Jump Diffusion Processes and their applications to Insurance and Finance

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Presentation to 15th International Conference
Computing in Economics and Finance

Sydney, July 15-17, 2009
Overview

- Jump diffusion processes in insurance and finance.

- The Laplace transform of the distribution of a jump diffusion process.

- Its application in insurance: aggregate claim amounts accumulated via a stochastic interest rate.


- Numerical illustration using transform analysis.
Jumps in the financial market

- In financial modelling, it has been observed that diffusion models are not robust enough to capture the appearance of jumps in underlying financial asset prices and interest rates due to primary events (i.e. the government’s fiscal and monetary policies, the release of corporate financial reports, the political and social decisions, the romours of mergers and acquisitions between firms, etc.).
Time value of claims

- In insurance modelling, Poisson processes have been used as a claim (jump) arrival process, where we assume that interest rates equal zero or deterministic. However in practice, it is not deterministic.
As a result, jump diffusion processes, which are simply speaking, the combinations of Poisson process and Brownian motion have gained their popularity for modelling in insurance and finance (Bertoin 1998; Sato 1999; Barndorff-Nielsen et al. 2001, and Cont and Tankov 2004).
The accumulated value of aggregate claim up to time $t$, $C_t$ is given by

$$C_t = \sum_{i=1}^{N_t} Y_i = Y_1 + Y_2 + \cdots + Y_{N_t-1} + Y_{N_t},$$

where $Y_i$, $i = 1, 2, \cdots$, are the claim amounts, $s_i$'s are time point at which claims occur ($s_i < t < \infty$) and $N_t$ is the number of claims up to time $t$, which follows a Poisson process with claim frequency rate $\rho$.

Note that it is (implicitly) assumed that interest rates equal zero.
Insurance modelling with deterministic interest rate

- Considering a constant risk-free force of interest rate, $\delta$, we have

$$M_t = \sum_{i=1}^{N_t} Y_i e^{\delta(t-s_i)} = Y_1 e^{\delta(t-s_1)} + Y_2 e^{\delta(t-s_2)} + \cdots + Y_1 e^{\delta(t-s_1)} + Y_2 e^{\delta(t-s_2)},$$

which can be expressed as

$$dM_t = \delta M_t dt + dC_t,$$

(*)

- The equation of (*) is the stochastic differential equation of a jump diffusion process $M_t$. 
Insurance modelling with stochastic interest rate

- In contrast, as it is not deterministic in practice, we now consider a stochastic interest rate to the aggregate claim amounts, i.e.

\[ dL_t = \delta L_t \, dt + \sigma \sqrt{L_t} \, dW_t + dC_t, \quad (**) \]

where \( L_t \) denotes the aggregate claim amounts accumulated via a stochastic interest rate and \( \sigma \geq 0 \). If we set \( \sigma = 0 \), it becomes \( M_t \), which is the aggregate claim amounts accumulated via a deterministic interest rate, \( \delta \). \( W_t \) is a standard Brownian motion.

- The equation of (**) is the stochastic differential equation of a jump diffusion process \( L_t \).
A default-free zero-coupon bond price paying $100 at time $t$ is given by

$$B(0, t) = \mathbb{E}[\$100 \times e^{-\int_0^t r_s ds} | r_0],$$

where $B(0, t)$ denotes the present value of $100$ at time $0$.

For interest rate process $r_r$, we can use celebrated Cox-Ingersoll-Ross (1985) model, i.e.

$$dr_t = (b - ar_t) dt + \sigma \sqrt{r_t} dW_t,$$

where $b \geq 0$, $a \geq 0$, $\sigma \geq 0$, $W_t$ is a standard Brownian motion.
A default-free zero-coupon bond pricing with jumps

- Considering the shocks that are primary events (i.e. the government’s fiscal and monetary policies, the release of corporate financial reports, the political and social decisions, the romours of mergers and acquisitions between firms, etc.), we can consider another process, denoted by $V_t$,

$$dV_t = (b - aV_t)dt + \sigma \sqrt{V_t} dW_t + dC_t,$$

where $C_t = \sum_{i=1}^{N_t} Y_i$ is a pure-jump process with $N_t$ being the number of jumps up to time $t$ and $Y_i$, $i = 1, 2, \cdots$, are their sizes.
Hence a default-free zero-coupon bond price paying 100 at time \( t \) is given by

\[
B(0, t) = \mathbb{E}[$100 \times e^{-\int_0^t V_s ds}$ | V_0],
\]

where

\[
dV_t = (b - aV_t)dt + \sigma \sqrt{V_t} dW_t + dC_t,
\]

which is the stochastic differential equation of a jump diffusion process \( V_t \).
Variations from a jump diffusion process $X_t$ for insurance application

From 

$$dX_t = c(b + aX_t)dt + \sigma \sqrt{X_t}dW_t + dC_t$$

to 

$$dM_t = \delta M_t dt + dC_t$$ (aggregate claim amounts accumulated via a deterministic interest rate),

$$dL_t = \delta L_t dt + \sigma \sqrt{L_t}dW_t + dC_t$$ (aggregate claim amounts accumulated via a stochastic interest rate),

where 

$$C_t = \sum_{i=1}^{N_t} Y_i.$$
Variations from a jump diffusion process $X_t$ for financial application

From $dX_t = c(b + aX_t)dt + \sigma \sqrt{X_t}dW_t + dC_t$ (for call option pricing)

to $dV_t = (b - aV_t)dt + \sigma \sqrt{V_t}dW_t + dC_t$, (interest rate process with jumps)

$dr_t = (b - ar_t) dt + \sigma \sqrt{r_t}dW_t$, (celebrated Cox-Ingersoll-Ross model),

where $C_t = \sum_{i=1}^{N_t} Y_i$. 
The generator of the process $(\Psi_t, X_t, t)$

- Define $\Psi_t = \int_0^t X_s ds$ then the generator of the process $(\Psi_t, X_t, t)$ acting on a function $f(\psi, x, t)$ belonging to its domain is given by

$$A f(\psi, x, t) = \lim_{dt \downarrow 0} \frac{E[f(\Psi_{t+dt}, X_{t+dt}, t+dt) \mid \Psi_t = a, X_t = b] - f(a, b, t)}{dt}$$

$$= \frac{\partial f}{\partial t} + x \frac{\partial f}{\partial \psi} + c(b + ax) \frac{\partial f}{\partial x} + \frac{1}{2} \sigma^2 x \frac{\partial^2 f}{\partial x^2}$$

$$+ \rho \left[ \int_{\psi}^{\infty} f(\psi, x + y, t) dG(y) - f(\psi, x, t) \right].$$
The Laplace transform of the distribution of $X_t$

A function $f (\psi, x, t)$ has to satisfy $Af = 0$ for it to be a martingale. So setting $f (\psi, x, t) = e^{-\nu x} e^{-\xi \psi} e^{B(t)}$, the Laplace transform of the distribution of $X_t$ is given by

$$E \left( e^{-\nu X_t} | X_0 \right) = \exp \left[ - \left\{ \frac{2ca\nu \exp(c\alpha)}{\sigma^2 \nu \{\exp(c\alpha) - 1\} + 2ca} \right\} X_0 \right]$$

$$\times \exp \left[ -\rho \int_0^t \left[ 1 - \hat{g} \left\{ \frac{2ca\nu \exp(c\alpha)}{\sigma^2 \nu \{\exp(c\alpha) - 1\} + 2ca} \right\} \right] ds \right] \times \left[ \frac{2ca}{\sigma^2 \nu \{\exp(c\alpha) - 1\} + 2ca} \right] \frac{2cb}{\sigma^2},$$

where $\nu \geq 0$, $\xi \geq 0$ and $\hat{g} (u) = \int_0^\infty e^{-uy} dG (y)$. 
and the Laplace transform of the distribution of $\Psi_t$ is given by $E\left(e^{-\xi\Psi_t|X_0}\right)$

$$
= \exp \left[ - \left\{ \frac{2\xi \left\{ 1 - \exp\left( -\sqrt{c^2a^2+2\sigma^2\xi}t \right) \right\}}{\left\{ (\sqrt{c^2a^2+2\sigma^2\xi-ca}) + (\sqrt{c^2a^2+2\sigma^2\xi+ca}) \exp\left( -\sqrt{c^2a^2+2\sigma^2\xi}t \right) \right\}} \right\} X_0 \right]
$$

$$
\exp \left[ -\rho \int_0^t \left[ 1 - \hat{g} \left\{ \frac{2\xi \left\{ 1 - \exp\left( -\sqrt{c^2a^2+2\sigma^2\xi}s \right) \right\}}{\left\{ (\sqrt{c^2a^2+2\sigma^2\xi-ca}) + (\sqrt{c^2a^2+2\sigma^2\xi+ca}) e^{-\sqrt{c^2a^2+2\sigma^2\xi}s} \right\}} \right\} \right] ds \right]
$$

$$
\times \left[ \frac{2\sqrt{c^2a^2+2\sigma^2\xi} \exp\left( -\frac{(\sqrt{c^2a^2+2\sigma^2\xi+ca})^2}{2} t \right)}{\left\{ (\sqrt{c^2a^2+2\sigma^2\xi-ca}) + (\sqrt{c^2a^2+2\sigma^2\xi+ca}) \exp\left( -\sqrt{c^2a^2+2\sigma^2\xi}t \right) \right\}} \right] \frac{2\sigma b}{\sigma^2}
$$

.$$
Assuming the mixture of two exponential jump size distribution, which is a special case of phase-type distributions, i.e. $g(y) = \beta_1 \alpha_1 e^{-\alpha_1 y} + \beta_2 \alpha_2 e^{-\alpha_2 y}, y > 0, \alpha_1 > \alpha_2 > 0$ and $\beta_1 + \beta_2 = 1$ (Asmussen, 2000), the Laplace transform of the distribution of $X_t$ is given by

$$
\mathbb{E}(e^{-\nu X_t} | X_0) = \exp \left[- \left\{ \frac{2ca\nu e^{cat}}{\sigma^2 \nu (e^{cat} - 1) + 2ca} \right\} X_0 \right]
$$

$$
\left\{ \frac{2ca(\alpha_1 + \nu e^{cat}) - \alpha_1 \sigma^2 \nu (1 - e^{cat})}{2ca(\alpha_1 + \nu)} \right\}^{-\beta_1 \frac{\rho}{ca}} \left\{ \frac{2ca(\nu e^{cat} + \alpha_1) + \alpha_1 \sigma^2 \nu (e^{cat} - 1)}{2ca(\alpha_1 + \nu)} \right\}^{\sigma^2 \left( \frac{\alpha_1 \beta_1}{\alpha_1 \sigma^2 + 2ca} \right) \frac{\rho}{ca}}
$$

$$
\times \left[ \frac{2ca}{\sigma^2 \nu (e^{cat} - 1) + 2ca} \right]^{\frac{2cb}{\sigma^2}}
$$
and the Laplace transform of the distribution of $\Psi_t$ is given by $\mathbb{E}\left(e^{-\xi\Psi_t|X_0}\right) = e^{-\rho t}$

\[
\times \exp\left[-\left\{ \frac{2\xi\left\{ 1 - \exp\left(-\sqrt{c^2a^2+2\sigma^2\xi}t\right) \right\}}{\left(\sqrt{c^2a^2+2\sigma^2\xi}-ca\right) + \left(\sqrt{c^2a^2+2\sigma^2\xi}+ca\right) \exp(-\sqrt{c^2a^2+2\sigma^2\xi}t)} \right\} X_0 \right]
\]

\[
\left\{ A_1^{(0,t)} \right\} \alpha_1 \frac{\sqrt{c^2a^2+2\sigma^2\xi}-ca}{\sqrt{c^2a^2+2\sigma^2\xi}} \frac{\rho}{\alpha_1\left(\sqrt{c^2a^2+2\sigma^2\xi}-ca\right) + 2\xi \sqrt{c^2a^2+2\sigma^2\xi}} \left\{ B_1^{(0,t)} \right\} \frac{\rho}{\alpha_1\left(\sqrt{c^2a^2+2\sigma^2\xi}-ca\right) + 2\xi \sqrt{c^2a^2+2\sigma^2\xi}}
\]

\[
\left\{ A_2^{(0,t)} \right\} \alpha_2 \frac{\sqrt{c^2a^2+2\sigma^2\xi}-ca}{\sqrt{c^2a^2+2\sigma^2\xi}} \frac{\rho}{\alpha_2\left(\sqrt{c^2a^2+2\sigma^2\xi}-ca\right) + 2\xi \sqrt{c^2a^2+2\sigma^2\xi}} \left\{ B_2^{(0,t)} \right\} \frac{\rho}{\alpha_2\left(\sqrt{c^2a^2+2\sigma^2\xi}-ca\right) + 2\xi \sqrt{c^2a^2+2\sigma^2\xi}}
\]

\[
\times \left\{ C^{(0,t)} \right\} \frac{2cb}{\sigma^2},
\]
The moments of a jump diffusion process, \( X_t \)

- If we differentiate the L.T. with respect to \( \nu \) and put \( \nu = 0 \), we can obtain the mean of \( X_t \), i.e.

\[
\mathbb{E}(X_t \mid X_0) = e^{cat} X_0 + \left( \frac{\beta_1 \rho}{\alpha_1} + \frac{\beta_2 \rho}{\alpha_2} + cb \right) \left( \frac{e^{cat} - 1}{ca} \right)
\]
and higher moments can be obtained by differentiating it further, i.e.

\[
\text{Var}(X_t \mid X_0) = \\
\left[ \left( \frac{2ca + \alpha_1 \sigma^2}{2} \right) \left( \frac{2ca e^{\text{cat}} + \alpha_1 \sigma^2 (e^{\text{cat}} - 1)}{2ca} + 1 \right) \left( \frac{\beta_1 \rho}{ca \alpha_1^2} - \frac{\sigma^2 \beta_1 \rho}{ca(\alpha_1 \sigma^2 + 2ca) \alpha_1} \right) \right] \left( \frac{e^{\text{cat}} - 1}{ca} \right) \\
+ \left( \frac{2ca + \alpha_2 \sigma^2}{2} \right) \left( \frac{2ca e^{\text{cat}} + \alpha_2 \sigma^2 (e^{\text{cat}} - 1)}{2ca} + 1 \right) \left( \frac{\beta_2 \rho}{ca \alpha_2^2} - \frac{\sigma^2 \beta_2 \rho}{ca(\alpha_2 \sigma^2 + 2ca) \alpha_2} \right) \left( \frac{e^{\text{cat}} - 1}{ca} \right).
\]
The moments of aggregate claim amounts accumulated via a stochastic interest rate, $L_t$

- Assume that $X_0 = 0$ and $c = 1$, $b = 0$ and $a = \delta$, in $dX_t = c(b + aX_t)dt + \sigma \sqrt{X_t}dW_t + dC_t$, then they are given by

$$\mathbb{E}(L_t) = \left( \frac{\beta_1\rho}{\alpha_1} + \frac{\beta_2\rho}{\alpha_2} \right) \left( \frac{e^{\delta t} - 1}{\delta} \right) = \left( \frac{\beta_1\rho}{\alpha_1} + \frac{\beta_2\rho}{\alpha_2} \right) \overline{s_t}$$

and

$$\text{Var}(L_t) = \left[ \left( \frac{2\delta + \alpha_1 \sigma^2}{2} \right) \left( \frac{2\delta e^{\delta t} + \alpha_1 \sigma^2 (e^{\delta t} - 1)}{2\delta} + 1 \right) \left( \frac{\beta_1\rho}{\delta \alpha_1^2} - \frac{\sigma^2 \beta_1\rho}{\delta (\alpha_1 \sigma^2 + 2\delta) \alpha_1} \right) \right] \overline{s_t},$$

where $dL_t = \delta L_t dt + \sigma \sqrt{L_t} dW_t + dC_t$ and $\overline{s_t} = \frac{e^{\delta t} - 1}{\delta}$.
The moments of aggregate claim amounts accumulated via a deterministic interest rate, $M_t$

- If we set $\sigma = 0$ in $dL_t = \delta L_t dt + \sigma \sqrt{L_t} dW_t + dC_t$, then they are given by

$$\mathbb{E}(M_t) = \left( \frac{\beta_1 \rho}{\alpha_1} + \frac{\beta_2 \rho}{\alpha_2} \right) \left( \frac{e^{\delta t} - 1}{\delta} \right) = \left( \frac{\beta_1 \rho}{\alpha_1} + \frac{\beta_2 \rho}{\alpha_2} \right) \bar{s}_{t|} \text{ (at } \delta),$$

and

$$\text{Var}(M_t) = \left( \frac{2\beta_1 \rho}{\alpha_1^2} + \frac{2\beta_2 \rho}{\alpha_2^2} \right) \left( \frac{e^{2\delta t} - 1}{2\delta} \right) = \left( \frac{2\beta_1 \rho}{\alpha_1^2} + \frac{2\beta_2 \rho}{\alpha_2^2} \right) \bar{s}_{t|} \text{ (at } 2\delta),$$

where $dM_t = \delta M_t dt + dC_t$. 
The moments of aggregate claim amounts ignoring interest rate, $C_t$

- If we set $\delta = 0$ and $\beta_2 = 0$ (i.e. $\beta_1 = 1$), then we can obtain

$$
\mathbb{E}(C_t) = \frac{\rho}{\alpha_1} t \quad \text{and} \quad \text{Var} (C_t) = \frac{2\rho}{\alpha_1^2} t,
$$

which are the moments of the compound Poisson process, where $C_t = \sum_{i=1}^{N_t} Y_i$. 
Comparison between the moments of $L_t$ and $M_t$

- As expected,
  \[\mathbb{E}(L_t) = \mathbb{E}(M_t)\]
  as \[\mathbb{E}\left[ \int_0^t \sigma \sqrt{X_s} dW_s \right] = 0\].

- But
  \[\text{Var}(L_t) \neq \text{Var}(M_t),\]
  where \[dL_t = \delta L_t dt + \sigma \sqrt{L_t} dW_t + dC_t\] and \[dM_t = \delta M_t dt + dC_t\].
Example 1

- The parameter values used to calculate the moments of the aggregate accumulated claim amounts are

\[ \alpha_1 = 0.01, \alpha_2 = 0.009, \beta_1 = 0.7, \beta_2 = 0.3, \delta = 0.05, \rho = 50 \text{ and } t = 1 \]

and the mean of the aggregate accumulated claim amounts is given by

\[ \mathbb{E}(L_t) = \mathbb{E}(M_t) = 5,298. \]

The calculations of variances of the aggregate accumulated claim amounts and their differences are shown in Table 1 by changing the values of diffusion coefficient of \( \sigma \).
Table 1.

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>Var($L_t$)</th>
<th>Var($M_t$)</th>
<th>Var($L_t$) − Var($M_t$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>1,125,700</td>
<td>1,125,700</td>
<td>0</td>
</tr>
<tr>
<td>0.5</td>
<td>1,126,400</td>
<td>1,125,700</td>
<td>700</td>
</tr>
<tr>
<td>0.7</td>
<td>1,127,000</td>
<td>1,125,700</td>
<td>1,300</td>
</tr>
<tr>
<td>0.8</td>
<td>1,127,500</td>
<td>1,125,700</td>
<td>1,800</td>
</tr>
<tr>
<td>1.0</td>
<td>1,128,400</td>
<td>1,125,700</td>
<td>2,700</td>
</tr>
</tbody>
</table>

The higher the value of diffusion coefficient of $\sigma$ is, the higher the variance of the aggregate accumulated claim amounts is. So if insurance companies use mean-variance principle for their premium calculations, they become higher than those calculated using a deterministic interest rate $\delta$ and it is necessary for insurance companies to charge higher premiums when the interest rate expected to be more volatile than as usual.
A default-free zero-coupon bond pricing using $V_t$

- Define $W_t = \int_0^t V_s ds$ and set $\xi = 1$, $c = 1$ and $a < 0$ in $\mathbb{E} \left( e^{-\xi \psi_t} | X_0 \right)$, then we can easily calculate zero-coupon bond price paying 100 at time $t$, i.e.

$$B(0, t) = \mathbb{E}[\$100 \times e^{-\int_0^t V_s ds} | V_0]$$

where $B(0, t)$ denotes the present value of 100 at time 0 and

$$dV_t = (b - aV_t) dt + \sigma \sqrt{V_t} dW_t + dC_t.$$
Example 2

- The parameter values used to calculate the prices of default-free zero-coupon bond are $V_0 = 0.05$, $\alpha_1 = 200$, $\alpha_2 = 250$, $\beta_1 = 0.7$, $\beta_2 = 0.3$, $a = 0.05$, $\rho = 3$, $\sigma = 0.8$, $b = 0.025$ and $t = 1$ and its price is given by

$$B(0, 1) = \mathbb{E}[100 \times e^{-\int_0^1 V_s ds} V_0] = 93.937,$$

where its counterpart using a deterministic interest rate $\delta$ is given by

$$100 \times e^{-0.05} = 95.123.$$
The calculations of prices of default-free zero-coupon bond by changing the values of **jump frequency rate** $\rho$ are shown in Table 2.1.

Table 2.1.

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>$B(0, 1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>94.557 (CIR case)</td>
</tr>
<tr>
<td>1</td>
<td>94.350</td>
</tr>
<tr>
<td>2</td>
<td>94.143</td>
</tr>
<tr>
<td>3</td>
<td>93.937</td>
</tr>
<tr>
<td>4</td>
<td>93.731</td>
</tr>
<tr>
<td>5</td>
<td>93.526</td>
</tr>
<tr>
<td>10</td>
<td>92.506</td>
</tr>
</tbody>
</table>

Having considered upward jumps only in $V_t$, we are expecting higher interest rate as time goes by. Therefore it is more attractive for investors to leave an
amount of money e.g. in a variable-rate savings account rather than purchasing a bond that pays guaranteed 100, regardless of the rate of interest at time \( t \). So the higher \( \rho \) is, the lower the default-free zero-coupon bond price is (see Table 2.1).
The calculations of prices of default-free zero-coupon bond by changing the values of the magnitude of jump sizes, i.e. $\alpha_1$ and $\alpha_2$ are shown in Table 2.2.

<table>
<thead>
<tr>
<th>$\alpha_1$</th>
<th>$\alpha_2$</th>
<th>$B(0, 1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2.5</td>
<td>56.486</td>
</tr>
<tr>
<td>20</td>
<td>25</td>
<td>88.682</td>
</tr>
<tr>
<td>200</td>
<td>250</td>
<td>93.937</td>
</tr>
<tr>
<td>2000</td>
<td>2500</td>
<td>94.495</td>
</tr>
</tbody>
</table>

Having considered upward jumps only in $V_t$, we are expecting higher interest rate as time goes by. Therefore it is more attractive for investors to leave an amount of money e.g. in a savings account rather than purchasing a bond that
pays guaranteed 100, regardless of the rate of interest at time \( t \). So the bigger the magnitude of positive jump is, less attractive purchasing a bond that pays guaranteed 100 is. So the smaller \( \alpha_1 \) and \( \alpha_2 \) are, the lower the default-free zero-coupon bond price is (see Table 2.2).
The calculations of prices of default-free zero-coupon bond by changing the values of diffusion coefficient of $\sigma$ are shown in Table 2.3.

Table 2.3.

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$B(0, 1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>93.428</td>
</tr>
<tr>
<td>0.1</td>
<td>93.437</td>
</tr>
<tr>
<td>0.5</td>
<td>93.640</td>
</tr>
<tr>
<td>0.8</td>
<td>93.937</td>
</tr>
<tr>
<td>10</td>
<td>98.806</td>
</tr>
<tr>
<td>$\infty$</td>
<td>100</td>
</tr>
</tbody>
</table>

As the same result as CIR case, the more volatile the interest rate is, that means more uncertainty for future consumption is, more attractive purchasing a bond that pays guaranteed 100 is. So the higher $\sigma$ is, the higher the default-free zero-coupon bond price is (see Table 2.3).
Further Research

- Considering negative (-ve) jumps in the jump diffusion process, $X_t$, i.e. 
  $$dX_t = c(b + aX_t)dt + \sigma \sqrt{X_t}dW_t + dC_t^{(1)} - dC_t^{(2)},$$
  where $C_t^{(j)} = \frac{N_t^{(j)}}{\sum_{i=1}^{N_t^{(j)}} Y_i^{(j)}}, j = 1, 2$.

- In practice, we might need to employ one of the heavy-tailed distributions for jump sizes, $G(y)$ to deal with extreme losses (or jumps).

- $c, b, a, \sigma$ can be time dependent, i.e. $c(t), b(t), a(t), \sigma(t)$ and they also can be stochastic. $N_t$ can be the Cox process, rather than the Poisson process, that has stochastic jump frequency rate $\rho(t)$. 