Locally risk-minimizing vs. $\Delta$-hedging in stochastic volatility models

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joint work with R. Poulsen (Kopenhagen) and K.R. Schenk-Hoppe (Leeds)
1 Stochastic volatility models

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4 Empirical results
The BS-model has several problems, the worst is that it proves itself to be wrong:
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Market prices of European call options with different strike’s imply different values for $\sigma$, which ought to be constant in BS model.
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More precisely: solve

$$C_{h(S_T)}^{BS}(\sigma) = C_{h(S_T)}^{obs}$$

The solution is called implied volatility $\sigma_{impl}$. 
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The solution is called implied volatility $\sigma_{\text{impl}}$.

$\sigma_{\text{impl}}$ in general depends on the current value of the stock, on time to maturity and on the payoff function $h$. It is in no way constant as the BS-model suggests.
Volatility smile:
Stochastic volatility models resolve some of the problems attached to the Black-Scholes model, and can generate smiles (as well as skews and smirks).

\[
dS(t) = S(t) \left( \mu dt + S(t)\gamma f(V(t)) \left[ \sqrt{1 - \rho^2} dW_1(t) + \rho dW_2(t) \right] \right)
\]

\[
dV(t) = V(t) \left( \beta(V(t)) dt + g(V(t)) dW_2(t) \right)
\]

Stochastic volatility models are not complete: There are contingent claims, which can not be hedged by a self financing investment strategy in stock and money market account.
<table>
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<th>Authors &amp; year</th>
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<td>$f(v) = v,$ $\beta(v) = 0,$ $g(v) = \sigma,$ $\rho = 0, \gamma = 0$</td>
<td>Local variance: Geometric Brownian motion. Options priced by mixing.</td>
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<td>Wiggins 1987</td>
<td>$f(v) = e^{v/2},$ $\beta(v) = \kappa(\theta - v)/v,$ $g(v) = \sigma,$ $\rho = 0, \gamma = 0$</td>
<td>Local volatility: Ornstein-Uhlenbeck in logarithms.</td>
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<td>Stein-Stein 1991</td>
<td>$f(v) =</td>
<td>v</td>
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<td>Heston 1993</td>
<td>$f(v) = \sqrt{v},$ $\beta(v) = \kappa(\theta - v)/v,$ $g(v) = \sigma/\sqrt{v},$ $\rho \in [-1, 1], \gamma = 0$</td>
<td>Local variance: CIR process. First model with correlation. Options priced by inversion of characteristic function.</td>
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<td>Romano-Touzi 1997</td>
<td>$f(v) = \sqrt{v},$ $\beta$ and $g$ are free, $\rho \in [-1, 1], \gamma = 0$</td>
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<td>SABR 2002</td>
<td>$f(v) = v,$ $\beta(v) = 0,$ $g(v) = \sigma,$ $\rho \in [-1, 1], \gamma \in [-1, 0]$</td>
<td>Level dependence in volatility. Options priced perturbation technique</td>
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**Table:** Specification of stochastic volatility models.
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dC - \Delta dS = ...dt + (C_S - \Delta) dS + C_V dV
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**Main problem caused by incompleteness**

The No-Arbitrage Principle alone does not determine a unique price for a derivative.
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- One can relax the condition of self financing, allowing non self financing strategies. These strategies come with a cost process

\[
\text{Cost}_\varphi(t) = V_\varphi(t) - \int_0^t \varphi^0(s)dB(s) - \varphi^1(s)dS(s),
\]

with

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V_\varphi(t) = \varphi^0(t) \cdot B(t) + \varphi^1(t) \cdot S(t)
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with \( V_\varphi(t) = \varphi^0(t) \cdot B(t) + \varphi^1(t) \cdot S(t) \)

- non self financing hedges always exist:

\[
\varphi^0(s) = 0, \quad \varphi^1(s) = 0 \quad \text{for all } s \in [0, T) \text{ and } \\
\varphi^0(T) = h(S_T), \quad \varphi^1(T) = 0.
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Obviously the non self financing hedge should be chosen reasonably:
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In order to comply with the last point traders often have to put new money into the market, this is called "bleeding". It wouldn’t be necessary if the BS model would indeed be the true model, in this case the $\Delta$-hedge would be self financing.
Risk-minimizing hedges

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Leaving aside the variances produced by the drift we obtain for the conditional variance of the cost process

$$\text{var}_t(d\text{Cost}) = ((C_S - \Delta)^2 S^2 S^{2\gamma} f^2(V) + C_V V^2 g^2(V) + 2(C_S - \Delta) C_V S S^{\gamma} f(V) V g(V) \rho) dt.$$
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This is a quadratic function of Δ, minimized by:

$$\Delta_{\min} = C_S + \rho \frac{g(V) V}{f(V) S^{1+\gamma}} C_V,$$
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This is a quadratic function of $\Delta$, minimized by

$$\Delta^{\text{min}} = C_S + \rho \frac{g(V) V}{f(V) S^{1+\gamma}} C_V,$$

If the agent is interested to minimize the variance in the cost process of his non self financing hedge, i.e. the risk of future bleeding, then he may go for this strategy.
The derivation above is rather ad hoc:

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- What happens to the drift terms?

A more formal approach is due to Foellmer and Schweitzer:

**Measure for uncertainty in the cost process**

$$R_\varphi(t) := \mathbb{E}[(C_\varphi(T) - C_\varphi(t))^2 | \mathcal{F}_t]$$

Note that the expectation is taken under the subjective probability measure
call a trading strategy \((\psi^0, \psi^1)\) an admissible continuation of \((\varphi^0, \varphi^1)\) if
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\psi^1(s) = \phi^1(s), \ s \leq t; \ \psi^0(s) = \phi^0(s), \ s < t; \ \text{and} \ V_{\psi}(T) = V_{\phi}(T) \ \mathbb{P}-\text{a.s. for all} \ t \in [0, T).
\]

**Definition**

We call \(\phi\) \(R\)-minimizing if for any \(t \in [0, T)\) and for any admissible continuation \(\psi\) of \(\phi\) from \(t\) on

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**Definition**

We call \(\varphi\) R-minimizing if for any \(t \in [0, T)\) and for any admissible continuation \(\psi\) of \(\varphi\) from \(t\) on

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\]

It turns out however, that in order to guarantee the existence of such a strategy the criterion has to be localized. This leads to the definition of locally risk minimizing hedging strategies:
$(\delta^0, \delta^1)$ is called a small perturbation if both $\delta^1$ and \[ \int_0^T |\delta^1(t)S(t)| \, dt \] are bounded and $\delta^0(T) = \delta^1(T) = 0$. 
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for \( (s, t] \subset [0, T] \), define the small perturbation 
\( \delta|_{(s,t]} := (\delta^0|_{[s,t)}, \delta^1|_{(s,t]} ) \) with \( \delta^0|_{[s,t]}(u, \omega) := \delta^0(u, \omega) \cdot 1_{[s,t]}(u) \) and \( \delta^1|_{(s,t]}(u, \omega) := \delta^1(u, \omega) \cdot 1_{(s,t]}(u) \)
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with \( \delta^0_{|[s,t]}(u, \omega) := \delta^0(u, \omega) \cdot 1_{[s,t]}(u) \)
and \( \delta^1_{|[s,t]}(u, \omega) := \delta^1(u, \omega) \cdot 1_{(s,t]}(u) \).

For \( \tau \) a partition of \([0, T]\) and \( \delta \) a small perturbation \Rightarrow define

\[ r^\tau(t, \varphi, \delta) := \sum_{t_i \in \tau} \frac{R_{\varphi+\delta_{|[t_i,t_{i+1}]}(t_i) - R_{\varphi}(t_i)} \left[ \mathbb{E} \left( \int_{t_i}^{t_{i+1}} S(t)^2 \|\sigma(t)\|^2 \, dt \big| \mathcal{F}_{t_i} \right) \right]}{1_{(t_i,t_{i+1}]}(t)}. \]
(δ⁰, δ¹) is called a small perturbation if both δ¹ and 
\[ \int_0^T |δ¹(t)S(t)| \, dt \] are bounded and δ⁰(T) = δ¹(T) = 0.

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and
\[ δ¹|(s,t)(u, ω) := δ¹(u, ω) \cdot 1_{(s,t]}(u) \]

for τ a partition of [0, T] and δ a small perturbation \(\Rightarrow\) define

\[ r^\tau(t, ϕ, δ) := \sum_{t_i \in τ} \frac{R_{ϕ+δ|(t_i,t_{i+1})}(t_i) - R_ϕ(t_i)}{\mathbb{E} \left( \int_{t_i}^{t_{i+1}} S(t)^2 \|σ(t)\|^2 \, dt \mid \mathcal{F}_{t_i} \right)} \cdot 1_{(t_i,t_{i+1}]}(t). \]
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Then \(\varphi\) is called \textit{locally risk-minimizing} if, for all \(\delta\),

\[ \liminf_{|\tau| \to 0} r^\tau(t, \varphi, \delta) \geq 0 \quad \mathbb{P}\text{-a.s. for all } t \in [0, T]. \]
How to find the locally risk minimizing hedge?
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- Complete the original market by introducing a second stock in such a way that the extended (and now complete) market has as its risk premium the one corresponding to the minimal martingale measure, $Q_{\text{min}}$. 
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A result by Quenez, Peng and ElKaroui guarantees that this gives indeed the locally risk minimizing hedge.
Theorem

In the stochastic volatility model discussed the risk-minimizing hedge of an $h$-claim holds

$$\xi_{\text{min}}(t) = C_S + \rho \frac{g(V(t))V(t)}{f(V(t))S(t)^{1+\gamma}} C_V \text{ units of the stock,} \quad (1)$$

where

$$C(t, S(t), V(t)) = e^{-r(T-t)E_t^{\text{min}}(h(S(t)))}.$$ 

with $E^{\text{min}}$ denoting expectation under the minimal martingale measure. The investment in the money market is given by $C - \xi_{\text{min}}(t)S(t)$. 

Christian-Oliver Ewald

Locally risk-minimizing vs. $\Delta$-hedging in stochastic volatility models
Note: The dynamics of stock and volatility under the minimal martingale measure (the one chosen for pricing) are

\[
\begin{align*}
    dS(t) &= S(t) \left( r dt + f(V(t))S(t)^\gamma \left[ \sqrt{1 - \rho^2} dW^{1,\text{min}}(t) + \rho dW^{2,\text{min}}(t) \right] \right) \\
    dV(t) &= V(t) \left( \left[ \beta(V(t)) - \rho \frac{g(V(t))}{f(V(t))}(\mu - r) \right] dt + g(V(t))dW^{2,\text{min}}(t). \right)
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The price of an option \( h(S_T) \) therefore will depend on the agents assessment of the expected return rate \( \mu \) of the stock, quite contrary to the BS model!
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Should we worry? No, prices are computed under martingale measures and therefore do not lead to arbitrage. Clearly the subjective assessment of the expected return rate influences the agents' assessment of the risk in the cost process.
Example: Heston model

\[ dS(t) = \mu S(t)dt + \sqrt{V(t)}(\sqrt{1-\rho^2}dW^1(t) + \rho dW^2(t)) \]
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The position in the stock for the locally risk-minimizing hedge is therefore given by

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\xi_{\text{min}}(t) = C_S + \rho \sigma \frac{C_V}{S(t)}
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Numerical results

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typical parameters: \( r = 0.04, \mu = 0.10, \theta = 0.0483, \kappa = 4.75, \sigma = 0.550, \rho = -0.569, S(0) = 100 \)
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\( S(0) = 100 \)
For arbitrary strategy compute the hedge error

\[
\text{hedge error} = 100 \times \frac{\sqrt{\text{var}^P(\text{cost}(T; n))}}{e^{-rT} \mathbb{E}^\text{min}(\lfloor S(T) - K \rfloor^+)}).
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in this model scenario we compared Heston-Delta hedging with locally risk minimizing hedging:
**Figure:** Hedge errors (i.e. standard deviation of cost relative to option value) for the ordinary delta and the locally risk-minimizing hedge strategies of a 1-year forward-at-the-money call option in the Heston model.
But maybe the real world is not Heston or it is and the parameters are wrong. We consider four potential scenarios:

- we use the wrong martingale measure (parameters $\kappa$ and $\theta$)
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How does the Delta-hedge and the locally risk minimizing hedge compare then?
Wrong martingale measure:
Wrong martingale measure:

<table>
<thead>
<tr>
<th>Martingale measure; $\mathbb{Q}$</th>
<th>Minimal</th>
<th>Misconceived minimal</th>
<th>Market</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{Q}$-parameters</td>
<td>$\theta$</td>
<td>$0.229^2$</td>
<td>$0.220^2$</td>
</tr>
<tr>
<td></td>
<td>$\kappa$</td>
<td>4.75</td>
<td>4.75</td>
</tr>
<tr>
<td>Hedge error</td>
<td></td>
<td>19.7</td>
<td>19.7</td>
</tr>
</tbody>
</table>

**Table:** Hedge error under different misspecifications of the volatility model.
Wrong martingale measure:

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<tbody>
<tr>
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<td>$0.229^2$</td>
<td>$0.220^2$</td>
<td>$0.289^2$</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>$4.75$</td>
<td>$4.75$</td>
<td>$2.75$</td>
</tr>
<tr>
<td>Hedge error</td>
<td>$19.7$</td>
<td>$19.7$</td>
<td>$20.3$</td>
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Quite robust!
Parameter Uncertainty: We run the data generating process with a specified set \((\mu, \kappa, \theta, \sigma, \rho)\) and for computing the locally risk minimizing hedge use random samples which are normally distributed around these (use results of Eraker about standard error of estimated parameters in Heston-model):
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<table>
<thead>
<tr>
<th>Expiry</th>
<th>Moneyness</th>
<th>Hedge frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>monthly</td>
</tr>
<tr>
<td>3M</td>
<td>At-the-money</td>
<td>0.1%</td>
</tr>
<tr>
<td></td>
<td>10% Out-of-the-money</td>
<td>1.1%</td>
</tr>
<tr>
<td>1Y</td>
<td>At-the-money</td>
<td>1.2%</td>
</tr>
<tr>
<td></td>
<td>10% Out-of-the-money</td>
<td>2.8%</td>
</tr>
</tbody>
</table>

**Table:** Effects of parameter uncertainty on locally risk-minimizing hedges. The table shows the relative increases in hedge error when the hedger uses parameters drawn from the distribution of Eraker’s estimator rather than the true parameter.
Using BS-Greeks for computing the locally risk minimizing hedge: $C_S$ and $C_V$ for Heston are not so handy, why not take the corresponding BS values. For these there exist nice formulas !?
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**Figure:** Hedge errors when using Black-Scholes resp. Heston Greeks.
Wrong data generating process:

We assume prices are generated by a SABR process while locally risk minimizing hedges are computed using the formulas for Heston.
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We assume prices are generated by a SABR process while locally risk minimizing hedges are computed using the formulas for Heston.

\[
\begin{align*}
    dS(t)/S(t) &= V(t)S^\gamma(t)dW^1(t), \\
    dV(t)/V(t) &= \nu dW^2(t).
\end{align*}
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\end{align*}
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- The SABR model can generate option prices (for a specific expiry) that are quite similar to those in the Heston model.
- Yet, the model is structurally quite different: The Skew is generated by a level effect rather than correlation.
Figure: 1-year implied volatilities in the Heston model (circles) and SABR model (solid line). Parameters for the Heston model are as specified in Table ?? (except for $r = \mu = 0$). SABR parameter settings are $V(0) = 1.92$, $\gamma = -1$ and $\nu = 0.2$. 
The investigation of the performance of the locally risk-minimizing hedge and the delta hedge in the (wrong) Heston model is carried out as follows: (1) simulate stock prices and volatilities from the SABR model, (2) for each path implement the Heston-based locally risk-minimizing strategy (using the initially calibrated parameters and the simulated Heston-sense local variance along each path) as well as a delta hedge and (3) implement the SABR model’s delta hedge (which, because of zero correlation of the Brownian motions, coincides with the locally risk-minimizing hedge) using the pricing formula given in Hagan et al. (Wilmott Magazine 1 (2002) ).
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<table>
<thead>
<tr>
<th>Hedge method</th>
<th>SABR RiskMin</th>
<th>Heston RiskMin</th>
<th>Heston Delta</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hedge error</td>
<td>13.3</td>
<td>13.9</td>
<td>18.4</td>
</tr>
</tbody>
</table>

**Table:** Hedge error under a misspecified data-generating process (SABR).
Empirical results

Back to the real world:

- Are there any gains from risk minimization based on a stochastic volatility model?
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- Does it matter which parametric model one uses?
Empirical results

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Data. Times series for spot and option prices for three different markets: The U.S. S&P 500 index, the European EUROSTOXX 50 index and the USD/EURO exchange rate (early 2004 to early 2008).
Stochastic volatility models
Risk-minimizing hedges
Numerical results
Empirical results

Christian-Oliver Ewald
Locally risk-minimizing vs. $\Delta$-hedging in stochastic volatility models
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**SABR** Fix $b$ at some value and chose $a_t$, $\eta$, and $\rho$ to minimize differences between the model’s and the market’s implied volatilities.
## Outline

- Stochastic volatility models
- Risk-minimizing hedges
- Numerical results
- Empirical results

<table>
<thead>
<tr>
<th></th>
<th>OTM put</th>
<th>ATM</th>
<th>OTM call</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>SPX</td>
<td>EUX</td>
<td>FX</td>
</tr>
<tr>
<td>Expiry 3M</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Average</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>Black-Scholes Delta</td>
<td>2.6</td>
<td>2.3</td>
<td>1.0</td>
</tr>
<tr>
<td>Heston Delta</td>
<td>2.9</td>
<td>3.2</td>
<td>1.0</td>
</tr>
<tr>
<td>Heston RiskMin</td>
<td>2.3</td>
<td>1.7</td>
<td>1.2</td>
</tr>
<tr>
<td>SABR Delta</td>
<td>3.1</td>
<td>2.8</td>
<td>1.4</td>
</tr>
<tr>
<td>SABR RiskMin</td>
<td>3.2</td>
<td>1.7</td>
<td>1.4</td>
</tr>
<tr>
<td>Std. deviation</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Black-Scholes Delta</td>
<td>17.5</td>
<td>14.8</td>
<td>16.3</td>
</tr>
<tr>
<td>Heston Delta</td>
<td>24.1</td>
<td>21.1</td>
<td>17.3</td>
</tr>
<tr>
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<td>12.3</td>
<td>10.2</td>
<td>17.8</td>
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<td>24.9</td>
<td>22.2</td>
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</tr>
<tr>
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<td>12.7</td>
<td>11.0</td>
<td>17.1</td>
</tr>
<tr>
<td>SABR Delta(b = 1/2))</td>
<td>23.8</td>
<td>25.8</td>
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Results.

The main message is that the risk-minimizing hedges offer a clear benefit:

- in markets where there is correlation between changes in the underlying and the instantaneous variance (skew in implied volatilities), the standard deviations of the profit-and-loss-ratios are reduced by between one-third and one-quarter.
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- in markets where there is correlation between changes in the underlying and the instantaneous variance (skew in implied volatilities), the standard deviations of the profit-and-loss-ratios are reduced by between one-third and one-quarter

- if correlation is close to zero (as in the exchange rate market), there is (as it should be) no gain from the suggested risk-minimization. But nothing is lost either
Literature: