An Optimal Investment Policy in Equity-Debt Financed Firms with Finite and Infinite Maturities

Kyoko YAGI¹,* , Ryuta TAKASHIMA², Katsushige SAWAKI³

¹ The University of Tokyo
² Chiba Institute of Technology
³ Nanzan University
1. Introduction (1)

- The interaction among firm’s investment and financing decisions under uncertainty by means of real option framework.

- The investment problems for the firm financed with
  - all-equity
    (McDonald and Siegel(1986), Dixit and Pindyck(1994))
  - equity and debt
1. Introduction (2)

- The infinite maturities of the investment and debt are assumed in order to simplify the problem.

- Example of finite option:
  
  **The investment of oil reserves development**
  - Offshore leases limit the time before the development (Paddock et al.(1988)).

  **The investment of the power plant**
  - It is difficult to postpone the construction investment for a long term because of ensuring the stable supply.
1. Introduction (3)

- When the investment problem as real case is analyzed, it is necessary to consider the maturity of the investment decision.
- It seems not only the investment timing but also financing is dependent on the maturity.

- Our objectives:
  - The optimal investment policy with finite option of the firm that is financed by issuing equity and debt
  - We discuss the effect of maturity on the investment timing, the firm value and the optimal capital structure.
2. The Model (1)

- A firm with an option to invest at any time.

- \( I \) : a fixed investment cost

- The firm partially finances the cost of investment with straight debt.
  - The instantaneous coupon payment of \( c \)

- \( \lambda \) : a constant corporate tax rate

- Coupon payments are tax-deductible.
2. The Model (2)

- Suppose the firm observes the demand shock $X_t$ for its product
  
  - $X_t$: a geometric Brownian motion
    
    $$dX_t = \mu X_t dt + \sigma X_t dW_t, \quad X_0 = x$$  \hspace{1cm} (1)

  - $\mu$: the risk-adjusted expected growth rate
  - $\sigma$: the volatility
  - $W_t$: a standard Brownian motion

- Once the investment option is exercised, the firm can receive the instantaneous profit
  
  $$\pi(X_t) = (1 - \lambda)(Q X_t - c)$$  \hspace{1cm} (2)

  - $Q$: the quality produced from the asset in place
2.1. All-equity Financing

Assume The investment is financed entirely with equity ( \( c = 0 \)).

2.1.1. Investment with Infinite Maturity (1)

Consider the investment with infinite maturity

This case has been investigated by using the basic model in real options theory.

\[ F_a(x) : \text{the value of investment option} \]

\[ x^a : \text{the optimal investment threshold} \]
2.1. All-equity Financing

2.1.1. Investment with Infinite Maturity (2)

- The value of investment option

\[
F_a(x) = \begin{cases} 
\left( \frac{x}{x^a} \right)^{\beta_1} (\epsilon(x^a) - I) & x < x^a \\
\epsilon(x) - I & x \geq x^a 
\end{cases}
\]  

- \( \beta_1 = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left( \frac{1}{2} - \frac{\mu}{\sigma^2} \right)^2 + \frac{2r}{\sigma^2}} > 1 \)
- \( r \) : the risk-free interest rate (\( r > \mu \))
- \( \epsilon(x) \) : the total post-investment profit

\[
\epsilon(x) = \frac{1 - \lambda}{r - \mu} Q x
\]  

- The investment threshold

\[
x^a = \frac{1}{1 - \lambda} \frac{\beta_1}{\beta_1 - 1} \frac{r - \mu}{Q} I
\]
2.1. All-equity Financing

2.1.2. Investment with Finite Maturity (1)

- Consider the investment with finite maturity
  - $T < \infty$

- The optimal investment rule
  - to exercise the investment option at the first passage time to an upper investment threshold $x_t^a$ by the maturity $T$.

- $\mathcal{T}_{t_1,t_2}$: the set of stopping times with respect to the filtration as $\{F_s; t_1 \leq s \leq t_2\}$

- $\tau \in \mathcal{T}_{0,T}$: the investment time
2.1. All-equity Financing

2.1.2. Investment with Finite Maturity (2)

- \( F_a(x, t) \) : the value of investment option

\[
F_a(x, t) = \sup_{\tau \in \mathcal{T}_{i,T}} E_t^a \left[ e^{-r(\tau-t)} \left( \int_{\tau}^{\infty} e^{-r(u-\tau)}(1-\lambda)QX_u du - I \right) \right] ^{-1}
\]  \hspace{1cm} (6)

- \( (x)^+ = \max(x, 0) \)

- \( x_t^a \) : the optimal investment threshold at time \( t \)

- The optimal investment time \( \tau_t^a \)

\[
\tau_t^a = \inf\{ \tau \in [t,T) \mid X_\tau \geq x_t^a \} \land T
\]  \hspace{1cm} (7)
2.1. All-equity Financing

2.1.2. Investment with Finite Maturity (3)

- From Bellman equation, the value of investment option satisfies the partial differential equation (PDE)

\[ \mathcal{A}F_a = 0 \quad \text{for } x < x_t^a \] (8)

\[ -\mathcal{A} = \frac{1}{2} \sigma^2 x^2 \frac{\partial^2}{\partial x^2} + \mu x \frac{\partial}{\partial x} + \frac{\partial}{\partial t} - r \]

- The boundary conditions

\[
\begin{cases}
F_a(X_T, T) = (\varepsilon(X_T) - I)^+

\lim_{x \uparrow x_t^a} F_a(x, t) = (\varepsilon(x_t^a) - I)^+, & t \in [0, T) \\
\lim_{x \uparrow x_t^a} \frac{\partial F_a}{\partial x}(x, t) = \frac{1 - \lambda}{r - \mu}, & t \in [0, T)
\end{cases}
\] (9)
2.2. Equity and Perpetual Debt Financing (1)

- The investment is financed with equity and debt with infinite maturity.

- $E_i(x; c)$: the total value of equity issued at investment time $t$

- $D_i(x; c)$: the total value of debt with the coupon payment of $c$, issued at investment time $t$

- $x^d$: the optimal default threshold
2.2. Equity and Perpetual Debt Financing (2)

- The equity value

\[ E_i(x; c) = \begin{cases} 
\epsilon(x) - \frac{(1 - \lambda)c}{r} - \left( \frac{x}{x^d} \right)^{\beta_2} \left( \epsilon(x^d) - \frac{(1 - \lambda)c}{r} \right), & x > x^d \\
0, & x \leq x^d
\end{cases} \]

- \[ \beta_2 = \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{\left( \frac{1}{2} - \frac{\mu}{\sigma^2} \right)^2 + \frac{2r}{\sigma^2}} < 0 \] (10)

- The default threshold

\[ x^d = \frac{r - \mu}{Q} \left( \frac{\beta_2}{\beta_2 - 1} \frac{c}{r} \right) \] (11)

- The debt value

\[ D_i(x; c) = \frac{c}{r} - \left( \frac{x}{x^d} \right)^{\beta_2} \left( \frac{c}{r} - (1 - \theta)\epsilon(x^d) \right), \quad x > x^d \] (12)

- \( \theta \) : the proportional bankruptcy cost \((0 \leq \theta \leq 1)\)
2.2. Equity and Perpetual Debt Financing (3)

- \( V_i(x; c) \): the firm value

\[
V_i(x; c) = E_i(x; c) + D_i(x; c)
\]

\[
= \epsilon(x) + \frac{\lambda c}{r} \left\{ 1 - \left( \frac{x}{x^d} \right)^{\beta_2} \right\} - \theta \epsilon(x^d) \left( \frac{x}{x^d} \right)^{\beta_2} \quad \text{for } x > x^d \quad (13)
\]

- The value of unlevered firm
- The expected present value of bankruptcy cost: \( BC(x; c) \)
- The expected present value of debt tax shields: \( TS(x; c) \)

- The optimal policy maximizing the firm value
2.2. Equity and Perpetual Debt Financing

2.2.1. Investment with Infinite Maturity (1)

- Consider the investment with infinite maturity
  \[ T = \infty \]

- This case has been introduced in Mauer and Sarkar (2005), and Sundaresan and Wang (2006).

- \( c^*(x) \): The optimal coupon payment of debt issued when \( X_t = x \) at time \( t \)

- The optimal capital structure is achieved by selecting the optimal coupon payment.

- \( x^i \): the optimal investment threshold
2.2. Equity and Perpetual Debt Financing

2.2.1. Investment with Infinite Maturity

- The optimal investment timing and the optimal coupon payment can be determined by maximizing the firm value, simultaneously.

- \( \tau \in \mathcal{T}_{0,\infty} \): the investment time

- \( F_i(x) \): the value of investment option

\[
F_i(x) = \sup_{\tau \in \mathcal{T}_{0,\infty}, c > 0} E_0^x \left[ e^{-r\tau} (V_i(X_{\tau}; c) - I) \right]
\]  \hspace{1cm} (14)

- The optimal investment time \( \tau^i \)

\[
\tau^i = \inf \{ \tau > 0 \mid X_{\tau} \geq x^i \}
\]  \hspace{1cm} (15)
2.2. Equity and Perpetual Debt Financing

2.2.1. Investment with Infinite Maturity

From Sundaresan and Wang (2006), the optimal coupon payment for any $x$

$$c^*(x) = \arg \max_{c>0} V_i(x; c) = \frac{r}{r - \mu} \frac{\beta_2 - 1}{\beta_2} \frac{Q_x}{h} > 0$$  \hspace{1cm} (16)$$

$$h = \left(1 - \beta_2 \left(1 - \theta + \frac{\theta}{\lambda}\right)\right)^{\frac{1}{\beta_2}}$$

The value of investment option

$$F_i(x) = \begin{cases} 
\left(\frac{x}{x^i}\right)^{\beta_1} (V_i(x^i; c^*(x^i)) - I) & x < x^i \\
V_i(x; c^*(x)) - I & x \geq x^i 
\end{cases}$$  \hspace{1cm} (17)$$
2.2. Equity and Perpetual Debt Financing

2.2.1. Investment with Infinite Maturity (4)

- The optimal investment threshold

\[ x^i = \psi x^a = \frac{\psi}{1 - \lambda} \frac{\beta_1}{\beta_1 - 1} \frac{r - \mu}{Q} I \]  

\[ - \psi = \left(1 + \frac{\lambda}{(1 - \lambda)h}\right)^{-1} < 1 \]
2.2. Equity and Perpetual Debt Financing

2.2.2. Investment with Finite Maturity (1)

- Consider the investment with finite maturity
  \[ T < \infty \]

- \( x_t^i \): the optimal investment threshold at time \( t \)

- \( F_i(x, t) \): the value of investment option
  \[
  F_i(x, t) = \sup_{\tau \in \mathcal{T}_{t,T}, c > 0} E_t^x \left[ e^{-r(\tau-t)}(V_i(X_\tau; c) - I)^+ \right] 
  \]  
  \[ (19) \]

- The optimal investment time
  \[
  \tau_t^i = \inf\{\tau \in [t, T) \mid X_\tau \geq x_\tau^i\} \land T 
  \]  
  \[ (20) \]
2.2. Equity and Perpetual Debt Financing

2.2.2. Investment with Finite Maturity (2)

- The value of investment option satisfies the PDE

\[ A F_i = 0 \quad \text{for} \quad x < x_t^i \quad (21) \]

- The boundary conditions

\[
\begin{align*}
  F_i(X_T, T) &= (V_i(X_T; c^*(X_T)) - I)^+ \\
  \lim_{x \uparrow x_t^i} F_i(x, t) &= (V_i(x_t^i; c^*(x_t^i)) - I)^+, \quad t \in [0, T) \\
  \lim_{x \uparrow x_t^i} \frac{\partial F_i}{\partial x}(x, t) &= \frac{dV_i}{dx}(x_t^i; c^*(x_t^i)), \quad t \in [0, T)
\end{align*}
\]
3. Numerical Analysis

- The value of investment option
- The values of equity and debt
- The investment threshold
- Tax shields
- Bankruptcy costs
- Optimal coupon payments
- Optimal leverage

using finite difference method (implicit method)
## Tab.1 Parameters

<table>
<thead>
<tr>
<th>Quantity for $X_t$</th>
<th>$Q = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial value of $X_t$</td>
<td>$x = 0.3$</td>
</tr>
<tr>
<td>Expected growth rate</td>
<td>$\mu = 0.01$</td>
</tr>
<tr>
<td>Volatility</td>
<td>$\sigma = 0.2$</td>
</tr>
<tr>
<td>Interest rate</td>
<td>$r = 0.05$</td>
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<tr>
<td>Investment cost</td>
<td>$I = 5$</td>
</tr>
<tr>
<td>Coupon payment</td>
<td>$c = 0.3$</td>
</tr>
<tr>
<td>Proportional bankruptcy cost</td>
<td>$\theta = 0.3$</td>
</tr>
<tr>
<td>Corporate tax rate</td>
<td>$\lambda = 0.3$</td>
</tr>
</tbody>
</table>
Fig. 1. Equity and debt value \( (c = 0.3) \)

\[ E_i(x; c), D_i(x; c) \]
Fig. 2. The value of investment option for the firm issuing equity and debt \( F_i(x, 0; c) \) (\( c = 0.3 \))
Fig. 3. Optimal investment threshold \((T = 50)\)

The degree of the threshold for coupon payments changes with time.
The investment timing depends not only on the investment threshold but also on the coupon payment. It is important to consider the maturity of the investment in the investment policy of the firm financed by equity and debt.
Fig. 5. The value and the ratio of tax shield

\[ V_i(x; c) = \epsilon(x) + \frac{TS(x; c)}{V_i(x_t; c)} - BC(x; c) \]

\[ TS(x_t^i; c) / V_i(x_t^i; c) \]

Although the value of tax shields decreases, the ratio of them to the firm value increases against time.
Fig. 6. The value and the ratio of bankruptcy cost

\[ V_i(x; c) = \epsilon(x) + TS(x; c) - BC(x; c) \]

\[ BC(x_t^i; c) \]

The threshold value of the investment with the short maturity is small.

The possibility of bankruptcy is high and the bankruptcy cost is also high.
Fig. 7. The value of investment option over coupon payment \( (x = 0.3) \)

\[ F^*_i(x, 0; c) \]

- \( T = 1 \)
- \( T = 5 \)
- \( T = 10 \)
Fig. 8. The optimal coupon payment, leverage, debt value and firm value

\[ c^*(x^i_t), \frac{D_i(x^i_t; c^*(x^i_t))}{V_i(x^i_t; c^*(x^i_t))}, D_i(x^i_t; c^*(x^i_t)), \frac{V_i(x^i_t; c^*(x^i_t))}{V_i(x^i_t; c^*(x^i_t))} \]

It is important to consider not only the maturity of the investment but also the one of the debt.
4. Concluding Remarks

- The optimal investment policy with finite option of the firm financed by issuing equity and debt
- We showed
  - When the maturity is shorter, the investment threshold and the coupon payment is smaller.

It is important to consider the maturity of investment.

- The optimal leverage is constant in time from the equity, debt and investment value

It is also important to consider the maturity of debt.
Future Work

- The issue of debt with finite maturity
- The investment problem of the firm with debt issued.
- The term structure of interest rate such as CIR model and Vasicek model
Thank you.

Kyoko YAGI, The University of Tokyo, 
E-mail: yagi@e.u-tokyo.ac.jp