Investment with Ambiguity and Regime-Switching Environment

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Ambiguity

- Knightian uncertainty: not able to attribute a precise probability to future outcomes.
- Ambiguity: several possible specifications $\mathbb{P}_1, \mathbb{P}_2, \ldots$.

cf. Risk: able to specify a unique probability on future outcomes.

Ambiguity aversion

Lotto, choosing number by computer or yourself?
Investment

- Irreversible, sunk cost.
- Business cycle.
Ambiguity

1. Knight (1921) distinguish between ambiguity and risk.
2. Ellsberg (1961) show that aversion to ambiguity plays a role in decision making.
Investment

1. Mcdonalds and Siegel(1986) apply financial option pricing techniques.
3. Heath and Tversky(1991) point out entrepreneurs tend to look at the best scenario when they face ambiguity since they have overconfidence.
4. Schröder(2007) solve an all or nothing problem and the irreversible investment problem using $\alpha$-MEU preferences, shows positive effect of ambiguity.
Regime-Switching

2. Fuh et al. (2003) explain empirical investigation like volatility smile, volatility clustering etc.
Firm's investment problem

- Risk neutral entrepreneur (or DM).
- Profit flow
  \[
  \frac{d\pi_t}{\pi_t} = \mu dt + \sigma dB_t, \quad \pi_0, \sigma > 0. \tag{2.1}
  \]
- Investment sunk cost = I.
- either decide to invest immediately, or abandon completely
  \[
  F_t = \max\{V_t - I, 0\} = \max\left\{E_t \left[\int_t^\infty e^{-\rho(s-t)}\pi_s ds\right] - I, 0\right\}. \tag{2.2}
  \]
- Assume \( \mu < \rho - \), otherwise expected project value could be infinite.
- If we know \( \mu, \sigma \) true parameter with no ambiguity, the standard expected value of an infinite profit stream
  \[
  V_t = \frac{\pi_t}{\rho - \mu}, \quad \text{Dixit and Pindyck (1994, p. 72)}. \tag{2.3}
  \]
Ambiguity

- **Density generators** $(\theta_t) \in \Theta \subset K = [-k, k]$
  - IID ambiguity. $(\theta_t)$ vary within the range $K$ in an independent and indistinguishable way).
  - $\kappa$-ignorance.
  - $k$: nonrandom, perceived degree of ambiguity.

- **The set of probability measure.**
  \[ P^\Theta = \{ Q^\theta | Q^\theta \text{ equivalent to the reference measure } P, \theta \in \Theta \} \]

- **Rectangular** : a necessary condition for intertemporal optimization problems to be dynamically consistent, i.e. for $0 \leq s \leq t \leq T$

\[ \min_{\theta \in \Theta} E_s^{Q^\theta}[x] = \min_{\theta \in \Theta} E_s^{Q^\theta} \left[ \min_{\theta' \in \Theta} E_t^{Q^{\theta'}}[x] \right]. \]
Regime-Switching Environment

- Consider a probability space $(\Omega, \mathcal{F}, P)$.
- A market-independent Poisson processes $N(t)$.
- With intensity $\lambda_i$, $i \in \{E, C\}$.
- When $N(t)$ jumps, regime $i \rightarrow$ regime $j \neq i$.
- $\mathcal{F}_t = \sigma\{B(s), N(s) \mid 0 \leq s \leq t\}$.
- 
  \[
  \frac{d\pi(t)}{\pi(t)} = \mu(t)dt + \sigma(t)dB(t), \tag{2.4}
  \]
  
  where

  \[
  \mu(t) = \begin{cases} 
  \mu_E, & \text{expansion}, \\
  \mu_C, & \text{contraction}, 
  \end{cases} \quad \sigma(t) = \begin{cases} 
  \sigma_E, & \text{expansion}, \\
  \sigma_C, & \text{contraction}, 
  \end{cases} \tag{2.5}
  \]

  and $\pi(0) > 0$. 
Profit flow under regime-switching environment and ambiguity

\[
\frac{d\pi_t}{\pi} = (\mu_t - \sigma_t \theta_t)dt + \sigma_t dB_t^\theta, \tag{2.6}
\]

where

\[
\mu_t = \begin{cases} 
\mu_E, & \text{Expansion}, \\
\mu_C, & \text{Contraction}.
\end{cases}, \quad \sigma_t = \begin{cases} 
\sigma_E, & \text{Expansion}, \\
\sigma_C, & \text{Contraction},
\end{cases} \tag{2.7}
\]

\[
B_t^\theta = B_t + \int_0^t \theta_s ds, \tag{2.8}
\]

and

\[
(\theta_t) \in \Theta \subset K = [-k, k]. \tag{2.9}
\]

Then by Itô’s formula, we get

\[
\pi(t) = \pi(0) \exp \left[ \int_0^t \mu(s) - \sigma(s)\theta(s) - \frac{1}{2} \sigma^2(s) ds + \int_0^t \sigma(s) dB(s)^\theta \right]. \tag{2.10}
\]
DM’s Problem

α-MEU

- α-Maxmin Expected Utility (α-MEU): convex combination of two extreme cases, i.e., the worst case and the best case.
- \(0 \leq \alpha \leq 1\): attitude of DM toward ambiguity.
- For \(i = E\) and \(C\),

\[
E^\alpha [f(x)] = \alpha \sup_{Q^\theta \in \mathcal{P}} E^Q_t [f(x) \mid \sigma(t) = \sigma_i] + (1 - \alpha) \inf_{Q^\theta \in \mathcal{P}} E^Q_t [f(x) \mid \sigma(t) = \sigma_i],
\]

(2.11)

where
- \(\mathcal{P}^\Theta\) : the set of possible probability measures,
- \(f : x \rightarrow \mathbb{R}\) is the stochastic payoff function,
- \(E^Q_t [: \mid \sigma(t) = \sigma_i]\) is the expectation with respect to \(Q^\theta\) conditional on \(\mathcal{F}(t)\) given that the current regime \(i\) is known.
DM’s Problem

Evaluating the investment

For $i = E$ and $C$,

$$V_i(\pi(0) | \alpha) = E^\alpha_0 \left[ \int_0^T e^{-\rho s} \pi(s) \, ds \right]$$

$$= \alpha \sup_{Q^\theta \in \mathcal{P}} E^Q_0 \left[ \int_0^T e^{-\rho s} \pi(s) \, ds \bigg| \sigma(0) = \sigma_i \right]$$

$$+ (1 - \alpha) \inf_{Q^\theta \in \mathcal{P}} E^Q_0 \left[ \int_0^T e^{-\rho s} \pi(s) \, ds \bigg| \sigma(0) = \sigma_i \right]$$

(2.12)

(2.13)
DM’s problem

Assumptions

- DM knows whether it is a contraction or an expansion.
- DM knows the present profit flow $\pi(0)$.

The DM evaluates at time 0

$$F_i(\pi(0)|\alpha) = \max\{ V_i(\pi(0)) | \alpha \} - I, \ 0 \}, \ i = E, \ C, \ (2.14)$$

the value of option to invest, where $I$ is the initial sunk cost of the project.

Critical present value of the profit flow

$\pi^*_i(0), \ i = E$ and $C$, satisfying

$$V_i(\pi(0)) | \alpha \} - I = 0$$
Theorem 1

When the profit flow of the project is given by Equation (2.10) subject to (2.7) and (2.9) the following equations is satisfied.

\[
\sup_{Q^\theta \in \mathcal{P}} E^{Q^\theta} \left[ \int_0^T e^{-\rho s} \pi(s) ds \right] \sigma(0) = \sigma_i
\]

\[
= \pi(0) \int_0^T \left[ \exp \left( (\rho - \mu + k ) s \right) \int_0^s \exp \left( (\mu_E - \mu - k (\sigma_E - \sigma_L)) u \right) f_1(s, u) du \right] ds,
\]

(3.15)

\[
\inf_{Q^\theta \in \mathcal{P}} E^{Q^\theta} \left[ \int_0^T e^{-\rho s} \pi(s) ds \right] \sigma(0) = \sigma_i
\]

\[
= \pi(0) \int_0^T \left[ \exp \left( (\rho - \mu - k ) s \right) \int_0^s \exp \left( (\mu_E - \mu - k (\sigma_E - \sigma_L)) u \right) f_1(s, u) du \right] ds,
\]

(3.16)
Theorem 1-continued

where \( f_i(s, u) \) is a PDF of \( \zeta(s) \) where \( \sigma(0) = \sigma_i \) such that

\[
f_H(s, u) \triangleq e^{-\lambda H s} \delta_0(s-u) + e^{-\lambda L s - u - \lambda_H u} \left( \frac{\lambda_H \lambda_L u}{s-u} \right)^{1/2} I_1 \left( 2(\lambda_H \lambda_L u(s-u))^{1/2} \right) + \lambda_H I_0 \left( 2(\lambda_H \lambda_L u(s-u))^{1/2} \right),
\]

\[
f_L(s, u) \triangleq e^{-\lambda L s} \delta_0(s-u) + e^{-\lambda L s - u - \lambda_H u} \left( \frac{\lambda_H \lambda_L u}{u} \right)^{1/2} I_1 \left( 2(\lambda_H \lambda_L u(s-u))^{1/2} \right) + \lambda_L I_0 \left( 2(\lambda_H \lambda_L u(s-u))^{1/2} \right),
\]

where \( I_a(z) \) is the modified Bessel function defined by

\[
I_a(z) \triangleq \left( \frac{z}{2} \right)^a \sum_{n=0}^{\infty} \frac{(z/2)^{2n}}{n! \Gamma(a+n+1)}.
\]
Corollary 1

The $\alpha$-expected value of the project in regime $i \in \{E, C\}$, for a decision maker who has a degree of optimism $\alpha \in [0, 1]$ is given by

$$V_i(\pi(0)|\alpha) = \pi(0) \left\{ \alpha \int_0^T \left[ \exp \left( (-\rho + \mu_C + k\sigma_C) s \right) \int_0^s \exp \left( (\mu_E + k\sigma_E - \mu_C - k\sigma_L) u \right) f_i(s, u) du \right] ds + (1 - \alpha) \int_0^T \left[ \exp \left( (-\rho + \mu_C - k\sigma_C) s \right) \int_0^s \exp \left( (\mu_E - k\sigma_E - \mu_C + k\sigma_L) u \right) f_i(s, u) du \right] ds \right\}$$

where $f_i$ are same as Equation (3.17), (3.18), respectively.
Value of the Investment

**Corollary 2 (One-state Model)**

When the dynamics of profit flow is given by Equation (2.6) subject to (2.9), the expected value of project $V^T(\pi(0)|\alpha)$ is represented as

$$V^T(\pi(0)|\alpha) = \pi(0) \left( \frac{\alpha(1 - \exp(-\rho - k\sigma - \mu)T)}{\rho - k\sigma - \mu} + \frac{(1 - \alpha)(1 - \exp(-\rho + k\sigma - \mu)T)}{\rho + k\sigma - \mu} \right)$$

(3.20)
**Value of the Investment**

**Corollary 3 (Perfectly pessimistic)**

The $\alpha$-expected value of the project in regime $i \in \{E, C\}$ for a decision maker who has a degree of optimism $\alpha = 0$ is given by

$$V_i(\pi(0)|\alpha = 0) = \pi(0) \int_0^T \left[ \exp \left( (-\rho + \mu_C - k\sigma_C) s \right) \int_0^s \exp \left( (\mu_E - \mu_C - k(\sigma_E - \sigma_C) u \right) f_i(s, u) du \right] ds,$$

(3.21)

where $f_i$ are same as Equation (3.17), (3.18), respectively.
**Figure:** The expected value of the investment $V(\pi_i(0))$ as a function of $k$ varying $\alpha$ when $T = 5$. Default parameters are $\lambda_E = 0.293$, $\lambda_C = 3.413$, $\mu_E = 0.056349723$, $\mu_C = 0.03069200$, $\sigma_E = 0.14$, $\sigma_C = 0.3$, $\rho = 0.1$, $\pi(0) = 1$, $l = 10$, $\mu = 0.049548513$, and $\sigma = 0.2$. 

**Numerical example**
Figure: The critical present value of the profit flow $\pi^*_i$ as a function of $k$ varying $\alpha$ when $T = 5$. Default parameters are

$\lambda_E = 0.293$, $\lambda_C = 3.413$, $\mu_E = 0.056349723$, $\mu_C = 0.03069200$, $\sigma_E = 0.14$, $\sigma_C = 0.3$, $\rho = 0.1$, $\pi(0) = 1$, $l = 10$, $\mu = 0.049548513$, and $\sigma = 0.2$. 
**Figure 3**

The expected value of the investment $V(\pi_i(0))$ as a function of $k$ varying $\alpha$ when $T = 450$. Default parameters are $\lambda_E = 0.293$, $\lambda_C = 3.413$, $\mu_E = 0.056349723$, $\mu_C = 0.03069200$, $\sigma_E = 0.14$, $\sigma_C = 0.3$, $\rho = 0.1$, $\pi(0) = 1$, $l = 10$, $\mu = 0.049548513$, and $\sigma = 0.2$. 
Figure: The critical present value of profit flow $\pi^*_i$ as a function of $k$ varying $\alpha$ when $T = 450$. Default parameters are $\lambda_E = 0.293$, $\lambda_C = 3.413$, $\mu_E = 0.056349723$, $\mu_C = 0.03069200$, $\sigma_E = 0.14$, $\sigma_C = 0.3$, $\rho = 0.1$, $\pi(0) = 1$, $l = 10$, $\mu = 0.049548513$, and $\sigma = 0.2$. 
Conclusion

1. Business cycle + Ambiguity
2. The longer investment period the greater value of investment.
3. The critical present value of the profit flow is regime-dependent.
4. Knowing whether it is an expansion or a contraction is very important when investment period is short.
5. The subjective attitude of DM toward ambiguity is important variable when investment period is long.
6. Ambiguity seeking tendency is mitigated if DM consider business cycle.
## Further Study

- Optimal stopping time.
- Learning.
- Smooth ambiguity.