Applying Filtered Historical Simulation to American Options: Evidence from S&P 100 Index Options

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Introduction

- As a successful innovation of option pricing, Black and Scholes (1973), many researchers have been attempting to depart from the perfect market assumptions when developing pricing models.
- One important extension of the Black and Scholes (1973) model is to take the stochastic volatility into account.
Introduction

Attributed to the successful application of GARCH model to financial time series (Bollerslev, Chou, and Kroner, 1992; Bollerslev, Engle, and Nelson, 1994), thousands of studies have been written to incorporate GARCH model into option pricing.

Duan (1995) and Heston and Nandi (2000) consider the pricing models based on the GARCH-type stochastic volatility under the assumption of Gaussian innovation.

Christoffersen, Heston and Jacobs (2006) obtain a pricing model that allows for a leverage effect and time-variant volatility and skewness by considering an asymmetric GARCH combined with an inverse Gaussian innovation.

Introduction

In this study, we modify and extend the FHS in empirical study:

- regarding the initial conditional volatility of the risk-neutral world as a parameter to be estimated.
- pricing American options with the Least-Square Monte-Carlo method (Longstaff and Schwartz, 2001).
- studying how the in-the-money options affect the calibration of the FHS model and how serious the pricing error may cause when the in-the-money options are excluded.
Asset price dynamics

Return dynamics under P measure: BEM(2008)

\[ r_t = \log(S_t / S_{t-1}) = \mu + \varepsilon_t \]

\[ \varepsilon_t = \sigma_t Z_t, Z_t \sim f(0,1) \]

\[ \sigma_t^2 = \omega + \beta \sigma_{t-1}^2 + \alpha \varepsilon_{t-1}^2 + \gamma l_{t-1} \varepsilon_{t-1}^2, \]

where \( l_t = 1 \) if \( \varepsilon_t < 0 \), \( l_t = 0 \) otherwise.

The parameter \( \theta = \{ \mu, \omega, \beta, \alpha, \gamma \} \) is estimated by using historical return data.
Asset Price Dynamics

Return dynamics under Q measure: BEM(2008)

\[ r_t = \log(S_t / S_{t-1}) = \mu^* + \varepsilon_t \]
\[ \varepsilon_t = \sigma^*_t Z_t, Z_t \sim f(0,1) \]
\[ \sigma^2_t = \omega^* + \beta^* \sigma^2_{t-1} + \alpha^* \varepsilon^2_{t-1} + \gamma^* I_{t-1} \varepsilon^2_{t-1} \]
\[ \sigma^2_0 = \sigma^2_P \]

where \( \sigma_P \) is the conditional volatility we estimated in \( P \) at time \( t \).

The parameter \( \theta^* = \{ \omega^*, \beta^*, \alpha^*, \gamma^* \} \) is estimated by using option market data.
FHS Algorithm

- Step 1.
  - At time $t$, we use $n$ historical log-returns of the underlying asset to estimate parameters $\theta$ in the GARCH model under $P$ measure.
  - Then we can obtain the empirical scaled or filtered innovations $z_j = \frac{\varepsilon_j}{\sigma_j}$.
FHS Algorithm

- Step 2.
  - Given a estimate of GARCH pricing parameter $\theta^*$, the return process under Q measure is simulated by using the GARCH pricing model.
FHS Algorithm

- Step 3.
  - The modeled put and call option prices are computed by using the Least Square Monte-Carlo technique of Longstaff & Schwartz (2001) via previous simulated return paths.
FHS Algorithm

Step 4.

- At time t, varying the pricing GARCH parameters $\theta^*$ to fit the cross-section of option price appropriately by minimizing the mean square error (MSE) between model prices and market prices.

- Once the MSE cannot be reduced by varying the pricing GARCH parameters $\theta^*$ or MSE is below some level, then the calibration is achieved.
Modified FHS Algorithm

- In our proposed method, a new parameter $\sigma_Q$ to modify the pricing model of BEM (2008),

$$r_t = \log(S_t / S_{t-1}) = \mu^* + \varepsilon_t$$

$$\varepsilon_t = \sigma_t Z_t, Z_t \sim f(0,1)$$

$$\sigma_t^2 = \omega^* + \beta^* \sigma_{t-1}^2 + \alpha^* \varepsilon_{t-1}^2 + \gamma^* I_{t-1} \varepsilon_{t-1}^2$$

$$\sigma_0^2 = \sigma_Q^2$$

- This study regards the $\sigma_Q$ as another parameter to be estimated.
Modified FHS Algorithm

- BEM (2008) suggest that the volatility dynamics of the underlying are not identical between the P measure and the Q measure when the return is not normally distributed.
- Then there is no reason why the initial conditional volatility of the risk-neutral world comes from the historical estimate.
- Our modification has the meaning of that the initial conditional volatility should be determined by the option market, instead of the historical GARCH estimate from empirical asset returns as in BEM (2008).
- It is found in this study that even such minor modification can significantly improve pricing accuracy.
The Pricing of American Options

- The American option price has no closed-form solution in general.
- Numerical methods include lattice method, finite difference method, Monte Carlo simulation method, et al.
- This study uses the Least Square Monte-Carlo (LSM) method of Longstaff and Schwartz (2001) because it is not only more efficient than other methods based on the Monte-Carlo framework, but also has more flexibility to apply with broad asset dynamics.
Least Square Monte-Carlo (LSM) (Longstaff and Schwartz, 2001)

- Assume that the option expires in $K$ periods, and specify the early exercise points as
  \[ 0 = t_0 < t_1 < t_2 < \ldots < t_K = T \]
- Simulate paths of the underlying asset dynamics under the risk-neutral measure.
- At the final expiration date of the option, the option holder will exercises the option if it is in-the-money, or let it expire if it is out-of-the-money.
Least Square Monte-Carlo (LSM) (Longstaff and Schwartz, 2001)

- At exercise point $t_i$ prior to the final expiration date, the option holder determine whether to exercise immediately or expand the life of the option by comparing between the early exercise value and the continuation value.

- Longstaff and Schwartz (2001) use the least square method (regression) to estimate the continuation value.
Least Square Monte-Carlo (LSM) (Longstaff and Schwartz, 2001)

- The dependent variable is a vector of the corresponding discounted cash flows received at time $t_i$ provided that the option is not exercised at time $t_{i-1}$.
- The independent variables include a set of basis functions generated by the asset prices at time $t_i$.
- The basis functions can be the weighted Laguerre, simple ordinary, Hermite, Legendre, Chebyshev, Gegenbauer, or Jacobi polynomials.
Least Square Monte-Carlo (LSM) (Longstaff and Schwartz, 2001)

- After calculating the conditional expectation value from continuation, we can justify whether to exercise immediately or expand the life of the option at time $t_i$.
- Discount the payoffs of all paths to time 0 and simply average them to obtain the risk-neutral price.
The Contracts

- S&P 100 index (OEX) Options
  - American-style
  - put and call
  - out-of-the-money (OTM) and in-the-money (ITM) OEX options
The Contracts

- Out-of-money European options were used in most of the empirical studies in the past literature.
- The reasons involve:
  - Put-call parity
  - Liquidity
Put-Call Parity

The put-call parity condition for European options on dividend paying stocks is:

\[ P_E + S_t e^{-q\tau} = C_E + X e^{-r\tau}, \]

The put-call parity condition for American options on dividend paying stocks is:

\[ S_t e^{-q\tau} - X \leq C_A - P_A \leq S_t - X e^{-r\tau}, \]
The Contracts

- Therefore, the ITM options may not redundant and excluding them from empirical calibration may cause a serious modeling error.
The Data

- Sample Selection
  - OptionMetrics
  - From 2 January 2003 to 30 December 2005 on each Wednesday
  - with price larger than 0.05,
  - with implied volatility less than 70%,
  - with time to maturity between 10 days and 360 days,
  - with the sum of open interest and trade volume not less than 100.

- Finally, this yields 19,282 observations.
S&P 100

Figure 1. The S&P 100 index from 2 January 2003 to 30 December 2005
Figure 2. The S&P 100 index daily log-returns from 2 Jan. 2003 to 30 Dec. 2005
Figure 3. The histogram and descriptive statistics of S&P 100 index daily log-returns from 2 January 2003 to 30 December 2005.
The Scaled Innovation

Figure 5. Daily log-returns of the S&P 100 index from 15 June 1989 to 30 April 2003 (3500 log-returns), annual conditional volatility $\sigma_t$, and scaled innovations $Z_t$. 
Experiment Designs

- Use 3500 historical return to obtain the model under P measure
- Use the mean of daily dividend yields as the constant dividend yields over the annual period while simulating the asset return paths.
- The riskless rate for each option contract is derived with the linearly interpolating method, using the term structure of zero-coupon default-free interest rates.
Experiment Designs

- While calibrating the pricing models, we add a stationary restriction, i.e. $\beta^*+\alpha^*+D\gamma^*<1$, to the GJR-GARCH model.
- Generate 20,000 simulated paths to calculate the option model prices.
- We use the 2nd order simple ordinary polynomials as the basis function in LSM method to calculate American option price.
## Calibration Results

Table 2. Estimation and calibration of the GJR-GARCH model

<table>
<thead>
<tr>
<th></th>
<th>$\omega \times 10^6$</th>
<th>$\beta$</th>
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<th>$\gamma$</th>
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<td>Mean</td>
<td>Std. dev.</td>
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<td>0.01</td>
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<td>0.05</td>
<td>0.94</td>
<td>0.00</td>
</tr>
<tr>
<td>FHS</td>
<td>Year</td>
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</tr>
<tr>
<td>2003</td>
<td>1.68</td>
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<td>2004</td>
<td>0.86</td>
<td>0.23</td>
<td>0.93</td>
<td>0.01</td>
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<tr>
<td>2005</td>
<td>0.68</td>
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<td>0.02</td>
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<tr>
<td>FHS(modified)</td>
<td>Year</td>
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<td></td>
<td></td>
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<td>2003</td>
<td>1.24</td>
<td>0.59</td>
<td>0.93</td>
<td>0.01</td>
</tr>
<tr>
<td>2004</td>
<td>0.86</td>
<td>0.22</td>
<td>0.92</td>
<td>0.01</td>
</tr>
<tr>
<td>2005</td>
<td>0.80</td>
<td>0.26</td>
<td>0.91</td>
<td>0.01</td>
</tr>
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</table>
Calibration Results

- The leverage parameter estimates in the FHS model and in our modified model are both higher than those in GJR-GARCH model under P measure.
- In addition, the t-test results in both cases are significant at 99% confidence level, which implies that the investors expect to have more compensation when the underlying asset declines under Q measure.
- On the other hand, the estimates of parameter $\beta$ in the original FHS model and in our modified model are both little smaller than that in GJR-GARCH model under P measure.
- Based on the above results, the volatility clustering phenomenon in the risk-neutral world seems to be less pronounced than that in the real world.
Calibration Results

- It is found that the average $\sigma_Q$ is much higher than $\sigma_P$ in 2003, and is lower than $\sigma_P$ in 2004 and 2005.
- The difference between $\sigma_Q$ and $\sigma_P$ is significant at 1% significance level.
- The absolute difference between $\sigma_Q$ and $\sigma_P$ is largest in 2003 and is smallest in 2004.
## Model Performance

### Table 3. In-sample & Out-of-sample pricing errors between the FHS & modified FHS models

<table>
<thead>
<tr>
<th></th>
<th>Total (in-sample)</th>
<th>Total (out-of-sample)</th>
<th>ITM (in-sample)</th>
<th>ITM (out-of-sample)</th>
<th>OTM (in-sample)</th>
<th>OTM (out-of-sample)</th>
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<tr>
<td><strong>Panel A: Aggregate valuation error across all year</strong></td>
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<td></td>
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<tr>
<td>FHS</td>
<td>0.99</td>
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<td>1.35</td>
<td>1.63</td>
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<td>FHS (modified)</td>
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<td>0.86</td>
<td>1.15</td>
<td>0.41</td>
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<td>t test (p-value)</td>
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<td>0.03</td>
<td>0.00</td>
<td>0.02</td>
<td>0.00</td>
<td>0.04</td>
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<td><strong>Panel B: Valuation errors by years</strong></td>
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<td>2003</td>
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<td></td>
<td></td>
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<tr>
<td>FHS</td>
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<td>1.89</td>
<td>1.89</td>
<td>2.38</td>
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<td>0.36</td>
<td>0.81</td>
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<tr>
<td>2004</td>
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<tr>
<td>FHS</td>
<td>0.72</td>
<td>0.84</td>
<td>1.04</td>
<td>1.15</td>
<td>0.59</td>
<td>0.72</td>
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<tr>
<td>FHS (modified)</td>
<td>0.61</td>
<td>0.94</td>
<td>0.94</td>
<td>1.29</td>
<td>0.47</td>
<td>0.80</td>
</tr>
<tr>
<td>2005</td>
<td></td>
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<tr>
<td>FHS</td>
<td>0.75</td>
<td>0.84</td>
<td>0.95</td>
<td>1.06</td>
<td>0.68</td>
<td>0.77</td>
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<tr>
<td>FHS (modified)</td>
<td>0.48</td>
<td>0.69</td>
<td>0.72</td>
<td>0.95</td>
<td>0.39</td>
<td>0.60</td>
</tr>
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</table>
In-Sample Comparison

- The overall performance of our modified FHS model is superior to the original FHS model in both the OTM and ITM options.
- The matched sample $t$ tests further confirm that the superiority is statistically significant at 1% significance level.
- The modified FHS model has absolute advantages against the original FHS model in 2003 and 2005. The two methods lead to similar results in 2004.
In-Sample Comparison

In order to see how the difference between $\sigma_Q$ and $\sigma_P$ can cause a serious pricing error, we randomly take two Wednesdays:

- 29 January 2003
- 28 January 2004

<table>
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<th>Date</th>
<th>28 January 2004</th>
<th>29 January 2003</th>
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<tbody>
<tr>
<td>$(\sigma_P, \sigma_Q)$</td>
<td>(0.005%, 0.005%)</td>
<td>(0.0113%, 0.0335%)</td>
</tr>
<tr>
<td>RMSE</td>
<td>(0.34, 0.33)</td>
<td>(3.07, 0.47)</td>
</tr>
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Figure 7. Forecasted volatility based on the pricing parameter estimates of both models on 28 January 2004. The initial volatilities are both 0.005% in the original FHS model and the modified FHS model.
Figure 9. Calibration results of the original FHS and modified FHS models on the cross-section of OEX options on 28 January 2004.
In-Sample Comparison

Figure 6. Forecasted volatility based on the pricing parameter estimates of both models on 29 January 2003. The initial volatilities are 0.0113% in the original FHS model and 0.0335% in the modified FHS model respectively.
In-Sample Comparison

Figure 8. Calibration results of the original FHS and modified FHS models on the cross-section of OEX options on 29 January 2003.
In-Sample Comparison

Therefore, we suggest that restricting the initial volatility to be an improper value under the Q measure would make a serious mispricing of the options and our modified FHS model can easily solve this problem.
Out-of-Sample Comparison

- The overall out-of-sample RMSE of our modified FHS model is 0.86, which is lower than that in the original FHS model, 1.28.
- The matched sample $t$ test reveals that such advantage is significant at 5% significance level.
- It is found the modified FHS model outperforms the original FHS model on both the OTM and ITM options.
- The modified FHS can easily beat the original FHS in 2003 and 2005. The two methods lead to similar results in 2004.
The OTM and the ITM Contracts

- We study how the model error will cause when the ITM options are excluded in the calibrating process of the American contracts.

<table>
<thead>
<tr>
<th>$\theta^\star_{\text{OTM}}$</th>
<th>$\omega^\star \times 10^5$</th>
<th>$\beta^\star$</th>
<th>$\alpha^\star \times 10^3$</th>
<th>$\gamma^\star$</th>
<th>$\sigma_\theta^2 \times 10^4$</th>
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<td>Mean</td>
<td>Std. dev.</td>
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<td>2004</td>
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<td>0.92</td>
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<td>2005</td>
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<td>0.01</td>
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<table>
<thead>
<tr>
<th>$\theta^\star$</th>
<th>$\omega^\star \times 10^5$</th>
<th>$\beta^\star$</th>
<th>$\alpha^\star \times 10^3$</th>
<th>$\gamma^\star$</th>
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<td>Mean</td>
<td>Std. dev.</td>
<td>Mean</td>
<td>Std. dev.</td>
<td>Mean</td>
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<td>2003</td>
<td>1.24</td>
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<td>0.93</td>
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<td>2004</td>
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<td>0.22</td>
<td>0.92</td>
<td>0.01</td>
<td>5.88</td>
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<td>2005</td>
<td>0.80</td>
<td>0.26</td>
<td>0.91</td>
<td>0.01</td>
<td>2.47</td>
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The OTM and the ITM Contracts

Table 5. Cross test of using the different pricing parameter estimates

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<td>in-sample</td>
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<td>0.86</td>
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Panel A: Aggregate valuation error across all year

2003
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2004
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<tr>
<td>σ*</td>
<td>0.61</td>
<td>0.94</td>
<td>0.94</td>
<td>1.29</td>
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2005
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<td>σ*</td>
<td>0.48</td>
<td>0.69</td>
<td>0.72</td>
<td>0.95</td>
<td>0.39</td>
<td>0.60</td>
<td></td>
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<tr>
<td>observations</td>
<td>6005</td>
<td>5857</td>
<td>1303</td>
<td>1271</td>
<td>4702</td>
<td>4586</td>
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</tr>
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</table>
The OTM and the ITM Contracts

- The $\theta^*_{\text{OTM}}$ has better in-sample performance on the OTM contracts, but poorer in-sample performance on ITM contracts. This finding suggests that neither the OTM nor the ITM contracts is redundant for pricing American-style options.

- Although $\theta^*$ is lost for the in-sample modeling quality for the OTM contracts (in order to conform both the OTM and ITM contracts), $\theta^*$ has brought enough information for the future forecasting for both the OTM and the ITM contracts.

- In light of the above analysis, we conclude that both the OTM and the ITM contracts should be included in the calibration process in order to obtain an accurate and robust result.
Conclusion

- This paper presents a modification of the FHS model proposed by BEM (2008).
- The modification is based on the fact that under the GARCH framework, the stochastic process of an asset under the real world and the risk-neutral world may differ.
- An extensive empirical analysis based on S&P 100 index options shows that the modified FHS outperforms the original FHS in both in-sample and out-of-sample data.
- We also suggest that both the OTM and ITM options are necessary while establishing a pricing model for American-style options.