Riding on the Smiles

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Objectives:

- Studying the calibration properties of several stochastic volatility models
- Provide some price approximations allowing to speed up the pricing process
Outline of the presentation:

2. Calibration of single asset multi-dimensional stochastic volatility models
3. Calibration of multi-asset multi-dimensional stochastic volatility models
4. How far is an academic from the market?
5. Price approximations
6. Conclusions
On the calibration of the Heston (1993) model

\[ \frac{dS_t}{S_t} = \sqrt{v_t} dW_t^1 \]
\[ dv_t = \kappa (\theta - v_t) dt + \sigma \sqrt{v_t} dW_t^2 \]
\[ dW_t^1 dW_t^2 = \rho dt \]

\( \rho \) controls the link between vol and asset returns

\[ \downarrow \]

The Skew or Leverage
Analytic and Financial properties

- Characteristic function of the asset returns

\[ \mathbb{E}_t \left[ e^{i\omega \log(S_{t+\tau})} \right] = e^{A(\tau) v_t + B(\tau) \log(S_t) + C(\tau)} \]

- \( A(\tau) \) solves a Riccati ODE: explicit solution!

- Quasi closed form option prices via Fast Fourier Transform (Carr and Madan 1999)

- Sensitivity analysis, vol of vol asymptotic expansion..

- Each parameter has a clear financial interpretation
Quoting vanilla options

The implied volatility $\sigma_{imp}$ is the quantity such that

$$C_{mkt}(t, T, S_t, K) = c_{bs}(t, T, S_t, K, \sigma_{imp}^2(T - t))$$

(1)
The Smiles
Important facts:

the skew is controlled by $\rho$

\[ \Downarrow \]

we have a term structure of skews

\[ \Downarrow \]

we should have different values for $\rho$

and

above $T - t > 0.1$ the smile is homogeneous
The important detail: the norm

Calibration of vanilla options (OTM), maturities available

\[
\min \frac{1}{N} \sum_{i=1}^{N} (C_{\text{model}}(t, T_i, K_i) - C_{\text{mkt}}(t, T_i, K_i))^2
\]

<table>
<thead>
<tr>
<th>error</th>
<th>$\rho$</th>
<th>$t_{\min}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.25E-07</td>
<td>-0.7095</td>
<td>0.05</td>
</tr>
<tr>
<td>2.06E-07</td>
<td>-0.7001</td>
<td>$0.1$</td>
</tr>
</tbody>
</table>

I don’t take the first maturity
• short term options have small (if no) impact on the solution
• the calibration seems to be good
• poor fit of short term options

What is the problem?

short term options have small time value w.r.t long term options

⇒ small/no impact on the objective
Selecting a good norm

\[
\min \frac{1}{N} \sum_{i=1}^{N} (\sigma_{imp}^{\text{model}}(t, T_i, K_i) - \sigma_{imp}^{\text{mkt}}(t, T_i, K_i))^2
\]

- more weight on short term options
- to fit the short term skew a low correlation is needed.
Calibration tests (Vol norm)

<table>
<thead>
<tr>
<th>error</th>
<th>$\rho$</th>
<th>$(T - t)_{\text{min}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00010773</td>
<td>-0.5562</td>
<td>0.05</td>
</tr>
<tr>
<td>4.31E-05</td>
<td>-0.6324</td>
<td>0.1</td>
</tr>
</tbody>
</table>

calibration date: 28/08/08

Maturities 0.06=19/09, 0.13=17/10 .. 4.31

- to fit the short term skew a low correlation is needed.
Why extending the Heston model?

- The dynamics of the implied volatility surface (vanilla options) and the Variance Swap curve are driven by several factors
- On the FX market the skew is stochastic
- We have a term structure of skew: short term skew $\neq$ long term skew
Double-Heston model

(Christoffensen, Heston, Jacobs 2007)

\[
\frac{dS_t}{S_t} = \sqrt{v_t^1} dZ_t^1 + \sqrt{v_t^2} dZ_t^2
\]

\[
dv_t^1 = \kappa^1 (\theta^1 - v_t^1) dt + \sigma^1 \sqrt{v_t^1} dW_t^1
\]

\[
dv_t^2 = \kappa^2 (\theta^2 - v_t^2) dt + \sigma^2 \sqrt{v_t^2} dW_t^2
\]

\[
dZ_t^1 dW_t^1 = \rho^1 dt
\]

\[
dZ_t^2 dW_t^2 = \rho^2 dt
\]

but

\[
\begin{array}{c}
\underbrace{dZ_t^1 dZ_t^2 = dW_t^1 dW_t^2 = dZ_t^1 dW_t^2 = dZ_t^2 dW_t^1} = 0
\end{array}
\]

AFFINITY
Recall the Duffie-Filipovic-Schachermayer (2003)’s condition

If \( X_t = (X^1_t, X^2_t)^\top \) is a vector affine square root process (thus positive):

\[
d \begin{pmatrix} X^1_t \\ X^2_t \end{pmatrix} = \ldots dt + \begin{pmatrix} \times & 0 \\ 0 & \times \end{pmatrix} d \begin{pmatrix} W^1_t \\ W^2_t \end{pmatrix}
\]

\[\downarrow\]

We have strong constraints on the diffusion

\[\downarrow\]

Strong constraints on the correlation!!

\[\downarrow\]

We can not correlate \( \nu^1_t \) and \( \nu^2_t \) in the Double-Heston
Main question

Is it possible to find an AFFINE model allowing for nontrivial correlation among factors

Choose a suitable State Space Domain!
Wishart multi-dim Stochastic Vol

- Extended by Da Fonseca, Grasselli and Tebaldi (2008)

\[
\frac{dS_t}{S_t} = r dt + Tr \left[ \sqrt{\Sigma_t} dZ_t \right]
\]

\[
d\Sigma_t = (\beta Q^\top Q + M \Sigma_t + \Sigma_t M^\top) dt + \sqrt{\Sigma_t} dW_t Q + Q^\top dW_t^\top \sqrt{\Sigma_t}
\]

- \( Z_t = \) Matrix Brownian Motion correlated with \( W_t \)

- \( Vol(S_t) = Tr [\Sigma_t] \) linear combination of the Wishart elements
\[ d\Sigma_t = (\beta Q^\top Q + M \Sigma_t + \Sigma_t M^\top) dt + \sqrt{\Sigma_t} dW_t Q + Q^\top dW_t^\top \sqrt{\Sigma_t} \]

\[ \Omega \Omega^\top = \beta Q^\top Q \] with \( \beta \) large enough (Gindikin’s condition)

\( M \) negative definite \( \iff \) mean reverting behavior

\( \Sigma_t \) SYMMETRIC MATRIX SQUARE ROOT PROCESS \((n \times n)\)

\( Q \) vol-of-vol.

\( (W_t; t \geq 0) \) is a matrix Brownian motion \((n \times n)\)
Correlation in the Wishart model

- $R \in M_n$ (identified up to a rotation) completely describes the correlation structure:

$$Z_t = W_t R^\top + B_t \sqrt{I - RR^\top}$$

$=$ Matrix Brownian motion!

- This choice is compatible with affinity of the model!!

- Other (few) choices are possible but harder to manage.
• The **Wishart Affine model** is solvable. That is, the conditional characteristic function can be written as:

\[
\mathbb{E}_t e^{i\omega \log(S_{t+\tau})} = e^{Tr[A(\tau)\Sigma_t] + B(\tau) \log(S_t) + C(\tau)}
\]

• *A(\tau)* solves a **Riccati ODE** that can be **linearized**! (Grasselli and Tebaldi 2008)
Stochastic correlation between stock returns and vol

\[ \text{Corr}_t (d\ln(S), d\text{Vol}(\ln(S))) = \rho_t = \frac{2 \text{Tr} [\Sigma_t RQ]}{\sqrt{\text{Tr} [\Sigma_t]} \sqrt{\text{Tr} [\Sigma_t Q\top Q]}} \]

- Stochastic correlation between the stock and its volatility

- Multi-dimensional correlation/volatility SHOULD allow for more complex skew effects
Calibration single-asset stochastic volatility models:

<table>
<thead>
<tr>
<th>Model</th>
<th>error</th>
<th>(\rho_1(\rho_{11}))</th>
<th>(\rho_2(\rho_{12}))</th>
<th>(\rho_{21})</th>
<th>(\rho_{22})</th>
</tr>
</thead>
<tbody>
<tr>
<td>Heston</td>
<td>0.00010773</td>
<td>-0.556</td>
<td>xxx</td>
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<td></td>
</tr>
<tr>
<td>BiHeston</td>
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<td>-0.866</td>
<td></td>
<td></td>
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<tr>
<td>Wishart</td>
<td>7.19E-05</td>
<td>-0.258</td>
<td>0.017</td>
<td>-0.343</td>
<td>-0.766</td>
</tr>
</tbody>
</table>
DAX calibration date: 28/08/08

Forward moneyness

Implied volatility

mkt 21/10/2008
model 21/10/2008
A Closer look at the $\sigma_{imp}$

Using perturbation method as in Benabid, Bensusan, El Karoui (2009) we can prove that

$$\sigma_{imp}^2 \sim \text{Tr}[\Sigma_t] + \frac{\text{Tr}[\frac{1}{2}(RQ + (RQ)^\top)\Sigma_t]}{\text{Tr}[\Sigma_t]}m_f$$  \hspace{1cm} (3)

A Double-Heston model would lead to

$$\sigma_{imp}^2 \sim v_1 + v_2 + \left(\frac{\rho_1\sigma_1}{2} + \frac{\rho_2\sigma_2}{2}\right)m_f$$  \hspace{1cm} (4)

- $\Sigma_{12}$ controls the slope of the skew and $\Sigma_{11} + \Sigma_{22}$ controls the level of the smile
- in the Double-Heston no factor to control the skew
Conclusions

• as far as we are interest with vanilla options the BiHeston and Wishart performs **equally**

• but the Wishart allows a better management of the **implied volatility risks**

• the **numerical cost** of the Wishart model is much more important. How to **speed up** the pricing process?
How to speed up the pricing process?
Recall the FFT methodology

\[
C(t, T, x) = \frac{B_{t,T}}{2\pi} \int_{-\infty+i\omega}^{+\infty+i\omega} e^{-i\omega k} \beta(\omega) \Phi_{Y_t}(\tau, \omega) d\omega
\]

\[
\Phi_{Y_t}(\tau, \omega) = \mathbb{E}_t \left[ e^{i\omega Y_{t+\tau}} \right] = e^{\text{Tr}[A(\tau)\Sigma_t] + b(\tau)Y_t + c(\tau)}
\]

\[
A(\tau) = A_{22}(\tau)^{-1} A_{21}(\tau),
\]

with

\[
\begin{pmatrix}
A_{11}(\tau) & A_{12}(\tau) \\
A_{21}(\tau) & A_{22}(\tau)
\end{pmatrix} = \exp \tau \begin{pmatrix}
M + i\omega Q^\top R^\top & -2Q^\top Q \\
\frac{i\omega(i\omega-1)}{2} I_n & -\left( M + i\omega Q^\top R^\top \right)^\top
\end{pmatrix},
\]
The Vol of Vol expansion

Perturb the vol-of-vol volatility matrix $Q$ by a factor $\alpha$

\[
\begin{pmatrix}
A_{11}(\tau) & A_{12}(\tau) \\
A_{21}(\tau) & A_{22}(\tau)
\end{pmatrix}
= \exp \left( \tau \begin{pmatrix}
M + \alpha i Q^T \omega R^T & -2\alpha^2 Q^T Q \\
-(\omega^2 + i\omega) I_n & -(M + \alpha i \omega Q^T R^T)^T
\end{pmatrix}
\right)
\]

Perturbation of this function w.r.t $\alpha$ (following Benabid, Bensusan, El Karoui (2009)) we can obtain that
Denote by $c_{bs}(t, T, x, \chi)$ the Black&Scholes price when $\chi = \sigma^2(T - t)$ is the integrated volatility.

\[
C(t, T, x) = c_{bs}(t, T, x, \chi_0) + \alpha \left( \text{Tr}[\tilde{A}^1 \Sigma_t] + \tilde{c}^1 \right) \partial_{x\chi}^2 c_{bs}(t, T, x, \chi_0) + \alpha^2 \left( \text{Tr}[\tilde{A}^{20} \Sigma_t] + \tilde{c}^{20} \right) \partial_{x^2 \chi}^2 c_{bs}(t, T, x, \chi_0) + \alpha^2 \left( \text{Tr}[\tilde{A}^{21} \Sigma_t] + \tilde{c}^{21} \right) \partial_{x^3 \chi}^3 c_{bs}(t, T, x, \chi_0)
\]

\[
+ \frac{\alpha^2}{2} \left( \text{Tr}[\tilde{A}^1 \Sigma_t] + \tilde{c}^1 \right)^2 \partial_{x^2 x\chi}^4 c_{bs}(t, T, x, \chi_0)
\]

with $\chi_0 = \left( \text{Tr}[A^0 \Sigma_t] + \tilde{c}^0 \right)$

All the deterministic matrices involve explicit integrals that can be computed efficiently.
Numerical results for price approximations
Dax 28/08/08 mat 0.06

forward moneyness

impl vol

model
approx 1order
approx 2order
The Multi-asset model

How to build a multi asset framework:

- Consistent with the smile in vanilla options
- With a general correlation structure
- Analytic as much as possible
Using Heston’s model

\[ dS_t^1 = S_t^1 r dt + S_t^1 \sqrt{V_t^1} dZ_t^1 \]
\[ dV_t^1 = \kappa_1 (\theta_1 - V_t^1) dt + \sigma_1 \sqrt{V_t^1} dW_t^1 \]
\[ dS_t^2 = S_t^2 r dt + S_t^2 \sqrt{V_t^2} dZ_t^2 \]
\[ dV_t^2 = \kappa_2 (\theta_2 - V_t^2) dt + \sigma_2 \sqrt{V_t^2} dW_t^2 \]

\[ dZ_t^1 dZ_t^2 = 0 \iff \text{Affinity of the model} \]

\[ \downarrow \]

\[ \frac{dS_t^1}{S_t^1} \frac{dS_t^2}{S_t^2} = 0 \]
The Wishart Affine Stochastic Correlation model

Da Fonseca, Grasselli and Tebaldi (RDR-2007):

**The model:** \( S_t = (S_t^1, \ldots, S_t^n)^\top \) and \( \Sigma_t \in M_{(n,n)} \)

\[
\begin{align*}
    dS_t &= diag[S_t] \left( \mu dt + \sqrt{\Sigma_t} dZ_t \right) \\
    d\Sigma_t &= \left( \Omega \Omega^\top + M \Sigma_t + \Sigma_t M^\top \right) dt + \sqrt{\Sigma_t} dW_t Q + Q^\top \left( dW_t \right)^\top \sqrt{\Sigma_t}
\end{align*}
\]

\( dZ_t \) is a vector BM \( (n,1) \) and \( dW_t \) is a matrix BM \( (n,n) \):

\[
\frac{dS_i}{S_i} \frac{dS_j}{S_j} = \Sigma_{ij} dt
\]
How to correlate $dZ$ and $dW$?

In Da Fonseca, Grasselli and Tebaldi (RDR-2007):

**Affinity** of the infinitesimal generator

$$dZ_t = dW_t \rho + \sqrt{1 - \rho^\top \rho} dB_t$$

where $\rho$ is a vector $(n,1)$ and $dB$ is a vector $BM(n,1)$.

- only $n$ parameters to specify the skew
- parsimonious model
- Characteristic function has an **exponential affine form**, it involves the computation of the exponential of a matrix.
Pricing plain vanilla options on single assets

- In the WASC model, the single assets evolve according to a Heston-like dynamics.

- Assets’ returns and volatilities are partially correlated:

\[
Corr_t \left( \text{Noise}(Y^1), \text{Noise}(Vol(S^1)) \right) = \frac{Q_{11}\rho_1 + Q_{21}\rho_2}{\sqrt{Q_{11}^2 + Q_{21}^2}}
\]

- Vol-Of-Vol(S_1) = \(2\sqrt{Q_{11}^2 + Q_{21}^2}\)

- Skew in the implied volatility is related with the correlation, cross-asset effects appear(systematic vs specific dependence)
Asymmetric conditional correlation and contagion effect

When prices fall inter-asset correlations grow, i.e. there’s negative covariation between shocks on returns and shocks on correlations.

\[ d < Y^i, \rho^{12} >_t \equiv \left( \sqrt{\frac{\sum_{t}^{ii}}{\sum_{t}^{jj}}} \left( 1 - (\rho_t^{12})^2 \right) Tr \left[ R_j Q \right] \right) dt \quad i, j = 1, 2. \]

\[ d \langle \rho^{12}, \Sigma^{11} \rangle_t = \sqrt{\frac{\sum_{t}^{11}}{\sum_{t}^{22}}} \left( 1 - (\rho_t^{12})^2 \right) \tilde{Q}_{12} (\tilde{Q}_{11} + \tilde{Q}_{22}) dt. \]

\( \tilde{Q} \) polar decomposition of \( Q \).
Calibration results in the multi-asset model

<table>
<thead>
<tr>
<th>Stock</th>
<th>error (WASC)</th>
<th>error (Heston)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dax</td>
<td>2.52E-05</td>
<td>1.105E-04</td>
</tr>
<tr>
<td>SP</td>
<td>1.39E-04</td>
<td>1.59E-04</td>
</tr>
</tbody>
</table>
DAX calibration date: 21/08/2008

Impied volatility

Forward moneyness

mkt 19/08/2008 model 19/08/2008
A closer loog at $\sigma_{imp}$

We can prove

$$\sigma_{imp}^{Dax} \sim \Sigma_{t}^{11} + (\rho_{1} Q_{11} + \rho_{2} Q_{21}) m_{f}$$

- vanilla option is a basket products!
Conclusions

- we build a model which is tractable
- allows for stochastic volatilities and stochastic correlation
- provide some results on calibration using single underlying options
Thanks for your attention!