sum_{j,k,l=1}^{d} \partial_{j} \partial_{k} \partial_{l} V_{0}(X_{u}^{(0)}, 0) A_{1u}^{k} A_{1u}^{j} A_{1u}^{l} du + 3 \sum_{j,k=1}^{d} \partial_{j} \partial_{k} V_{0}(X_{u}^{(0)}, 0) A_{1u}^{k} A_{2u}^{l} du \right) \\
+ 3 \sum_{j,k=1}^{d} \partial_{j} \partial_{k} \partial_{l} V_{0}(X_{u}^{(0)}, 0) A_{1u}^{k} A_{1u}^{j} A_{2u}^{l} du + 3 \sum_{j=1}^{d} \partial_{j} \partial_{l} V_{0}(X_{u}^{(0)}, 0) A_{2u}^{l} du \\
+ 3 \sum_{j=1}^{d} \partial_{j} \partial_{2} V_{0}(X_{u}^{(0)}, 0) A_{1u}^{j} du + \partial_{2}^{2} V_{0}(X_{u}^{(0)}, 0) du \]

18
\[ A_{tt} = \int_0^t Y_1 Y_u^{-1} \left( \sum_{j,k,l,m=1}^d \partial_j \partial_k \partial_l \partial_m V_0(X_u^{(0)}, 0) A_{11u}^j A_{12u}^k A_{21u}^l A_{22u}^m du + 12 \sum_{j=1}^d \partial_j \partial_k V(X_u^{(0)}) A_{11u}^j A_{12u}^k A_{22u}^k dW_u \right) \]

Note that each \( A_{ik} \) \( i = 1, \ldots, d, k = 1, 2, 3, 4 \) has all finite moments due to a grading structure. For the detail of the following definition and theorem, see pp.45-47 in Bichteler, Gravereaux and Jacod [1].

**Definition 1** A grading of \( \mathbb{R}^d \) is a decomposition \( \mathbb{R}^d = \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_q} \) with \( d = d_1 + \cdots + d_q \). The coordinates of a point in \( \mathbb{R}^d \) are always arranged in an increasing order along the subspace \( \mathbb{R}^{d_i} \), and we set \( M_0 = 0 \) and \( M_l = d_1 + \cdots + d_l \) for \( 1 \leq l \leq q \). We say that a mapping \( V \) on \( \mathbb{R}^d \) is graded according to the grading \( \mathbb{R}^d = \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_q} \) if \( V^i(y) \) depends upon only the coordinates \( (y^k)_{1 \leq k \leq M_l} \) when \( M_{l-1} \leq i \leq M_l \).

**Theorem 1** Consider the stochastic differential equation of the form

\[ dY_t = V_0(Y_t, t) dt + V(Y_t, t) dW_t; \ Y_0 = y_0 \in \mathbb{R}^d \quad (30) \]

where coefficients \( V_0 : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d \) and \( V : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d \otimes \mathbb{R}^d \) have a Lipschitz lower triangular structure, and are graded according to \( \mathbb{R}^d = \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_q} \) with respect to \( Y \). Moreover for \( F(y, t) = V_0(y, t) \) or \( V(y, t) \), we assume \( F \) is differentiable in \( y \) in \( \mathbb{R}^d \) and

1. \( |F(0, t)| \leq Z_t \)
2. \( |D_y F(y, t)| \leq \tilde{Z}_t (1 + |y|^\theta) \)
3. \( |\partial_j F^i(y, t)| \leq \zeta \) if \( M_{r-1} \leq i \leq M_r \) for some \( r \leq q \)

where \( \zeta, \theta > 0 \) are constants, and \( Z, \tilde{Z} \) are predictable processes such that \( ||Z||_p \) and \( ||\tilde{Z}||_p \) are finite for all \( p \geq 1 \). Then (30) have a unique solution \( Y \), and for
every \( p \geq 1 \) there are constants \( c_p \) and \( \gamma_p \) depending only upon \((\zeta, \theta, \{||\hat{Z}_r||_r\}_{r \geq 1})\), such that

\[
||Y_T||_p \leq c_p(y_0 + ||Z||_{\gamma_p}).
\]

Applying Theorem 1 to the system of stochastic differential equations consists of \( A_{1t}^k (i = 1, \ldots, d, k = 1, 2, 3, 4) \) and any products of them, we obtain the following lemma.

**Lemma 5** Each coefficient of the expansion \( A_{1t}^k(i = 1, \ldots, d, k = 1, 2, 3, 4) \) has all finite moments.

**(proof)** Consider the system of stochastic differential equations which \( A_{1t}^1, \ldots, A_{1t}^d, A_{2t}^1, \ldots, A_{2t}^d, A_{3t}^1, \ldots, A_{3t}^d, \ldots \) follow, then it is easily shown that the coefficients of the equation have a grading structure and satisfy the conditions in Theorem 1. Hence the coefficients \( A_{1t}^k \) have all finite moments. \( \square \)

Here, we redefine \( \hat{g}_1 = \{\hat{g}_{1t}; t \in \mathbb{R}^+\} \) and \( Z^{(\xi)} = \{Z^{(\xi)}_t; t \in \mathbb{R}^+\} \) as the stochastic processes

\[
\hat{g}_{1t} = \int_0^t \hat{V}(X^{(0)}_u, u)dW_u,
\]

and

\[
Z^{(\xi)}_t = \exp\{i\xi\hat{g}_{1t} + \frac{\xi^2}{2} \xi_t\},
\]

respectively where

\[
\hat{V}(x, t) = (\partial g(X^{(0)}_t))^y [Y_T^{-1}V(x)].
\]

We define \( \eta_{1,1}^1, \eta_{2,1}^1, \eta_{3,1}^1, \eta_{3,2}^1, \eta_{3,3}^1, \eta_{4,1}^1, \eta_{4,2,1}^1, \eta_{4,2,2}^1, \eta_{5,2}^1, \eta_{5,3}^1, \) and \( \eta_{6,3}^{i,k,l} \) as

\[
\eta_{1,1}^1(t) := E[A_{1t}^1Z_t], \quad \eta_{2,1}^1(t) := E[A_{2t}^1Z_t], \quad \eta_{3,1}^1(t) := E[A_{3t}^1A_{1t}^1Z_t], \quad \eta_{3,2}^1(t) := E[A_{3t}^1A_{2t}^1Z_t], \quad \eta_{3,3}^1(t) := E[A_{3t}^1A_{1t}^1A_{2t}^1Z_t], \\
\eta_{4,1}^1(t) := E[A_{4t}^1Z_t], \quad \eta_{4,2,1}^1(t) := E[A_{4t}^1A_{1t}^1Z_t], \quad \eta_{4,2,2}^1(t) := E[A_{4t}^1A_{2t}^1Z_t], \\
\eta_{4,3}^k(t) := E[A_{4t}^1A_{4t}^kZ_t], \quad \eta_{4,4}^1(t) := E[A_{1t}^1A_{2t}^1A_{3t}^1Z_t], \quad \eta_{5,2}^1(t) := E[A_{5t}^1A_{3t}^1A_{2t}^1Z_t], \\
\eta_{5,3}^i(t) := E[A_{5t}^iA_{2t}^1A_{3t}^1Z_t], \quad \eta_{6,4}^i(t) := E[A_{6t}^iA_{2t}^1A_{3t}^1A_{4t}^1Z_t], \quad \eta_{6,5}^{i,k,l}(t) := E[A_{6t}^iA_{2t}^1A_{3t}^1A_{4t}^1A_{5t}^1Z_t], \\
\eta_{6,6}^{i,k,l,m}(t) := E[A_{6t}^iA_{2t}^1A_{3t}^1A_{4t}^1A_{5t}^1A_{6t}^1Z_t], \quad \eta_{6,7}^{i,k,l,m,n}(t) := E[A_{6t}^iA_{2t}^1A_{3t}^1A_{4t}^1A_{5t}^1A_{6t}^1A_{7t}^1Z_t].
\]

(31)

We derive the system of ordinary differential equations of \( \eta \).

In the followings, for simplicity, we assume that \( V_0 \) doesn’t depend on \( \epsilon \), and write \( V_0(x, \epsilon) \) as \( V_0(x) \).

Consider the evaluation of \( \eta_{2,1}^1(T) = E[A_{2T}^1Z_T^{(\xi)}] \) which appears in the \( \epsilon \)-order. Applying Ito’s formula to \( A_{2t}^1Z_t^{(\xi)} \), we have

\[
d(A_{2t}^1Z_t^{(\xi)}) = A_{2t}^1dZ_t^{(\xi)} + Z_t^{(\xi)}dA_{2t}^1 + dA_{2t}^1dZ_t^{(\xi)}
\]

\[
= \left\{2(\xi)\sum_{i' = 1}^d A_{i'2t}^1Z_t^{(\xi)}\hat{V}(X^{(0)}_t, t)\partial_{i'}V^{(0)}(X^{(0)})\right\} + \sum_{i' = 1}^d A_{i'2t}^1Z_t^{(\xi)}\partial_{i'}V_0^{(0)}(X^{(0)})
\]

\[
\text{(31)}
\]
\[ + \sum_{i'=1}^{d} \sum_{k'=1}^{d} A_{i'i'}^{(\xi)} A_{i'i'}^{(\xi)} Z_{i'i'}^{(\xi)} \partial_{i'} \partial_{k'} V_{0}^{i'}(X_{t}^{i'}(0)) \right \} dt \\
+ \left \{ (i\xi) A_{i'i'}^{(\xi)} V(X_{t}^{i'}(0), t) + 2 \sum_{i'=1}^{d} A_{i'i'}^{(\xi)} \partial_{i'} V^{i'}(X_{t}^{i'}(0)) \right \} dW_{t} \]

Since the second and third terms are martingales, taking the expectation on both sides, we have the following ordinary differential equation of \( \eta_{2,1} \):

\[
\frac{d}{dt} \eta_{2,1}(t) = 2(i\xi) \sum_{i'=1}^{d} \eta_{i',1}(t) \hat{V}(X_{t}^{i'}(0), t) \partial_{i'} V^{i'}(X_{t}^{i'}(0))' \\
+ \sum_{i'=1}^{d} \eta_{i',2}(t) \partial_{i'} V_{0}^{i'}(X_{t}^{i'}(0)) + \sum_{i'=1}^{d} \sum_{k'=1}^{d} \eta_{i',k'}^{(i')}(t) \partial_{i'} \partial_{k'} V_{0}^{i'}(X_{t}^{i'}(0))
\]

Here, \( \eta_{i',1}(i = 1, \cdots, d) \) appearing in the right hand side of above ODE are evaluated in the similar manner:

\[
d(A_{i'i'}^{(\xi)} Z_{i'i'}^{(\xi)} = A_{i'i'}^{(\xi)} dZ_{i'i'}^{(\xi)} + Z_{i'i'}^{(\xi)} dA_{i'i'}^{(\xi)} + dA_{i'i'}^{(\xi)} dZ_{i'i'}^{(\xi)}
\]

\[
+ \left \{ (i\xi) Z_{i'i'}^{(\xi)} \hat{V}(X_{t}^{i'}, t) V^{i'}(X_{t}^{i'}(0))' + \sum_{i'=1}^{d} A_{i'i'}^{(\xi)} \partial_{i'} V_{0}^{i'}(X_{t}^{i'}(0)) \right \} dt \\
+ \left \{ (i\xi) A_{i'i'}^{(\xi)} \hat{V}(X_{t}^{i'}, t) + Z_{i'i'}^{(\xi)} V^{i'}(X_{t}^{i'}(0)) \right \} dW_{t}.
\]

hence, we have

\[
\frac{d}{dt} \eta_{i',1}(t) = (i\xi) \hat{V}(X_{t}^{i'}, t) V^{i'}(X_{t}^{i'}(0))' + \sum_{i'=1}^{d} \eta_{i',i'}^{(i')}(t) \partial_{i'} V_{0}^{i'}(X_{t}^{i'}(0)).
\]

\( \eta_{i',k} \) and other higher order terms can be evaluated in the same way.

The key observation is that each ODE does not involve any higher order terms, and only lower or the same order terms appear in the right hand side of the ODE. So, one can easily solve (analytically or numerically) the system of ODEs and evaluate expectations.

**Proposition 2** For \( \eta_{i,m,k} \) defined in (31), the following system of ordinary differential equations is hold:

\[
\frac{d}{dt} \eta_{i,1}(t) = (i\xi) \hat{V}(X_{t}^{i}, t) V^{i}(X_{t}^{i}(0))' + \sum_{i'=1}^{d} \eta_{i,i'}^{(i)}(t) \partial_{i'} V_{0}^{i}(X_{t}^{i}(0))
\]

\[
\frac{d}{dt} \eta_{i,1}(t) = (i\xi) \sum_{i'=1}^{d} \eta_{i,i'}^{(i)}(t) \hat{V}(X_{t}^{i'}, t) \partial_{i'} V^{i'}(X_{t}^{i'}(0))' \\
+ \sum_{i'=1}^{d} \eta_{i,i'}^{(i)}(t) \partial_{i'} V_{0}^{i'}(X_{t}^{i'}(0)) + \sum_{i'=1}^{d} \sum_{k'=1}^{d} \eta_{i,k'}^{(i)}(t) \partial_{i'} \partial_{k'} V_{0}^{i'}(X_{t}^{i'}(0))
\]

\[
\frac{d}{dt} \eta_{i,k}(t) = (i\xi) \left \{ \eta_{i,k}^{(i)}(t) \hat{V}(X_{t}^{i}, t) V^{i}(X_{t}^{i}(0))' + \eta_{i,1}(t) \hat{V}(X_{t}^{i}, t) V^{k}(X_{t}^{i}(0))' \right \} \\
+ V^{i}(X_{t}^{i}(0)) V^{k}(X_{t}^{i}(0))' + \sum_{i'=1}^{d} \sum_{k'=1}^{d} \eta_{i,k'}^{(i)}(t) \partial_{i'} V_{0}^{i'}(X_{t}^{i'}(0)) \partial_{k'} V_{0}^{k'}(X_{t}^{i'}(0))
\]

21
\[
\frac{d}{dt} \eta^i_{3,1}(t) = (i\xi) \left\{ 3 \sum_{i'=1}^d \eta^i_{2,1}(t) \bar{V}(X_i^{(0)}(t), t) \partial_t V^i(X_i^{(0)})' \right.
\]
\[+ 3 \sum_{i'=1}^d \sum_{k'=1}^d \eta^i_{2,2}(t) \bar{V}(X_i^{(0)}(t), t) \partial_{i'} \partial_{k'} V^i(X_i^{(0)})' \} + \sum_{i'=1}^d \eta^i_{3,1}(t) \partial_t V^i_0(X_i^{(0)}) + 3 \sum_{i'=1}^d \sum_{k'=1}^d \eta^i_{3,2}(t) \partial_{i'} \partial_{k'} V^i_0(X_i^{(0)})
\]
\[+ \sum_{i'=1}^d \sum_{k'=1}^d \eta^i_{3,3}(t) \partial_{i'} \partial_{k'} V^i_0(X_i^{(0)}) \right\}
\]
\[
\frac{d}{dt} \eta^i_{3,2}(t) = (i\xi) \left\{ \eta^i_{2,2}(t) \bar{V}(X_i^{(0)}(t), t) V^i(X_i^{(0)}(t))' + 2 \sum_{i'=1}^d \eta^i_{2,2}(t) \bar{V}(X_i^{(0)}(t), t) \partial_{i'} V^k(X_i^{(0)})' \} + \right.
\]
\[+ \sum_{i'=1}^d \eta^i_{3,2}(t) \partial_t V^i_0(X_i^{(0)}) + \sum_{k'=1}^d \eta^i_{3,2}(t) \partial_{k'} V^k_0(X_i^{(0)}) \right\}
\]
\[
\frac{d}{dt} \eta^i_{3,3}(t) = (i\xi) \left\{ \eta^i_{2,2}(t) \bar{V}(X_i^{(0)}(t), t) V^i(X_i^{(0)}(t))' + \eta^i_{2,2}(t) \bar{V}(X_i^{(0)}(t), t) V^k(X_i^{(0)})' + \eta^i_{2,2}(t) \bar{V}(X_i^{(0)}(t), t) V^l(X_i^{(0)})' \right.
\]
\[+ \eta^i_{1,1}(t) V^k(X_i^{(0)}(t)) V^l(X_i^{(0)}(t))' + \eta^i_{1,1}(t) V^i(X_i^{(0)}(t)) V^l(X_i^{(0)}(t))' + \eta^i_{1,1}(t) V^i(X_i^{(0)}(t)) V^k(X_i^{(0)}(t))' \right.
\]
\[+ \sum_{i'=1}^d \eta^i_{3,3}(t) \partial_{i'} V^i_0(X_i^{(0)}) + \sum_{k'=1}^d \eta^i_{3,3}(t) \partial_{k'} V^k_0(X_i^{(0)}) + \sum_{l'=1}^d \eta^i_{3,3}(t) \partial_{l'} V^l_0(X_i^{(0)}) \right\}
\]
\[
\frac{d}{dt} \eta^i_{4,1}(t) = (i\xi) \left\{ 4 \sum_{i'=1}^d \eta^i_{3,1}(t) \bar{V}(X_i^{(0)}(t), t) \partial_t V^i(X_i^{(0)})' + 12 \sum_{i'=1}^d \sum_{k'=1}^d \eta^i_{3,2}(t) \bar{V}(X_i^{(0)}(t), t) \partial_{i'} \partial_{k'} V^i(X_i^{(0)})' \right.
\]
\[+ 4 \sum_{i'=1}^d \sum_{k'=1}^d \sum_{l'=1}^d \eta^i_{3,3}(t) \bar{V}(X_i^{(0)}(t), t) \partial_{i'} \partial_{k'} \partial_{l'} V^i(X_i^{(0)})' \} + \sum_{i'=1}^d \eta^i_{4,1}(t) \partial_t V^i_0(X_i^{(0)}) + 4 \sum_{i'=1}^d \sum_{k'=1}^d \eta^i_{4,2}(t) \partial_{i'} \partial_{k'} V^i_0(X_i^{(0)}) \right.
\]
\[+ 3 \sum_{i'=1}^d \sum_{k'=1}^d \eta^i_{4,2}(t) \partial_{i'} \partial_{k'} V^i_0(X_i^{(0)}) + 4 \sum_{i'=1}^d \sum_{k'=1}^d \sum_{l'=1}^d \eta^i_{4,3}(t) \partial_{i'} \partial_{k'} \partial_{l'} V^i_0(X_i^{(0)}) \right.
\]
\[+ \sum_{k'=1}^d \sum_{l'=1}^d \sum_{m'=1}^d \eta^i_{4,4}(t) \partial_{i'} \partial_{k'} \partial_{l'} \partial_{m'} V^i_0(X_i^{(0)}) \right\}
\]
\[
\frac{d}{dt} \eta^i_{4,2,1}(t) = (i\xi) \left\{ \eta^i_{3,1}(t) \bar{V}(X_i^{(0)}(t), t) V^i(X_i^{(0)}(t))' + 3 \sum_{i'=1}^d \eta^i_{3,2}(t) \bar{V}(X_i^{(0)}(t), t) \partial_{i'} V^k(X_i^{(0)})' \right.
\]
\[+ 3 \sum_{i'=1}^d \sum_{k'=1}^d \eta^i_{3,3}(t) \bar{V}(X_i^{(0)}(t), t) \partial_{i'} \partial_{k'} V^k(X_i^{(0)})' \} + \sum_{i'=1}^d \eta^i_{4,2}(t) \bar{V}(X_i^{(0)}(t), t) \partial_{i'} V^i_0(X_i^{(0)})' + 3 \sum_{i'=1}^d \sum_{k'=1}^d \eta^i_{4,2}(t) \bar{V}(X_i^{(0)}(t), t) \partial_{i'} \partial_{k'} V^k_0(X_i^{(0)})' \right\}
\]
\[
\frac{d}{dt} q^i_{4,2,2}(t) = \left\{ \begin{aligned}
& \sum_{i' = 1}^{d} \eta^l_{4,3}(t) \frac{\partial}{\partial \eta} V^{i'(X_t^0)} + \sum_{k' = 1}^{d} \eta^l_{4,2,1}(t) \frac{\partial}{\partial \eta} V^{k'(X_t^0)} \\
& + 3 \sum_{i' = 1}^{d} \sum_{k' = 1}^{d} \eta^{i',k'}_{4,3}(t) \frac{\partial}{\partial \eta} V^{i'(X_t^0)} + \sum_{i' = 1}^{d} \sum_{k' = 1}^{d} \eta^{i',k',l}_{4,3}(t) \frac{\partial}{\partial \eta} \frac{\partial}{\partial \eta} V^{k'(X_t^0)}
\end{aligned} \right\} \\
\frac{d}{dt} q^i_{4,2,3}(t) = \left\{ \begin{aligned}
& \sum_{i' = 1}^{d} \eta^{i',k'}_{4,3}(t) \frac{\partial}{\partial \eta} V^{i'(X_t^0)} + \sum_{k' = 1}^{d} \eta^{i',k'}_{4,2,2}(t) \frac{\partial}{\partial \eta} V^{k'(X_t^0)}
\end{aligned} \right\} \\
\frac{d}{dt} q^i_{4,4,1}(t) = \left\{ \begin{aligned}
& \sum_{i' = 1}^{d} \eta^{i',k',l}_{4,3}(t) \frac{\partial}{\partial \eta} V^{i'(X_t^0)} + \sum_{k' = 1}^{d} \eta^{i',k',l}_{4,2,3}(t) \frac{\partial}{\partial \eta} V^{k'(X_t^0)} + \sum_{i' = 1}^{d} \sum_{k' = 1}^{d} \eta^{i',k',l}_{4,4,1}(t) \frac{\partial}{\partial \eta} V^{k'(X_t^0)}
\end{aligned} \right\} \\
\frac{d}{dt} q^i_{4,4,3}(t) = \left\{ \begin{aligned}
& \sum_{i' = 1}^{d} \eta^{i',k',l}_{4,3}(t) \frac{\partial}{\partial \eta} V^{i'(X_t^0)} + \sum_{k' = 1}^{d} \eta^{i',k',l}_{4,2,3}(t) \frac{\partial}{\partial \eta} V^{k'(X_t^0)} + \sum_{i' = 1}^{d} \sum_{k' = 1}^{d} \eta^{i',k',l}_{4,4,3}(t) \frac{\partial}{\partial \eta} V^{k'(X_t^0)}
\end{aligned} \right\} \\
\frac{d}{dt} q^i_{5,2}(t) = \left\{ \begin{aligned}
& \sum_{i' = 1}^{d} \eta^{i',k',l}_{4,3}(t) \frac{\partial}{\partial \eta} V^{i'(X_t^0)} + \sum_{k' = 1}^{d} \eta^{i',k',l}_{4,2,3}(t) \frac{\partial}{\partial \eta} V^{k'(X_t^0)} + \sum_{i' = 1}^{d} \sum_{k' = 1}^{d} \eta^{i',k',l}_{4,4,3}(t) \frac{\partial}{\partial \eta} V^{k'(X_t^0)}
\end{aligned} \right\}
\]
\[
\frac{d}{dt} \eta_{5,3,1}^{i,k,l}(t) = (i\xi) \left\{ \eta_{4,2,1}^{i,k,l}(t)\bar{V}(X_t^{(0)}, t)V^i(X_t^{(0)})' + \eta_{4,2,1}^{i,l}(t)\bar{V}(X_t^{(0)}, t)V^k(X_t^{(0)})' \right\}
\]
\[
+ \frac{d}{dt} \eta_{5,3,2}^{i,k,l}(t) = (i\xi) \left\{ \eta_{4,3}^{i,k,l}(t)\bar{V}(X_t^{(0)}, t)V^i(X_t^{(0)})' + \eta_{4,3}^{i,l}(t)\bar{V}(X_t^{(0)}, t)V^k(X_t^{(0)})' \right\}
\]
\[
+ \frac{d}{dt} \eta_{5,4}^{i,k,l,m}(t) = (i\xi) \left\{ \eta_{4,3}^{i,k,l,m}(t)\bar{V}(X_t^{(0)}, t)V^i(X_t^{(0)})' + \eta_{4,3}^{i,m}(t)\bar{V}(X_t^{(0)}, t)V^k(X_t^{(0)})' \right\}
\]
\begin{align*}
+ & \eta_{3,3}^{k,m}(t)V^i(X_t^{(0)})V^k(X_t^{(0)})' + \eta_{3,3}^{k,m}(t)V^i(X_t^{(0)})V^l(X_t^{(0)})' + \eta_{3,3}^{l,m}(t)V^k(X_t^{(0)})V^l(X_t^{(0)})' \\
+ & 2 \sum_{i'=1}^{d} \eta_{3,3}^{k,i',l}(t)V^i(X_t^{(0)})\partial_{i'}V^m(X_t^{(0)})' + 2 \sum_{i'=1}^{d} \eta_{3,3}^{i,i',l}(t)V^k(X_t^{(0)})\partial_{i'}V^m(X_t^{(0)})' \\
+ & 2 \sum_{i'=1}^{d} \eta_{3,3}^{i,k,l}(t)V^l(X_t^{(0)})\partial_{i'}V^m(X_t^{(0)})' \\
+ & \sum_{i'=1}^{d} \eta_{4,4}^{i,k,l,m}(t)\partial_{i'}V_0^m(X_t^{(0)}) + \sum_{k'=1}^{d} \eta_{4,4}^{i,k,l,m}(t)\partial_{k'}V_0^m(X_t^{(0)}) \\
+ & \sum_{m'=1}^{d} \eta_{4,4}^{i,k,l,m'}(t)\partial_{m'}V_0^m(X_t^{(0)}) + \sum_{i'=1}^{d} \sum_{k'=1}^{d} \eta_{4,4}^{i,k,l,m}(t)\partial_{i'}\partial_{k'}V_0^m(X_t^{(0)}) \\
\frac{d}{dt} \eta_{3,3}^{i,k,l,m,n}(t) & = (i\xi) \left\{ \sum_{i'=1}^{d} \eta_{4,4}^{i,k,l,m,n}(t)\tilde{V}(X_t^{(0)},t)\tilde{V}^i(X_t^{(0)})' + \eta_{4,4}^{i,k,l,m,n}(t)\tilde{V}(X_t^{(0)},t)V^k(X_t^{(0)})' \\
+ & \eta_{4,4}^{i,k,l,m,n}(t)\tilde{V}(X_t^{(0)},t)V^l(X_t^{(0)})' + \eta_{4,4}^{i,k,l,m,n}(t)\tilde{V}(X_t^{(0)},t)V^m(X_t^{(0)})' \\
+ & \sum_{i'=1}^{d} \eta_{5,5}^{i,k,l,m,n}(t)\partial_{i'}V_0^m(X_t^{(0)}) + \sum_{k'=1}^{d} \eta_{5,5}^{i,k,l,m,n}(t)\partial_{k'}V_0^m(X_t^{(0)}) \\
+ & \sum_{m'=1}^{d} \eta_{5,5}^{i,k,l,m,n}(t)\partial_{m'}V_0^m(X_t^{(0)}) + \sum_{i'=1}^{d} \sum_{k'=1}^{d} \eta_{5,5}^{i,k,l,m,n}(t)\partial_{i'}\partial_{k'}V_0^m(X_t^{(0)}) \\
\frac{d}{dt} \eta_{3,3}^{i,k,l}(t) & = (i\xi) \left\{ \sum_{i'=1}^{d} \eta_{4,3,3}^{i,k,l}(t)\tilde{V}(X_t^{(0)},t)\partial_{i'}V^i(X_t^{(0)})' + \sum_{k'=1}^{d} \eta_{4,3,3}^{i,k,l}(t)\tilde{V}(X_t^{(0)},t)\partial_{k'}V^k(X_t^{(0)})' \\
+ & 2 \sum_{i'=1}^{d} \eta_{3,3}^{i,k,l}(t)\tilde{V}(X_t^{(0)},t)\partial_{i'}V^l(X_t^{(0)})' + 2 \sum_{i'=1}^{d} \eta_{3,3}^{i,k,l}(t)\tilde{V}(X_t^{(0)},t)\partial_{i'}V^m(X_t^{(0)})' \\
+ & 2 \sum_{i'=1}^{d} \eta_{3,3}^{i,k,l}(t)\tilde{V}(X_t^{(0)},t)\partial_{i'}V^m(X_t^{(0)})' \\
+ & \sum_{i'=1}^{d} \eta_{4,3}^{i,k,l}(t)\partial_{i'}V_0^m(X_t^{(0)}) + \sum_{k'=1}^{d} \eta_{4,3}^{i,k,l}(t)\partial_{k'}V_0^m(X_t^{(0)}) \\
+ & \sum_{m'=1}^{d} \eta_{4,3}^{i,k,l}(t)\partial_{m'}V_0^m(X_t^{(0)}) + \sum_{i'=1}^{d} \sum_{k'=1}^{d} \eta_{4,3}^{i,k,l}(t)\partial_{i'}\partial_{k'}V_0^m(X_t^{(0)}) \\
\sum_{i'=1}^{d} \sum_{k'=1}^{d} \eta_{5,5}^{i,k,l}(t)\partial_{i'}\partial_{k'}V_0^m(X_t^{(0)}) + \sum_{i'=1}^{d} \sum_{k'=1}^{d} \eta_{5,5}^{i,k,l}(t)\partial_{i'}\partial_{k'}V_0^m(X_t^{(0)}) \right\}
\end{align*}
The asymptotic expansion of density of $G^{(c)}$ up to $\epsilon^3$-order is given by

$$ f_{G^{(c)}}(x) = f_{g_{1\tau}}(x) + \epsilon \left\{ \sum_{i=1}^{3} C_i H_i(x; \Sigma_T) f_{g_{1\tau}}(x) \right\} + \epsilon^2 \left\{ \sum_{i=1}^{6} C_{2\tau} H_i(x; \Sigma_T) f_{g_{1\tau}}(x) \right\} + \epsilon^3 \left\{ \sum_{i=1}^{9} C_{3\tau} H_i(x; \Sigma_T) f_{g_{1\tau}}(x) \right\} + o(\epsilon^3). $$

where

$$ C_{1\tau} = \Sigma_T a_{1}^{1.1,1,1}, $$

$$ C_{2\tau} = \Sigma_T a_{1}^{2.1,0.1,1} + \frac{1}{2} \chi_T^2 a_{1}^{2,2,2,2,0}, $$

$$ C_{3\tau} = \Sigma_T a_{1}^{3.1,0.0,1,1} + \frac{1}{2} \chi_T^2 a_{1}^{3.2,1,2,0} + \frac{1}{6} \chi_T^3 a_{1}^{3,3,3,0}. $$

$a_{1}^{m,k}$ are given by (29), and expectations in (29) are obtained as the solutions to the system of ordinary differential equations given in Proposition 2.
4.2 Asymptotic Expansion of Option Prices

We apply the asymptotic expansion to option pricing. We consider the plain vanilla option on the underlying asset \( g(X_T^{(c)}) \) whose dynamics is given by (1). For example, the call option price with strike \( K \) and maturity \( T \) is given by

\[
C(K, T) = \epsilon P(0, T) \int_{-k^{(c)}}^{\infty} (x + k^{(c)}) f_{G^{(c)}}(x) dx
\]

where \( k^{(c)} = \frac{G^{(0)} - K}{P(0, T)} \), \( P(0, T) \) denotes the price at time 0 of a zero coupon bond with maturity \( T \), and \( f_{G^{(c)}} \) is the normal asymptotic expansion of density of \( G^{(c)} \) given by (28). In particular, using the result of the previous subsection, the approximated price to the option up to the fourth order can be expressed as

\[
C(K, T) = \epsilon P(0, T) \int_{-k^{(c)}}^{\infty} (x + k^{(c)}) f_{G^{(c)}}(x) dx + \epsilon^2 P(0, T) \int_{-k^{(c)}}^{\infty} (x + k^{(c)}) \left\{ \sum_{l=1}^{3} C_{1l} H_l(x; \Sigma_T) \right\} f_{G^{(c)}}(x) dx + \epsilon^3 P(0, T) \int_{-k^{(c)}}^{\infty} (x + k^{(c)}) \left\{ \sum_{l=1}^{6} C_{2l} H_l(x; \Sigma_T) \right\} f_{G^{(c)}}(x) dx + \epsilon^4 P(0, T) \int_{-k^{(c)}}^{\infty} (x + k^{(c)}) \left\{ \sum_{l=1}^{9} C_{3l} H_l(x; \Sigma_T) \right\} f_{G^{(c)}}(x) dx + o(\epsilon^4).
\]

Integrals appeared in the right hand side can be calculated using following formulas related to the Hermite polynomial

\[
\int_{-\infty}^{\infty} H_k(x; \Sigma) f_{G^{(c)}}(x) dx = \Sigma H_{k-1}(-y; \Sigma) f_{G^{(c)}}(y) \quad (k \geq 1),
\]

\[
\int_{-\infty}^{\infty} x H_k(x; \Sigma) f_{G^{(c)}}(x) dx = -\Sigma g H_{k-1}(-y; \Sigma) f_{G^{(c)}}(y) + \Sigma^2 H_{k-2}(-y; \Sigma) f_{G^{(c)}}(y) \quad (k \geq 2).
\]

4.3 Log- Normal Asymptotic Expansion

Suppose that the underlying asset process \( S^{(c)} \) follows

\[
dS_t^{(c)} = g(X_t^{(c)}) dW_t; \quad S_0^{(c)} = s_0
\]

\[
dX_t^{(c)} = V_0(X_t^{(c)}, \epsilon) dt + \epsilon V(X_t^{(c)}) dW_t; \quad X_0^{(c)} = x_0 \in \mathbb{R}^d.
\]

Define \( \hat{X}^{(c)} \) as

\[
\hat{X}_t^{(c)} = \log \left( \frac{S_t^{(c)}}{S_0} \right).
\]

Then, we have

\[
\hat{X}_t^{(c)} = -\frac{\sigma^2}{2} \int_0^t g(X_u^{(c)})^2 du + \int_0^t g(X_u^{(c)}) \sigma dW_u,
\]

and note that

\[
\hat{X}_T^{(0)} \sim N(\hat{\mu}_T, \hat{\Sigma}_T),
\]

28
where
\[
\hat{\mu}_T = -\frac{|\hat{\sigma}|^2}{2} \int_0^T g(X_u^{(0)})^2 du = -\frac{1}{2} \hat{\Sigma}_T,
\]
\[
\hat{\Sigma}_T = |\hat{\sigma}|^2 \int_0^T g(X_u^{(0)})^2 du.
\]

Moreover, an asymptotic expansion of \(\hat{X}_T^{(\epsilon)}\) up to \(\epsilon^N\)-order is expressed as
\[
\hat{X}_T^{(\epsilon)} = \hat{X}_T^{(0)} + \sum_{n=1}^N \epsilon^n \hat{A}_n T + o(\epsilon^N),
\]
where \(\hat{A}_n = \frac{\partial^n \hat{X}_T^{(\epsilon)}}{\partial \epsilon^n} |_{\epsilon=0}\). Note that \(S^{(\epsilon)}\) is expanded around a log-normal distribution since \(\hat{X}_T^{(0)}\) has a Gaussian distribution.

Next, define \(Z_t^{(\epsilon)}\) as
\[
Z_t^{(\epsilon)} = \exp \left( i \xi \int_0^t g(X_u^{(0)}) \hat{\sigma} dW_u \right).
\]

Then, the result in the previous subsection is applied to deriving the density function of \(\hat{X}_T^{(\epsilon)}\) if \(G^{(\epsilon)}\) is replaced by \(\hat{X}_T^{(\epsilon)}\).

Similar to the normal case, the log-normal asymptotic expansion of the price of the call option on \(\hat{X}_T^{(\epsilon)}\) is given by
\[
C(K, T) = P(0, T) \int_{\log \frac{K}{s_0}}^\infty e^{x} \sigma_2^2(x) f_{\hat{X}_T^{(\epsilon)}}(x) dx.
\]

## 5 Numerical Examples

### 5.1 \(\lambda\)-SABR model

To test the validity of the expansion, we first consider the European plain-vanilla call and put prices under the following \(\lambda\)-SABR model [24] (interest rate=0\%):
\[
\begin{align*}
    dS^{(\epsilon)}(t) &= \epsilon \sigma^{(\epsilon)}(t) S^{(\epsilon)}(t)^\beta dW_1^1, \\
    d\sigma^{(\epsilon)}(t) &= \lambda(\theta - \sigma^{(\epsilon)}(t)) dt + \epsilon_1 \sigma^{(\epsilon)}(t) dW_1^2 + \epsilon_2 \sigma^{(\epsilon)}(t) dW_2^2,
\end{align*}
\]
where \(\epsilon_1 = \rho \nu, \epsilon_2 = (\sqrt{1-\rho^2})\nu\). (The correlation between \(S\) and \(\sigma\) is \(\rho \in [-1, 1]\).)

Approximated prices by the asymptotic expansion method are calculated up to the fifth order. Note that all the solutions to differential equations are obtained analytically. Benchmark values are computed by Monte Carlo simulations. \(\epsilon\) is set to be one and other parameters used in the test are given in Table 1:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>(S(0))</th>
<th>(\lambda)</th>
<th>(\sigma(0))</th>
<th>(\beta)</th>
<th>(\rho)</th>
<th>(\theta)</th>
<th>(\nu)</th>
<th>(T)</th>
</tr>
</thead>
<tbody>
<tr>
<td>i</td>
<td>100</td>
<td>0.1</td>
<td>3.0</td>
<td>0.5</td>
<td>-0.7</td>
<td>3.0</td>
<td>0.3</td>
<td>10</td>
</tr>
<tr>
<td>ii</td>
<td>100</td>
<td>0.1</td>
<td>3.0</td>
<td>0.5</td>
<td>-0.7</td>
<td>3.0</td>
<td>0.1</td>
<td>10</td>
</tr>
<tr>
<td>iii</td>
<td>100</td>
<td>0.1</td>
<td>3.0</td>
<td>0.5</td>
<td>-0.7</td>
<td>3.0</td>
<td>0.3</td>
<td>1</td>
</tr>
</tbody>
</table>
In Monte Carlo simulations for benchmark values, we use Euler-Maruyama scheme as a discretization scheme with 1024, 1024, and 512 time steps for case i, ii, and iii respectively, and generate $10^8$ paths in each simulation.

For the case of $\beta = 1$ in the $\lambda$-SABR model, we can apply the log-normal asymptotic expansion method given in the previous section. To test the efficiency of the high order log-normal asymptotic expansion method, we consider the European plain-vanilla call and put prices under the following parameters (and $\epsilon = 1$ as well as in the previous examples) with different maturities:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$S(0)$</th>
<th>$\lambda$</th>
<th>$\sigma(0)$</th>
<th>$\beta$</th>
<th>$\rho$</th>
<th>$\theta$</th>
<th>$\nu$</th>
<th>$T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>iv</td>
<td>100</td>
<td>0.1</td>
<td>0.3</td>
<td>1.0</td>
<td>-0.7</td>
<td>0.3</td>
<td>0.3</td>
<td>10</td>
</tr>
<tr>
<td>v</td>
<td>100</td>
<td>0.1</td>
<td>0.3</td>
<td>1.0</td>
<td>-0.7</td>
<td>0.3</td>
<td>0.3</td>
<td>20</td>
</tr>
<tr>
<td>vi</td>
<td>100</td>
<td>0.1</td>
<td>0.3</td>
<td>1.0</td>
<td>-0.7</td>
<td>0.3</td>
<td>0.3</td>
<td>30</td>
</tr>
</tbody>
</table>

We calculate approximated prices by the log-normal asymptotic expansion method up to the fourth order. Benchmark prices are computed by Monte Carlo simulations. In the simulations, we adapt the second order discretization scheme given by Ninomiya-Victoir [34] with 128, 256, 256 time steps respectively.

Results are in Table 3 and Table 4.

From the results, in each case, the higher order asymptotic expansion or log-normal asymptotic expansion almost always improve the accuracy of approximation by the lower expansions. Improvement is significant especially in long-term cases in which the lower order asymptotic expansions cannot approximate the price well.
### Table 3:

<table>
<thead>
<tr>
<th>Case</th>
<th>Strike(C/P)</th>
<th>1st 2nd 3rd 4th 5th</th>
<th>1st 2nd 3rd 4th 5th</th>
</tr>
</thead>
<tbody>
<tr>
<td>i</td>
<td>50 Put</td>
<td>13.109 4.576 5.000 2.313 1.007 0.260</td>
<td>37.20% 38.14% 17.64% 8.14% 1.98%</td>
</tr>
<tr>
<td></td>
<td>60 Put</td>
<td>16.618 4.544 4.648 1.931 0.938 0.195</td>
<td>27.34% 27.97% 11.62% 5.65% 1.17%</td>
</tr>
<tr>
<td></td>
<td>70 Put</td>
<td>20.482 4.241 4.322 1.585 0.844 0.149</td>
<td>20.71% 21.10% 7.74% 4.12% 0.73%</td>
</tr>
<tr>
<td></td>
<td>80 Put</td>
<td>24.720 3.965 4.020 1.269 0.778 0.117</td>
<td>16.04% 16.26% 5.14% 3.15% 0.47%</td>
</tr>
<tr>
<td></td>
<td>90 Put</td>
<td>29.347 3.710 3.738 0.980 0.735 0.094</td>
<td>12.64% 12.74% 3.34% 2.51% 0.32%</td>
</tr>
<tr>
<td></td>
<td>100 Call</td>
<td>34.375 3.472 3.472 0.712 0.712 0.077</td>
<td>10.10% 10.10% 2.07% 2.07% 0.22%</td>
</tr>
<tr>
<td></td>
<td>110 Call</td>
<td>29.811 3.246 3.217 0.459 0.704 0.063</td>
<td>10.89% 10.79% 1.54% 2.36% 0.21%</td>
</tr>
<tr>
<td></td>
<td>120 Call</td>
<td>25.659 3.026 2.971 0.220 0.711 0.050</td>
<td>11.79% 11.58% 0.86% 2.77% 0.19%</td>
</tr>
<tr>
<td></td>
<td>130 Call</td>
<td>21.914 2.809 2.728 -0.010 0.731 0.035</td>
<td>12.82% 12.45% -0.04% 3.33% 0.16%</td>
</tr>
<tr>
<td></td>
<td>140 Call</td>
<td>18.571 2.591 2.487 -0.230 0.762 0.018</td>
<td>13.95% 13.39% -1.24% 4.10% 0.10%</td>
</tr>
<tr>
<td></td>
<td>150 Call</td>
<td>15.615 2.370 2.246 -0.441 0.804 -0.002</td>
<td>15.18% 14.38% -2.83% 5.15% -0.02%</td>
</tr>
<tr>
<td>ii</td>
<td>50 Put</td>
<td>1.682 -0.914 0.030 0.475 0.182 -0.016</td>
<td>-54.33% 1.81% 28.25% 10.84% -0.92%</td>
</tr>
<tr>
<td></td>
<td>60 Put</td>
<td>2.067 -1.056 0.129 0.445 0.103 -0.003</td>
<td>-40.52% 4.94% 17.06% 3.95% -0.13%</td>
</tr>
<tr>
<td></td>
<td>70 Put</td>
<td>3.950 -1.047 0.214 0.364 0.061 0.008</td>
<td>-26.51% 5.41% 9.22% 1.55% 0.20%</td>
</tr>
<tr>
<td></td>
<td>80 Put</td>
<td>5.883 -0.825 0.254 0.258 0.048 0.013</td>
<td>-14.03% 4.32% 4.39% 0.82% 0.23%</td>
</tr>
<tr>
<td></td>
<td>90 Put</td>
<td>8.631 -0.390 0.237 0.150 0.047 0.016</td>
<td>-4.52% 2.75% 1.74% 0.54% 0.18%</td>
</tr>
<tr>
<td></td>
<td>100 Call</td>
<td>12.450 0.166 0.166 0.048 0.048 0.018</td>
<td>1.33% 1.33% 0.39% 0.39% 0.14%</td>
</tr>
<tr>
<td></td>
<td>110 Call</td>
<td>7.577 0.664 0.037 -0.050 0.053 0.022</td>
<td>8.76% 0.49% -0.67% 0.70% 0.29%</td>
</tr>
<tr>
<td></td>
<td>120 Call</td>
<td>4.131 0.927 -0.153 -0.149 0.062 0.027</td>
<td>22.43% -3.70% -3.60% 1.49% 0.65%</td>
</tr>
<tr>
<td></td>
<td>130 Call</td>
<td>2.008 0.894 -0.367 -0.217 0.086 0.033</td>
<td>44.52% -18.27% -10.79% 4.30% 1.64%</td>
</tr>
<tr>
<td></td>
<td>140 Call</td>
<td>0.887 0.663 -0.522 -0.205 0.136 0.030</td>
<td>74.77% -58.78% -23.16% 15.35% 3.36%</td>
</tr>
<tr>
<td></td>
<td>150 Call</td>
<td>0.372 0.396 -0.548 -0.104 0.189 -0.009</td>
<td>106.35% -147.29% -27.82% 50.82% -2.34%</td>
</tr>
<tr>
<td>iii</td>
<td>50 Put</td>
<td>0.633 -0.038 0.094 0.061 0.015 0.005</td>
<td>-6.05% 14.84% 9.64% 2.33% 0.85%</td>
</tr>
<tr>
<td></td>
<td>60 Put</td>
<td>1.335 -0.063 0.111 0.058 0.013 0.006</td>
<td>-4.74% 8.32% 4.34% 0.97% 0.42%</td>
</tr>
<tr>
<td></td>
<td>70 Put</td>
<td>2.571 -0.072 0.121 0.048 0.011 0.006</td>
<td>-2.79% 4.72% 1.87% 0.45% 0.22%</td>
</tr>
<tr>
<td></td>
<td>80 Put</td>
<td>4.579 -0.046 0.124 0.034 0.010 0.005</td>
<td>-1.00% 2.71% 0.75% 0.22% 0.12%</td>
</tr>
<tr>
<td></td>
<td>90 Put</td>
<td>7.608 0.019 0.119 0.019 0.008 0.004</td>
<td>0.25% 1.57% 0.26% 0.11% 0.05%</td>
</tr>
<tr>
<td></td>
<td>100 Call</td>
<td>11.857 0.111 0.111 0.008 0.008 0.004</td>
<td>0.94% 0.94% 0.07% 0.07% 0.03%</td>
</tr>
<tr>
<td></td>
<td>110 Call</td>
<td>7.430 0.197 0.096 -0.004 0.008 0.003</td>
<td>2.65% 1.29% -0.05% 0.10% 0.05%</td>
</tr>
<tr>
<td></td>
<td>120 Call</td>
<td>4.289 0.244 0.074 -0.015 0.009 0.004</td>
<td>5.70% 1.74% -0.36% 0.20% 0.09%</td>
</tr>
<tr>
<td></td>
<td>130 Call</td>
<td>2.260 0.239 0.046 -0.027 0.009 0.003</td>
<td>10.57% 2.03% -1.21% 0.40% 0.14%</td>
</tr>
<tr>
<td></td>
<td>140 Call</td>
<td>1.080 0.192 0.017 -0.036 0.009 0.002</td>
<td>17.77% 1.62% -3.30% 0.88% 0.19%</td>
</tr>
<tr>
<td></td>
<td>150 Call</td>
<td>0.466 0.129 -0.004 -0.036 0.010 0.001</td>
<td>27.62% -0.75% -7.81% 2.13% 0.13%</td>
</tr>
<tr>
<td>Case</td>
<td>Strike(C/P)</td>
<td>MC</td>
<td>Log Normal A.E.(Difference)</td>
</tr>
<tr>
<td>------</td>
<td>-------------</td>
<td>------------</td>
<td>-----------------------------</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Log-Norm 1st 2nd 3rd 4th</td>
</tr>
<tr>
<td>iv</td>
<td>50 Put</td>
<td>9.429</td>
<td>-0.896 0.250 0.470 -0.223 0.021</td>
</tr>
<tr>
<td></td>
<td>60 Put</td>
<td>13.095</td>
<td>-0.187 0.168 0.449 -0.215 0.028</td>
</tr>
<tr>
<td></td>
<td>70 Put</td>
<td>17.307</td>
<td>0.678 0.045 0.431 -0.203 0.034</td>
</tr>
<tr>
<td></td>
<td>80 Put</td>
<td>22.041</td>
<td>1.620 -0.099 0.414 -0.190 0.039</td>
</tr>
<tr>
<td></td>
<td>90 Put</td>
<td>27.272</td>
<td>2.577 -0.253 0.397 -0.177 0.045</td>
</tr>
<tr>
<td></td>
<td>100 Call</td>
<td>32.971</td>
<td>3.503 -0.416 0.379 -0.163 0.051</td>
</tr>
<tr>
<td></td>
<td>110 Call</td>
<td>29.110</td>
<td>4.367 -0.589 0.360 -0.149 0.057</td>
</tr>
<tr>
<td></td>
<td>120 Call</td>
<td>25.655</td>
<td>5.149 -0.773 0.338 -0.135 0.063</td>
</tr>
<tr>
<td></td>
<td>130 Call</td>
<td>22.576</td>
<td>5.837 -0.972 0.315 -0.120 0.069</td>
</tr>
<tr>
<td></td>
<td>140 Call</td>
<td>19.842</td>
<td>6.424 -1.186 0.289 -0.104 0.076</td>
</tr>
<tr>
<td></td>
<td>150 Call</td>
<td>17.420</td>
<td>6.912 -1.416 0.261 -0.088 0.083</td>
</tr>
<tr>
<td>v</td>
<td>50 Put</td>
<td>15.350</td>
<td>0.961 -0.125 0.782 -0.523 0.148</td>
</tr>
<tr>
<td></td>
<td>60 Put</td>
<td>20.207</td>
<td>1.990 -0.391 0.823 -0.513 0.153</td>
</tr>
<tr>
<td></td>
<td>70 Put</td>
<td>25.499</td>
<td>3.062 -0.664 0.857 -0.495 0.153</td>
</tr>
<tr>
<td></td>
<td>80 Put</td>
<td>31.184</td>
<td>4.134 -0.937 0.884 -0.472 0.150</td>
</tr>
<tr>
<td></td>
<td>90 Put</td>
<td>37.228</td>
<td>5.175 -1.207 0.908 -0.446 0.145</td>
</tr>
<tr>
<td></td>
<td>100 Call</td>
<td>43.598</td>
<td>6.168 -1.474 0.928 -0.417 0.137</td>
</tr>
<tr>
<td></td>
<td>110 Call</td>
<td>40.267</td>
<td>7.101 -1.741 0.946 -0.387 0.129</td>
</tr>
<tr>
<td></td>
<td>120 Call</td>
<td>37.208</td>
<td>7.967 -2.009 0.962 -0.356 0.119</td>
</tr>
<tr>
<td></td>
<td>130 Call</td>
<td>34.399</td>
<td>8.763 -2.278 0.977 -0.323 0.107</td>
</tr>
<tr>
<td></td>
<td>140 Call</td>
<td>31.818</td>
<td>9.487 -2.551 0.990 -0.289 0.095</td>
</tr>
<tr>
<td></td>
<td>150 Call</td>
<td>29.447</td>
<td>10.142 -2.829 1.003 -0.255 0.082</td>
</tr>
<tr>
<td>vi</td>
<td>50 Put</td>
<td>19.801</td>
<td>2.280 -0.889 1.143 -0.592 0.182</td>
</tr>
<tr>
<td></td>
<td>60 Put</td>
<td>25.471</td>
<td>3.371 -1.248 1.254 -0.581 0.154</td>
</tr>
<tr>
<td></td>
<td>70 Put</td>
<td>31.500</td>
<td>4.459 -1.594 1.351 -0.560 0.120</td>
</tr>
<tr>
<td></td>
<td>80 Put</td>
<td>37.847</td>
<td>5.520 -1.927 1.437 -0.535 0.081</td>
</tr>
<tr>
<td></td>
<td>90 Put</td>
<td>44.476</td>
<td>6.541 -2.246 1.515 -0.505 0.039</td>
</tr>
<tr>
<td></td>
<td>100 Call</td>
<td>51.357</td>
<td>7.512 -2.555 1.586 -0.474 -0.005</td>
</tr>
<tr>
<td></td>
<td>110 Call</td>
<td>48.465</td>
<td>8.430 -2.856 1.652 -0.442 -0.051</td>
</tr>
<tr>
<td></td>
<td>120 Call</td>
<td>45.780</td>
<td>9.291 -3.150 1.715 -0.409 -0.098</td>
</tr>
<tr>
<td></td>
<td>130 Call</td>
<td>43.281</td>
<td>10.097 -3.439 1.774 -0.376 -0.147</td>
</tr>
<tr>
<td></td>
<td>140 Call</td>
<td>40.954</td>
<td>10.848 -3.724 1.831 -0.342 -0.197</td>
</tr>
<tr>
<td></td>
<td>150 Call</td>
<td>38.782</td>
<td>11.545 -4.007 1.886 -0.309 -0.248</td>
</tr>
</tbody>
</table>
5.2 Currency Option under a Libor Market Model of Interest Rates and a Stochastic Volatility of a Spot Exchange Rate

In this subsection, we apply our methods to pricing options on currencies under Libor Market Models(LMMs) of interest rates and a stochastic volatility of the spot foreign exchange rate(Forex). Due to limitation of space, only the structure of the stochastic differential equations of our model is described here. For details of the underlying model, see Takahashi and Takehara [53].

5.2.1 Cross-Currency Libor Market Models

Let \((\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{0 \leq t \leq T^* < \infty})\) be a complete probability space with a filtration satisfying the usual conditions. We consider the following pricing problem for the call option with maturity \(T \in (0, T^*)\) and strike rate \(K > 0\):

\[
V^C(0; T, K) = P_d(0, T) \times \mathbb{E}^P \left[ (S(T) - K)^+ \right] = P_d(0, T) \times \mathbb{E}^P \left[ (F_T(T) - K)^+ \right] 
\]

(32)

where \(V^C(0; T, K)\) denotes the value of an European call option at time 0 with maturity \(T\) and strike rate \(K\), \(S(T)\) denotes the spot exchange rate at time \(t \geq 0\) and \(F_T(t)\) denotes the time \(t\) value of the forex forward rate with maturity \(T\).

Similarly, for the put option we consider

\[
V^P(0; T, K) = P_d(0, T) \times \mathbb{E}^P \left[ (K - S(T))^+ \right] = P_d(0, T) \times \mathbb{E}^P \left[ (K - F_T(T))^+ \right]. 
\]

(33)

It is well known that the arbitrage-free relation between the forex spot rate and the forex forward rate are given by \(F_T(t) = S(t) \frac{P_d(t,T)}{P_f(t,T)}\) where \(P_d(t,T)\) and \(P_f(t,T)\) denote the domestic and foreign zero coupon bonds with maturity \(T\) respectively. \(\mathbb{E}^P[\cdot]\) denotes an expectation operator under EMM(Equivalent Martingale Measure) \(P\) whose associated numeraire is the domestic zero coupon bond maturing at \(T\).

For these pricing problems, a market model and a stochastic volatility model are applied to modeling interest rates’ and the spot exchange rate’s dynamics respectively.

We first define domestic and foreign forward interest rates as \(f_d(t) = \left( \frac{P_d(t,T)}{P_d(t,T+j)} - 1 \right) \frac{1}{\tau_j}\) respectively, where \(j = n(t), n(t) + 1, \ldots, N\), \(\tau_j = T_j + 1 - T_j\), and \(P_d(t,T_j)\) and \(P_f(t,T_j)\) denote the prices of domestic/foreign zero coupon bonds with maturity \(T_j\) at time \(t \leq T_j\) respectively; \(n(t) = \min\{i : t \leq T_i\}\). We also define spot interest rates to the nearest fixing date denoted by \(f_{d,n(t)-1}(t)\) and \(f_{f,n(t)-1}(t)\) as \(f_{d,n(t)-1}(t) = \left( \frac{1}{P_d(t,T_{n(t)})} - 1 \right) \frac{1}{(T_{n(t)} - t)}\) and \(f_{f,n(t)-1}(t) = \left( \frac{1}{P_f(t,T_{n(t)})} - 1 \right) \frac{1}{(T_{n(t)} - t)}\). Finally, we set \(T = T_{N+1}\) and will abbreviate \(F_T(t)\) to \(F_{N+1}(t)\) in what follows.

Under the framework of the asymptotic expansion in the standard cross-currency libor market model, we have to consider the following system of stochastic differential equations(henceforth called S.D.E.s) under the domestic terminal measure \(P\) to price options. For detailed arguments on the framework of these S.D.E.s see [53].

As for the domestic and foreign interest rates we assume forward market
models; for \( j = n(t) - 1, n(t), n(t) + 1, \ldots, N, \)
\[
\begin{align*}
 f_{d_j}^{(c)}(t) &= f_{d_j}(0) + \epsilon^2 \sum_{i=j+1}^{N} \int_{0}^{t} g_{d_i}^{0,(c)}(u) \gamma_{d_i}(u) f_{d_j}^{(c)}(u) du + \epsilon \int_{0}^{t} f_{d_j}^{(c)}(u) \gamma_{d_j}(u) dW_u, \tag{34} \\
 f_{f_j}^{(c)}(t) &= f_{f_j}(0) - \epsilon^2 \sum_{i=0}^{j} \int_{0}^{t} g_{f_i}^{0,(c)}(u) \gamma_{f_j}(u) f_{f_j}^{(c)}(u) du + \epsilon^2 \sum_{i=0}^{N} \int_{0}^{t} g_{d_i}^{0,(c)}(u) \gamma_{f_j}(u) f_{f_j}^{(c)}(u) du \\
&\quad - \epsilon^2 \int_{0}^{t} \sigma^{(c)}(u) \gamma_{f_j}(u) f_{f_j}^{(c)}(u) du + \epsilon \int_{0}^{t} f_{f_j}^{(c)}(u) \gamma_{f_j}(u) dW_u, \tag{35}
\end{align*}
\]
where
\[
g_{d_j}^{0,(c)}(t) := \frac{-\tau f_{d_j}^{(c)}(t)}{1 + \tau f_{d_j}^{(c)}(t)} \gamma_{d_j}(t), \quad g_{f_j}^{0,(c)}(t) := \frac{-\tau f_{f_j}^{(c)}(t)}{1 + \tau f_{f_j}^{(c)}(t)} \gamma_{f_j}(t);
\]
x' denotes the transpose of x, \( \hat{J}_{j+1} := \{0, 1, \ldots, j\} \), and W is a \( d' \)-dimensional standard Wiener process under the domestic terminal measure \( P \); \( \gamma_{d_j}(s), \gamma_{f_j}(s) \)
are \( d' \)-dimensional vector-valued functions of time-parameter s; \( \hat{\sigma} \) denotes a \( d' \)-dimensional constant vector satisfying \( ||\hat{\sigma}|| = 1 \) and \( \sigma(t) \), the volatility of the spot exchange rate, is specified to follow a \( \mathbb{R}^{n(t)} \)-valued general time-inhomogeneous Markovian process as follows:
\[
\sigma(t) = \sigma(0) + \int_{0}^{t} \mu(u, \sigma^{(c)}(u)) du + \epsilon^2 \sum_{j=1}^{N} \int_{0}^{t} g_{d_j}^{0,(c)}(u) \omega(u, \sigma^{(c)}(u)) du + \epsilon \int_{0}^{t} \omega(u, \sigma^{(c)}(u)) dW_u, \tag{36}
\]
where \( \mu(s, x) \) and \( \omega(s, x) \) are functions of s and x.

Finally, we consider the process of the forex forward \( F_{N+1}(t) \). Since \( F_{N+1}(t) \equiv F_{T_{N+1}}(t) \) can be expressed as \( F_{N+1}(t) = S(t) \frac{P_{T_{N+1}}(t)}{P_{T_{N+1}}(0)} \), we easily notice that it is a martingale under the domestic terminal measure. In particular, it satisfies the following stochastic differential equation
\[
F_{T}^{(c)}(t) = F_{T}(0) + \epsilon \int_{0}^{t} \sigma_{F}^{(c)}(u) F^{(c)}(u) dW_u, \tag{37}
\]
where
\[
\sigma_{F}^{(c)}(t) := \sum_{j=0}^{N} \left( g_{f_j}^{0,(c)}(t) - g_{d_j}^{0,(c)}(t) \right) + \sigma^{(c)}(t).
\]

5.2.2 Numerical Examples

We here specify our model and parameters, and confirm the effectiveness of our method in this cross-currency framework.

First of all, the processes of domestic and foreign forward interest rates and of the volatility of the spot exchange rate are specified. We suppose \( d = 4 \), that is the dimension of a Brownian motion is set to be four; it represents the uncertainty of domestic and foreign interest rates, the spot exchange rate, and its volatility. Note that in this framework correlations among all factors are allowed. We also suppose \( S(0) = 100 \).

Next, we specify a volatility process of the spot exchange rate in (36) with
\[
\begin{align*}
\mu(s, x) &= \kappa(\theta - x), \\
\omega(s, x) &= \omega x, \tag{38}
\end{align*}
\]
Table 5: Initial domestic/foreign forward interest rates and their volatilities

<table>
<thead>
<tr>
<th>Case</th>
<th>( f_d )</th>
<th>( \gamma_d^* )</th>
<th>( f_f )</th>
<th>( \gamma_f^* )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i)</td>
<td>0.05</td>
<td>0.12</td>
<td>0.05</td>
<td>0.12</td>
</tr>
<tr>
<td>(ii)</td>
<td>0.02</td>
<td>0.3</td>
<td>0.05</td>
<td>0.12</td>
</tr>
<tr>
<td>(iii)</td>
<td>0.05</td>
<td>0.12</td>
<td>0.02</td>
<td>0.3</td>
</tr>
<tr>
<td>(iv)</td>
<td>0.02</td>
<td>0.3</td>
<td>0.02</td>
<td>0.3</td>
</tr>
</tbody>
</table>

where \( \theta \) and \( \kappa \) represent the level and speed of its mean-reversion respectively, and \( \omega \) denotes a volatility vector on the volatility. In this section the parameters are set as follows; \( \epsilon = 1, \sigma(0) = \theta = 0.1, \) and \( \kappa = 0.1; \omega = \omega^* \bar{v} \) where \( \omega^* = 0.3 \) and \( \bar{v} \) denotes a four dimensional constant vector given below.

We further suppose that initial term structures of domestic and foreign forward interest rates are flat, and their volatilities also have flat structures and are constant over time: that is, for all \( j, f_d(0) = f_d, f_f(0) = f_f, \gamma_d(t) = \gamma_d^* \) and \( \gamma_f(t) = \gamma_f^* \). Here, \( \gamma_d^* \) and \( \gamma_f^* \) are constant scalars, and \( \bar{\gamma}_d \) and \( \bar{\gamma}_f \) denote four dimensional constant vectors. Moreover, given a correlation matrix \( C \) among all four factors, the constant vectors \( \bar{\gamma}_d, \bar{\gamma}_f, \bar{\sigma} \) and \( \bar{v} \) can be determined to satisfy \( ||\bar{\gamma}_d|| = ||\bar{\gamma}_f|| = ||\bar{\sigma}|| = ||\bar{v}|| = 1 \) and \( VV^t = C \) where \( V := (\bar{\gamma}_d, \bar{\gamma}_f, \bar{\sigma}, \bar{v}) \).

In this subsection, we consider four different cases for \( f_d, \gamma_d^*, f_f \) and \( \gamma_f^* \) as in Table 5. For correlations, four sets of parameters are considered: In the case “Corr.1”, all the factors are independent; In “Corr.2”, there exists only the correlation of -0.5 between the spot exchange rate and its volatility (i.e. \( \bar{\sigma} \bar{v} = -0.5 \)) while there are no correlations among the others; In “Corr.3”, the correlation between interest rates and the spot exchange rate are allowed while there are no correlations among the others; the correlation between domestic ones and the spot forex is 0.5(\( \bar{\gamma}_d^* \bar{\sigma} = 0.5 \)) and the correlation between foreign ones and the spot forex is -0.5(\( \bar{\gamma}_f^* \bar{\sigma} = -0.5 \)): Finally in “Corr.4”, more intricately correlated structure is considered; \( \bar{\gamma}_d^* \bar{\sigma} = 0.5, \bar{\gamma}_f^* \bar{\sigma} = -0.5 \) between interest rates and the spot forex; and \( \bar{\sigma} \bar{v} = -0.5 \) between the spot forex and its volatility. It is well known that (both of exact and approximate) evaluation of the long-term options is a hard task in the case with complex structures of correlations such as in “Corr.3” or “Corr.4”.

Lastly, we make an assumption that \( \gamma_d(t) = \gamma_d^* \) and \( \gamma_f(t) = \gamma_f^* \), volatilities of the domestic and foreign interest rates applied to the period from \( t \) to the next fixing date \( T_{n(t)} \), are equal to be zero for arbitrary \( t \in [t, T_{n(t)}] \).

In Table 6-9 and Figure 1, we compare our estimations of the values of call and put options by an asymptotic expansion up to the fourth order to the benchmarks estimated by \( 10^6 \) trials of Monte Carlo simulation which is discretized by Euler-Maruyama scheme with time step 0.05 and applied the Antithetic Variable Method. For the moneynesses(defined by \( K/F_{N+1(0)} \)) less than one, the prices of put options are shown; otherwise, the prices of call options are displayed.

As seen in these tables and figure, in general the estimators show more accuracy as the order of the expansion increases. Especially, for the deep OTM options the fourth order approximation performs much better and is stabler than the approximation with lower orders.
<table>
<thead>
<tr>
<th>Case</th>
<th>Moneyness</th>
<th>1st</th>
<th>2nd</th>
<th>3rd</th>
<th>4th</th>
<th>Diff.</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i)</td>
<td></td>
<td>0.0021</td>
<td>0.0136</td>
<td>0.051</td>
<td>0.1504</td>
<td>0.3865</td>
</tr>
<tr>
<td>(ii)</td>
<td></td>
<td>2nd</td>
<td>-0.102</td>
<td>-0.163</td>
<td>-0.217</td>
<td>-0.209</td>
</tr>
<tr>
<td>(iii)</td>
<td></td>
<td>3rd</td>
<td>0.2001</td>
<td>0.2571</td>
<td>0.3089</td>
<td>0.386</td>
</tr>
<tr>
<td>(iv)</td>
<td></td>
<td>4th</td>
<td>-0.181</td>
<td>-0.129</td>
<td>-0.031</td>
<td>0.1267</td>
</tr>
<tr>
<td>(v)</td>
<td></td>
<td>Diff.</td>
<td>1st</td>
<td>0.0304</td>
<td>0.0645</td>
<td>0.1239</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2nd</td>
<td>-0.104</td>
<td>-0.177</td>
<td>-0.268</td>
<td>-0.359</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3rd</td>
<td>0.198</td>
<td>0.2435</td>
<td>0.2579</td>
<td>0.2356</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4th</td>
<td>-0.183</td>
<td>-0.143</td>
<td>-0.082</td>
<td>-0.024</td>
</tr>
<tr>
<td>(vi)</td>
<td></td>
<td>Corr.</td>
<td>1st</td>
<td>0.0329</td>
<td>0.0789</td>
<td>0.1764</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4th</td>
<td>-0.197</td>
<td>-0.142</td>
<td>-0.04</td>
<td>0.1192</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3rd</td>
<td>0.213</td>
<td>0.2616</td>
<td>0.277</td>
<td>0.2534</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4th</td>
<td>-0.271</td>
<td>-0.199</td>
<td>-0.056</td>
<td>0.1764</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Diff.</td>
<td>A.E.</td>
<td>0.2864</td>
<td>0.3732</td>
<td>0.4574</td>
</tr>
<tr>
<td></td>
<td></td>
<td>A.E.</td>
<td>0.2832</td>
<td>0.353</td>
<td>0.3803</td>
<td>0.3523</td>
</tr>
<tr>
<td></td>
<td></td>
<td>A.E.</td>
<td>-0.274</td>
<td>-0.219</td>
<td>-0.133</td>
<td>-0.051</td>
</tr>
<tr>
<td>Moneyness</td>
<td>MC</td>
<td>2nd</td>
<td>-8E-04</td>
<td>0.0182</td>
<td>0.0777</td>
<td>0.2238</td>
</tr>
<tr>
<td>-----------</td>
<td>----------</td>
<td>-----</td>
<td>---------</td>
<td>---------</td>
<td>---------</td>
<td>---------</td>
</tr>
<tr>
<td>MC</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1st</td>
<td>0.0329</td>
<td>0.0789</td>
<td>0.1764</td>
<td>0.3688</td>
<td>0.7221</td>
<td>1.3266</td>
</tr>
<tr>
<td>2nd</td>
<td>-0.005</td>
<td>0.0101</td>
<td>0.0648</td>
<td>0.2055</td>
<td>0.508</td>
<td>1.0786</td>
</tr>
<tr>
<td>3rd</td>
<td>0.0983</td>
<td>0.1716</td>
<td>0.291</td>
<td>0.489</td>
<td>0.8238</td>
<td>1.3882</td>
</tr>
<tr>
<td>1st</td>
<td>0.036</td>
<td>0.057</td>
<td>0.08</td>
<td>0.1058</td>
<td>0.1339</td>
<td>0.158</td>
</tr>
<tr>
<td>A.E.</td>
<td>0.1137</td>
<td>0.2071</td>
<td>0.3669</td>
<td>0.6403</td>
<td>1.1075</td>
<td>1.8916</td>
</tr>
<tr>
<td>Diff.</td>
<td>-0.005</td>
<td>-0.018</td>
<td>-0.041</td>
<td>-0.071</td>
<td>-0.099</td>
<td>-0.114</td>
</tr>
<tr>
<td>A.E.</td>
<td>0.009</td>
<td>0.0311</td>
<td>0.0561</td>
<td>0.0777</td>
<td>0.0873</td>
<td>0.0824</td>
</tr>
<tr>
<td>Case (iv)</td>
<td>Corr.2</td>
<td>0.0087</td>
<td>0.05</td>
<td>0.1589</td>
<td>0.3929</td>
<td>0.8411</td>
</tr>
<tr>
<td>Case (i)</td>
<td>Corr.3</td>
<td>Moneyness</td>
<td>MC</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>---------</td>
<td>--------</td>
<td>------------</td>
<td>-----</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Moneyness</td>
<td></td>
<td>0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1 1.1 1.2 1.3 1.4 1.5 1.6 1.7 1.8 1.9 2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1st</td>
<td>0.3084</td>
<td>0.294</td>
<td>0.2991</td>
<td>0.4108</td>
<td>0.7602</td>
<td>1.5013</td>
</tr>
<tr>
<td>2nd</td>
<td>-0.32</td>
<td>-0.409</td>
<td>-0.469</td>
<td>-0.481</td>
<td>-0.444</td>
<td>-0.382</td>
</tr>
<tr>
<td>3rd</td>
<td>-0.076</td>
<td>-0.019</td>
<td>0.0125</td>
<td>0.0211</td>
<td>0.0177</td>
<td>0.0121</td>
</tr>
<tr>
<td>4th</td>
<td>0.0027</td>
<td>0.0201</td>
<td>0.0821</td>
<td>0.2509</td>
<td>0.6367</td>
<td>1.3919</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Case (ii)</th>
<th>Corr.3</th>
<th>Moneyness</th>
<th>MC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Moneyness</td>
<td></td>
<td>0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1 1.1 1.2 1.3 1.4 1.5 1.6 1.7 1.8 1.9 2</td>
<td></td>
</tr>
<tr>
<td>1st</td>
<td>0.1501</td>
<td>0.2846</td>
<td>0.5155</td>
</tr>
<tr>
<td>3rd</td>
<td>0.3816</td>
<td>0.3594</td>
<td>0.3452</td>
</tr>
<tr>
<td>3rd</td>
<td>0.3789</td>
<td>0.3393</td>
<td>0.2631</td>
</tr>
<tr>
<td>4th</td>
<td>-0.088</td>
<td>-0.008</td>
<td>0.0341</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Case (iii)</th>
<th>Corr.3</th>
<th>Moneyness</th>
<th>MC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Moneyness</td>
<td></td>
<td>0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1 1.1 1.2 1.3 1.4 1.5 1.6 1.7 1.8 1.9 2</td>
<td></td>
</tr>
<tr>
<td>1st</td>
<td>-0.137</td>
<td>-0.023</td>
<td>0.1426</td>
</tr>
<tr>
<td>1st</td>
<td>0.1967</td>
<td>0.3448</td>
<td>0.5453</td>
</tr>
<tr>
<td>1st</td>
<td>-0.44</td>
<td>-0.565</td>
<td>-0.664</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Case (iv)</th>
<th>Corr.3</th>
<th>Moneyness</th>
<th>MC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Moneyness</td>
<td></td>
<td>0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1 1.1 1.2 1.3 1.4 1.5 1.6 1.7 1.8 1.9 2</td>
<td></td>
</tr>
<tr>
<td>3rd</td>
<td>0.5061</td>
<td>0.51</td>
<td>0.5373</td>
</tr>
<tr>
<td>4th</td>
<td>-0.165</td>
<td>-0.05</td>
<td>0.0249</td>
</tr>
<tr>
<td></td>
<td>1st</td>
<td>2nd</td>
<td>3rd</td>
</tr>
<tr>
<td>---</td>
<td>-----</td>
<td>-----</td>
<td>-----</td>
</tr>
<tr>
<td>MC</td>
<td>0.008 0.0478 0.1569 0.3962 0.8563 1.6506 2.8984 4.7016 7.1253 10.182 7.7353 5.8173 4.3474 3.2373 2.4089 1.7966 1.3448 1.0114 0.7649 0.5809</td>
<td>0.1437 0.1527 0.1506 0.1426 0.1333 0.124 0.1138 0.1004 0.0819 0.0622 0.0462 0.0398 0.0386 0.04 0.0366 0.0221 -0.004 -0.038 -0.076 -0.109</td>
<td>0.0005 0.0316 0.0544 0.066 0.0667 0.0595 0.0495 0.0418 0.038 0.0403 0.0477 0.0599 0.0699 0.0776 0.0817 0.0814 0.0782 0.0707 0.0566 0.0368</td>
</tr>
<tr>
<td>A.E.</td>
<td>-0.029 0.0612 0.2091 0.4663 0.9216 1.6949 2.919 4.7102 7.1422 10.231 7.8355 5.9805 4.5715 3.5145 2.7272 2.1409 1.6994 1.3576 1.0816 0.849</td>
<td>-0.007 -0.043 -0.138 -0.337 -0.689 -1.225 -1.932 -2.722 -3.448 -3.951 -4.126 -3.965 -3.551 -3.012 -2.459 -2.009 -1.556 -1.235 -0.986 -0.795</td>
<td>-0.013 -0.062 -0.191 -0.459 -0.923 -1.598 -2.41 -3.189 -3.741 -3.951 -3.833 -3.498 -3.073 -2.638 -2.224 -1.842 -1.504 -1.216 -0.98 -0.794</td>
</tr>
</tbody>
</table>
References


[38] Osajima, Y. [2006], “The Asymptotic Expansion Formula of Implied Volatility for Dynamic SABR Model and FX Hybrid Model,” Preprint, Graduate School of Mathematical Sciences, the University of Tokyo.

[39] Osajima, Y. [2007], “General Asymptotics of Wiener Functionals and Application to Mathematical Finance,” Preprint, Graduate School of Mathematical Sciences, the University of Tokyo.


