A Binomial Model of Asset and Option Pricing with Heterogeneous Beliefs

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Outline

Basic Idea and Motivation

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Recent literatures on behavior finance (Thaler (2002), Shefrin (2005)), Heterogeneous agent models (Hommes (2006), LeBaron (2006) and Chiarella, Dieci and He (2008) and asset pricing under heterogeneous beliefs (Abel (1989, 2002), Jouni and Napp (2006,2007), David (2008)) have shown that investors’ behavioral biases and heterogeneous beliefs have significant impact on equilibrium asset prices.

On the other hand, evolutionary finance (Kogan etal (2008) and Blume and Easly (2006)) literature found that under the assumption of log-utility in a complete market, heterogeneity in investors’ beliefs cannot persist, “irrational” investors will be driven out of the market after a long horizon.
Using a rather simple binomial model, this paper examines the impact of heterogeneity on asset and option prices.

Investors have heterogeneous beliefs regarding the probability and future returns in each state of the economy.

We construct a consensus belief from the subjective beliefs to characterize market equilibrium.

We show that if the consensus belief prevails as the “objective” belief, then investors’ wealth share process are all martingale. Also heterogeneity can persist and have interesting effect on investors’ wealth share processes, asset prices, equilibrium risk-free and option prices.
(H1) Assume \( U(x) = \ln(x) \).

(H2) Assume that stock price follow a multi-period Cox-Ross-Rubinstein model. This means, given information at time \( t \), the cum-price of the risky asset at time \( t + 1 \) has the following probability distribution,

\[
S(t + 1) = \begin{cases} 
S(t) \ u(t, t + 1), & p(t, t + 1); \\
S(t) \ d(t, t + 1), & 1 - p(t, t + 1). 
\end{cases}
\]

with \( d(t, t + 1) < R_f(t) < u(t, t + 1) \) where \( R_f(t) = 1 + r_f(t) \) is the return of the riskless asset over the period \([t, t + 1]\).

(H3) Assume that \( I \) investors have their own subjective beliefs of the probability and rate of return at each state. Let \( p_i(t) = \mathbb{E}_t^i(p(t, t + 1)), u_i(t) = \mathbb{E}_t^i(u(t, t + 1)) \) and \( d_i(t) = \mathbb{E}_t^i(d(t, t + 1)) \) denote investor \( i \)'s belief of \( p(t, t + 1), u(t, t + 1) \) and \( d(t, t + 1) \), respectively. \( \mathcal{B}_i(t) : = (p_i(t), u_i(t), d_i(t)) \) denotes investor \( i \)'s set of subjective belief of \( S(t + 1) \) at time \( t \).
Portfolio Strategies

Lemma
Let $\bar{u}_i = u(t) - R_f(t)$ and $\bar{d}_i = d(t) - R_f(t)$ be the excess rate of return in the up and down states, respectively over the period $[t, t + 1]$ under investor $i$’s perspective. The optimal proportion of wealth of investor $i$ invested in the risky asset over the period $[t, t + 1]$ is given by

$$\hat{\omega}_i = R_f \frac{\bar{u}_i \ p_i + \bar{d}_i \ (1 - p_i)}{-\bar{u}_i \ \bar{d}_i}$$
Consensus belief and Market equilibrium

Definition

A belief $\mathcal{B}_m(t) = (p_m(t), u_m(t), d_m(t))$, defined by the probability and return of the risky asset associated with the up and down state respectively, is called a consensus belief at time $t$ if the market equilibrium price for the risky asset and risk free rate under the heterogeneous beliefs are also those under the homogeneous belief $\mathcal{B}_m(t)$. 
Proposition

(i) Let $w_i(t) = \frac{W_i(t)}{W_m(t)}$ denote investor $i$ share of the market at time $t$. The consensus belief at time $t$, $\mathcal{B}_m(t)$, is given by

$$p_m = \sum_i w_i p_i, \quad u_m = \bar{u}_m + R_f, \quad d_m = \bar{d}_m + R_f$$

where

$$\bar{u}_m = \left( \sum_{i=1}^I w_i \frac{1-p_i}{1-p_m} \bar{u}_i^{-1} \right)^{-1},$$

$$\bar{d}_m = \left( \sum_{i=1}^I w_i \frac{p_i}{p_m} \bar{d}_i^{-1} \right)^{-1}$$
(ii) The equilibrium risk free rate is given by
\[ \frac{1}{R_f} = \frac{1 - p_m}{d_m} + \frac{p_m}{u_m} = \mathbb{E}_t^m \left[ \frac{1}{R(t + 1)} \right], \]
where \( R(t + 1) = S(t + 1)/S(t) \).

(iii) The individual and market consensus belief of the state prices or the risk neutral probabilities of the up and down state at time \( t \) are given by
\[
q_{i,u}(t) = \frac{-\bar{d}_i}{\bar{u}_i - \bar{d}_i}, \quad q_{i,d}(t) = \frac{\bar{u}_i}{\bar{u}_i - \bar{d}_i}, \quad i = 1, 2, \ldots, l, m.
\]
(iv) In equilibrium, the stock price at time $t$ is given by

$$S(t) = \frac{\mathbb{E}_t^{Q_i}(S(t+1))}{R_f} = \frac{\mathbb{E}_t^i(ZS(t+1))}{R_f}, \quad i = 1, 2, \cdots, I, m$$

where

$$Z = \begin{cases} \frac{q_u(t)}{p}, & p; \\ \frac{q_d(t)}{1-p}, & 1-p \end{cases}$$

is the Randon-Nikodym derivative that change the probability measure from $i$ to $Q_i$, and often referred to as the “pricing kernel” in the asset pricing literatures.
Example 1

Consider a benchmark belief $B_o(t) = (p_o(t), u_o(t), d_o(t))$, and assume that investors’ subjective beliefs diverge from the benchmark belief. We do this by imposing a Mean Preserved Spread (MPS) on investors’ beliefs. Let investor $i$’s belief be given by $B_i(t) = (u_o(t) + \epsilon_{iu}, d_o(t) + \epsilon_{id})$ where $\epsilon_{iu} \sim \text{Unif}(-\theta_u, \theta_u)$, similarly $\epsilon_{id} \sim \text{Unif}(-\theta_d, \theta_d)$. Therefore investors’ divergence of opinion regarding the stock returns in both up and down states are i.i.d. We let number of investors $I = 10$ and run 5000 simulations for each pair of $(\theta_u, \theta_d)$. Furthermore, we assume investors agree on the probability, hence $p_i = p_o$ for all $i$. 
Impact of heterogeneity on consensus belief

\[ \theta_u \in [0.01, 0.2], \quad \theta_d = 0 \]

\[ \theta_u = 0, \quad \theta_d \in [0.01, 0.1] \]

\[ \theta_u \in [0.01, 0.2], \quad \theta_d = 0.1 \]
Equilibrium risk-free rate, $\mathbb{E}(r_f)$ (black), $\sigma(r_f)$ (red)

$\theta_u \in [0.01, 0.2], \theta_d = 0$

$\theta_u = 0, \theta_d \in [0.01, 0.1]$

$\theta_u \in [0.01, 0.2], \theta_d = 0.1$
Assume investors’ wealth are evenly distributed, then the market as a consensus investor believes that divergence of opinion regarding future asset return in the upstate (downstate) is negatively (positively) related to expected future stock return and the equilibrium risk-free rate. Higher divergence of opinions leads to higher volatility for both the expected future asset return and the equilibrium risk-free rate.
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\((\theta_u, \theta_d) = (0.01, 0.1)\) \hspace{1cm} \((\theta_u, \theta_d) = (0.1, 0.1)\) \hspace{1cm} \((\theta_u, \theta_d) = (0.2, 0.1)\)

**Figure:** Impact of divergence of opinion on the distribution of expected future stock return under the market consensus belief.
Dynamic Impact of Heterogeneous beliefs

**Proposition**

*Under the market consensus belief $B_m(p_m(t), u_m(t), d_m(t))$, the wealth share of every investor $i$ who survives up to time $t$, that is $w_i(t) > 0$,*

$$\mathbb{E}_t^m \left[ \frac{w_i(t + 1)}{w_i(t)} \right] = 1$$
Proposition

Under investor $i$’s subjective belief $B_i(p_i(t), u_i(t), d_i(t))$, investor $i$ survives at time $t + 1$ and

$$
\mathbb{E}_i^* \left[ \frac{w_i(t + 1)}{w_i(t)} \right] \geq 1.
$$

Equality will hold if and only if $B_i(t) = B_m(t)$. 
A two-parameter model

We assume that each investor $i$ believes that the asset price dynamics will conform to the Black-Sholes model in continuous-time, however they have different beliefs about the drift ($\mu_i$) and volatility parameter ($\sigma_i$). Motivation is to allow us to price options and bonds and compare with benchmark results. We assume that the beliefs $\mathcal{B}_i(\mu_i, \sigma_i)$ are formed in the follow way,

$$u_i = \exp(\sigma_i \sqrt{\Delta t}), \quad d_i = \frac{1}{u_i}, \quad \text{and} \quad p_i = \frac{\exp(\mu_i \Delta t) - d_i}{u_i - d_i}$$

where $\Delta t$ is the time increment between any two successive periods. It is well known that this discrete-time model approaches the following continuous-time model when $\Delta t \to 0$,

$$\frac{dS(t)}{S(t)} = \mu_i \ dt + \sigma_i \ dW(t)$$
Consider 2 investors who have the same belief about the instantaneous expected return for the risky asset, i.e. $\mu_1 = \mu_2 = \mu_0$. However, they have different levels of confidence, let $\sigma_1 = \sigma_o(1 - \delta)$ and $\sigma_2 = \sigma_o(1 + \delta)$, where $\delta > 0$. Obviously, investor 1 is more confident about his/her estimate of instantaneous expected return than investor 2. We let $T = 1$, $\Delta t = 1/250$, $\mu_0 = 0.07$, $\sigma_o = 0.1654$, $\delta = 0.5$, $w_1(0) = w_2(0) = 0.5$ and investigate the distribution and first 4 moments of wealth share of investor 1 $w_1(T)$, log asset prices $\ln(S(T))$ and risk-free rate $r_f(T)$. 
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<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Std Dev</th>
<th>Skewness</th>
<th>Kurtosis</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_1(T)$ ($\delta = 0$)</td>
<td>0.4914</td>
<td>0.0506</td>
<td>-0.3790</td>
<td>3.1612</td>
</tr>
<tr>
<td>ln($S(T)$) ($\delta = 0$)</td>
<td>0.0437</td>
<td>0.1244</td>
<td>-0.1558</td>
<td>2.9639</td>
</tr>
<tr>
<td>$r_f(T)$ ($\delta = 0$)</td>
<td>0.0575</td>
<td>0.0011</td>
<td>-0.9621</td>
<td>4.6820</td>
</tr>
</tbody>
</table>

$w_1(T)$  
ln($S(T)$)  
r_f(T)
Consider any self-financing portfolio which invests $\omega(t)$ in the risky asset and $(1 - \omega(t))$ in the risk-free asset at time $t$, then the expected value (under the consensus belief of the portfolio at time $t + 1$, $H(t + 1)$) when benchmarked again the risky asset is

\[
\mathbb{E}_t^m \left( \frac{H(t+1)}{S(t+1)} \right) = \mathbb{E}_t^m \left( \frac{H(t)\omega(t)R(t+1) + H(t)(1 - \omega(t))R_f(t+1)}{S(t+1)} \right)
\]

\[
= \omega(t) + (1 - \omega(t))\mathbb{E}_t^m \left( \frac{H(t)R_f(t+1)}{S(t)R(t+1)} \right) = \frac{H(t)}{S(t)}
\]

that is the value of any contingent claim with a payoff function $V(.,.)$ is given by

\[
V(S(t), t) = S(t) \mathbb{E}_t^m \left( \frac{V(S(T), T)}{S(T)} \right).
\]
Zero-coupon bond prices and Call option prices

Figure: Impact of heterogeneity on bond and option prices. Dark line represent the bond and call option prices under the consensus belief when investors have different level of confidence as in Example 2. Light line represent the bond and call option under the homogeneous benchmark belief which is also the average belief.
Market expects a lower (higher) future return and equilibrium risk-free rate when investor’s opinions diverge more regarding future stock return for the upstate than for the downstate.

Dynamically, under the market consensus beliefs, every investor’s wealth share process is a martingale. Therefore, in the long-run each investor’s wealth share is expected to remain the same, hence heterogeneity in the market will be persistent.
In a modified version of the model, we characterize investors by their beliefs of the instantaneous expected return and volatility of the risky asset. When investors have common beliefs regarding the expected return, the distribution of future wealth is negatively skewed for the more confident investor. The distribution for future risk-free rate is also heavily skewed to the left.

Heterogeneity causes the zero-coupon bond and call option prices to deviate from the homogeneous benchmark values.