Switching Agent Based Models and Random Coefficient Auto-Regressive Models.

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Abstract

Several Agent Based Models have been proposed based on heterogeneity and bounded rationality to explain stylized facts. However, the complexity of these models does not allow to determine the dynamic of asset returns. We describe an Agent Based Model where each agent makes decisions by using a sum of two signals. The first is related to fundamental information while the second is related to traders’ idiosyncratic noise. This description gives a model with a switching phenomena between two groups namely fundamentalist and noise traders. We show that, if the price impact function is log-linear, the dynamic of asset returns belongs to Random Coefficient Autoregressive Models RCAM(p) with a unit root in average. The latter dynamic has the ability to generate the main properties of financial time series.

Keywords: Stylized facts, Agent Based Models, Behavioral Finance.

1 Introduction

Modeling financial markets has been always an interest for academic researchers. Several models have been developed for this purpose. The literature can be classified in three groups depending on the hypothesis made about agents’ decision making. The first group, classical financial models, assumes that agents are fully rational. So their decision respects Von Neumann-Morgenstern or Savage axioms. It is supposed also that agents have the same views with respect to future expected returns (homogeneous expectation). These assumptions, even if they are not too realistic, allow to obtain tractable models. Homogeneous rational expectation with frictionless assumption imply that markets are efficient, in other words asset prices always reflect all available information. There is no way to earn positive excess returns by using the available information. The arrival of news is the only factor which drives security prices. These assumptions would not matter if equilibrium prices which result from these models match characteristics of financial time series. Unfortunately, this is not the case. Since 1980, some patterns have been
detected seeming to contrast Efficient Market theory (excess of volatility, momentum effects, long time reversal effects). Also, with the homogeneous rational expectation, there is a low or zero trading volume since traders hold the same portfolio while in practice this is not the case. Trading volume is not small and is positively correlated with volatility. These facts have motivated some researchers to relax one or two of the hypotheses mentioned above. Behavioral Finance relaxes the individual rationality of traders. It is supposed that there are two groups of traders: Rational and Noise traders. Noise traders may survive in the market for a long time because arbitrage opportunities are costly and risky. So Limits to Arbitrage is the first building of Behavioral Finance. The second is based on the psychology field and explains why irrational investors may deviate from subjective expected utility by having correlated trades. We refer to Barberis and Thaler [3] for a review of Behavioral Finance foundation. The dynamic prices obtained with these models reproduce generally momentum and long time reversal effect, see for instance Barberis et al. [4] or Daniel et al. [9].

The last group is the microscopic (Agent Based) financial markets. A motivation of this approach may be found in Konte [12] where a justification to combine both theories is given. Indeed, a market model with a violation of Law of One Price is proposed such that anomalies may be interpreted as chance or randomness of errors on the one hand and as Psychology and Limits to arbitrage on the other hand. The idea was to show why it is difficult to discriminate between these two arguments and so why it is useful to reconcile both theories by using dynamic models. In the latter, homogeneity and rationality are replaced respectively by heterogeneity and bounded rationality. This class of models is solvable using computational tools since each trader can not determine her future expectation returns which depend on the other ones. Often heterogeneous interacting agents or interacting rules are considered see for example [13], [6], [5], [2]. There are other models that work directly on agent’s decision making without the question of learning. Models are based on herding, communication or other psychological effects, see [7], [10], [8].

Our main is to propose an Agent Based Model similar to Alfrano and Lux [1] where the market is populated by fundamentalist and noise traders with a switching phenomena. Here, the probability for a trader to switch from one group to another one depends on the size of the last returns. The simplicity of the model allows to determine the dynamic of asset returns but also the key element which generates stylized facts.

The paper is divided in four additional sections. Section II presents the Agent Based Model and then determines the dynamic of asset prices. Section III is consecrated to simulations. Stylized facts such that volatility clustering, fat tails, little or insignificant asset returns’ autocorrelation are generated. Section IV gives some possible extensions and comments of the model. The last section concludes.

1. Some rational explanations have been given, see for example Rubinstein [15], Malkiel [14]
2 Description of the class.

We consider a market model where the frequency of transactions is high. The transactions appear at $0 = t_0, t_1, \ldots, t_{n-1}, t_n = T$ where $t_k - t_{k-1}$ is small (minutes, hours). In this case, we may neglect the dividend process and the interest rate model. Asset prices move due to arrival of news say $(i_t)$ following a normal distribution $N(0, \sigma_i^2)$. At each period $t-1$, all agents indexed by $k = 1, \ldots, M$ receive the fundamental information $i_t$. Then they assess it by making some idiosyncratic errors. It is natural to suppose that the bias is more important when the market is stormy than stable.

So to integrate this point, we assume that the trader $k$ evaluates the signal $i_t$ by

$$\epsilon_t^k = i_t + \nu_t^k r_{t,p_k} \tag{1}$$

where $\nu_t^k \sim N(0, \sigma_k^2)$ is the idiosyncratic noise of the trader $k$ weighted by $r_{t,p_k}$. The latter is given by

$$r_{t,p_k} = \sum_{i=1}^{p_k} \alpha_i^k r_{t-i} \quad \text{with} \quad r_{t-i} = \ln S_{t-i} - \ln S_{t-i-1}. \tag{2}$$

Equation (1) says that the bias in assessing $i_t$ is an increasing function of the last asset returns. It captures the empirical fact that traders tend to extrapolate more the information when there is a trend or more generally when the current volatility is high. We impose the parameters $\sigma_k, \alpha_i^k, i = 1, \ldots, p_k$ to be positive and small so that

$$\sigma_k^2 r_{t,p_k}^2 < \sigma_i^2. \tag{3}$$

The strict inequality says that the error forecast must be generally small with respect to the fundamental news. This condition introduces at the same time bounded rationality in our model. Indeed, when traders are fully rational, then $i_t$ and $\epsilon_t^k$ are equal at any time $t$. So their decisions are to buy if the information $i_t$ is positive (good news) and to sell in the opposite case (bad news). Here, the two signals may have opposite signs if $|\nu_t^k r_{t,p_k}| > |i_t|$ see equation (1). This event has however a small probability to be realized due to (3) since $\text{Var}(\nu_t^k r_{t,p_k}) < \text{Var}(i_t)$. So agents’ decision follows mainly the sign of the information with some exceptions. We may call fundamentalists at time $t$, those such that $\epsilon_t^k$ and $i_t$ have the same sign and noise traders the remaining group. An agent may be in the first group at time $t$ and then switch latter to the second group and vice-versa. So the model may be seen as an switching Agent Based Model where the probability for a trader $k$ to go from one group to another depends on the parameters.
In our setup, there is no communication and the terms \((\nu_k^t)\) and \((\nu_l^t)\) are supposed to be mutually independent for two different traders \((k \neq l)\). Also we suppose that the information flow \((i_t)\) is mutually independent of \((\nu_k^t)\). That means traders’ bias \(\nu_k^t r_{t,p_k}\) is only dependent on the market’s state through past asset prices for any trader \(k\).

The next step is to determine the excess demand. We suppose that the decision of agent \(k = 1, \cdots, M\) is proportional to his/her received signal from \(i_t\). That means \(n_{k,t} = c_k \epsilon_k^t \in \mathbb{R}\) where \(c_k\) is a positive constant. So the excess demand is given by

\[
E(t) = \sum_{k=1}^{M} n_{k,t} = \sum_{k=1}^{M} c_k \epsilon_k^t = c_i t + \sum_{k=1}^{M} c_k \nu_k^t (r_{t,p_k}),
\]

where \(c = \sum_{k=1}^{M} c_k\).

The impact of the excess demand in asset prices is determined by using \(\ln S_t - \ln S_{t-1} = \frac{E(t)}{\lambda}\)

where \(\lambda\) is a positive constant representing the market depth. We take it as \(\lambda = c = \sum_{k=1}^{M} c_k\) to normalize the coefficient of \(i_t\). A more general constant may be taken without loss of generality for example \(\lambda = \beta c, \beta \in \mathbb{R}_+\).

We have by using equation (4)

\[
\ln S_t = \ln S_{t-1} + \frac{E(t)}{\lambda} = \ln S_{t-1} + i_t + \sum_{k=1}^{M} \frac{c_k}{c} \nu_k^t r_{t,p_k}.
\]

Each term \(r_{t,p_k}\) may be rewritten using equation (2) as

\[
r_{t,p_k} = \alpha_1^k \ln S_{t-1} + \sum_{i=2}^{p_k} \ln S_{t-i} (\alpha_1^k - \alpha_i^k) - \alpha_1^k \ln S_{t-p_k-1}.
\]

By replacing \(r_{t,p_k}\) by its value, equation (5) becomes

\[
\ln S_t = \ln S_{t-1} \left(1 + \sum_{k=1}^{M} \frac{c_k}{c} \alpha_1^k \nu_k^t\right) + i_t + \sum_{k=1}^{M} \frac{c_k}{c} \nu_k^t \left( \sum_{i=2}^{p_k} \ln S_{t-i} (\alpha_1^k - \alpha_i^k) - \alpha_1^k \ln S_{t-p_k-1} \right).
\]
Note that the right side of the above equation is a linear combination of log asset prices. So if we set $J_{\text{max}} = \max\{p_k + 1, k = 1 \cdot \cdot \cdot M\}$, we may rewrite the latter equation as follows

$$\ln S_t = \ln S_{t-1} \left(1 + \sum_{k=1}^{M} \frac{c_k}{c} \alpha_k^1 \nu_t^{k} \right) + \ i_t + \sum_{l=2}^{J_{\text{max}}} \ln S_{t-l} X^l_t$$

where $X^l_t = \sum_{k=1}^{m_l} D_{i_k} \nu_t^{i_k}$ with $m_l \leq M$, $i_k \in \{1, \cdot \cdot \cdot , M\}$ satisfying $i_k \neq i_m$ for $k \neq m$ and $D_{i_k}$ is a constant depending on the parameters $\alpha_k^1$, $(\alpha_k^1 - \alpha_{i_k-1}^1)$, $\alpha_{i_k}^1$.

If we set $X^1_t = 1 + \sum_{k=1}^{M} \frac{c_k}{c} \alpha_k^1 \nu_t^{k}$, we obtain the following equation

$$\ln S_t = \sum_{l=1}^{J_{\text{max}}} \ln S_{t-l} X^l_t + \ i_t. \quad (7)$$

The processes $(X^l_t), l = 1, \cdot \cdot \cdot , J_{\text{max}}$ follow normal distributions and are mutually independent of $(i_t)$. Also they are such that $E(X^1_t) = 1$, $E(X^l_t) = 0$ for $l \geq 2$ and $\text{cov}(X^l_t, X^{l_2}_t) > 0$, for $l_1, l_2 \geq 1$. These points come from assumptions made about $i_t$ and $\nu_t^{k}$, $k = 1, \cdot \cdot \cdot , M$.

We obtain a class of models similar to autoregressive models where coefficients are random variables. This kind of models is known as random coefficient autoregressive models RCAM(p). Namely we have stochastic unit root models since $E(X^1_t) = 1$, $E(X^l_t) = 0$ for $l \geq 2$. The timing of events which gives this dynamic is the following. At time $t-1$, the information flow $i_t$ is announced for all traders. The next step is the formation of expectations. Based on the past asset prices (state of the market) each investor $k$ forms his/her expectation about the distribution of the next period’s price by interpreting the new information $i_t$. He/She makes an idiosyncratic error $\nu_t^{k}$ and then obtains the signal $\epsilon_t^{k}$. The latter is used to make decision by taking $c_k \epsilon_t^{k}$ where $c_k > 0$. The excess demand is after determined and the previous price is adjusted by using equation (5), see figure [1].

Before making the simulations, we emphasize three points that are addressed generally to Agent Based Models. The model uses a lot of parameters represented by $\sigma_k$, $c_k$, $\alpha_j^k$ for $k = 1, \cdot \cdot \cdot , M$ and $j = 1, \cdot \cdot \cdot , p_k$, see figure [1]. However if we change the inputs, the dynamic of asset returns conserves the same structure since we remain also in the class of Random Coefficient Autoregressive Models RCAM(p). Only the laws of $X^l_t, l = 1, \cdot \cdot \cdot , p$ are modified. The second remark is related to the role of $M$ (number of traders). The model is not influenced basically by this
parameter since traders are supposed to be independent. So we conserve also the same dynamic RCAM(p) with a modification of the laws of \( X_l^j, l = 1, \cdots, p \) and perhaps of the parameter \( p \) too. The last remark is related to Agent Based Models using impact functions. Here, the parameter \( \lambda \) does not modify the structure of asset returns. We may replace it by another constant say \( \lambda = \beta c, \beta > 0 \) without loss of generality. Only the size of asset returns is transformed. So the Agent Based Model is stable with respect to its entries.

3 Simulation.

To make the simulations, we need the parameters of each trader \( k \) that means \( \sigma_k \) for the idiosyncratic random variable \( \nu_k^t \), \( \alpha_k^j \), \( j = 1, \cdots, p_k \) for \( r_{t,p_k} \) weight and \( c_k \) for the decision making, see figure 1. But we know that in the aggregation level, the dynamic of asset prices follows RCAM(p). So the simulations can be made directly by using equation (7) and by specifying the law of \( X_l^t, l = 1, \cdots, p \). This point reduces hugely the execution time necessary to generate a time series since we need not to simulate the behavior of each trader as generally made in Agent Based Models. We investigate the stylized facts listed in Yoon [16] by considering a RCAM(2) defined by

\[
\ln S_t = \ln S_{t-1} \left( 1 + u_t \right) + \ln S_{t-2} v_t + i_t. \tag{8}
\]

To obtain realistic values for asset returns, we take small variances for all random variables corresponding to a big \( \beta \) in the expression of \( \lambda = \beta c \). Namely we set \( u_t \sim \mathcal{N}(0, (0.02)^2) \), \( v_t \sim \mathcal{N}(0, (0.02)^2) \) and \( i_t \sim \mathcal{N}(0, (0.02)^2) \). The covariances between variables are \( \text{cov}(u_t, v_t) = 0.005^2 \) and \( \text{cov}(u_t, i_t) = \text{cov}(v_t, i_t) = 0 \) for \( t = 1, \cdots, T \). The number of step is \( T = 20000 \). We take \( \ln S_0 = \ln S_1 = 0.05 \).

Table 1 gives the time series’ characteristic of \( r_t = \Delta \ln S_t = \ln S_t - \ln S_{t-1} \) and \( |r_t| = |\ln S_t - \ln S_{t-1}| \). We see that the kurtosis is big for \( (r_t) \) with respect to the normal distribution. The null hypothesis based on the Jacque Berra test is rejected with a p-value smaller than 0.001. Also the mean and the standard deviation of \( |r_t| \) are close as in Yoon [16].

Figure 2 shows the estimated autocorrelation functions of \( (r_t) \) and \( \text{sign}(r_t) \). We see that their sizes are both small showing that the power of predictability of asset returns is weak when linear forecasting rules are used.

2. The sign function is defined by \( \text{sign}(x) = 1 \) if \( x > 0 \), \( \text{sign}(x) = -1 \) if \( x < 0 \) and \( \text{sign}(0) = 0 \).
Figure 3 shows the estimated autocorrelation functions of $|r_t|^q$ for $q = 0.25, 0.5, 0.75, 1, 1.25, 1.5, 1.75$. These values decrease to zero with some degree of persistence giving volatility clustering, see Figure 2 where the time series $(r_t)$ is plotted. The persistence of the time series $|r_t|^q$ is more pronounced when $q = 0.25$ as found in some series like exchange rates. Also, the long memory property is more important for $q = 1$ than $q = 2$ as found in financial time series.

In the next step, we test the robustness of RCAM(p). As mentioned above, when the inputs are modified, the laws of $X^l_t$ and the parameter $p$ may change. We simulate different scenarios by modifying the correlation and the variance of $X^l_t$, $l = 1, \cdots, p$. We modify also the parameter $T$ and $p$. All results are generally similar and reproduce the main properties of real financial data. We give only one illustration of these points.

$$\ln S_t = \sum_{l=1}^p \ln S_{t-l} X^l_t + i_t, \; p = 3 \quad (9)$$

Here, $[i_t, X^1_t, X^2_t, X^3_t]'$ follows a multivariate normal distribution $\mathcal{N}(0, A)$ where

$$A = \begin{pmatrix}
0.02^2 & 0 & 0 & 0 \\
0 & 0.01^2 & 0.004^2 & 0.003^2 \\
0 & 0.004^2 & 0.005^2 & 0.002^2 \\
0 & 0.003^2 & 0.002^2 & 0.03^2
\end{pmatrix}$$

The results are shown in the figures 4 and 5 where we take $T = 10000$. We remark that the same properties are reproduced as in the figures 2 and 3. In table 2 we represent some statistics of the time series $(r_t)$ and $(|r_t|)$. We see that the results are similar to table 1.

4 Extension and comments.

The previous model may be extended in two ways. First, we use only the stability of the normal distribution with respect to a sum of independent variables. So we may replace it by any symmetric $\alpha$ stable laws. But since we are concerning with asset returns, it is useful to have a finite variance.

Secondly, equation (1) says that the bias of traders is dependent only on last asset returns and so integrates the representative bias. Of course, we may add some others factors if we want to enlarge the class of models. For example, we may suppose that agents’ misspecification depends
also on the size of asset prices. Indeed, Kahneman and Tversky \[11\] show experimentally the anchoring bias which stipulates that agents by making decisions use first a reference point and then make an adjustment. That means even if we have a given sample of returns, the decision of traders may switch if we use different levels of asset prices that generates the same asset returns. So it makes sense to integrate in the bias of traders a term related to the reference point. To model this point, we may replace for example equation (1) by
\[ \epsilon_t^k = i_t + \nu_t^k (r_{t,p_k} + s_{t,q_k}) \] (10)

where
\[ r_{t,p_k} = \sum_{i=1}^{p_k} \alpha_i^k r_{t-1}^i, \quad s_{t,q_k} = \sum_{i=1}^{q_k} \beta_i^k \ln S_{t-1} \text{ with } \sigma_k^2 (r_{t,p_k} + s_{t,q_k})^2 < \sigma_i^2. \] (11)

The terms \( \nu_t^k r_{t,p_k} \) and \( \nu_t^k s_{t,q_k} \) represent respectively the idiosyncratic noise coming from representative and anchoring biases of traders. The right term of the equation (11) introduces bounded rationality as in the equation (3). In this framework, the model used in Yoon \[16\] may be obtained by taking \( \forall k, r_{t,p_k} = r_{t-1} \) and \( s_{t,q_k} = \ln S_{t-2} \). This corresponds to the case \( p_k = 1 \) with \( \alpha_1^k = 1 \) and \( q_k = 2 \) with \( \beta_1^k = 0, \beta_2^k = 1 \). The dynamic of asset prices is then
\[ \ln S_t = \ln S_{t-1} \left( 1 + u_t \right) + i_t \] (12)

where \( u_t = \sum_{k=1}^{M} c_k \nu_t^k \).

Yoon \[16\] has noted that stochastic unit root models (equation 12) have the ability to reproduce the main properties of financial time series. But the model is not connected to any Agent Based Model. Here we may investigate the crucial point that generates stylized facts. When agents only use the signal \( i_t \), then they are all fully rational. So the decision for the trader \( k \) is given by \( c_t i_t \). In this case, asset returns follow a random walk. However, the condition that agents use only \( i_t \) is compelling and may be relaxed. We may suppose that agents can not evaluate exactly the fundamental signal \( i_t \) and then make some over-valuation or under-valuation but randomly and independently of the state of the market i.e \( \epsilon_t^k = i_t + \nu_t^k \). In this case, equation (5) becomes
\[ \ln S_t = \ln S_{t-1} + \frac{E(t)}{\lambda} = \ln S_{t-1} + i_t + \sum_{k=1}^{M} \frac{c_k}{c} \nu_t^k = \ln S_{t-1} + \gamma_t \]

3. The switching phenomena appears due to a new reference point.
where $\gamma_t = i_t + \sum_{k=1}^{M} \Omega_t \nu_t^k$ is a white noise due to assumptions made with respect to $i_t$ and $\nu_t^k$ for $k = 1, \ldots, M$. We obtain also a random walk theory in the aggregation level and the model can not generates stylized facts. So the only way expecting to obtain the latter is to introduce feedback effects. We show that considering a model where independent agents are affected by the representativeness bias ($\epsilon_t^k = i_t + \nu_t^k r_{t,p_k}$) can reproduce the main properties of financial time series. The extended model $\epsilon_t^k = i_t + \nu_t^k (r_{t,p_k} + s_{t,q_k})$ taking account the anchoring bias generates the same properties with more flexibility. In both cases, we have a structure of random coefficient auto-regressive models RCAM $(p)$ for the dynamic of asset prices with a unit root in average. This dynamic is obtained by an interaction of both neo-classical and behavioral finance arguments. Traders follow mainly the sign of the information (efficient market theory) but make also extrapolations (psychological bias), see equation (3) or (11). So the market efficiency question is difficult to analyze. The comparison for example between the process $(S_t)$ given by the equation (12) or (7) and the process $(Z_t)$ given by $\ln Z_t = \ln Z_{t-1} + i_t$ is not easy in a statistical sense. In a continuous time framework, this difficulty has been illustrated in [12] where $Z_t$ and $S_t$ are replaced respectively by a diffusion process $(X_t)$ and a diffusion process $(Y_t)$ similar to $(X_t)$ but with a jump part having zero mean. If we suppose that these processes model two identical asset prices, the question is how can we explain the violation of the Law of One Price. Even if the formulation avoids the joint hypothesis since assets are similar, it does not allow to have a categoric response. The market may be considered as efficient since in mean asset prices reflect their fundamental values. The over-valuations and under-valuations arrive randomly. On the other hand, the inefficiencies that appear may be explained by the presence of noise traders plus limits to arbitrage.
5 Conclusion.

There are several Agent Based Models which have been developed explaining stylized facts. Some of them used interacting rules and others work on a public signal received by all traders. If we want to know what kind of asset dynamics such models may produce, it is useful to introduce some simplifications. In our model, we follow the second approach. Traders are independent and their behavior is determined by two signals. The first is based on the information flow \( (i_t) \) which is shared by all traders. The second is specific to each trader (idiosyncratic noise) and corresponds to the error made by trying to assess the fundamental news. We link this second signal to the last asset returns to highlight that the bias is more important when the market is volatile than stable. To obtain a realistic model, we impose a small variance for the noise part with respect to the fundamental news. That means traders follow mainly the sign of the information with some exceptions giving so two groups of traders called fundamentalist and noise traders. There is a random switching phenomena between these two groups. A fundamentalist may become a noise trader and conversely. We show that the presence of heterogeneity and feedback with respect to the last asset prices is the main factor which explains stylized facts and then supports the findings of the previous agent based models. Furthermore, we may have an idea of the asset price dynamics of such models. We show under some simplifications on the impact function and agents’ decision making that the equilibrium prices belong to Random Coefficient Autoregressive Models with a unit root in average. The normal distributions used in this study may be replaced without loss of generality by any symmetric stable laws. Also the class of models that is obtained may be enlarged if we integrate the level of asset prices in the bias made by traders to reflect the anchoring bias.

At last Stochastic Unit Root Models or diffusions with a jump part having zero mean may be useful to model asset prices. Indeed they integrate the arguments of both paradigms so that it is difficult to say if the market is efficient or not independently of the joint hypothesis. So I conjecture the debate between Neo-classical and Behavioral Finance is related to this point. Discount asset prices reflect their fundamental values or follow a martingale not in level but in mean. This passage from level to mean in the definition implies also a transition from homogeneous rational models to interacting agent based models.
Références

Figure 1. The timing events of the Agent Based Model.
**Figure 2.** Some properties of the time series based on the equation (8).
<table>
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<th>time series</th>
<th>Mean</th>
<th>Variance</th>
<th>minimum</th>
<th>maximum</th>
<th>skewness</th>
<th>kurtosis</th>
<th>JB statistic</th>
<th>p</th>
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<td>0.23</td>
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</table>

Table 1. Sample statistics of $(r_t)$ and $(|r_t|)$ based on the equation (8).

<table>
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<tr>
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Table 2. Sample statistics of $(r_t)$ and $(|r_t|)$ based on the equation (9).
Figure 3. Estimated autocorrelation functions of $|r_t|^q$, for $q = 0.25, 0.5, 0.75, 1, 1.25, 1.5, 1.75$, based on the equation (8).
Figure 4. Some properties of the time series ($r_t$) based on the equation (9).
Figure 5. Estimated autocorrelation functions of $|r_t|^q$, for $q = 0.25, 0.5, 0.75, 1, 1.25, 1.5, 1.75$ based on the equation (9).