Dynamic Operational Risk: modelling dependence and combining different sources of information.

Statistical Modelling and Operational Risk

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Outline: Part I

1. Operational Risk Review
   - Basel II and Operational Risk

2. Dynamic Operational Risk and Dependence Modelling
   - Loss Distributional Approach (LDA)
   - Dependence Modelling in LDA Models
   - Stochastic Model (Peters, Shevchenko, Wüthrich, 2009)
   - Dependence via Copulas

3. Bayesian Model, Simulation Methodology and Examples
   - Bayesian Model
   - Markov chain Monte Carlo
   - Simulation Study
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**BASEL II ACCORD - (2004)**

**Pillar 1: Minimum Capital Requirements**
- **Credit Risk**
  1. Standardised Approach
  2. Foundation Internal Ratings Based (IRB)
  3. Advanced Internal Ratings Based (IRB)
- **Market Risk**
  1. Standardised Approach
  2. Internal Models Approach (IMA)
- **Operational Risk**
  1. Basic Indicator Approach
  2. Standardised Approach
  3. Advanced Measurement Approach (AMA)

**Pillar 2: Supervisory Review**
1. Institutions capital adequacy
2. Internal assessment processes

**Pillar 3: Market Discipline**
1. Disclosure of information

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These are linked in Australian Regulation (APRA) – APS115
Operational risk for a financial institution:

"the risk of loss resulting from inadequate or failed internal processes, people and systems or from external events."

(Basel Banking Supervision, 2006, p144)

- Includes - Legal Risk
- Excludes - Reputational Risk
Operational Risk (OpRisk) is significant in most financial institutions.

OpRisk severities can be extreme, for example:

- 1995: Barings Bank (loss GBP 1.3 billion)
- 1996: Sumitomo Corporation (loss USD 2.6 billion)
- 2001: September 11
- 2001: Enron (loss USD 2.2 billion)
- 2002: Allied Irish (loss GBP 450m)
- 2004: National Australia Bank (AUD 360m)
- Societe Generale (Euro 4.9 billion)
Business Lines and Risk Types - Brief Overview (4)

**Business Lines**
- Corporate Finance
- Payment and Settlement
- Trading and Sales
- Agency Services
- Retail Banking
- Asset Management
- Commercial Banking
- Retail Brokerage

**Risk Types**
- **Internal Fraud**
  1. intentional misreporting
  2. employee theft
  3. insider trading
- **External Fraud**
- **Employment Practices and Workplace Safety**
  1. workers comp. claims
  2. OH&S violations
  3. discrimination claims
- **Clients, Products and Business Practices**
  1. improper trading activities
  2. money laundering
  3. unauthorised products
- **Damage to Physical Assets**
  1. terrorism
  2. vandalism
  3. earthquakes
  4. fires
  5. floods.
- **Business Disruption and System Failure**
  1. hardware & software failures
  2. telecommunication problems
  3. utility outages
  4. computer viruses
- **Execution, Delivery and Process Management**
  1. data entry errors
  2. management failures
  3. incomplete legal docs
Overview

- **Insurance**: Internal Loss Data (scaling, bias), External Loss Data (scaling, bias), Expert Opinions (business judgements on # losses and $)
- **Dependence Factors**: Risk Frequency and Severity distributions
- **Control/Risk Indicators**: Risk Annual Loss Distributions
- **Monte Carlo Simulations**
- **Total Annual Loss Distributions (over all risks)**
- **Annual Capital Charge**: (Expected and Unexpected Losses) Capital Allocation, Scenario Analysis

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Loss Distribution Approach model

- Annual loss in risk cell (business line/event type) is a compound random variable,
  \[ Z_t^{(j)} = \sum_{s=1}^{N_t^{(j)}} X_s^{(j)}(t). \]
  \[ t \in \mathbb{N} \text{ discrete time (annual units) \& } j^{th} \text{ risk cell} \]

- Total annual loss distribution in year \( t \in \mathbb{N} \) is a J-fold convolution
  \[ f_{Z_t^{(1)}(Z_1)} \ast f_{Z_t^{(2)}(Z_2)} \ast \cdots \ast f_{Z_t^{(J)}(Z_J)}. \]
  \[ Z_t = \sum_{j=1}^{J} Z_t^{(j)}. \]

for OpRisk \( J = 56. \)
Loss Distribution Approach model (2)

**Frequency**

- $N_t^{(j)}$ r.v. for annual number of events, with $N_t^{(j)} \sim P^{(j)} \left( \cdot | \lambda_t^{(j)} \right)$

- $P^{(j)} \left( \cdot | \lambda_t^{(j)} \right)$ is a frequency counting distribution, which also depends on time dependent parameter(s) $\lambda_t^{(j)}$

**Severity**

- $X_s^{(j)} (t)$ r.v. for severity in year $t$, with $X_s^{(j)} (t) \sim f^{(j)} \left( \cdot | \chi_t^{(j)} \right)$

- $f^{(j)} \left( \cdot | \chi_t^{(j)} \right)$ is a severity distribution with parameter(s) $\chi_t^{(j)}$
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Frequency Dependence:

- Between frequencies $N_t^{(j)}$ and $N_t^{(i)}$:
  - **Copula methods.** Note copula methods for discrete random variables needs care.
  - **Common shocks** - Model events affecting many cells at the same time. Results in dependence between frequencies of the risks if superimposed with cell internal events.

- **Dependence between event times of different risks:** e.g. the 1$^{st}$ event time of the $j^{th}$ risk correlated to the 1$^{st}$ event time of the $i^{th}$ risk, etc.
Severity Dependence:

- Dependence between loss amounts: e.g. first loss amount of the $j^{th}$ risk correlated to first loss of $i^{th}$ risk, second loss in $j^{th}$ risk correlated to second loss in $i^{th}$ risk, etc).
- Can be difficult to interpret e.g. high frequency versus low frequency risks.

Annual Loss Dependence:

- Dependence between annual losses directly via copula methods.
- Can create irreconcilable problems with modelling insurance for OpRisk which directly involves event times.
- Problematic to quantify correlations using historical data, and the LDA model will lose its structure.
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Stochastic Model

- **Stochastically evolving in time risk profiles:**
  \[ \lambda_t = \left( \lambda_t^{(1)}, \ldots, \lambda_t^{(J)} \right) \] and \[ \psi_t = \left( \psi_t^{(1)}, \ldots, \psi_t^{(J)} \right) \] modelled by random vectors \( \Lambda_t \) and \( \Psi_t \).

**Example:**

- **Stochastic modelling of risk profiles is intuitive.**
  - Assume the annual number of events for the \( j^{th} \) risk, in year \( t \), is a random variable with Poisson distribution
    \[ \text{Poi} \left( \Lambda_t^{(j)} = \lambda_t^{(j)} \right). \]
  - Conditional on \( \Lambda_t^{(j)} \), the expected number of events per year is \( \Lambda_t^{(j)} \).
  - This changes for different banks and different risks and also changes from year to year for a risk in the same bank \( \Rightarrow \) stochastic model for risk profiles.
Evolution of $\Lambda_t^{(j)}$, modelled with deterministic (trend, seasonality) and stochastic components. (Actuaries - mixed Poisson model.)

Given sequence $(\Lambda_1, \Psi_1), \ldots, (\Lambda_{T+1}, \Psi_{T+1})$ it is naive to assume risk profiles of all risks are independent.

- Dependence present due to changes in politics, regulations, law, economy, technology (sometimes called drivers or external risk factors) that jointly impact on many risk cells at each time instant.

- Dependence between risks introduced through dependence between their risk profiles $\Lambda_t$ and $\Psi_t$. 
Stochastic Model

- $J$ risk processes, each with annual loss in year $t$, $Z_t^{(j)}$
- $Z_t^{(j)}$ modelled by severity $X_s^{(j)}(t)$ and frequency $N_t^{(j)}$.
- Frequency and severity risk profiles are modelled by random vectors $\Lambda_t = (\Lambda_1^{(1)}, \ldots, \Lambda_J^{(J)})$ and $\Psi_t = (\Psi_1^{(1)}, \ldots, \Psi_J^{(J)})$.
- $\Lambda_t$ and $\Psi_t$ are parameterized by risk characteristics $\theta_{\Lambda} = (\theta_{\Lambda_1^{(1)}}, \ldots, \theta_{\Lambda_J^{(J)}})$ and $\theta_{\Psi} = (\theta_{\Psi_1^{(1)}}, \ldots, \theta_{\Psi_J^{(J)}})$.
- Dependence between risk profiles is parameterized by $\theta_\rho$.
- Given $\theta$, between different years, the risk profiles for frequencies and severities as well as the number of losses and actual losses are independent.
Stochastic Model (4)

Given risk characteristics \( \theta = (\theta_\Lambda, \theta_\Psi, \theta_\rho) \):

1. The random vectors,

\[
\begin{align*}
(\Psi_1, \Lambda_1, N_1^{(j)}, X_s^{(j)} (1); j = 1, \ldots, J, s \geq 1) \\
\vdots \\
(\Psi_{T+1}, \Lambda_{T+1}, N_{T+1}^{(j)}, X_s^{(j)} (T + 1); j = 1, \ldots, J, s \geq 1)
\end{align*}
\]

are independent.

2. The risk profile random vectors \((\Psi_1, \Lambda_1), \ldots, (\Psi_{T+1}, \Lambda_{T+1})\) are i.i.d. from a joint distribution with marginal distributions \(\Lambda_t^{(j)} \sim G\left(\cdot|\theta_\Lambda^{(j)}\right),\) \(\Psi_t^{(j)} \sim H\left(\cdot|\theta_\Psi^{(j)}\right)\) and \(2J\)-dimensional copula \(C(\cdot|\theta_\rho)\).

3. Given \(\Lambda_t = \lambda_t\) and \(\Psi_t = \psi_t\): the compound random variables \(Z_t^{(1)}, \ldots, Z_t^{(J)}\) are independent with \(N_t^{(j)}\) and \(X_1^{(j)} (t), X_2^{(j)} (t), \ldots\) independent; frequencies \(N_t^{(j)} \sim P\left(\cdot|\lambda_t^{(j)}\right)\) and severities \(X_s^{(j)} (t) \sim F\left(\cdot|\psi_t^{(j)}\right), s \geq 1.\)
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Copula models

- **Copula function** $C(\cdot)$ used to model dependence between the risk profiles, $(\Psi_1, \Lambda_1), \ldots, (\Psi_{T+1}, \Lambda_{T+1})$.

- **Gaussian copula**: no upper or lower tail dependence

  $$c (u_1, \ldots, u_d | \Sigma) = \frac{f^\Sigma_n \left( F_n^{-1}(u_1), \ldots, F_n^{-1}(u_d) \right)}{\prod_{i=1}^{d} f_n \left( F_n^{-1}(u_i) \right)}.$$

- **Clayton copula**: lower tail dependence

  $$c (u_1, \ldots, u_d | \rho) = \left( 1 - d + \sum_{i=1}^{d} (u_i)^{-\rho} \right)^{-d-\frac{1}{\rho}} \prod_{i=1}^{d} \left( (u_i)^{-\rho-1} \{ (i-1) \rho + 1 \} \right),$$

  where $\rho > 0$ is a dependence parameter.

- **Gumbel copula**: upper tail dependence

  $$C (u_1, \ldots, u_d | \rho) = \exp \left\{ - \left( \sum_{i=1}^{d} (- \log (u_i))^\rho \right)^{\frac{1}{\rho}} \right\},$$

  where $\rho \geq 1$ is a dependence parameter.
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The following are regulatory requirements:

- Basel II AMA (see BIS, p.152) states
  "Any operational risk measurement system must have certain key features to meet the supervisory soundness standard set out in this section. These elements must include the use of internal data, relevant external data, scenario analysis and factors reflecting the business environment and internal control systems".
**Example:** Modelling frequencies for multiple risk cells

1. Consider J risk cells each with fixed, deterministic volume $V^{(j)}$.

2. **Priors:** Risk characteristic $\Theta_{\Lambda} = (\Theta_{\Lambda}^{(1)}, \ldots, \Theta_{\Lambda}^{(J)})$ has $J$-dimensional prior density $\pi(\theta_{\Lambda})$. Copula parameters $\Theta_{\rho}$ have prior density $\pi(\theta_{\rho})$.
   - $\Theta_{\Lambda}$ and $\Theta_{\rho}$ are independent.

3. Given $\Theta_{\Lambda} = \theta_{\Lambda}$ and $\Theta_{\rho} = \theta_{\rho}$: $(\Lambda_1, N_1), \ldots, (\Lambda_{T+1}, N_{T+1})$ are i.i.d. and the intensities $\Lambda_t = (\Lambda_t^{(1)}, \ldots, \Lambda_t^{(J)})$ have a $J$-dimensional conditional density with marginal distributions $\Lambda_t^{(j)} \sim G(\cdot | \theta_{\Lambda}^{(j)}) = \Gamma(\alpha^{(j)}, \alpha^{(j)}/\theta_{\Lambda}^{(j)})$ and the copula $c(\cdot | \theta_{\rho})$. Thus the joint density of $\Lambda_t$ is given by

$$
\pi(\lambda_t | \theta_{\Lambda}, \theta_{\rho}) = c\left(G(\lambda_t^{(1)} | \theta_{\Lambda}^{(1)}), \ldots, G(\lambda_t^{(J)} | \theta_{\Lambda}^{(J)} | \theta_{\rho})\right) \prod_{j=1}^{J} \pi(\lambda_t^{(j)} | \theta_{\Lambda}^{(j)}),
$$
4. Given $\Theta_{\Lambda} = \theta_{\Lambda}$ and $\Lambda_t = \lambda_t$: the number of claims are independent with $N_t^{(j)} \sim \text{Poi}(\nu^{(j)}\lambda_t^{(j)}), j = 1, \ldots, J$.

5. There are scenario analysis data $\Delta_k = (\Delta_k^{(1)}, \ldots, \Delta_k^{(J)}), k = 1, \ldots, K$ on $\theta_{\Lambda}$. Given $\Theta_{\Lambda} = \theta_{\Lambda}$: $\Delta_k$ and $(\Lambda_t, N_t)$ are independent for all $k$ and $t$; and $\Delta_k^{(j)}$ are all independent with $\Delta_k^{(j)} \sim \Gamma(\xi^{(j)}, \xi^{(j)}/\theta_{\Lambda}^{(j)})$.

**Prior Structure:** $\pi(\theta_{\Lambda})$ and $\pi(\theta_{\rho})$.

- Risk characteristics $\Theta_{\Lambda}^{(j)}$: independent $\Theta_{\Lambda}^{(j)} \sim \Gamma(a^{(j)}, b^{(j)})$.
  - If the company has a bad risk profile in risk class $j$ then the risk profile in risk class $i$ need not necessarily also be bad.

- **Copula parameter** - uniform prior (ranges $[-1,1], (0,30]$ and $[1,30]$ for Gaussian, Clayton and Gumbel copulas respectively).
Posterior density. The marginal posterior density of random vector \((\Theta_\Lambda, \Theta_\rho)\) given data of counts \(N_1 = n_1, \ldots, N_T = n_T\) and scenario analysis data \(\Delta_1 = \delta_1, \ldots, \Delta_K = \delta_K\) is

\[
\pi(\theta_\Lambda, \theta_\rho | n_1:T, \delta_1:K) = \prod_{t=1}^T \int \pi(\theta_\Lambda, \theta_\rho, \lambda_t | n_1:T, \delta_1:K) d\lambda_t
\]

\[
\propto \prod_{t=1}^T \left( \int \prod_{j=1}^J \exp \left\{ -V(j) \lambda_t^{(j)} \right\} \frac{(V(j) \lambda_t^{(j)})^{n_t^{(j)}}}{n_t^{(j)}!} \pi(\lambda_t | \Theta_\Lambda, \Theta_\rho) d\lambda_t \right)
\]

\[
\times \prod_{k=1}^K \prod_{j=1}^J \left( \frac{(\xi^{(j)}/\theta_\Lambda^{(j)})^{\xi^{(j)}}}{\Gamma(\xi^{(j)})} (\delta_k^{(j)})^{\xi^{(j)}-1} \exp \left\{ -\delta_k^{(j)} \xi^{(j)}/\theta_\Lambda^{(j)} \right\} \right)
\]

\[
\times \prod_{j=1}^J \frac{(b(j)^{a(j)})^{\theta_\rho^{(j)}}}{\Gamma(a(j))} (\theta_\Lambda^{(j)})^{\theta_\rho^{(j)}-1} \exp \left\{ -b(j) \theta_\Lambda^{(j)} \right\} \pi(\theta_\rho).
\]
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Markov chain Sampling Methodology

- Derived univariate full conditional distributions:
  \[
  \pi(\theta^{(j)}_\Lambda | \theta^{(-j)}_\Lambda, \lambda_{1:T}, n_{1:T}, \delta_{1:K}, \theta_\rho) \\
  \pi(\lambda_t^{(j)} | \theta_\Lambda, \lambda^{(-t,-j)}_{1:T}, n_{1:T}, \delta_{1:K}, \theta_\rho) \\
  \pi(\theta_\rho | \theta_\Lambda, \lambda_{1:T}, n_{1:T}, \delta_{1:K})
  \]

- Developed deterministic scan univariate Slice Sampler for posterior \( \pi(\theta_\Lambda, \lambda_t, \theta_\rho | n_{1:T}, \delta_{1:K}) \) (discard Markov chain samples corresponding to "nuisance" parameters \( \lambda_t, \theta_\rho \)).
  - Slice sampling provides "black box" approach for sampling the target distribution only known up to normalization.

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Obtaining a sample using a univariate Slice sampler:

1. Sample $u$ from a uniform distribution

$$U \left[ 0, \pi \left( \theta_{\Lambda,l}^{(j)}, \lambda_{1:T,l}, n_{1:T}, \delta_{1:K}, \theta_\rho \right) \right].$$

2. Sample $\tilde{\theta}_{\Lambda}^{(j)}$ uniformly from the intervals (level set)

$$A = \left\{ \theta_{\Lambda}^{(j)} : \pi \left( \theta_{\Lambda}^{(j)}, \lambda_{1:T,l}, n_{1:T}, \delta_{1:K}, \theta_\rho \right) > u \right\}.$$
Figure: Markov chain created for $\Theta_\Lambda$ and auxiliary random variable $U$, $(u_1, \theta_{\Lambda,1}), \ldots, (u_{l-1}, \theta_{\Lambda,l-1}), (u_l, \theta_{\Lambda,l}), \ldots$ has stationary distribution with the desired marginal distribution $p(\theta_{\Lambda})$. 

\[ p(\theta) \]
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Simulation Study

Simulation Settings

- True parameter values $\Theta_\Lambda^{(1)}$ and $\Theta_\Lambda^{(2)}$ set as $\theta_{true}^{(1)} = 5$ and $\theta_{true}^{(2)} = 5$
- Scenario analysis data on the true parameters set as an underestimate in risk profile 1 with $\Delta_1^{(1)} = 2$ and an overestimate for risk profile 2 with $\Delta_1^{(2)} = 8$.
- Model parameters set to $\xi^{(1)} = \xi^{(2)} = 2$, $\alpha^{(1)} = \alpha^{(2)} = 2$, and prior distribution parameters $a^{(1)} = a^{(2)} = 2$, $b^{(1)} = b^{(2)} = 2.5$. 

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Statistical Modelling and Operational Risk
Simulation Study (2)

- One scenario analysis data point is assumed for each risk.
- **Accuracy of the parameter estimates vs number of observations studied.**
  - Simulation results involve 20 independent data sets, each of length 20 years - *each data set simulations are performed for subsets of the data going for 1, 2, 5, 10, 15 and 20 years.*
- **Average performance of posterior estimates reported.**
Simulation Study (3)

Sampler Settings

- Markov chains run for 50,000 iterations with 10,000 iterations discarded as burnin.
- Simulation time depends on the number of risk profiles, the number of observations and expert opinions and the length of the Markov chain.
- Coded in Fortran and run on 2.40GHz Intel Core2.
  - Typical run with 5 years of data and 1 expert in the bivariate case for 50,000 simulations took approximately 50sec.
  - Approximately 43min for the case of ten risk profiles.
Simulation Study (4)

- **Joint**: results from MCMC samples from
  \[ \pi (\theta_\Lambda, \lambda_{1:T} | \mathbf{n}_{1:T}, \delta_{1:K}, \theta_\rho) \]  
  with correct copula model and copula parameter used in the sampler.

- **Marginal**: results from MCMC samples from
  
  \[ \pi (\theta_\Lambda, \lambda_{1:T} | \mathbf{n}_{1:T}, \delta_{1:K}, \theta_\rho) = \prod_{j=1}^{J} \pi (\theta_\Lambda^{(j)} | \mathbf{n}_{1:T}, \delta_{1:K}, \theta_\rho) \]

  which is the posterior in the case of independence.

  *(Marginal estimation where single risk cell data is analyzed separately)*

- **Benchmark**: verification case - assume perfect knowledge of the realized random process \( \Lambda_{1:T} \) and perform inference on \( \Theta_\Lambda \).
  - Samples taken from \( \pi (\theta_\Lambda | \lambda_{1:T}, \mathbf{n}_{1:T}, \delta_{1:K}, \theta_\rho) \) conditional on the true simulated realizations of random variables \( \Lambda_{1:T} \).
### Table: Average estimates of posterior mean and standard deviation of $\Theta^{(1)}_\Lambda$ for 20 data sets. Data are generated using different copula models as specified. The true values are $\theta^{(1)}_{true} = \theta^{(2)}_{true} = 5$. 

<table>
<thead>
<tr>
<th>Year</th>
<th>1</th>
<th>2</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
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<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Independent</td>
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<tr>
<td>Marginal</td>
<td>3.72 (2.04)</td>
<td>4.10 (1.98)</td>
<td>4.08 (1.62)</td>
<td>4.64 (1.42)</td>
<td>5.13 (1.31)</td>
<td>5.24 (1.27)</td>
</tr>
<tr>
<td>Benchmark</td>
<td>4.32 (1.88)</td>
<td>4.50 (1.67)</td>
<td>4.84 (1.46)</td>
<td>5.17 (1.26)</td>
<td>5.19 (1.12)</td>
<td>5.21 (1.02)</td>
</tr>
<tr>
<td>Joint</td>
<td>3.91 (2.01)</td>
<td>4.41 (1.72)</td>
<td>4.37 (1.56)</td>
<td>4.76 (1.33)</td>
<td>5.10 (1.21)</td>
<td>4.95 (1.05)</td>
</tr>
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<td>4.09 (1.97)</td>
<td>4.06 (1.61)</td>
<td>4.48 (1.37)</td>
<td>5.07 (1.29)</td>
<td>5.04 (1.13)</td>
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<tr>
<td>Gaussian copula with ($\rho = 0.9$)</td>
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<td>Benchmark</td>
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<td>5.00 (0.84)</td>
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<td>4.96 (1.08)</td>
<td>4.90 (0.93)</td>
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<td>4.43 (2.10)</td>
<td>4.54 (1.74)</td>
<td>4.47 (1.36)</td>
<td>4.75 (1.22)</td>
<td>4.72 (1.08)</td>
</tr>
<tr>
<td>Clayton copula with ($\rho = 10$)</td>
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<tr>
<td>Benchmark</td>
<td>4.32 (1.98)</td>
<td>4.46 (1.70)</td>
<td>4.86 (1.41)</td>
<td>5.08 (1.16)</td>
<td>5.16 (1.01)</td>
<td>5.11 (0.88)</td>
</tr>
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<td>Joint</td>
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<td>4.21 (1.80)</td>
<td>4.54 (1.56)</td>
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<td>4.17 (1.62)</td>
<td>4.63 (1.41)</td>
<td>4.74 (1.22)</td>
<td>4.72 (1.07)</td>
</tr>
<tr>
<td>Gumbel copula with ($\rho = 3$)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Benchmark</td>
<td>4.32 (1.98)</td>
<td>4.46 (1.70)</td>
<td>4.86 (1.41)</td>
<td>5.08 (1.16)</td>
<td>5.16 (1.01)</td>
<td>5.11 (0.88)</td>
</tr>
<tr>
<td>Joint</td>
<td>4.33 (2.06)</td>
<td>4.21 (1.80)</td>
<td>4.54 (1.56)</td>
<td>4.96 (1.23)</td>
<td>5.01 (1.05)</td>
<td>4.98 (0.93)</td>
</tr>
<tr>
<td>Marginal</td>
<td>3.84 (2.08)</td>
<td>3.76 (1.87)</td>
<td>4.17 (1.62)</td>
<td>4.63 (1.41)</td>
<td>4.74 (1.22)</td>
<td>4.72 (1.07)</td>
</tr>
</tbody>
</table>
Bayesian Model
Markov chain Monte Carlo
Simulation Study

Figure: Scatter plot of \((\Theta^{(1)}_\Lambda, \Theta^{(2)}_\Lambda)\) from \(\pi(\theta_\Lambda|n_{1:20}, \delta_{1:1}, \lambda_{1:20}, \theta_\rho)\) with Gaussian, Clayton and Gumbel copulas \(C(\cdot|\theta_\rho = \rho)\) between frequency risk profiles. Top row: strong correlation. Bottom row: weak correlation.
For Further Reading I


For Further Reading IV


For Further Reading V


For Further Reading


Peters G. W. and Teruads V. (2007). Low probability large consequence events. *Australian Centre for Excellence in Risk Analysis project no. 06/02*.


For Further Reading XIII

