Consensus Investor and Intertemporal Asset Pricing with Heterogeneous Beliefs

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Literature and Motivation

- Classical economic and finance theory: identical beliefs and free information
  - CAPM: Sharpe (1964), Lintner (1965), Mossin (1966)
  - Pricing of contingent claims: Black and Scholes (1973), Merton (1977)
- Two new directions:
This paper

- **Set-up**
  - Complete market
  - Individual: optimal problem of maximizing the expected utility of his terminal wealth
  - Heterogeneity: common information interpreted in different ways

- **Results**
  - Consensus investor: difference and connection between heterogeneous and homogeneous agent cases
  - Impact of heterogeneous beliefs on the equilibrium
  - Survivance conditions of different investors
Set-up: Trading Assets

- A continuous-time complete market with risk-neutral probability space \((\Omega, \mathcal{F}, \tilde{Q})\).
- One riskless asset with net zero supply, whose price process is
  \[
  dS_0(t) = r(t)S_0(t)dt, \quad S_0(0) = 1. \tag{1}
  \]
- One risky asset with price process under the risk-neutral measure \(\tilde{Q}\)
  \[
  dS(t) = S(t) \left[ r(t)dt + \sigma(t)d\tilde{W}(t) \right], \quad S(0) > 0, \tag{2}
  \]
  where \(\sigma > 0\).
Set-up: Individual Portfolio

- Suppose there are $H$ agents indexed by $h = 1, \ldots, H$ and the total wealth of agent $h$ is $X^h(t)$ at time $t$.
- Agent $h$ holds $N^h(t)$ shares of the risky asset at time $t$ and the value invested in the risky asset by agent $h$ is
  \[ \pi^h(t) = N^h(t)S(t). \]  
  (3)
- Transaction costs and consumption are not considered.
- The value invested in the riskless asset by agent $h$ is
  \[ X^h(t) - \pi^h(t). \]  
  (4)
- The evolution of the total wealth can be described as
  \[
  \begin{cases}
  \quad dX^h(t) = r(t)X^h(t)dt + \pi^h(t)\sigma(t)d\tilde{W}(t), \\
  \quad X^h(0) = x^h_0 > 0,
  \end{cases}
  \]  
  (5)

where $\pi^h(t) = N^h(t)S(t)$ is called a portfolio of agent $h$. 
Set-up: Individual Optimal Problem

- Each agent $h$ has
  - a subjective belief $Q^h$
  - a von Neumann-Morgenstern utility function for terminal wealth of the form

  $$E^h(U^h(X^h_T)),$$

  where $E^h$ is the expectation based on the subjective belief $Q^h$ of agent $h$, and

- an optimal problem

  $$\max \ E^h\left(U^h(X^h(T))\right)$$

  s.t.

  $$dX^h(t) = r(t)X^h(t)dt + \pi^h(t)\sigma(t)d\tilde{W}(t),$$

  $$X^h(0) = x^h_0 > 0.$$
Individual Optimal Problem

- $Q^h$ is equivalent to $\tilde{Q}$, that is, there is an Radon-Nikodym derivative $Z^h_T (\geq 0)$ of $Q^h$ with respect to $\tilde{Q}$, that is

$$dQ^h = Z^h_T d\tilde{Q} \quad \text{on} \quad \mathcal{F}_T. \quad (8)$$

- $(7)$ is equivalent to

$$(OP^h) : \quad \begin{align*}
\max & \quad \tilde{E} \left( Z^h_T U^h(X^h(T)) \right) \\
\text{s.t.} & \quad dX^h(t) = r(t)X^h(t)dt + \pi^h(t)\sigma(t)d\tilde{W}(t), \\
& \quad X^h(0) = x_0^h > 0,
\end{align*}$$
Questions:

- What is the price? What are the difference and connection between heterogeneous and homogeneous agent cases?
- Who can survive in the market?
Market clearing condition: net zero supply of the riskless asset

\[ \pi^*(t) := \sum_{h=1}^{H} \hat{\pi}^h(t) = \sum_{h=1}^{H} \hat{X}^h(t) =: X^*(t), \quad t \in [0, T]. \quad (9) \]

\( \{r_t\}_{t \in [0, T]} \) is called an equilibrium riskless rate process relative to heterogeneous beliefs \( \{Z^h_T\}_{h=1,...,H} \).
Question-1: Difference and Connection

Aggregate Individual Problem
Question-1: Difference and Connection

Aggregate Individual Problem

Consensus Investor Problem
Question-1: Difference and Connection

Aggregate Individual Problem

⇑️ ?

Consensus Investor Problem
**Individual Problem**

\[
\max_{\pi^h} \tilde{E}(Z_T^h U^h(X_T^h))
\]

s.t. \( dX_t^h = r_t X_t^h dt + \pi_t^h \sigma_t d\tilde{W}_t \)

\( X_0^h = x_0^h \)

**Market Clearing Condition**

\[
\sum_{h=1}^H \hat{\pi}^h = \sum_{h=1}^H \hat{X}^h
\]

**Optimal Solution:** \( (r_t, (\hat{\pi}^h)_{h=1}^H) \)
Individual Problem

\[
\max_{\pi^h} \tilde{E}(Z_T U^h(X_T^h))
\]

\[s.t. \quad dX_t^h = r_t X_t^h dt + \pi_t^h \sigma_t d\tilde{W}_t\]

\[X_0^h = x_0^h\]

Market Clearing Condition

\[\sum_{h=1}^{H} \hat{\pi}_h = \sum_{h=1}^{H} \hat{X}_h\]

Optimal Solution: \(\left(r_t, (\hat{\pi}_h^h)_{h=1}^{H}\right)\)
Individual Problem

\[
\max_{\pi^h} \tilde{E}(Z_T^h U^h(X_T^h)) \\
\text{s.t. } dX_t^h = r_t X_t^h dt + \pi^h_t \sigma_t d\tilde{W}_t \\
X_0^h = x_0^h
\]

Consensus Investor Problem

\[
\max_{\pi^c} \tilde{E}(Z_T^c U^c(X_T^c)) \\
\text{s.t. } dX_t^c = r_t X_t^c dt + \pi_t^c \sigma_t d\tilde{W}_t \\
X_0^c = \sum_{h=1}^H x_0^h
\]

Consensus Characteristic

\[
Z_T^c
\]

Aggregate Utility

\[
U^c(X)
\]

Market Clearing Condition

\[
\sum_{h=1}^H \hat{\pi}^h = \sum_{h=1}^H \hat{X}^h
\]

Optimal Solution: \( (r_t, (\hat{\pi}^h)_{h=1}^H) \)

\[
\hat{\pi}^c = \hat{X}^c
\]

Optimal Solution: \( (r_t, (\hat{\pi}^c) ) \)
Individual Problem

\[
\max_{\pi^h} \tilde{E}(Z_T^h U^h(X_T^h))
\]

s.t. \( dX_t^h = r_t X_t^h dt + \pi_t^h \sigma_t d\tilde{W}_t \)

\( X_0^h = x_0^h \)

Consensus Investor Problem

\[
\max_{\pi^c} \tilde{E}(Z_T^c U^c(X_T^c))
\]

s.t. \( dX_t^c = r_t X_t^c dt + \pi_t^c \sigma_t d\tilde{W}_t \)

\( X_0^c = \sum_{h=1}^H x_0^h \)

Consensus characteristic: \( Z_T \)

Aggregate utility: \( U^c(X) \)

Optimal Solution: \( \left( r_t, (\hat{\pi}^h)_{h=1}^H \right) \)

Optimal Solution: \( \left( r_t, (\hat{\pi}^c) \right) \)

\[\sum_{h=1}^H \hat{\pi}^h = \sum_{h=1}^H \hat{X}^h \]

Same equilibrium riskless rate: \( r_t \)

Same marginal utility:

\[ Z_T^h U_x^h(\hat{X}_T^h) = Z_T^c U_x^c(\hat{X}_T^c) \]

Same total wealth and portfolio:

\[ \sum_{h=1}^H \hat{X}^h = \hat{X}^c = X^*, \quad \sum_{h=1}^H \hat{\pi}^h = \hat{\pi}^c = \pi^* \]
Individual Problem

\[
\max_{\pi^h} \tilde{E}(Z_T^h U^h(X_T^h))
\]

s.t.
\[
\begin{align*}
\frac{dX_t^h}{dt} &= r_t X_t^h \, dt + \pi_t^h \sigma_t \, d\tilde{W}_t \\
X_0^h &= x_0^h
\end{align*}
\]

Consensus Investor Problem

\[
\max_{\pi^c} \tilde{E}(Z_T^c U^c(X_T^c))
\]

s.t.
\[
\begin{align*}
\frac{dX_t^c}{dt} &= r_t X_t^c \, dt + \pi_t^c \sigma_t \, d\tilde{W}_t \\
X_0^c &= \sum_{h=1}^H x_0^h
\end{align*}
\]

Markets Clearing Condition

\[
\sum_{h=1}^H \hat{\pi}^h = \sum_{h=1}^H \hat{X}^h
\]

Optimal Solution: \( (r_t, (\hat{\pi}^h)_{h=1}^H) \)

\[
\frac{\partial}{\partial \hat{X}_T^h} Z_T^h U^h(X_T^h) = \frac{\partial}{\partial \hat{X}_T^c} Z_T^c U^c(X_T^c)
\]

Optimal Solution: \( (r_t, (\hat{\pi}^c) ) \)

\[
\sum_{h=1}^H \hat{X}^h = \hat{X}^c = X^*, \quad \sum_{h=1}^H \hat{\pi}^h = \hat{\pi}^c = \pi^*
\]

Same equilibrium riskless rate: \( r_t \)

Same marginal utility: \( Z_T^h U^h(\hat{X}_T^h) = Z_T^c U^c(\hat{X}_T^c) \)

Same total wealth and portfolio: \( \sum_{h=1}^H \hat{X}^h = \hat{X}^c = X^*, \quad \sum_{h=1}^H \hat{\pi}^h = \hat{\pi}^c = \pi^* \)
Examples

Example 1. If the individual utility functions are of exponential type, more precisely if $U^h_x(X) = a^h \exp(-X/\gamma^h)$, then $U^c_x(X) = a^* \exp(-X/\gamma^*)$ for $\gamma^* = \sum_{h=1}^H \gamma^h$, $a^* = \prod_{h=1}^H (a^h/\lambda^h)^{\gamma^h/\gamma^*}$ and

$$Z^c_T = \prod_{h=1}^H (Z^h_T)^{\gamma^h/\gamma^*}.$$ 

Example 2. If the individual utility functions are of power type, more precisely if $U^h_x(X) = a^h(\gamma^h + \eta X)^{-1/\eta}$ for $\eta \neq 0$, then $U^c_x(X) = a^*(\gamma^* + \eta X)^{-1/\eta}$ for $\gamma^* = \sum_{h=1}^H \gamma^h$, $a^* = \left(\sum_{h=1}^H (a^h/\lambda^h)^{\eta}\right)^{1/\eta}$ and

$$Z^c_T = \left(\sum_{h=1}^H \tau^h_1 (Z^h_T)^{-\eta}\right)^{-1/\eta},$$

$$= \left(\sum_{h=1}^H \tau^h_2 (Z^h_T)^{\eta}\right)^{1/\eta},$$

for $\tau^h_1 = \frac{\gamma^h + \eta \bar{X}^h_T}{\gamma^* + \eta \bar{X}^*}$ and $\tau^h_2 = \frac{(a^h/\lambda^h)^{\eta}}{(a^*)^{\eta}}.$
Within the Framework of Logarithmic Utility Functions

Assumptions:

- Each individual agent $h$ knows the variance ($\sigma(t)$) of the risky asset exactly.
- Agent $h$ estimates its return $\mu^h(t)$ with error $e^h(t)$, that is $\mu^h(t) = r(t) - e^h(t)$.
- Define $\varepsilon^h(t) = \sigma^{-1}(t)e^h(t)$.
- For each agent $h$

$$
\mu^h(t) - \sigma^2(t) > 0, \quad (10)
$$
$$
\tilde{E} \left[ \exp \left( \frac{1}{2} \int_0^T (\varepsilon^h(t))^2 dt \right) \right] < \infty, \quad (11)
$$
$$
\tilde{E} \left[ \exp \left( \frac{1}{2} \int_0^T (\varepsilon^h(t) + \sigma(t))^2 dt \right) \right] < \infty. \quad (12)
$$
Within the Framework of Logarithmic Utility Functions

- Individual belief of agent $h$ can be described as $dQ^h = Z_T^h d\tilde{Q}$, where $Z_T^h$ is the positive density process of $Q^h$ with respect to $\tilde{Q}$ and satisfies

$$dZ^h(t) = -Z^h(t)\varepsilon^h(t) d\tilde{W}(t), \quad (13)$$

where

$$\varepsilon^h(t) = \sigma^{-1}(t)(r(t) - \mu^h(t)).$$
Within the Framework of Logarithmic Utility Functions

- Individual belief of agent $h$ can be described as $dQ^h = Z^h_T d\tilde{Q}$, where $Z^h_T$ is the positive density process of $Q^h$ with respect to $\tilde{Q}$ and satisfies

  \[ dZ^h(t) = -Z^h(t)\varepsilon^h(t)d\tilde{W}(t), \]  

  \[ (13) \]

  where
  \[
  \varepsilon^h(t) = \sigma^{-1}(t)(r(t) - \mu^h(t)).
  \]

- The optimal problem of agent $h$ can be described by

  \[
  \begin{align*}
  \max & \quad \tilde{E} \left( Z^h_T \log (X^h_T) \right) \\
  \text{s.t.} & \quad dX^h(t) = r(t)X^h(t)dt + \pi^h(t)\sigma(t)d\tilde{W}(t) \\
  & \quad X^h(0) = x^h_0.
  \end{align*}
  \]

\[ (14) \]
Consensus Investor

Proposition

The utility function of the consensus investor is

\[ U^c(x) = x^*_0 \log(x), \quad \text{where} \quad x^*_0 = \sum_{h=1}^{H} x_0^h, \quad (15) \]

and the consensus belief

\[ Z_T^c = \exp \left( \int_0^T -\frac{1}{2} (\varepsilon^c(t))^2 \, dt - \varepsilon^c(t) d\tilde{W} \right), \quad (16) \]

with

\[ \varepsilon^c(t) = \sum_{h=1}^{H} \frac{\hat{X}^h(t)}{X^*(t)} \varepsilon^h(t). \quad (17) \]
Equilibrium

Proposition

Under the equilibrium condition \( \pi^*(t) = X^*(t) \), the equilibrium riskless rate is

\[
r(t) = \mu^c(t) - \sigma^2(t)
\]

where

\[
\mu^c(t) = \sum_{h=1}^{H} \frac{\hat{X}^h(t)}{X^*(t)} \mu^h(t).
\]
Equilibrium

Proposition

Under the equilibrium condition $\pi^*(t) = X^*(t)$, the equilibrium riskless rate is

$$r(t) = \mu^c(t) - \sigma^2(t)$$  \hspace{1cm} (18)

where

$$\mu^c(t) = \sum_{h=1}^{H} \frac{\hat{X}^h(t)}{X^*(t)} \mu^h(t).$$  \hspace{1cm} (19)

Remarks:

- In the single agent economy with the belief of agent $h$, the equilibrium riskless rate is

  $$r^h(t) = \mu^h(t) - \sigma^2(t).$$  \hspace{1cm} (20)

- In heterogeneous case, the equilibrium riskless rate (18) can be rewritten as

  $$r(t) = \sum_{h=1}^{H} \frac{\hat{X}^h(t)}{X^*(t)} r^h(t).$$  \hspace{1cm} (21)
Example: three agents

- Same initial wealth: \( x_0^h = 0.1 \) for \( h = 1, 2, 3 \)
- Different constant beliefs: \( \mu^1 \equiv 0.05 < \mu^2 \equiv 0.08 < \mu^3 \equiv 0.10 \)
- Volatility: \( \sigma = 0.1 \)

\[
\begin{align*}
    r^1 &= \mu^1 - \sigma^2 \\
    r^2 &= \mu^2 - \sigma^2 \\
    r^3 &= \mu^3 - \sigma^2 \\
    r &= \mu^c - \sigma^2
\end{align*}
\]
Asset Pricing

Proposition

Consider a contingent claim with $\mathcal{F}_t$-measurable payoff $P_T$ at maturity date $T$. Then the asset price is

$$P_t = \tilde{E} \left[ \exp \left( - \int_t^T r(t) dt \right) P_T \mid \mathcal{F}_t \right]. \quad (22)$$

Then (22) can be decomposed as

$$\tilde{E} \left[ \exp \left( - \int_t^T r(t) dt \right) P_T \mid \mathcal{F}_t \right] = \sum_{h=1}^H \frac{\hat{X}^h(t)}{X^*(t)} \tilde{E}^h \left[ \exp \left( - \int_t^T r^h(t) \right) P_T \mid \mathcal{F}_t \right]. \quad (23)$$

Here $\tilde{E}^h (h = 1, \ldots, H)$ represents the expectation relative to the equivalent martingale measure $d\tilde{Q}^h = M_t^h \ d\tilde{Q}$ where the density of the associated martingale measure satisfies

$$M_t^h = \exp \left( - \int_0^t \frac{1}{2} (\theta^h(s))^2 ds - \int_0^t \theta^h(s) d\tilde{W} \right),$$

with $\theta^h = \sigma^{-1}(r - r^h)$. 
Definition

Agent $h$ ($h = 1, 2$) is said to experience relative extinction in the long run if

$$\lim_{T \to \infty} \frac{\hat{X}_T^h}{\hat{X}_T^{h'}} = 0 \quad \text{a.s.} \quad \text{where } h, h' = 1, 2 \text{ and } h \neq h'.$$ (24)

An agent is said to survive relatively in the long run if relative extinction does not occur.
Proposition

If $\mu^1$, $\mu^2$ and $\sigma$ are constant, $\mu^1 < \mu^2$ and letting $u = \frac{2\sigma^2}{\mu^2 - \mu^1}$, then only one of the agents survives in the long run. In particular,

1. when $u > 1$, then agent 1 survives;
2. when $u < 1$, assuming the initial wealth of agent $h$ ($h = 1, 2$) is $x^h_0$ and letting

$$p = p \left( \frac{x^1_0}{x^2_0}, u \right) = \frac{F \left( \frac{x^1_0}{x^2_0}, u \right)}{F(\infty, u)},$$

then

(i) agent 1 can survives with probability $p \left( \frac{x^1_0}{x^2_0}, u \right)$;
(ii) agent 2 can survives with probability $1 - p \left( \frac{x^1_0}{x^2_0}, u \right)$. 
Question-2: Survivance

\[ F(y, u) = -\frac{y^{-u}}{1 + y} - uy^{-u} \text{LerchPhi}(-y, 1, -u), \quad (26) \]

and \( \lim_{y \to \infty} F(y, u) = F(\infty, u) \) exists for \( u \in (0, 1) \).

\[ p = p \left( \frac{x_0^1}{x_0^2}, u \right) = \frac{F \left( \frac{x_0^1}{x_0^2}, u \right)}{F(\infty, u)} \quad (27) \]
Survivance

\[
F(y, u) = -\frac{y}{1+y} - y\cdot\text{LerchPhi}(-y, 1, -u),
\]
and

\[
\lim_{y \to \infty} F(y, u) = F(\infty, u) \text{ exists for } u \in (0, 1).
\]

Some remarks:

- When \( \sigma = 0 \), the more initial wealth dominance of agent \( h \), the higher probability for agent \( h \) to survive.

Figure: \( \sigma = 0 \)
Some remarks:

- The more volatile, the more survival space for the pessimistic agent.

Figure: $\frac{x_0^1}{x_0^2} = 1$
Simulation - Case 1: low volatile

<table>
<thead>
<tr>
<th>belief ($\mu'$)</th>
<th>initial wealth ($x_0'$)</th>
<th>Survival Prob</th>
</tr>
</thead>
<tbody>
<tr>
<td>Agent 1</td>
<td>0.05</td>
<td>0.1</td>
</tr>
<tr>
<td>Agent 2</td>
<td>0.10</td>
<td>0.2</td>
</tr>
<tr>
<td>Volatility ($\sigma$)</td>
<td>0.1</td>
<td></td>
</tr>
</tbody>
</table>

\[ u = \frac{2\sigma^2}{(\mu^2 - \mu^1)} \]

\[ 0.4 (< 1) \]

(a) Time series

(b) Histogram of survivance probability
Simulation - Case 2: high volatile

<table>
<thead>
<tr>
<th>Agent 1</th>
<th>belief ($\mu'$)</th>
<th>initial wealth ($x^i_0$)</th>
<th>Survival Prob</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>0.1</td>
<td>100%</td>
<td></td>
</tr>
</tbody>
</table>

| Agent 2 | 0.10 | 0.2 | 0 |

Volatility ($\sigma$): 0.2

$$u = \frac{2\sigma^2}{(\mu^2 - \mu^1)}$$

1.6 ($> 1$)

(c) Time series

(d) Histogram of survival probability
Conclusion and Future Work

- Existence of consensus investor.
- Aggregate utility functions and consensus characteristics for HARA utility function.
- Impact of heterogeneous beliefs and wealth on price.
- Survival of heterogeneous belief agents.
- Future work:
  - Learning process in heterogeneous beliefs.
  - Consensus beliefs and asset pricing for general utility functions.
  - Switching processing between different belief agents.