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Outline

• Introduction
  Stylized facts of financial markets

• Multi-fractal models in Finance
  Combinatorial and iterative MF processes

• Bivariate Multi-fractal models
  Introduction
  Estimation of BMF models:
    likelihood approach
    Simulation based inference particle filter

• Empirical Application on Risk Management
  Value-at-Risk;
  Expected Shortfall;

• Concluding remarks
1. Introduction

1.1. Stylized facts of financial markets:

- Non-Normal distributed (leptokurtotic)
  Skewness and Kurtosis.

- Volatility clustering

- Long-term dependence: Autocorrelation function ACF
  \[
  \lim_{\tau \to \infty} \frac{\rho(\tau)}{C_{\tau^2d-1}} = 1, \\
  \lim_{\lambda \to 0} \frac{f(\lambda)}{C^{d/|\lambda|^{-\alpha}}} = 1, \quad \Rightarrow \quad J. \ Geweke \ and \ S. \ Porter-Hudak \ (1983).
  \]

  Efficient markets hypotheses (EMH).
Probability density function (Dow Jones)

- empirical
- Standard Normal

Cumulative distribution function

- Empirical
- Standard Normal

Probability density function (US/UK)

- empirical
- Standard Normal

Cumulative distribution function
2. Fractals and Multi-Fractal Models

2.1 Introduction of Multi-Fractal Model of Asset Returns

Multi-Fractal Model of Asset Returns (MMAR) by Mandelbrot et. al (1997), assumes returns $r_t$ follow a compound process:

$$r_t = B_H[\theta(t)]$$ (1)

- $B_H[\cdot]$ fractional Brownian motion with index $H$;
- $\theta(t)$ cumulative distribution function of a multi-fractal measure;
- $B_H[\cdot]$ and $\theta(t)$ are independent.

The simplest way to create a multi-fractal measure is the “binomial multi-fractal”,

...
A simple example "binomial multi-fractal"
A simple example “binomial multi-fractal” with $m_0 = 0.6$
Markov-switching multifractal model


\[ r_t = \sigma_t \cdot u_t \]  \hspace{1cm} (2)

with \( u_t \sim N(0, 1) \) and instantaneous volatility being determined by the product of \( k \) volatility components or multipliers \( M_t^{(1)}, M_t^{(2)}, \ldots, M_t^{(k)} \) and a constant scale factor \( \sigma \):

\[ \sigma_t^2 = \sigma^2 \prod_{i=1}^{k} M_t^{(i)} \]  \hspace{1cm} (3)

The transition probabilities are specified by Calvet and Fisher (2004) as:

\[ \gamma_i = 1 - (1 - \gamma_k)^{(b^i - k)} \]  \hspace{1cm} i = 1, \ldots, k, \hspace{1cm} (4)

with parameters \( \gamma_k \in [0, 1] \) and \( b \in (1, \infty) \). Each volatility component is renewed at time \( t \) with probability \( \gamma_i \) depending on its rank within the hierarchy of multipliers and it remains unchanged with probability \( 1 - \gamma_i \).
Markov-switching multifractal model

A discrete version of MF process adopts the binomial distribution:

\[ M_t^{(i)} \sim \{m_0, 2-m_0\}, \quad 1 \leq m_0 < 2. \]

Lux (2006) further introduces a continuous version of multi-fractal process, i.e.

\[ M_t^{(i)} \sim LN(-\lambda, \sigma_m^2). \] (5)

Normalisation via, \( E[M_t^{(i)}] = 1 \), and it leads to

\[ \exp(-\lambda + 0.5\sigma_m^2) = 1 \quad \Rightarrow \quad \sigma_m = \sqrt{2\lambda}. \] (6)

Note that the admissible parameter space for the location parameter \( \lambda \) is \( \lambda \in [0, \infty) \).

The borderline cases:
\( \lambda = 0 \) for Lognormal case (the volatility process collapses to a constant),
\( m_0 = 1 \) for the Binomial case.

The power-law behaviour of the autocovariance function:

\[ \text{Cov}(\vert r_t \vert^q, \vert r_{t+\tau} \vert^q) \propto \tau^{2d(q)-1}. \] (7)
3. Bivariate Multi-fractal (BMF) models

3.1 Calvet/Fisher/Thompson model

Calvet/Fisher/Thompson model assumes volatility is composed of heterogenous frequencies. For each frequency \( i \), the local volatility components \( M_t \) are:

\[
M_t = \begin{bmatrix}
M_{1,t}^{(i)} \\
M_{2,t}^{(i)}
\end{bmatrix}
\] (8)

\( M_t \): 2 \times n matrix, and each column contains a particular volatility component at corresponding cascade level \( M_t^{(i)} \).

\[
g(M_t) = \prod_{i=1}^{n} M_t^{(i)},
\] (9)

Calvet/Fisher/Thompson’s approach assumes that each time series follows univariate MF process in Eq. (2), and specifies the bivariate time series \( r_{q,t} \) (2 \times 1 vector) as:

\[
r_{q,t} = \sigma_q \otimes [g(M_t)]^{1/2} \otimes u_{q,t}.
\] (10)

\( \otimes \): element by element multiplication, \( \sigma_q, u_{q,t} \) are 2 \times 1 vectors \( u_{q,t} \): bivariate standard Normal distribution.
3.1 Calvet/Fisher/Thompson model

$M_{1,t}$ takes value $m_1 \in (1, 2)$ and $2 - m_1$, and $M_{2,t}$ takes value $m_2 \in (1, 2)$ and $2 - m_2$. Thus the random vector $M_t$ has four possible values, whose probabilities are determined by a $2 \times 2$ matrix:

$$\begin{bmatrix}
\frac{1+\rho_m}{4} & \frac{1-\rho_m}{4} \\
\frac{1-\rho_m}{4} & \frac{1+\rho_m}{4}
\end{bmatrix}$$

with $\rho_m$ being the correlation between volatility components $M_{1,t}$ and $M_{2,t}$ under the distribution of $M_t$, and $\rho_m \in [-1, 1]$. The model focuses on the specification $\rho_m = 1$ for simplicity.

The transition probabilities $\gamma_i$, which is specified as:

$$\gamma_i = 1 - (1 - \gamma_n)^{(b^{-n})}, \quad i = 1, \ldots n, \quad (11)$$

with parameters $\gamma_n \in [0, 1]$ and $b \in (1, \infty)$.

Furthermore, arrivals across two series are characterized by a correlation parameter $\lambda_m \in [0, 1]$. New arrivals are independent if $\lambda_m = 0$ and simultaneous if $\lambda_m = 1$. 
3.2 Liu/Lux model

For each frequency $i$, the local volatility components $M_t$ are:

\[ M_t = \begin{bmatrix} M_{1,t}^{(i)} \\ M_{2,t}^{(i)} \end{bmatrix} \] (12)

$M_t$: $2 \times n$ matrix, and each column contains a particular volatility component at corresponding cascade level $M_t^{(i)}$.

$q = 1, 2$ refers to the two time series respectively, having $n$ levels of their volatility cascades. $\otimes$ denotes element by element multiplication, $\sigma_q$ is the scale parameter (unconditional standard deviation); $u_{q,t}$ is a $2 \times 1$ vector whose elements follow a bivariate standard Normal distribution, with an unknown correlation parameter $\rho$. In our model, we assume for the column vector $M_t$ that

\[ g'(M_t) = \prod_{i=1}^{k} M_t^{(i)} \prod_{j=k+1}^{n} M_{q,t}^{(j)}, \] (13)

The bivariate asset returns $r_{q,t}$ are assumed:

\[ r_{q,t} = \sigma_q \otimes \left[ g'(M_t) \right]^{1/2} \otimes u_{q,t}. \] (14)
3.2 Liu/Lux model

In our model, we allow two starting cascades within each time series (the multi-fractal process starts again after the joint cascade level $k$):

$$\gamma_i = 2^{-(n-i)}, \quad \text{for} \quad i = 1, \ldots k + 1, \ldots n. \quad (15)$$

The discrete version: Binomial model:

$$M_t^{(i)} \sim \{m_0, \ 2 - m_0\}, \quad 1 \leq m_0 < 2.$$  

The continuous version: Lognormal model, i.e.

$$-\log_2 M_t^{(i)} \sim N(\lambda, \sigma_m^2).$$

Normalisation via, $E[M_t^{(i)}] = 0.5$, and it leads to

$$\exp[-\lambda \ln 2 + 0.5\sigma_m^2 (\ln 2)^2] = 0.5, \quad \Rightarrow \quad \sigma_m^2 = 2(\lambda - 1)/\ln 2. \quad (16)$$

The economic intuition: the observed correlation between different markets/assets can either be due to common news processes, or to common factors, such as the business cycle or technology shocks.
Simulation of the Bivariate Binomial Multi-Fractal Model

simulation time series 1

simulation time series 2
ACF for the Simulation of the Bivariate Binomial Multi-Fractal Model

ACF for r1

ACF for r2

ACF

time lag

ACF

time lag

ACF

ACF

absolute return
row return
approx. ACF
Maximum Likelihood Estimation

Let $r_t$ be the set of joint return observations, the explicit likelihood function is:

$$f(r_1, \cdots, r_T; \Theta) = \prod_{t=1}^{T} f(r_t | r_1, \cdots, r_{t-1})$$

$$= \prod_{t=1}^{T} \left[ f(r_t | M_t = m^i) \cdot \sum_{i=1}^{4^n} P(M_t = m^i | r_1, \cdots, r_{t-1}) \right]$$

$$= \prod_{t=1}^{T} f(r_t | M_t = m^i) \cdot (\pi_{t-1}A).$$

The transition matrix $A$ whose components $A_{ij}$ (note: here $i, j = \{1, 2 \ldots 4^n\}$) equals to

$$P(M_{t+1} = m^j | M_t = m^i).$$
Maximum Likelihood Estimation

The density of the innovation $r_t$ conditional on $M_t$ is:

$$f(r_t \mid M_t = m^i) = \frac{F_N \left\{ r_t \div \left[ \sigma \otimes \eta^{1/2} \right] \right\}}{\sigma \otimes \eta^{1/2}}.$$  \hspace{1cm} \text{(18)}

$F_N\{\cdot\}$: the bivariate standard Normal density function;

\div: element-by-element division;

\otimes: element by element multiplication;

$\eta = g(M_t)$ for the Calvet/Fisher/Thompson model;

$\eta = g'(M_t)$ for the Liu/Lux model.
The last unknown element in the likelihood function is $\pi_t$, which is the conditional probability, defined as

$$\pi_t^i = P(M_t = m^i \mid r_1, \ldots, r_t),$$

(19)

Due to

$$\sum_{i=1}^{4^n} \pi_t^i = 1$$

Bayesian updating:

$$\pi_{t+1} = \frac{f(r_{t+1} \mid M_{t+1} = m^i) \otimes (\pi_t A)}{\sum f(r_{t+1} \mid M_{t+1} = m^i) \otimes (\pi_t A)}.$$  

(20)
Simulation Based Estimation: particle filter

It is straightforward if $P(M_{t+1} = m^i \mid M_t = m^j)$ has a reasonable size of finite discrete elements as the previous calculation can be computed explicitly:

$$\sum_{j=1}^{4^n} P(M_{t+1} = m^i \mid M_t = m^j) P(M_t = m^j \mid r_t).$$  \hspace{1cm} (21)

We evaluate eq. (20) by combining the conditional density with eq. (21) up to proportionality (for $R = 4^n$):

$$\pi_{t+1}^i \propto f(r_{t+1} \mid M_{t+1} = m^i) \sum_{j=1}^{R} P(M_{t+1} = m^i \mid M_t = m^j) \pi_t^j.$$ \hspace{1cm} (22)

One step ahead conditional probability is

$$\pi_{t+1}^i \propto f(r_{t+1} \mid M_{t+1} = m^i) \frac{1}{B} \sum_{b=1}^{B} P(M_{t+1} = m^i \mid M_t = m^{(b)}).$$ \hspace{1cm} (23)
Simulation Based Estimation: particle filter

Sampling/Importance Resampling (SIR):

1. Simulate the Markov chain one-step-ahead to obtain $M_{t+1}^{(1)}$ given $M_t^{(1)}$. Repeat $B = 1000$ times to generate draws $M_{t+1}^{(1)}, \ldots M_{t+1}^{(B)}$.

2. This preliminary step only uses information available at date $t$, and must therefore be adjusted to account for the new return. Draw a random number $q$ from 1 to $B$ with the probability of:

$$w_j = \frac{f(r_{t+1} \mid M_{t+1} = m^{(j)})}{\sum_{i=1}^{B} f(r_{t+1} \mid M_{t+1} = m^{(i)})}, \quad j = 1, 2, \ldots, B. \quad (24)$$

3. Repeat $B$ times and obtain $B$ draws to get new $M_{t+1}^{(1)}, \ldots M_{t+1}^{(B)}$ which have been adjusted to account for the new returns.
Simulation Based Estimation: particle filter

One step ahead density hence becomes:

\[
f(r_t | r_1, \cdots, r_{t-1}) = \sum_{i=1}^{R} f(r_t | M_t = m^{(i)}) P(M_t = m^{(i)} | I_{t-1})
\]

\[
\approx \frac{1}{B} \sum_{b=1}^{B} f(r_t | M_t = m^{(b)}).
\] (25)

The approximate likelihood function is given below:

\[
g(r_1, \cdots, r_T; \Theta) = \prod_{t=1}^{T} f(r_t | r_1, \cdots, r_{t-1})
\]

\[
\approx \prod_{t=1}^{T} \left[ \sum_{b=1}^{B} f(r_t | M_t = m^{(b)}) \right].
\] (26)
3.5 Specification of cascade levels

A heuristic method:

(1) We make its equal-weight portfolio for a collection of assets.

(2) By using the GPH approach by J. Geweke and S. Porter-Hudak (1983):

\[
\log[I(\lambda_j)] = \alpha + \beta \log[4\sin^2(\lambda_j/2)] + \epsilon_t,
\]

with the \( j \)th periodogram \( I(\lambda_j) \), the empirical estimate \( \hat{\beta} \) for the absolute returns of equally-weighted portfolio is calculated.

(3) Based on the empirical estimates with different numbers of cascade levels, \( N \) simulations are conducted for each asset; and long memory parameter \( \beta \) is calculated for each simulated equally-weighted portfolio.

(4) We then select the case of the cascade level whose mean value of \( \hat{\beta} \) is close to the empirical GPH estimator \( \beta \).
4. Empirical applications in risk management

4.1 Data description

- **Stock exchange indices:**
  Dow Jones Composite 65 Average Index
  NIKKEI 225 Average Index (*DOW/NIK*, Jan. 1969 - Dec. 2008);

- **Foreign exchange rates:**

- **Interest rates:**
### 4.2 Empirical estimates

Table 1: SML estimates for Calvet/Fisher/Thompson model (in-sample data)

<table>
<thead>
<tr>
<th></th>
<th>$\hat{m}_1$</th>
<th>$\hat{m}_2$</th>
<th>$\hat{\sigma}_1$</th>
<th>$\hat{\sigma}_2$</th>
<th>$\hat{\rho}$</th>
<th>$\hat{\lambda}$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Dow/Nik</strong></td>
<td>1.435</td>
<td>1.375</td>
<td>0.924</td>
<td>1.211</td>
<td>0.288</td>
<td>0.373</td>
</tr>
<tr>
<td></td>
<td>(0.022)</td>
<td>(0.030)</td>
<td>(0.034)</td>
<td>(0.037)</td>
<td>(0.024)</td>
<td>(0.020)</td>
</tr>
<tr>
<td><strong>US/DM</strong></td>
<td>1.430</td>
<td>1.415</td>
<td>0.797</td>
<td>0.672</td>
<td>0.276</td>
<td>0.470</td>
</tr>
<tr>
<td></td>
<td>(0.025)</td>
<td>(0.022)</td>
<td>(0.036)</td>
<td>(0.037)</td>
<td>(0.015)</td>
<td>(0.029)</td>
</tr>
<tr>
<td><strong>TB1/TB2</strong></td>
<td>1.371</td>
<td>1.447</td>
<td>0.357</td>
<td>0.411</td>
<td>0.804</td>
<td>0.519</td>
</tr>
<tr>
<td></td>
<td>(0.029)</td>
<td>(0.032)</td>
<td>(0.043)</td>
<td>(0.042)</td>
<td>(0.020)</td>
<td>(0.024)</td>
</tr>
</tbody>
</table>

Note: The number of cascade levels $n = 8$ as in Calvet et al (2006).
### 4.2 Empirical estimates

Table 2: SML estimates for Liu/Lux model (in-sample data)

<table>
<thead>
<tr>
<th></th>
<th>$\hat{m}_0$</th>
<th>$\hat{\sigma}_1$</th>
<th>$\hat{\sigma}_2$</th>
<th>$\hat{\rho}$</th>
</tr>
</thead>
<tbody>
<tr>
<td><em>Dow/Nik</em></td>
<td>1.335</td>
<td>0.952</td>
<td>1.093</td>
<td>0.141</td>
</tr>
<tr>
<td></td>
<td>(0.019)</td>
<td>(0.019)</td>
<td>(0.020)</td>
<td>(0.014)</td>
</tr>
<tr>
<td><em>US/DM</em></td>
<td>1.473</td>
<td>0.778</td>
<td>0.690</td>
<td>0.302</td>
</tr>
<tr>
<td></td>
<td>(0.024)</td>
<td>(0.019)</td>
<td>(0.017)</td>
<td>(0.012)</td>
</tr>
<tr>
<td><em>TB1/TB2</em></td>
<td>1.592</td>
<td>0.282</td>
<td>0.307</td>
<td>0.812</td>
</tr>
<tr>
<td></td>
<td>(0.028)</td>
<td>(0.013)</td>
<td>(0.028)</td>
<td>(0.026)</td>
</tr>
</tbody>
</table>

Note: We use the number of joint cascade by matching empirical and simulated GPH long memory estimates, they are $j = 2$ for *Dow/Nik*, $j = 3$ for *US/DM*, $j = 5$ for *TB1/TB2*, respectively.
4.3 Constant Correlation GARCH model

For $N$ assets returns $r_t = \{r_{1,t}, \ldots, r_{N,t}\}$, the multivariate GARCH model can be defined as follows:

$$ r_t = \mu_t + \varepsilon_t; \quad \varepsilon_t = H_t^{1/2} \cdot z_t, $$

$(27)$

$H_t^{1/2}$: a $N \times N$ positive definite matrix. $N \times 1$ random vector $\{z_t\}$ is i.i.d, with $E[z_t] = 0, Var[z_t] = I_N$.

$$ E[r_t | \Omega_{t-1}] = \mu_t; \quad Var(r_t | \Omega_{t-1}) = H_t^{1/2}(H_t^{1/2})', $$

$H_t = h_{ij,t}$ ($i, j = 1, 2, \ldots, N$). The vectorized conditional-variance matrix (VEC(1,1) model):

$$ h_t = C + A\varepsilon_{t-1}^2 + Gh_{t-1}, $$

$(28)$

where

$$ \varepsilon_t = vech(\varepsilon_t\varepsilon_t)', \quad h_t = vech(H_t), $$

CC-GARCH (1, 1):

$$ h_{ii,t} = \omega_i + \alpha_i \varepsilon_{t-1}^2 + \beta_i h_{ii,t-1} \quad i = 1, 2, \ldots, N, $$

$(29)$

$$ h_{ij,t} = \rho_{ij} \sqrt{h_{i,t}h_{j,t}} \quad \forall i \neq j. $$
### 4.3 Constant Correlation GARCH model

Table 3: CC-GARCH(1, 1) model estimates (in-sample data)

<table>
<thead>
<tr>
<th></th>
<th>$\hat{\mu}_1$</th>
<th>$\hat{\mu}_2$</th>
<th>$\hat{\omega}_1$</th>
<th>$\hat{\omega}_2$</th>
<th>$\hat{\alpha}_1$</th>
<th>$\hat{\alpha}_2$</th>
<th>$\hat{\beta}_1$</th>
<th>$\hat{\beta}_2$</th>
<th>$\hat{\rho}_{12}$</th>
</tr>
</thead>
<tbody>
<tr>
<td><em>Dow/Nik</em></td>
<td>-0.02</td>
<td>0.01</td>
<td>0.06</td>
<td>0.09</td>
<td>0.16</td>
<td>0.09</td>
<td>0.77</td>
<td>0.92</td>
<td>0.07</td>
</tr>
<tr>
<td></td>
<td>(0.01)</td>
<td>(0.00)</td>
<td>(0.01)</td>
<td>(0.02)</td>
<td>(0.03)</td>
<td>(0.02)</td>
<td>(0.01)</td>
<td>(0.01)</td>
<td>(0.02)</td>
</tr>
<tr>
<td><em>US/DM</em></td>
<td>0.01</td>
<td>0.01</td>
<td>0.18</td>
<td>0.14</td>
<td>0.10</td>
<td>0.13</td>
<td>0.87</td>
<td>0.82</td>
<td>0.19</td>
</tr>
<tr>
<td></td>
<td>(0.01)</td>
<td>(0.00)</td>
<td>(0.01)</td>
<td>(0.02)</td>
<td>(0.01)</td>
<td>(0.03)</td>
<td>(0.01)</td>
<td>(0.02)</td>
<td>(0.03)</td>
</tr>
<tr>
<td><em>TB1/TB2</em></td>
<td>-0.02</td>
<td>-0.01</td>
<td>0.13</td>
<td>0.09</td>
<td>0.07</td>
<td>0.09</td>
<td>0.85</td>
<td>0.83</td>
<td>0.71</td>
</tr>
<tr>
<td></td>
<td>(0.01)</td>
<td>(0.00)</td>
<td>(0.03)</td>
<td>(0.02)</td>
<td>(0.03)</td>
<td>(0.03)</td>
<td>(0.01)</td>
<td>(0.02)</td>
<td>(0.03)</td>
</tr>
</tbody>
</table>

Note: The ML estimation of the CC-GARCH(1, 1) model is implemented via the GAUSS module ‘Fanpac’ provided by Aptech$^\text{TM}$ Systems Inc.
4.4 Value-at-Risk

Value-at-Risk (VaR):

- Specified target horizon;
- Statistical confidence level;
- The worst loss.


VaR at the $h$-period horizon is defined as the $\alpha$ quantile of the conditional probability distribution of $r_{t:t+h}$:

$$ Pr (r_{t:t+h} \leq VaR_{t:t+h}^\alpha | I_t) = \alpha. $$

(30)
4.4 Value-at-Risk

1. After having estimated the parameters with in-sample data, we invoke once more the particle filter algorithm by starting at $t = 0$ with drawing $M_{0}^{(1)}, \ldots, M_{0}^{(B)}$ from the initial condition $\pi_0$ and iterating it to obtain $\hat{M}_1^{(1)}, \ldots, \hat{M}_1^{B}$. For $t \geq 1$, we proceed with the importance sampler via:

2. Draw a random number $q$ from 1 to $B$ with the probability of $P(q = b) = \frac{f(r_t | \hat{M}_t^{(b)})}{\sum_{i=1}^{B} f(r_t | \hat{M}_t^{(i)})}$.

3. $M_t^{(1)} = \hat{M}_t^{(q)}$ is then selected, repeat Step 2 $B$ times and obtain $B$ draws with $M_t^{(1)}, \ldots, M_t^{(B)}$.

4. After the last iteration of the in-sample series (say time $t$) by reaching $M_t$, we simulate the Markov chain one-step-ahead to obtain $\hat{M}_{t+1}^{(1)}$ given $M_t^{(1)}$, repeat $B$ times to generate draws $\{\hat{M}_{t+1}^{(b)}\}_{b=1}^{B}$, i.e., $\hat{M}_{t+1}^{(1)}, \ldots, \hat{M}_{t+1}^{(B)}$, which are used for one-step ahead forecast, i.e., to move from $t$ to the forecast for $t + 1$.

5. For $h$-period forecasts given information up to time $t$, we iterate the particles obtained from importance resampling at time $t$ $h$ times to obtain $h$-period ahead volatility draws $\hat{M}_{t+h}^{(1)}, \ldots, \hat{M}_{t+h}^{(B)}$. For all cases, we use $B = 10000$ simulated draws.

6. For the next one-step particles we move from $t$ to $t + 1$ applying the SIR via Step 2 and Step 3 to obtain importance sampler $\{M_{t+1}^{(b)}\}_{b=1}^{B}$, then iterate the Markov chain to generate draws $\{\hat{M}_{t+2}^{(b)}\}_{b=1}^{B}$ to $\{\hat{M}_{t+h+1}^{(b)}\}_{b=1}^{B}$ given $\{M_{t+1}^{(b)}\}_{b=1}^{B}$.
Table 4: Failure rates for multi-period Value-at-Risk forecasts (Liu/Lux model)

<table>
<thead>
<tr>
<th></th>
<th>One day horizon</th>
<th>Five days horizon</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>DOW  NIK  EW  HG</td>
<td>DOW  NIK  EW  HG</td>
</tr>
<tr>
<td><strong>Stocks</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\alpha = 10%$</td>
<td>0.1060 0.1191$^+$ 0.1096 0.0954</td>
<td>0.0912 0.1214$^+$ 0.1079 0.0932</td>
</tr>
<tr>
<td>$\alpha = 5%$</td>
<td>0.0428 0.0592 0.0554 0.0495</td>
<td>0.0438 0.0562 0.0534 0.0468</td>
</tr>
<tr>
<td>$\alpha = 1%$</td>
<td>0.0069 0.0175$^+$ 0.0074 0.0184$^+$</td>
<td>0.0062 0.0128 0.0167 0.0062</td>
</tr>
<tr>
<td><strong>FXs</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\alpha = 10%$</td>
<td>0.0984 0.1005 0.0905 0.1161</td>
<td>0.1026 0.0961 0.0982 0.1077</td>
</tr>
<tr>
<td>$\alpha = 5%$</td>
<td>0.0501 0.0477 0.0415 0.0604</td>
<td>0.0508 0.0452 0.0515 0.0560</td>
</tr>
<tr>
<td>$\alpha = 1%$</td>
<td>0.0063 0.0090 0.0048 0.0141</td>
<td>0.0039 0.0071 0.0127 0.0048</td>
</tr>
<tr>
<td><strong>Bonds</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\alpha = 10%$</td>
<td>0.0898 0.1117 0.0970 0.0715$^*$</td>
<td>0.0793 0.1077 0.1049 0.0974</td>
</tr>
<tr>
<td>$\alpha = 5%$</td>
<td>0.0602 0.0513 0.0500 0.0481</td>
<td>0.0436 0.0587 0.0517 0.0373$^*$</td>
</tr>
<tr>
<td>$\alpha = 1%$</td>
<td>0.0116 0.0083 0.0080 0.0031$^*$</td>
<td>0.0044 0.0125 0.0189$^+$ 0.0019$^*$</td>
</tr>
</tbody>
</table>

Note: This table shows the failure rate (proportion of observations above the VaR). Stocks are Dow Jones Composite 65 Average Index (DOW) and NIKKEI 225 Stock Average Index (NIK); FXs are Foreign Exchange rate of U.S. Dollar (US) and German Mark (DM) to British Pound; Bonds are the U.S. 1-Year and 2-Year Treasury Constant Maturity Rate (TB1, TB2 respectively). EW denotes Equal-Weight portfolio, HG denotes Hedge, a zero investment portfolio. + and $^*$ denote too risky and too conservative VaR, respectively.
Table 5: Failure rates for multi-period Value-at-Risk forecasts (Calvet/Fisher/Thompson model)

<table>
<thead>
<tr>
<th></th>
<th>One day horizon</th>
<th>Five days horizon</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>DOW</td>
<td>NIK</td>
</tr>
<tr>
<td><strong>Stocks</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \alpha = 10% )</td>
<td>0.1046</td>
<td>0.0910</td>
</tr>
<tr>
<td>( \alpha = 5% )</td>
<td>0.0601</td>
<td>0.0538</td>
</tr>
<tr>
<td>( \alpha = 1% )</td>
<td>0.0104</td>
<td>0.0068</td>
</tr>
<tr>
<td><strong>FXs</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \alpha = 10% )</td>
<td>0.0904</td>
<td>0.0937</td>
</tr>
<tr>
<td>( \alpha = 5% )</td>
<td>0.0609</td>
<td>0.0460</td>
</tr>
<tr>
<td>( \alpha = 1% )</td>
<td>0.0068</td>
<td>0.0104</td>
</tr>
<tr>
<td><strong>Bonds</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \alpha = 10% )</td>
<td>0.0933</td>
<td>0.1081</td>
</tr>
<tr>
<td>( \alpha = 5% )</td>
<td>0.0438</td>
<td>0.0499</td>
</tr>
<tr>
<td>( \alpha = 1% )</td>
<td>0.0046</td>
<td>0.0044</td>
</tr>
</tbody>
</table>

Note: This table shows the failure rate (proportion of observations above the VaR). Stocks are Dow Jones Composite 65 Average Index (DOW) and NIKKEI 225 Stock Average Index (NIK); FXs are Foreign Exchange rates of U.S. Dollar (US) and German Mark (DM) to British Pound; Bonds are the U.S. 1-Year and 2-Year Treasury Constant Maturity Rate (TB1, TB2 respectively). EW denotes Equal-Weight portfolio, HG denotes Hedge, a zero investment portfolio. + and * denote too risky and too conservative VaR, respectively.
<table>
<thead>
<tr>
<th></th>
<th>One day horizon</th>
<th>Five days horizon</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>DOW</td>
<td>NIK</td>
</tr>
<tr>
<td><strong>Stocks</strong></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
| $\alpha = 10\%$ | 0.1040 | 0.1066 | 0.0927 | 0.1023 | 0.1079 | 0.1096 | 0.1007 | 0.7360*
| $\alpha = 5\%$ | 0.0576 | 0.0528 | 0.0469 | 0.0500 | 0.0471 | 0.0415 | 0.0504 | 0.0321*
| $\alpha = 1\%$ | 0.0221$^+$ | 0.0169$^+$ | 0.0116 | 0.0197$^+$ | 0.0090 | 0.0021$^*$ | 0.0028$^*$ | 0.0019$^*$ |
| **FXs** |      |      |     |     |      |      |     |     |
| $\alpha = 10\%$ | 0.0906 | 0.0917 | 0.1210$^+$ | 0.1050 | 0.0962 | 0.1039 | 0.1030 | 0.0843*
| $\alpha = 5\%$ | 0.0423 | 0.0456 | 0.0594 | 0.0411 | 0.0471 | 0.0401 | 0.0518 | 0.0378*
| $\alpha = 1\%$ | 0.0059 | 0.0196$^+$ | 0.0172$^+$ | 0.0203$^+$ | 0.0093 | 0.0101 | 0.0158$^+$ | 0.0204$^+$ |
| **Bonds** |      |      |     |     |      |      |     |     |
| $\alpha = 10\%$ | 0.1079 | 0.0947 | 0.1007 | 0.7360$^*$ | 0.0962 | 0.1039 | 0.1030 | 0.0843*
| $\alpha = 5\%$ | 0.0542 | 0.0415 | 0.0504 | 0.0321$^*$ | 0.0471 | 0.0401 | 0.0518 | 0.0378*
| $\alpha = 1\%$ | 0.0090 | 0.0021$^*$ | 0.0028$^*$ | 0.0019$^*$ | 0.0042 | 0.0048 | 0.0021$^*$ | 0.0018$^*$ |

Note: This table shows the failure rate (proportion of observations above the VaR) based on the bivariate CC-GARCH(1, 1) model. Stocks are Dow Jones Composite 65 Average Index (DOW) and NIKKEI 225 Stock Average Index (NIK); FXs are Foreign Exchange rates of U.S. Dollar (US) and German Mark (DM) to British Pound; Bonds are the U.S. 1-Year and 2-Year Treasury Constant Maturity Rate (TB1, TB2 respectively). EW denotes Equal-Weight portfolio, HG denotes Hedge, as zero investment portfolio. $^+$ and $^*$ denote too risky and too conservative VaR, respectively.
4.5 Expected Shortfall

- ‘Tail risk’: VaR reports percentiles of profit-loss distributions, disregarding any loss beyond the VaR level;
- Sub-additive: the risk of the total position is less than or equal to the sum of the risk of individual portfolios.

Expected Shortfall (ES):
- Specified target horizon;
- Statistical confidence level;
- The expected losses conditional on the loss exceeding the $VaR_{t:t+h}^\alpha$.

Hence, Expected Shortfall is given by:

$$ES_{t:t+h}^\alpha = E[(\tilde{r}_{t:t+h} | \tilde{r}_{t:t+h} \leq VaR_{t:t+h}^\alpha) | I_t].$$

(31)
Table 7: Multi-period Expected shortfall forecasts (Liu/Lux model)

<table>
<thead>
<tr>
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<th>One day horizon</th>
<th>Five days horizon</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>DOW</td>
<td>NIK</td>
</tr>
<tr>
<td>10%</td>
<td>(2.21)</td>
<td>1.31</td>
</tr>
<tr>
<td>5%</td>
<td>(2.84)</td>
<td>1.64</td>
</tr>
<tr>
<td>1%</td>
<td>(4.36)</td>
<td>2.45</td>
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</table>

<table>
<thead>
<tr>
<th></th>
<th>US</th>
<th>DM</th>
<th>EW</th>
<th>HG</th>
<th>US</th>
<th>DM</th>
<th>EW</th>
<th>HG</th>
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</thead>
<tbody>
<tr>
<td>10%</td>
<td>(1.84)</td>
<td>1.98</td>
<td>(2.51)</td>
<td>2.90</td>
<td>(4.11)</td>
<td>4.56</td>
<td>(4.23)</td>
<td>4.34</td>
</tr>
<tr>
<td>5%</td>
<td>(2.34)</td>
<td>2.61</td>
<td>(3.08)</td>
<td>3.69</td>
<td>(5.27)</td>
<td>5.73</td>
<td>(5.40)</td>
<td>5.44</td>
</tr>
<tr>
<td>1%</td>
<td>(3.57)</td>
<td>4.31</td>
<td>(4.34)</td>
<td>5.70</td>
<td>(9.15)</td>
<td>8.47</td>
<td>(9.16)</td>
<td>7.95</td>
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<table>
<thead>
<tr>
<th></th>
<th>TB1</th>
<th>TB2</th>
<th>EW</th>
<th>HG</th>
<th>TB1</th>
<th>TB2</th>
<th>EW</th>
<th>HG</th>
</tr>
</thead>
<tbody>
<tr>
<td>10%</td>
<td>(2.11)</td>
<td>1.87</td>
<td>(4.26)</td>
<td>3.46</td>
<td>(1.09)</td>
<td>1.90</td>
<td>(4.55)</td>
<td>4.33</td>
</tr>
<tr>
<td>5%</td>
<td>(2.81)</td>
<td>2.66</td>
<td>(5.67)</td>
<td>4.15</td>
<td>(1.54)</td>
<td>2.30</td>
<td>(6.04)</td>
<td>5.48</td>
</tr>
<tr>
<td>1%</td>
<td>(4.87)</td>
<td>4.41</td>
<td>(9.53)</td>
<td>7.03</td>
<td>(2.69)</td>
<td>4.22</td>
<td>(10.10)</td>
<td>8.17</td>
</tr>
</tbody>
</table>

Note: This table reports the Expected Shortfall (ES) forecast based on bivariate MF model, the numbers in parentheses are the empirical realized ES values. Numbers inside parentheses are empirical ES, and numbers outside parentheses are corresponding ES obtained by forecast. Bold numbers show those cases for which we cannot reject identity of the empirical and simulated ES, i.e. the empirical value falls into the range between the 2.5 to 97.5 percent quantile of the simulated ones.
Table 8: Multi-period Expected shortfall forecasts (Calvet/Fisher/Thompson model)

<table>
<thead>
<tr>
<th></th>
<th>One day horizon</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th>Five days horizon</th>
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<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>DOW</td>
<td>NIK</td>
<td>EW</td>
<td>HG</td>
<td>DOW</td>
<td>NIK</td>
<td>EW</td>
<td>HG</td>
<td>DOW</td>
<td>NIK</td>
<td>EW</td>
</tr>
<tr>
<td>10%</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(2.21)</td>
<td>(1.80)</td>
<td><strong>1.75</strong></td>
<td>(3.02)</td>
<td>2.55</td>
<td>(2.60)</td>
<td><strong>2.29</strong></td>
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<tr>
<td>5%</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(2.84)</td>
<td>(2.28)</td>
<td>(2.39)</td>
<td><strong>2.17</strong></td>
<td>(3.89)</td>
<td>3.19</td>
<td>(3.25)</td>
</tr>
<tr>
<td>1%</td>
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<td></td>
<td></td>
<td></td>
<td>(4.36)</td>
<td>(2.82)</td>
<td>(4.40)</td>
<td>3.12</td>
<td>(6.48)</td>
<td>4.80</td>
<td>(5.09)</td>
</tr>
<tr>
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<td><strong>US</strong></td>
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<td></td>
</tr>
<tr>
<td>10%</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(1.84)</td>
<td>(2.27)</td>
<td>(1.79)</td>
<td><strong>1.77</strong></td>
<td>(2.51)</td>
<td><strong>2.87</strong></td>
<td>(2.44)</td>
</tr>
<tr>
<td>5%</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(2.34)</td>
<td>(3.00)</td>
<td>(2.26)</td>
<td><strong>2.27</strong></td>
<td>(3.08)</td>
<td>3.66</td>
<td>(2.99)</td>
</tr>
<tr>
<td>1%</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(3.57)</td>
<td>(4.97)</td>
<td>(3.39)</td>
<td><strong>3.48</strong></td>
<td>(4.34)</td>
<td>5.70</td>
<td>(4.22)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td><strong>TB1</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10%</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(2.11)</td>
<td>(1.47)</td>
<td>(2.30)</td>
<td>1.14</td>
<td>(4.26)</td>
<td>2.48</td>
<td>(1.09)</td>
</tr>
<tr>
<td>5%</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(2.81)</td>
<td>(1.82)</td>
<td>(3.03)</td>
<td>1.41</td>
<td>(5.67)</td>
<td>3.01</td>
<td>(1.54)</td>
</tr>
<tr>
<td>1%</td>
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<td></td>
<td></td>
<td></td>
<td>(4.87)</td>
<td>(2.60)</td>
<td>(5.02)</td>
<td>2.06</td>
<td>(9.53)</td>
<td>4.19</td>
<td>(2.69)</td>
</tr>
</tbody>
</table>

Note: This table reports the Expected Shortfall (ES) forecast based on Calvet/Fisher/Thompson model, the numbers in parentheses are the empirical realized ES values. Numbers inside parentheses are empirical ES, and numbers outside parentheses are corresponding ES obtained by forecast. Bold numbers show those cases for which we cannot reject identity of the empirical and simulated ES, i.e. the empirical value falls into the range between the 2.5 to 97.5 percent quantile of the simulated ones.
Table 9: Multi-period Expected shortfall forecasts (CC-GARCH model)

<table>
<thead>
<tr>
<th></th>
<th>One day horizon</th>
<th>Five days horizon</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>DOW</td>
<td>NIK</td>
</tr>
<tr>
<td>10%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>DOW</td>
<td>(2.21)</td>
<td>1.81</td>
</tr>
<tr>
<td>NIK</td>
<td>(2.84)</td>
<td>2.37</td>
</tr>
<tr>
<td>EW</td>
<td>(4.36)</td>
<td>4.10</td>
</tr>
<tr>
<td>HG</td>
<td>(8.81)</td>
<td>9.34</td>
</tr>
</tbody>
</table>

|                |      |      |      |      |      |      |      |      |
| US             |      |      |      |      |      |      |      |      |
| DM             | (1.84)| 2.31 | (1.79)| 1.99 | (2.51)| 3.04 | (2.44)| 3.07 |
| EW             | (2.34)| 2.91 | (2.26)| 2.43 | (3.08)| 3.72 | (2.99)| 3.75 |
| HG             | (3.57)| 4.55 | (3.39)| 3.49 | (4.34)| 5.40 | (4.22)| 5.43 |

|                |      |      |      |      |      |      |      |      |
| TB1            |      |      |      |      |      |      |      |      |
| TB2            | (2.11)| 1.54 | (2.30)| 0.87 | (4.26)| 1.76 | (1.09)| 1.78 |
| EW             | (2.81)| 1.83 | (3.03)| 1.02 | (5.67)| 2.08 | (1.54)| 2.10 |
| HG             | (4.87)| 2.42 | (5.02)| 1.34 | (9.53)| 2.73 | (2.69)| 2.76 |

|                |      |      |      |      |      |      |      |      |
| TB1            |      |      |      |      |      |      |      |      |
| TB2            | (2.11)| 1.54 | (2.30)| 0.87 | (4.26)| 1.76 | (1.09)| 1.78 |
| EW             | (2.81)| 1.83 | (3.03)| 1.02 | (5.67)| 2.08 | (1.54)| 2.10 |
| HG             | (4.87)| 2.42 | (5.02)| 1.34 | (9.53)| 2.73 | (2.69)| 2.76 |

Note: This table reports the Expected Shortfall (ES) forecast based on CC-GARCH(1, 1) model, the numbers in parentheses are the empirical realized ES values. Bold numbers show those cases for which we cannot reject identity of the empirical and simulated ES, i.e. the empirical value falls into the range between the 2.5 to 97.5 percent quantile of the simulated ones.
5. Concluding Remarks

- Review of multifractal models in finance.
- Bivariate multi-fractal models with parsimonious setting.
- Estimation issues: Maximum likelihood (ML), Simulation based ML. Monte Carlo studies
- Empirical applications for financial risk management.
- Further studies in this direction are likely to offer new insights:
Thank you.