Density Forecasts of Crude-Oil Prices Using Option-Implied and ARCH-Type Models

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Overview

Crude Oil Density Forecasts:

- ▶ Forecasts of the crude oil price probability distribution
- ▶ Necessary input to estimate risk measures such as VaR
- ▶ The standard approach is to compute them using time-series models that rely on past returns (for instance GARCH)
- ▶ Affected by factors that are difficult to incorporate in a time-series model

In This Paper:

- ▶ We compare GARCH-based forecasts with those obtained by option prices and calibration functions
- ▶ Our empirical results favor an option-based, non-parametric calibration, density forecast method

Why options?

Option Prices:

- ▶ Are determined by aggregate market expectations about the risk-neutral dynamics of the underlying asset until the maturity of the contract
- ▶ Given a set of observed option prices, option-implied expectations can be extracted

Informational Efficiency of Option-Implied Forecasts:

- ▶ Option-implied volatility has enjoyed great success in terms of forecasting (Poon and Granger, 2003)
- ▶ Option-implied betas³ and option-implied correlations⁴ are also found useful in terms of forecasting

³See Christoffersen et al (2008).

⁴See Campa and Chang (1998), Walter and Lopez (2000), Skintzi and Refenes (2005) and Engle and Kelly (2009).

Risk-Neutral (Forecast) Densities

- ▶ The risk-neutral density (RND, henceforth) can be inferred using a set of observed European option prices
- ▶ Central banks routinely monitor RNDs in order to gauge market sentiment

Advantages:

- ▶ Have forward-looking features by construction
- ▶ Contain information that is absent from the record of historical returns

Disadvantages:

- ▶ Risk-neutral densities refer to an artificial world where investors do not require compensations for bearing risks
- ▶ Appropriate adjustments are required to correct these forecast densities and define "real-world" specifications

Real-World Forecast Densities

Previous Research:

- ▶ Anagnou-Basioudis et al (2005) find that they perform similar to GARCH-based forecasts
- ▶ Their risk-adjustment methods are however problematic and their results *in-sample*
- ▶ Liu et al (2007) prefer "real-world" densities over GARCH-based specifications but their results are also *in-sample*
- ▶ Shackleton et al (2007) provide the first *out-of-sample* evidence that "real-world" densities perform better for forecast horizons longer than one week

Contributions (1)

Crude Oil Forecast Densities:

- ▶ The current literature has only focused on models that rely solely on past returns
- ▶ Particularly, fat-tailed GARCH specifications are suggested in Giot and Laurent (2003), Hung, Lee and Liu (2008) and Asem, Costello and Gardner (2008) for constructing VaR forecasts

We also construct forecast densities and VaR related quantities but use option prices instead of only GARCH-based estimates

Contributions (2)

Accuracy of Option-Implied Forecasts:

- ▶ The only genuine evidence regarding the out-of-sample forecast performance of option-implied densities are provided by Shackleton et al (2007) for the case of the S&P500 index
- ▶ The superior performance of option-implied densities may not hold for other, less liquid, assets

Towards this end, we add further empirical evidence in the related literature

Contributions (3)

Piece-wise Evaluation of "Real-World" Densities:

- ▶ End users usually have interests only in certain segments of the target density
- ▶ A risk-adjustment method may improve the overall performance of the risk-neutral forecasts but still fail to correct forecasts for the region of interest
- ▶ Thus, it may be inappropriate to generally recommend an option-based approach

For this reason we pay special focus on testing for potential mis-specification in various density segments

Historical Densities

- ▶ Historical forecast densities, i.e. forecast densities obtained using only past returns, are constructed using a standard GARCH(1,1) specification
- ▶ In particular, mean and conditional variance of the returns are modeled as follows,

$$r_t = \mu + e_t, \quad (1)$$

$$e_t = \sqrt{h_t} z_t, \quad z_t \sim i.i.d. D(0, 1) \quad (2)$$

$$h_t = \omega + \alpha e_{t-1}^2 + \beta h_{t-1}, \quad (3)$$

with $D(0, 1)$ being either the standard normal or the standardized Student-t probability density.

- ▶ The target densities are obtained by simulating the price process for the forecast horizon of interest

Risk-Neutral Densities

- ▶ The risk-neutral density (RND) of a particular asset at some future horizon T can be estimated using a panel of N European call option prices $CALL(K_1), \dots, CALL(K_N)$ with strike prices $K_1, \dots, K_N$ and maturity T
- ▶ We employ the flexible GB2 density to approximate the RND. The corresponding parameter vector is estimated by minimizing sum of squared price errors, i.e.

$$\sum_{i=1}^N (CALL(K_i) - CALL_{GB2}(K_i | \theta_{GB2}))^2, \quad (4)$$

where $CALL_{GB2}(K_i | \theta_{GB2})$ is the theoretical (fitted) price of a call option given by a GB2 density with a parameter vector θ_{GB2}

From Risk-Neutral to Real-World Densities

Two alternative approaches can be used in order to convert risk-neutral densities into "real-world" specifications

Economic Theory:

- ▶ Several studies find that empirical pricing kernels are not consistent with standard utility theory for the S&P500
- ▶ Additional complications arise because our underlying asset is crude oil, so extra assumptions are needed

Statistical Calibrations:

- ▶ A more straightforward approach is to treat RNDs as mis-specified and attempt to properly re-calibrate them
- ▶ Two calibrations functions are employed in this paper: the Beta function as in Fackler and King (1990) and a non-parametric approach as in Diebold et al (1999) and Shackleton et al (2007)

Calibration Functions

- ▶ Let $f_Q(x)$ and $F_Q(x)$ denote the risk-neutral density and cumulative distribution function of the time T price of the underlying asset X_T
- ▶ Following Bunn (1984), David (1984) and Fackler and King (1990) also define the calibration function $C(u)$ as the real-world cumulative distribution of the random variable $U = F_Q(X_T)$. The real-world density, $f_P(x)$, and cumulative distribution function, $F_P(x)$, of X_T are then:

$$F_P(x) = C(F_Q(x)) \quad (5)$$

$$f_P(x) = \frac{dF_P(x)}{dx} = \frac{dC(F_Q(x))}{dx} = \frac{dC}{dF} \frac{dF}{dx} = c(F_Q(x))f_Q(x) \quad (6)$$

RWD Estimation: Parametric Calibration Function (1)

Parametric Specification:

- ▶ Beta distribution as in Fackler and King (1990), Anagnou-Basioudis et al (2005), Liu et al (2007) and Shackleton et al (2007)
- ▶ Flexible, two-parameter, specification with cumulative density, $C(u)$, and probability density function, $c(u)$, defined as:

$$C(u|\alpha, \beta) = \frac{1}{B(\alpha, \beta)} \int_0^u t^{\alpha-1} (1-t)^{\beta-1} dt \quad (7)$$

$$c(u|\alpha, \beta) = \frac{u^{\alpha-1} (1-u)^{\beta-1}}{B(\alpha, \beta)}, \quad B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)} \quad (8)$$

RWD Estimation: Parametric Calibration Function (2)

- ▶ Every time a RND is available, a RWD specification can be generated through eq. (6) if the calibration function is known

Estimation Procedure:

- 1) We collect all past risk-neutral forecast densities $f_{Q,i}$, given by the GB2 parameter vector θ_i , and corresponding realizations of the target quantity X_{i+1}
- 2) We estimate the α and β parameters of the Beta density by maximum likelihood, i.e. by finding the parameters that maximize the following quantity:

$$\log(L(X_1, X_2, \dots, X_t)) = \sum_{i=0}^{t-1} \log(f_{P,i}(X_{i+1} | \theta_i, \alpha, \beta)) \quad (9)$$

RWD Estimation: Non-Parametric Calibration Function

- For the non-parametric estimation of the calibration function we follow Shackleton et al (2007)

Estimation Procedure:

- 1) We collect all past realizations of the variable $u_i = F_{Q,i}(X_{i+1})$
- 2) We define a new series $z_i = \Phi^{-1}(u_i)$ where $\Phi(\cdot)$ is the inverse of the cumulative distribution function of the standard normal density
- 3) We fit a normal kernel in order to obtain the kernel density $h(z)$, and empirical distribution function, $H(z)$, of $z_0, z_1, \dots, z_{t-1}$ The real-world cumulative and probability density function are then:

$$F_P(x) = C(F_Q(x)) \quad \text{and} \quad f_P(x) = \frac{f_Q(x)h(z)}{\phi(z)}, \quad (10)$$

where $\phi(\cdot)$ is the probability density function of the standard normal

Properties of Correctly Specified Forecast Densities

- ▶ Let $f_t(u)$ denote a forecast density produced at time t and X_{t+1} the corresponding realization. Also define the following quantities:

$$PIT_t = \int_{-\infty}^{X_{t+1}} f_t(u) du \text{ and } z_t = \Phi^{-1}(PIT_t) , \quad (11)$$

where $\Phi^{-1}(\cdot)$ is the inverse standard normal cumulative distribution function

- ▶ As discussed in Diebold, Gunter and Tay (1998) and Berkowitz (2001), if $f(u)$ is correctly specified, the PIT_t series should be *i.i.d. Unif(0,1)* and the z_t series should be *i.i.d. N(0,1)*

Goodness-of-fit tests (1)

We first test the independence and standard normality (or uniformity) assumption separately (Diebold et al, 1998)

Standard normality assumption:

- ▶ Kolmogorov-Smirnov test

Independence assumption:

- ▶ Following Diebold et al (1998), we investigate autocorrelation pattern in the first four moments of the z_t series
- ▶ In particular, a test of no autocorrelation in the k^{th} moment is conducted by regressing $(z_t - \bar{z}_t)^k$ on 6 and 12 lags, for $k = 1, 2, 3, 4$.
- ▶ A Lagrange Multiplier test is then conducted using the statistic $(T - l)R^2$, where l is the number of lags

Goodness-of-fit tests (2)

- ▶ A joint test for independence and standard normality in the z_t series is conducted using the LR3 test of Berkowitz (2001), i.e. by estimating the following model,

$$z_t - \mu = \rho(z_{t-1} - \mu) + \epsilon_t, \quad \epsilon \sim N(0, \sigma^2) \quad (12)$$

- ▶ The unknown parameter vector $\theta = [\mu, \sigma^2, \rho]$ can be estimated via maximum likelihood
- ▶ A hypothesis that the z_t series follows a uncorrelated Gaussian process with zero mean and unit variance against a Gaussian AR(1) alternative can be tested using a standard LR test
- ▶ Specifically, the following statistic:
 $LR3 = -2(L(0, 1, 0) - L(\hat{\mu}, \hat{\sigma}^2, \hat{\rho}))$, has a χ^2 distribution with three degrees of freedom under the null

Goodness-of-fit tests (3)

Piecewise and interval density evaluation:

- ▶ Let $R_{i,t}$ denote a forecast density region of interest containing $p_{i,t}$ probability mass, i.e. $Pr(X_{t+1} \in R_{i,t}) = p_{i,t}$
- ▶ Let $HIT_{t+1} = 1$ if $X_{t+1} \in R_{i,t}$ and 0 otherwise
- ▶ Assuming that $p_{i,t}$ is constant, then the HIT_t series should be *i.i.d. Bernoulli* ($p_{i,t}$)
- ▶ We follow Patton (2006) and estimate a logit regression on the HIT_t series with four explanatory variables, i.e. a constant and the number of previous hits in the last 1, 5 and 10 observations
- ▶ A likelihood ratio test is then conducted to check whether the constant is equal to $p_{i,t}$ and that all other variables are insignificant

Likelihood Comparisons

- ▶ The out-of-sample likelihood values can be used to rank competing forecast models (Bao et al, 2007; Shackleton et al, 2007)
- ▶ In particular, given a sequence of n ex-ante density forecasts $f_t(x)$ and n ex-post realizations of the target quantity X_{t+1} the out-of-sample log-likelihood (OLL) is defined as:

$$OLL = \sum_{t=0}^{n-1} \log(f_t(X_{t+1})) \quad (13)$$

- ▶ The correct forecast model has the highest expected log-likelihood
- ▶ The highest statistic indicates the specification that is closer to the true target density using the Kullback-Leibler Information Criterion as a loss function

Data Issues: Futures Data

- ▶ New York Mercantile Exchange (NYMEX)
- ▶ January 1987-December 2006
- ▶ GARCH models are estimated using the 5 years of past data
- ▶ Returns are computed from the nearest expiration contract except when the time until expiration is less than four days, when we switch to the next contract

Data Issues: Options Data

- ▶ NYMEX, April 1990-December 2006
- ▶ American options written on the shortest delivery futures contract
- ▶ Barone-Adesi and Whaley (1987) formula to compute the prices of European options
- ▶ We select contracts with appr. 28 days to expire
- ▶ All risk-neutral densities are non-overlapping
- ▶ Three years of data reserved for initial calibrations, appr. 14 years left for forecasting

Some Remarks

Summary of competing forecast models:

- ▶ GARCH model with either standard normal (GARCH-N) or standardized Student- t innovations (GARCH-T)
- ▶ Risk-neutral density forecasts (RND)
- ▶ Real-world densities, obtained by RND forecasts adjusted using either a parametric (RWD-P) or a non-parametric (RWD-NP) calibration function

Summary of evaluation criteria:

1) G.O.F. Tests:

- ▶ Separate tests of standard normality and independence
- ▶ Joint tests of standard normality and independence
- ▶ Mis-specification tests in particular density segments

2) Likelihood Criteria

- ▶ Out-of-sample likelihood levels

Normality Tests

TABLE 1
Normality Tests

Panel A. Descriptive Statistics				
	Mean	Variance	Skewness	Kurtosis
RND	0.181	0.93	-0.213	2.023
GARCH-N	0.056	0.925	-0.235	2.439
GARCH-T	0.011	0.919	-0.241	2.35
RWD-P	0.06	0.945	-0.219	2.057
RWD-NP	0.059	0.84	-0.172	2.498
Panel B. Kolmogorov-Smirnov Test				
	KS Statistic	P-Value		
RND	0.139	0.003		
GARCH-N	0.097	0.085		
GARCH-T	0.076	0.289		
RWD-P	0.093	0.109		
RWD-NP	0.05	0.791		

- If a method produces well specified forecasts, then
 $z_t \sim N(0,1)$

Independence Tests

TABLE 2
Autocorrelation Tests

	1st Moment		2nd Moment		3rd Moment		4th Moment	
<i>Lags</i>	<i>6</i>	<i>12</i>	<i>6</i>	<i>12</i>	<i>6</i>	<i>12</i>	<i>6</i>	<i>12</i>
RND	0.18	0.25	0.24	0.26	0.51	0.49	0.46	0.68
GARCH-N	0.27	0.3	0.25	0.3	0.74	0.82	0.78	0.87
GARCH-T	0.33	0.35	0.22	0.17	0.83	0.82	0.76	0.61
RWD-P	0.18	0.22	0.18	0.21	0.68	0.42	0.33	0.36
RWD-NP	0.19	0.14	0.11	0.13	0.84	0.26	0.22	0.24

NOTE: P-values of the LM tests corresponding to the null of no autocorrelation in the first four moments of the z_t series.

- Independence tests for the z_t series produced by each of the forecast models

Joint Tests

TABLE 3

Joint Tests of Normality and Independence					
	$\hat{\mu}$	$\hat{\sigma}^2$	$\hat{\rho}$	LR3	P-value
RND	0.19	0.91	-0.001	6.526	0.089
GARCH-N	0.062	0.908	-0.009	1.366	0.714
GARCH-T	0.026	0.897	-0.025	1.122	0.77
RWD-P	0.07	0.921	-0.01	1.336	0.721
RWD-NP	0.068	0.82	-0.03	3.879	0.275

- ▶ Joint tests of normality and independence of the z_t series produced by each of the forecast models
- ▶ If the forecast model are correctly specified then we have that $\mu = 0$, $\sigma^2 = 1$ and $\rho = 0$

TABLE 4
HIT TESTS

Panel A. Equal Density Mass					
Region	RND	GARCH-N	GARCH-T	RWD-P	RWD-NP
$PIT_t \in (0, 0.2)$	0.414	0.479	0.29	0.13	0.163
$PIT_t \in (0.2, 0.4)$	0.69	0.861	0.94	0.541	0.378
$PIT_t \in (0.4, 0.6)$	0.621	0.214	0.095	0.332	0.4
$PIT_t \in (0.6, 0.8)$	0.816	0.63	0.792	0.828	0.509
$PIT_t \in (0.8, 1)$	0.003	0.804	0.736	0.236	0.813
Panel B. Unequal Density Mass					
Region	RND	GARCH-N	GARCH-T	RWD-P	RWD-NP
$PIT_t \in (0, 0.15)$	0.415	0.66	0.78	0.48	0.289
$PIT_t \in (0.15, 0.35)$	0.836	0.837	0.87	0.772	0.99
$PIT_t \in (0.35, 0.65)$	0.027	0.18	0.25	0.14	0.8
$PIT_t \in (0.65, 0.85)$	0.491	0.365	0.095	0.231	0.57
$PIT_t \in (0.85, 1)$	0.1	0.273	0.28	0.431	0.434
Panel C. Tails and Center					
Region	RND	GARCH-N	GARCH-T	RWD-P	RWD-NP
$PIT_t \in (0, 0.25)$	0.724	0.727	0.76	0.289	0.23
$PIT_t \in (0.25, 0.75)$	0.043	0.165	0.242	0.047	0.632
$PIT_t \in (0.75, 1)$	0.001	0.053	0.097	0.014	0.417
$PIT_t \in (0, 0.25) \cup (0.75, 1)$	0.042	0.17	0.135	0.013	0.178

Likelihood Comparisons

TABLE 5
Out-of-Sample Log-Likelihoods

	RND	GARCH-N	GARCH-T	RWD-P	RWD-NP
Period A	-154.4586	-153.2976	-152.0864	-155.1178	-152.7213
Period B	-224.2925	-221.8704	-221.6892	-222.5730	-219.1821
Entire Sample	-378.7511	-375.1680	-373.7756	-377.6908	-371.9035

- The highest statistic indicates the method that produces forecast densities that are closer to the true target density using the KLIC as a loss function

Summary and Conclusions

Forecast Setting:

- ▶ Crude oil price densities, monthly horizon, 1993-2006 inclusive
- ▶ Comparisons between ARCH-type forecast methods and option-implied specifications

Empirical Results:

- ▶ Our preferred density forecast method is the RWD-NP specification
- ▶ Forecast densities based on the RWD-NP model rank favorably in goodness-of-fit tests
- ▶ They also have the highest out-of-sample log-likelihood values, which suggest the RWD-NP forecasts are closer to the true target density
- ▶ On the contrary, the parametric calibration function we employ fails to correct all sources of error

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