Numerical Evaluation of American Options Written on Two Underlying Assets using the Fourier Transform Approach

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McKean (1965) first solved the American pricing problem as a free boundary value problem.

The pricing PDE is transformed to the corresponding ODE using the incomplete Fourier Transforms.


We attempt to extend this approach to options with more than one underlying asset.

Here we use Jamshidian’s (1992) approach to set up the PDE as an inhomogeneous problem and present the corresponding general soln.

Fourier transforms are applied to the PDE for the density function.
The Problem Statement

- Under the risk-neutral probability measure, $S_1$ and $S_2$ are driven by the GBM processes,

$$
dS_1 = (r - q_1)S_1 dt + \sigma_1 S_1 d\tilde{W}_1
$$
$$
dS_2 = (r - q_2)S_2 dt + \sigma_2 S_2 d\tilde{W}_2.\tag{1.1}
$$

- By letting $S_i = e^{x_i}$ for $i = 1, 2$, we represent the American option price as $C(\tau, x_1, x_2)$, where the price satisfies the inhomogeneous PDE,

$$
\frac{\partial C}{\partial \tau} = \mathcal{L}C - rC - 1_{\{\text{cont}\}} \mathcal{L}c(x_1, x_2),\tag{1.2}
$$

- $\mathcal{L}$ is the Dynkin operator of (1.1) defined in equation (1.3) below.
The Problem Statement

The Dynkin operator is defined as,

\[
\mathcal{L} = (r - q_1 - \frac{1}{2}\sigma_1^2) \frac{\partial}{\partial x_1} + (r - q_2 - \frac{1}{2}\sigma_2^2) \frac{\partial}{\partial x_2} + \frac{1}{2}\sigma_1^2 \frac{\partial^2}{\partial x_1^2} \\
+ \rho \sigma_1 \sigma_2 \frac{\partial^2}{\partial x_1 \partial x_2} + \frac{1}{2}\sigma_2^2 \frac{\partial^2}{\partial x_2^2}.
\]  
(1.3)

We solve (1.2) subject to ICs and BCs,

\[
C(0, x_1, x_2) = c(x_1, x_2), \quad -\infty < x_1, x_2 < \infty \quad \text{(1.4)}
\]

\[
C(\tau, 0, 0) = 0, \quad 0 \leq \tau \leq T, \quad \text{(1.5)}
\]

1 and smooth pasting conditions imposed to ensure continuity at the early exercise boundary.

2 These will be specified for particular payoff functions.
By using Duhamel’s principle, the general solution of the PDE (1.2) can be represented as,

\[ C(\tau, x_1, x_2) = C_E(\tau, x_1, x_2) + C_P(\tau, x_1, x_2), \]  

where,

\[ C_E(\tau, x_1, x_2) = e^{-r\tau} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} c(u_1, u_2) \times U(\tau, x_1, x_2; u_1, u_2) \, du_1 \, du_2, \]

\[ C_P(\tau, x_1, x_2) = \int_{0}^{\tau} e^{-r(\tau - \xi)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi, u_1, u_2) \times U(\tau - \xi, x_1, x_2; u_1, u_2) \, du_1 \, du_2 \, d\xi. \]

The 1\textsuperscript{st} part of (1.6) is the European Option component and the 2\textsuperscript{nd} is the Early Exercise premium.

\( U(\tau, x_1, x_2; u_1, u_2) \) is the Green’s function satisfying the Kolmogorov PDE.
The transformed PDE for the density function becomes,

\[
\frac{\partial U}{\partial \tau} = \left( r - q_1 - \frac{1}{2} v_1 \right) \frac{\partial U}{\partial x_1} + \left( r - q_2 - \frac{1}{2} v_2 \right) \frac{\partial U}{\partial x_2} \\
+ \frac{1}{2} \sigma_1^2 \frac{\partial^2 U}{\partial x_1^2} + \rho \sigma_1 \sigma_2 \frac{\partial^2 U}{\partial x_1 \partial x_2} + \frac{1}{2} \sigma_2^2 \frac{\partial^2 U}{\partial x_2^2}.
\] (1.7)

This is solved subject to the initial condition,

\[
U(\tau; x_1, x_2; x_{1,0}, x_{2,0}) = \delta(e^{x_1} - e^{x_{1,0}})\delta(e^{x_2} - e^{x_{2,0}}).
\]

In solving equation (1.7), we first transform to the corresponding ODE by using transform techniques.

We give key results of the Fourier Transform approach.
The Fourier Transforms

- The Fourier Transform of a function, $U(\tau, x_1, x_2)$, is defined as,
  \[
  \mathcal{F}\{U(\tau, x_1, x_2)\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\eta_1 x_1 + i\eta_2 x_2} U(\tau, x_1, x_2) dx_1 dx_2
  = \hat{U}(\tau, \eta_1, \eta_2).
  \tag{1.8}
  \]

- The inverse of the Fourier Transform is represented as,
  \[
  \mathcal{F}^{-1}\{\hat{U}(\tau, \eta_1, \eta_2)\} = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i\eta_1 x_1 - i\eta_2 x_2} \hat{U}(\tau, \eta_1, \eta_2) d\eta_1 d\eta_2
  = U(\tau, x_1, x_2).
  \tag{1.9}
  \]

- Given these definitions, we apply to the PDE (1.7) for the density function.
Applying the Fourier Transform to the PDE

Proposition

The Fourier Transform of the PDE (1.7) satisfies the ODE,

\[ \frac{\partial \hat{U}}{\partial \tau} (\tau, \eta_1, \eta_2) + \left[ i\eta_1 \kappa_1 + i\eta_2 \kappa_2 + \frac{1}{2} \sigma_1^2 \eta_1^2 + \eta_1 \eta_2 \rho \sigma_1 \sigma_2 \right. \\
\left. + \frac{1}{2} \sigma_2^2 \eta_2^2 \right] \hat{U}(\tau, \eta_1, \eta_2) = 0. \] (1.10)

Equation (1.10) is solved subject to the initial condition,

\[ \hat{U}(0, \eta_1, \eta_2) = e^{i\eta_1 x_{1,0} + i\eta_2 x_{2,0}}. \] (1.11)
Inverting the Fourier Transform

The Inverse Fourier Transform of (1.10) is given by,

\[
U(\tau, x_1, x_2) = \frac{1}{2\pi \sigma_1 \sigma_2 \tau \sqrt{1 - \rho^2}} \exp \left\{ \frac{-1}{2\tau(1 - \rho^2)} \left[ \left( \frac{x_1 - x_{1,0} + \kappa_1 \tau}{\sigma_1} \right)^2 - 2\rho \left( \frac{x_1 - x_{1,0} + \kappa_1 \tau}{\sigma_1} \right) \left( \frac{x_2 - x_{2,0} + \kappa_2 \tau}{\sigma_2} \right) + \left( \frac{x_2 - x_{2,0} + \kappa_2 \tau}{\sigma_2} \right)^2 \right] \right\}
\]

(1.12)

Equation (1.12) is the bivariate transition density function of the two stochastic processes, \(x_1\) and \(x_2\).

Given this, we now have full representation of the American call option price, equation (1.6).

We present the American Spread and Max Option examples.
The American Spread call is an option written on the difference of two underlying assets.

The American Spread Option problem is solved subject to initial and boundary conditions

\[ c(x_1, x_2) = \max(0, e^{x_1} - e^{x_2} - K) \] which is the payoff at maturity,

\[ C(\tau, -\infty, x_2) = 0 \] which is an absorbing state.

\[ C(\tau, B(\tau, x_2), x_2) = c(B(\tau, x_2), x_2), \implies \text{value matching condition for continuity at the early exercise boundary.} \]

\[ \lim_{x_1 \to B(\tau, x_2)} \frac{\partial C}{\partial x_1} = 0 \]

\[ \implies \text{the smooth pasting condition imposed to avoid arbitrage opportunities.} \]

\(^1\text{Note } B(\tau, x_2) = \ln b(\tau, S_2)\)
There are different forms of Spread options trading on the markets which includes:

- **Fixed income markets**
  - Derived from differences in interest rates in different countries

- **Agricultural Futures Markets**
  - Soybean complex spread (CBOT)
  - Underlying comprises of the futures contracts of soybean oil and soybean meal.

- **Energy markets**
  - Used to quantify the cost of production of refined products from the complex raw materials used to produce them,
Proposition

The early exercise surface of the American Spread Call option solves,

\[ B(\tau, x_2) - e^{x_2} - K = C(\tau, B(\tau, x_2), x_2) \]  \hspace{1cm} (1.13)

Corresponding price

\[ C(\tau, x_1, x_2) = C_E(\tau, x_1, x_2) + C_P(\tau, x_1, x_2). \]  \hspace{1cm} (1.14)

- A non-linear Volterra integral eqn; solve numerically using extd Simpson’s rule.
- Root finding yields free boundary at each time step.
To validate our approach, we use Monte Carlo algorithm of Ibañez & Zapatero (2004).

- The American option is treated like a Bermudan Option.
- The early exercise surface is tracked by using DP.
- Valuation algorithm heavily dependent on the value-matching condition, that is,

\[ V(\tau, b(\tau, S_2), S_2) = v(b(\tau, S_2), S_2), \] (1.15)

- Discretize the time domain such that \( \tau_i = ih \) for \( i = 0, 1, \ldots, M \).
- At each time step, use knowledge of both the price and exercise surface at the previous time steps.
- So the value of the American option can be treated like a European option
- Thus we can apply the plain vanilla Monte Carlo simulation.
- At maturity, \( \tau_0 \), the American option value is \( (S_1,\tau_0 - S_2,\tau_0 - K)^+ \).
Numerical Implementation of the American Spread Option Using Monte Carlo Algorithm cont...

Figure: Possible Paths of $S_1$
Numerical Results

- Parameters: $K = 5$, $r = 5\%$, $q_1 = 3\%$, $q_2 = 2\%$, $\sigma_1 = 40\%$, $\sigma_2 = 49\%$, $\rho = 0.5$, and $T = 1$. 

Early Exercise Surface of the American Spread Option Using Fourier Transform Approach
Numerical Results cont...

![Early Exercise and Continuation Regions](image-url)

- Early Exercise Region
- Continuation Region

$B(\tau, S_2)$

$B(T, S_2)$

$B(0, S_2)$
Numerical Results cont...

Early Exercise Surface Differences

Relative Differences

Chiarella & Ziveyi: 15th International Conference on Computing in Economics and Finance
Numerical Results cont...

When $S_1 = 35$, $S_2 = 30$ and $K = 5$, $\Rightarrow$ Relative Price Difference $= 7.796\%$.

When $S_1 = 60$, $S_2 = 50$ and $K = 5$, $\Rightarrow$ Relative Price Difference $= 6.364\%$. 
The American max call option is a contact whose payoff is derived from the maximum of two assets.

The American max option has two exercise regions which we call $B_1(\tau, x_2)$ and $B_2(\tau, x_1)$.

The American max option problem is solved subject to:
- The payoff $\implies c(x_1, x_2) = \max[\max(e^{x_1}, e^{x_2}) - K, 0]$,
- Absorbing state $C(\tau, -\infty, -\infty) = 0$
- Value matching condition,

$$C(\tau, B_1(\tau, x_2), B_2(\tau, x_1)) = c(B_1(\tau, x_2), B_2(\tau, x_1)),$$
Proposition

- The early exercise surface of the American max call option solves,

\[
B_1(\tau, x_2) - K = C_E(\tau, B_1(\tau, x_2), x_2) + C_P(\tau, B_1(\tau, x_2), x_2) \\
B_2(\tau, x_1) - K = C_E(\tau, x_1, B_2(\tau, x_1)) + C_P(\tau, x_1, B_2(\tau, x_1)),
\]

and the terminal values of the two boundary functions are defined as,

\[
B_1(0, x_2) = \max \left[e^{x_2}, \max \left(K, \frac{r}{q_2}K\right)\right] \quad (1.16) \\
B_2(0, x_1) = \max \left[e^{x_1}, \max \left(K, \frac{r}{q_1}K\right)\right]. \quad (1.17)
\]

- Corresponding price

\[
C(\tau, x_1, x_2) = C_E(\tau, x_1, x_2) + C_P(\tau, x_1, x_2). \quad (1.18)
\]
Numerical Results for the Max Option

- Parameters $K = 100$, $r = 5\%$, $q_1 = 18\%$, $q_2 = 18\%$, $\sigma_1 = 20\%$, $\sigma_2 = 20\%$ and $T = 3$.
- Assume 5 exercise opportunities.

Figure: The 1st Exercise Opportunity when $\rho = -0.5 \& 0.5$
Numerical Results cont...

Figure: The 2\textsuperscript{nd} Second Exercise Opportunity when $\rho = -0.5$ & 0.5

Figure: The 3\textsuperscript{rd} Exercise Opportunity when $\rho = -0.5$ & 0.5
Numerical Results cont...

Figure: The $4^{th}$ Exercise Opportunity when $\rho = -0.5 \& 0.5$

Figure: The $5^{th}$ Exercise Opportunity when $\rho = -0.5 \& 0.5$
Numerical Results cont...

Figure: Price Profile of the Max Option
Conclusion

- We have managed to derive and solve the integral representation of American options written on two underlying assets.
- Details of how to solve the free boundary at each time step.
- Presented two examples for the Spread call and Max option.