A Markov Regime Switching Model in Option Pricing

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2 Option Prices as Fourier Integrals
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3 Markov Regime Switching Model
   - Markov volatility
   - Laplace transform of $U_T$
   - The case N=2
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4 Numerical Result
   - Monte Carlo simulation
   - Volatility smile
The classical Black-Scholes formula involves
- expected rate of return
- volatility

Both are assumed to be deterministic constant

The implied volatility should change with respect to the maturity and the strike price of the option

Black-Scholes model fails to reflect the volatility smile
The stock price $S$ is assumed to satisfy

$$dS_t = \mu S_t \, dt + V_t S_t \, dW_t,$$

where $W$ is a standard Brownian motion.

$$dS_t = rS_t \, dt + V_t S_t \, dW_t,$$

where $r$ is the risk-free rate and $W$ is a standard Brownian motion under the pricing measure $\mathbb{P}$, independent of $V$.

“Physical” measure vs “Risk-neutral” measure
No-arbitrage price of a European option

\[ \mathbb{E} g(X) = \int g(x) \, d\mu_X(x) \]

The distribution of \( X \), \( \mu_X \) is either unknown or too complicated.

The Fourier transform

\[ \hat{\mu}_X(w) = \mathbb{E} e^{iwX} \]

is known and relatively easy to compute.
### Parseval’s Theorem

If $g \in L^1$ and $\mu$ is a signed measure with $|\mu| < \infty$ then

$$\int g(x) \, d\mu_X(x) = \frac{1}{2\pi} PV \int_{-\infty}^{\infty} \hat{g}(-w) \hat{\mu}_X(w) \, dw,$$

- $\hat{g}(w) = \int e^{iwx} g(x) \, dx$ is the Fourier transform of $g$
- "PV" means a "principal value" integral

$$PV \int_{-\infty}^{\infty} h(w) \, dw = \lim_{M \to \infty} \int_{-M}^{M} h(w) \, dw.$$

- Is the function $g$ Lebesgue integrable?
We introduce a “damping factor” to replace $g$ with an integrable function.

The price of a European option can be written

$$\int g(x) \, d\mu_X(x) = \int e^{-\alpha x} g(x) e^{\alpha x} \, d\mu_X(x)$$

Apply Parseval’s theorem:

$$\frac{1}{2\pi} \text{PV} \int_{-\infty}^{\infty} \hat{g}(-\alpha)(-w) \hat{\mu_X}(\alpha)(w) \, dw$$

Find $\alpha$ such that $g(-\alpha)(x)$ is Lebesgue integrable and $\mu^{(\alpha)}$ has finite mass.
Let

\[ U_t = \int_0^t V_s^2 \, ds \]

or integrated realised variance over \([0, t]\).

Since \(dS_t = rS_t \, dt + V_t S_t \, dW_t\),

\[ S_t = S_0 \exp \left( rt - \frac{U_t}{2} + \int_0^t V_s \, dW_s \right) \]

\[ e^{-rT} \mathbb{E}(K - S_T)_+ = e^{-rT} \mathbb{E} \left[ \mathbb{E}[(K - S_T)_+ | V] \right] \]

\[ = e^{-rT} \mathbb{E}[K - S_0 \exp(rT - \frac{1}{2} U_T + \sqrt{U_T}W_1)]_+ \]

\[ = \mathbb{E}g(U_T) \]

\[ g(u) = \mathbb{E}[e^{-rT}K - S_0 \exp(-\frac{u}{2} + \sqrt{u}W_1)]_+ \]
Theorem 1

Let $U_T$ and $S_T$ be as above, and let $\nu$ be the distribution of $U_T$, so that

$$\nu^{(\alpha)}(u) = E e^{(\alpha + iu)U_T}.$$ 

Suppose that $E e^{\alpha^* U_T} < \infty$ for some $\alpha^* > 0$. Then, for any $0 < \alpha < \alpha^*$,

$$e^{-rT} E (K - S_T)_+ = \frac{1}{2\pi} PV \int_{-\infty}^{\infty} g_1^{(-\alpha)}(-u) \nu^{(\alpha)}(u) \, du,$$

where, if $\bar{k} = Ke^{-rT} / S_0$,

$$g_1^{(-\alpha)}(-u) = \begin{cases} S_0 \bar{k}^{(1 + \sqrt{1 + 8\alpha + 8iu})/2} / (\alpha + iu) \sqrt{1 + 8\alpha + 8iu} & \text{if } Ke^{-rT} < S_0 \\ S_0 (\bar{k} - 1) / \alpha + iu + S_0 \bar{k}^{(1 - \sqrt{1 + 8\alpha + 8iu})/2} / (\alpha + iu) \sqrt{1 + 8\alpha + 8iu} & \text{if } Ke^{-rT} \geq S_0. \end{cases}$$
The total mass $|\nu^{(\alpha)}|$ is finite by assumption $\mathbb{E}e^{\alpha^* U_T} < \infty$

Define $g_1(x) = \mathbb{E}(Ke^{-rT} - S_0e^{-\frac{x}{2} + \sqrt{x}W_1}) + 1\{x \geq 0\}$

$$
\int_{-\infty}^{\infty} |g_1^{(-\alpha)}(x)| \, dx = \int_{-\infty}^{\infty} e^{-\alpha x} |g_1(x)| \, dx < \infty
$$

because $\alpha > 0$ and $0 \leq g_1(x) \leq Ke^{-rT}$

$G_1(y) = \int_{\mathbb{R}} g_1^{(-\alpha)}(x - y) \, d\nu^{(\alpha)}(x)$

is continuous at $y = 0$ by dominated convergence theorem.
The regime switching model was first introduced by Hamilton in 1989 to describe a regime switching time series.

Di Masi *et al.* discuss mean-variance hedging for regime switching European option pricing.

It is the Black-Scholes exponential Brownian motion model along with a hidden Markov process.

This allows volatility to possibly take two or more different values depending on the state of Markov chain.
• $V$ is a continuous-time Markov chain with finite states.

• $v_j$ is the value of $V$ in state $j$.

• $\lambda_{jk}$ is the transition rate from state $j$ to $k$.

• $S$ is a spot price.

• $K$ is a strike price.

• $r$ is a risk-free rate.

• $T$ is an option maturity.

• $W$ is a standard Brownian motion under the pricing measure $\mathbb{P}$. 
Two approaches:

1. Find explicit expression for $\mu U_T$.
   - It is feasible for $N=2$.
   - Finding the distribution for $N=3$ or more states seems to be complicated.
   - Fuh et al. approximate the occupational time distributions in given state using its stationary distribution.

2. Apply Parseval’s theorem.
   - The Fourier transform $\hat{\mu}_U(w) = \mathbb{E} e^{iwU_T}$ is easy to compute.
We focus on the Laplace transform of

\[ U_T = \int_0^T V_s^2 \, ds, \]

Given \( V \),

\[ \int_0^T V_s \, dW_s \sim N(0, \int_0^T V_s^2 \, ds). \]

Note that

\[ U_T = v_1^2 J_1^T + v_2^2 J_2^T + \cdots + v_n^2 J_n^T \]

where \( J_t^j \) is the time spent by \( V \) in state \( j \) over \([0, t] \).
Define

\[ \lambda_j = \sum_{k \neq j} \lambda_{jk} \]

\[ L_j(r, T) = \mathbb{E}[\exp\{-rU_T\} | V_0 = v_j^2] \]

\[ \vec{L}(r, T) = (L_1(r, T), \ldots, L_N(r, T))' \]

\[ \Lambda(r) = \begin{pmatrix}
-\lambda_1 - rv_1^2 & \lambda_{12} & \ldots & \lambda_{1N} \\
\lambda_{21} & -\lambda_2 - rv_2^2 & \ldots & \lambda_{2N} \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_{N1} & \lambda_{N2} & \ldots & -\lambda_N - rv_N^2
\end{pmatrix} \]

\[ \Lambda = \Lambda(0), \quad D = \text{diag}(v_1^2, \ldots, v_N^2), \quad \vec{1} = (1, \ldots, 1)', \quad \vec{v} = D\vec{1}' = (v_1^2, \ldots, v_N^2)'. \]
Proposition 2

For $\Re(r) > 0$,

$$\bar{L}(r, T) = e^{\Lambda(r) T} \bar{1},$$

and, for $\Re(s)$ large enough,

$$\int_0^{\infty} e^{-sT} \bar{L}(r, T) \,dT = (sI - \Lambda(r))^{-1} \bar{1}.$$
Apply Proposition 2:

\[
\int_0^\infty e^{-sT}L_1(r, T) \, dT = \frac{s + rv_2^2 + \lambda_{12} + \lambda_{21}}{(s + rv_1^2 + \lambda_{12})(s + rv_2^2 + \lambda_{21}) - \lambda_{12}\lambda_{21}}
\]

Note that \( U_T = \int_0^T V_s^2 \, ds = v_1^2 J_T + v_2^2 (T - J_T) \)

The Laplace transform of \( J_T \) is the one above with \((v_1^2, v_2^2) = (1, 0)\).

We need to invert

\[
\frac{s + \lambda_{12} + \lambda_{21}}{(r + s + \lambda_{12})(s + \lambda_{21}) - \lambda_{12}\lambda_{21}}
\]

in \( r \) and \( s \) to get the law of \( J_T \):

\[
P(J_T \in dx) = e^{-\lambda_{12}T}1_{\{x=T\}} + \lambda_{12} e^{-\lambda_{12}x-\lambda_{21}(T-x)}
\times \left[ \lambda_{21} x \ 0F_1(2; \lambda_{12}\lambda_{21}x(T-x)) + 0F_1(1; \lambda_{12}\lambda_{21}x(T-x)) \right] \ 1_{\{0<x<T\}} \ dx
\]
Using Proposition 2 again,

\[
\int_0^\infty e^{-sT} L_1(r, T) dT = \frac{s^2 + \left( r(v_2^2 + v_3^2) + \lambda_{12} + \lambda_{13} + \lambda_2 + \lambda_3 \right) s + r^2(v_2^2v_3^2) + r\left( v_2^2(\lambda_{13} + \lambda_3) + v_3^2(\lambda_{12} + \lambda_2) \right)}{D(r, s)} + c_1
\]

where

\[
D(r, s) = \left( (s + rv_1^2 + \lambda_1)(s + rv_2^2 + \lambda_2) - \lambda_{12}\lambda_{21} \right)(s + rv_3^2 + \lambda_3) - \left( \lambda_{13}(s + rv_2^2 + \lambda_2) + \lambda_{12}\lambda_{23} \right)\lambda_{31} - \left( \lambda_{13}\lambda_{21} + (s + rv_1^2 + \lambda_1)\lambda_{23} \right)\lambda_{32}
\]

\[
c_1 = \lambda_{12}(\lambda_{23} + \lambda_3) + \lambda_{13}(\lambda_{32} + \lambda_2) + \lambda_2\lambda_3 - \lambda_{23}\lambda_{32}
\]
Fourier transform of the distribution of $U_T$

$$\hat{\nu}(\alpha)(u) = E e^{(\alpha + iu)U_T}$$

Define $\psi_j(r, T) = E[e^{-rJ_T} | V_0 = j]$

Thus, $\hat{\nu}(\alpha)(u) = e^{(\alpha + iu)v_2^2T}\psi_1((\alpha + iu)(v_2^2 - v_1^2), T)$

To get $\psi_1$, invert

$$\frac{s + \lambda_{12} + \lambda_{21}}{(s + r + \lambda_{12})(s + \lambda_{21}) - \lambda_{12}\lambda_{21}}$$

in $s$ only.
Recall that the Brownian motion and the volatility are independent.

It is not required to simulate the Brownian motion, since

\[ e^{-rT} \mathbb{E}[(K - S_T)_+] = \mathbb{E}[e^{-rT} \mathbb{E}(K - S_0 e^{rT - \frac{UT}{2} + \sqrt{UT}Z})_+ | V]. \]

The conditional expectation on the right is simply the Black-Scholes formula for a put with \( T, K \) and \( \sigma = \sqrt{UT/T} \).

Therefore it is sufficient to simulate the Black-Scholes formula for such a put and take the average.
Two shortcomings:

1. The general method applies only when $V$ is independent of $W$.

2. The computing of the Fourier integral requires some trial and error to find a good path of integration.
   - Good results are obtained for $0 < \alpha < 10$
<table>
<thead>
<tr>
<th>T</th>
<th>Exact</th>
<th>Parseval</th>
<th>Monte Carlo</th>
</tr>
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<tr>
<td></td>
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<tr>
<td></td>
<td></td>
<td></td>
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<tr>
<td>K=80</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>0.50</td>
<td>0.14842</td>
<td>0.14842</td>
<td>0.14822 (±0.00055)</td>
</tr>
<tr>
<td>1.00</td>
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<td>0.61403</td>
<td>0.61264 (±0.00147)</td>
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<td>2.00</td>
<td>1.60916</td>
<td>1.60916</td>
<td>1.60917 (±0.00254)</td>
</tr>
<tr>
<td>3.00</td>
<td>2.34793</td>
<td>2.34793</td>
<td>2.34847 (±0.00294)</td>
</tr>
<tr>
<td>K=100</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.50</td>
<td>2.90070</td>
<td>2.90092</td>
<td>2.89899 (±0.00341)</td>
</tr>
<tr>
<td>1.00</td>
<td>4.41744</td>
<td>4.41765</td>
<td>4.41763 (±0.00487)</td>
</tr>
<tr>
<td>2.00</td>
<td>6.25551</td>
<td>6.25551</td>
<td>6.25523 (±0.00567)</td>
</tr>
<tr>
<td>3.00</td>
<td>7.17574</td>
<td>7.17574</td>
<td>7.17008 (±0.00563)</td>
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<tr>
<td>K=120</td>
<td></td>
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<tr>
<td>0.50</td>
<td>17.58961</td>
<td>17.58959</td>
<td>17.58825 (±0.00167)</td>
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<td>2.00</td>
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<td>16.08429</td>
<td>16.08301 (±0.00639)</td>
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<tr>
<td>3.00</td>
<td>15.79802</td>
<td>15.79800</td>
<td>15.79693 (±0.00689)</td>
</tr>
<tr>
<td>Time (sec)</td>
<td>5.502</td>
<td>0.440</td>
<td>13.923</td>
</tr>
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</table>

Put option prices for $N = 2$

$S_0 = 100, \nu_1 = 0.1, \nu_2 = 0.3, \lambda_{12} = 1, \lambda_{21} = 1, r = 0.05$
<table>
<thead>
<tr>
<th>T</th>
<th>Parseval</th>
<th>Monte Carlo</th>
</tr>
</thead>
<tbody>
<tr>
<td>K=80</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.50</td>
<td>0.15100</td>
<td>0.15154 (±0.00049)</td>
</tr>
<tr>
<td>1.00</td>
<td>0.62574</td>
<td>0.62582 (±0.00123)</td>
</tr>
<tr>
<td>2.00</td>
<td>1.58298</td>
<td>1.58471 (±0.00194)</td>
</tr>
<tr>
<td>3.00</td>
<td>2.26221</td>
<td>2.26279 (±0.00216)</td>
</tr>
<tr>
<td>K=100</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.50</td>
<td>3.22523</td>
<td>3.22592 (±0.00312)</td>
</tr>
<tr>
<td>1.00</td>
<td>4.81814</td>
<td>4.81971 (±0.00395)</td>
</tr>
<tr>
<td>2.00</td>
<td>6.48258</td>
<td>6.48377 (±0.00413)</td>
</tr>
<tr>
<td>3.00</td>
<td>7.21571</td>
<td>7.21609 (±0.00397)</td>
</tr>
<tr>
<td>K=120</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.50</td>
<td>17.66806</td>
<td>17.65049 (±0.02018)</td>
</tr>
<tr>
<td>1.00</td>
<td>16.85663</td>
<td>16.83267 (±0.02739)</td>
</tr>
<tr>
<td>2.00</td>
<td>16.35320</td>
<td>16.34778 (±0.03336)</td>
</tr>
<tr>
<td>3.00</td>
<td>15.90152</td>
<td>15.86252 (±0.03560)</td>
</tr>
<tr>
<td>Time (sec)</td>
<td>7.907</td>
<td>18.299</td>
</tr>
</tbody>
</table>

Put option prices with $N = 3$

$S_0 = 100, \nu_1 = 0.1, \nu_2 = 0.2, \nu_3 = 0.3,$

$\lambda_{12} = \lambda_{13} = \lambda_{21} = \lambda_{23} = \lambda_{31} = \lambda_{32} = 1, \ r = 0.05$
N=2; \nu_1=0.1; \nu_2=0.3; \lambda_{12}=\lambda_{21}=1

Volatility smile
Option Prices as Fourier Integrals

Markov Regime Switching Model

Numerical Result

Monte Carlo simulation

Volatility smile

\[ N=2; v_1=0.2; v_2=0.6; \lambda_{12}=\lambda_{21}=1 \]

Strike prices, \( K \)

Implied vol
Mean-variance hedging of options on stocks with Markov volatilities. 

Fourier inversion formulas in option pricing and insurance. 

Option pricing in a Black-Scholes model with Markov switching. 

A simple option formula for general jump-diffusion and other exponential Lévy processes. 