Optimal Investment and Consumption Decision of Family with Life Insurance

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Introduction

- There are two economic agents in family: Parents and Children.
- Parents receives deterministic income until fixed time horizon $T$ but his lifetime is uncertain.
- If parents die before $T$, the children has no income until $T$ and they choose the optimal consumption and portfolio with remaining wealth combining the insurance benefit.
Introduction

- We consider utility functions of parents and children separately.
- Before the death time of parents, the object of family is to maximize weighted average of utility of parents and utility of children.
- Using HARA utility we imposed the condition that instantaneous consumption rate of parents and children should be above given lower bounds, consumption floor, respectively.
- We analyze how the change of weight and other parameters such as lower bound of consumption, income process, hazard rate affect the optimal policies using various numerical examples.
History

- Yaari (1965): optimal consumption and life insurance premium choice problem of individual whose lifetime is uncertain and bounded.
- Pliska and Ye (2007): optimal life insurance and consumption for a income earner whose lifetime is random and unbounded but without investment.
- Ye (2007): optimal life insurance, consumption and portfolio choice problem under uncertain lifetime using martingale method as we used to solve our problem.
- Huang et al. (2008): optimal life insurance, consumption and portfolio choice problem under uncertain lifetime with stochastic income process. They focussed on the effect of correlation between the dynamics of financial capital and human capital.
Financial market composed by two assets

Brownian motion
- $W_t$: a standard Brownian motion on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$
- $\{\mathcal{F}_t\}_{t=0}^T$ is the $\mathbb{P}$-augmentation of the natural filtration generated by $W_t$.

Financial Market
It is assumed that there are one risk-free asset and one risky asset.
- Risk-free asset: $dS_t^0 = rS_t^0 dt$
- Risky asset: $dS_t^1 = \mu S_t^1 dt + \sigma S_t^1 dW_t$
- $r, \mu, \sigma$: constants
Definitions

- \( \pi(t) \): amount invested in the risky asset \( S_t^1 \) at time \( t \)
- \( c_p(t) \): consumption rate of parents at time \( t \)
- \( c_c(t) \): consumption rate of children at time \( t \)
- \( l(t) \): life insurance premium at time \( t \)
- \( w_t \): deterministic income of parents
- market-price-of-risk: \( \theta \triangleq \frac{\mu - r}{\sigma} \)
- discount process: \( \zeta_t \triangleq \exp \left( - \int_0^t (\lambda y + s + r) ds \right) \)
- exponential martingale process: \( Z_t \triangleq \exp \left\{ -\theta W_t - \frac{1}{2} \theta^2 t \right\} \)
- pricing kernel(state-price-density) process: \( H_t \triangleq \zeta_t Z_t \)
Uncertain life time

Law of mortality $\lambda_{y+t}$

Let $\lambda_{y+t}$ denotes an instantaneous force of mortality curve (hazard rate), where $y$ is the age of the breadwinner at initial time of the model. Then the conditional probability of survival, from age $y$ to $y + t$, under the law of mortality defined by $\lambda_{y+t}$ can be computed by

$$ t p_y \triangleq e^{- \int_0^t (\lambda_{y+s}) ds}. $$

(1)
Life insurance benefit

- Family pays $l(t)$ dollars at time $t$ as life insurance premium.
- Receives lump sum insurance benefit $l(\tau)/\lambda_{y+\tau}$ at the death time of parents $\tau$.
- Define $X_t$ as family's wealth at time $t$ until $\tau_m \triangleq \min[\tau, T]$.
- $M(t) \triangleq X_t + l(t)/\lambda_{y+t}$: total legacy when the parents die at time $t$ with wealth $X_t$.

Family’s wealth dynamics under $\mathbb{P}$

The family’s wealth dynamics $X_t$ follows the following SDE’s

$$dX_t = [rx_t + (\mu - r)\pi(t) - c_p(t) - c_c(t) - l(t) + w_t]dt + \sigma \pi(t) dW_t, \quad 0 \leq t \leq \tau_m.$$
Family’s wealth dynamics

Equivalent martingale measure

For a given $T$, we define the equivalent martingale measure

$$\tilde{\mathbb{P}}(A) \triangleq \mathbb{E}[Z_T 1_A], \quad \text{for } A \in \mathcal{F}_T.$$ 

By Girsanov’s theorem, $\tilde{W}_t \triangleq W_t + \theta t$, $0 \leq t \leq T$, is a standard Brownian motion under the new measure $\tilde{\mathbb{P}}$.

Family’s wealth dynamics under $\tilde{\mathbb{P}}$

The wealth process (2) before $\tau_m$ can be rewritten as

$$dX_t = [rX_t - c_p(t) - c_c(t) - l(t) + w_t]dt + \sigma \pi(t) d\tilde{W}_t$$

$$= [(r + \lambda_{y+t})X_t - c_p(t) - c_c(t) - \lambda_{y+t} M(t) + w_t]dt + \sigma \pi(t) d\tilde{W}_t,$$

for $0 \leq t \leq \tau_m$. 
Budget constraint

We have the following budget constraint

\[ \mathbb{E}_t \left[ \int_t^T H_s c_p(s) ds + \int_t^T H_s c_c(s) ds + \int_t^T \lambda_{y+s} H_s M(s) ds + H_T X_T \right] \leq H_t(X_t + b_t), \quad (4) \]

where

\[ b_t \triangleq \tilde{\mathbb{E}}_t \left[ \int_t^T w_s \frac{\zeta_s}{\zeta_t} ds \right]. \]
Utility functions of parents and children

We consider utility function of parents $u_p(c)$ and utility function of children $u_c(c)$ separately,

$$u_p(c) \triangleq \frac{(c - R_p)^{1-\gamma_p}}{1 - \gamma_p}$$ (5)

$$u_c(c) \triangleq \frac{(c - R_c)^{1-\gamma_c}}{1 - \gamma_c}.$$ (6)

$\gamma_p > 0 (\gamma_p \neq 1)$ and $\gamma_c > 0 (\gamma_c \neq 1)$ are the parents’ and children’s coefficients of relative risk aversion, respectively, and $R_p \geq 0$ and $R_c \geq 0$ represent constant consumption floors of parents and children, respectively.
Optimization problem of family

Expected utility at time $t$

$\omega_1 \geq 0$ and $\omega_2 \geq 0$ are constant weights of utility functions for parents and children, respectively, until the time $\tau_m$ satisfying

$$\omega_1 + \omega_2 = 1.$$ (7)

The family’s expected utility function $U(t, X_t; c_p, c_c, \pi, I)$ with the initial endowment $X_t$ at time $t$, $t < \tau_m$, is given by

$$U(t, X_t; c_p, c_c, \pi, I) = \mathbb{E}_t \left[ \int_t^{\tau_m} e^{-\delta(s-t)} \left\{ \omega_1 u_p(c_p(s)) + \omega_2 u_c(c_c(s)) \right\} ds \right] + \int_{\tau_m}^{T} e^{-\delta(s-t)} u_c(c_c(s)) ds,$$ (8)

where $\delta > 0$ is a constant subjective discount rate.
Optimization problem of family

Definition of value function

\[ V(t, X_t) \triangleq \sup_{(c_p, c_c, \pi, l) \in A(t, X_t)} U(t, X_t; c_p, c_c, \pi, l) \]  

(10)

subject to the budget constraint (4), where \( A(t, X_t) \) is the admissible class of the quadruple \( (c_p, c_c, \pi, l) \) at time \( t \) for which the family’s expected utility function (8) is well-defined.
Assumption 3.1

We define the Merton’s constant $K_i$, $i = p, c$, and assume that it is always positive, that is,

$$K_i \triangleq r + \frac{\delta - r}{\gamma_i} + \frac{\gamma_i - 1}{2\gamma_i^2} \theta^2 > 0, \quad i = p, c.$$
Value function for $\tau_m \leq t \leq T$

Lemma 3.1

For $\tau_m \leq t \leq T$, there are no labor income and no life insurance premium. Then the value function is $V(t, X_t) = e^{\delta t} \Phi(t, X_t)$, where

$$\Phi(t, X_t) \triangleq e^{-\delta t} g(t)^{\gamma_c} \left\{ X_t - \frac{R_c}{r} \left( 1 - e^{-r(T-t)} \right) \right\}^{1-\gamma_c}$$

and

$$g(t) \triangleq \frac{1 - e^{-K_c(T-t)}}{K_c}.$$  

And the optimal policies are

$$c_c^*(t) = \frac{1}{g(t)} \left\{ X_t - \frac{R_c}{r} \left( 1 - e^{-r(T-t)} \right) \right\} + R_c, \quad (11)$$

$$\pi^*(t) = \frac{\theta}{\sigma \gamma_c} \left\{ X_t - \frac{R_c}{r} \left( 1 - e^{-r(T-t)} \right) \right\}. \quad (12)$$
Dual value function $\tilde{J}(\nu, t)$

For a Lagrange multiplier $\nu > 0$, let us define a dual value function

$$\tilde{J}(\nu, t) \triangleq \sup_{(c_p, c_c, X_T, M) \in A(t, X_t)} \left\{ \mathbb{E}_t \left[ \omega_1 \int_t^T e^{-\left(\delta s + \int_t^s \lambda_{y+u}du\right)} u_p(c_p(s))ds ight. ight.$$  

$$+ \omega_2 \int_t^T e^{-\left(\delta s + \int_t^s \lambda_{y+u}du\right)} u_c(c_c(s))ds + \int_t^T \lambda_{y+s}e^{-\int_t^s \lambda_{y+u}du} \Phi(M(s), s)ds \\ 

$$ - \nu e^{\int_0^t \lambda_{y+u}du} \mathbb{E}_t \left( \int_t^T H_s c_p(s)ds + \int_t^T H_s c_c(s)ds + \int_t^T \lambda_{y+s} H_s M(s)ds + H_T X_T \right) \left. \right\}.$$
The Optimization Problem

Legendre transform inverse formula

We can derive the value function $V(t, X_t)$ from $\tilde{J}(\nu, t)$ by Legendre transform inverse formula

$$V(t, X_t) = e^{\delta t} \inf_{Y_{t^\nu} > 0} \left[ \tilde{J}(\nu, t) + \nu e^{\int_0^t \lambda_y + u} du H_t(X_t + b_t) \right].$$

Optimal Lagrange multiplier $\nu^*$ satisfies the following equation

$$X_t + b_t = \omega_1^p \left( Y_{t^\nu^*} \right) - \frac{1}{\gamma_p} \int_t^T e^{-\int_t^s (\lambda_y + u + K_p)} du ds$$

$$+ \omega_2^c \left( Y_{t^\nu^*} \right) - \frac{1}{\gamma_c} \int_t^T e^{-\int_t^s (\lambda_y + u + K_c)} du ds$$

$$+ (Y_{t^\nu^*}) - \frac{1}{\gamma_c} \left\{ g(t) - \int_t^T e^{-\int_t^s (\lambda_y + u + K_c)} du ds \right\}$$

$$+ R_p \int_t^T e^{-\int_t^s (\lambda_y + u + r)} du ds + \frac{R_c}{r} \left( 1 - e^{-r(T-t)} \right).$$

(13)
The Optimization Problem

Value function

Theorem 3.1

\[ V(X_t, t) = \omega_1 \frac{1}{\gamma_p} \frac{\gamma_p}{1 - \gamma_p} (Y_{t^*}^\nu) \frac{\gamma_p - 1}{\gamma_p} \int_t^T e^{-\int_t^s (\lambda y + u + K_p) du} ds \]

\[ + \omega_2 \frac{1}{\gamma_c} \frac{\gamma_c}{1 - \gamma_c} (Y_{t^*}^\nu) \frac{\gamma_c - 1}{\gamma_c} \int_t^T e^{-\int_t^s (\lambda y + u + K_c) du} ds \]

\[ + \frac{\gamma_c}{1 - \gamma_c} (Y_{t^*}^\nu) \frac{\gamma_c - 1}{\gamma_c} \left\{ g(t) - \int_t^T e^{-\int_t^s (\lambda y + u + K_c) du} ds \right\} \]

\[ + Y_{t^*}^\nu \left\{ X_t + b_t - R_p \int_t^T e^{-\int_t^s (\lambda y + u + r) du} ds - \frac{R_c}{r} (1 - e^{-r(T-t)}) \right\} , \]

where \( Y_{t^*}^\nu \) satisfies equation (13).
Theorem 3.2

For $t < \tau_m$, the optimal policies are

\[ c^*_p(t) = \omega_1^{\gamma_p} (Y^*_t)^{-\frac{1}{\gamma_p}} + R_p, \quad c^*_c(t) = \omega_2^{\gamma_c} (Y^*_t)^{-\frac{1}{\gamma_c}} + R_c, \]

\[ \pi^*(t) = \frac{\theta}{\sigma \gamma_c} \left\{ X_t + b_t - R_p \int_t^T e^{-\int_t^s (\lambda y + u + r) du} ds - \frac{R_c}{r} \left( 1 - e^{-r(T-t)} \right) \right\} \]

\[ + \frac{\theta}{\sigma} \frac{\gamma_c - \gamma_p}{\gamma_p \gamma_c} \omega_1^{\gamma_p} (Y^*_t)^{-\frac{1}{\gamma_p}} \int_t^T e^{-\int_t^s (\lambda y + u + K_p) du} ds, \]

\[ \frac{l^*(t)}{\lambda y + t} = b_t - R_p \int_t^T e^{-\int_t^s (\lambda y + u + r) du} ds - \omega_1^{\gamma_p} (Y^*_t)^{-\frac{1}{\gamma_p}} \int_t^T e^{-\int_t^s (\lambda y + u + K_p) du} ds \]

\[ - \left( \omega_2^{\gamma_c} - 1 \right) (Y^*_t)^{-\frac{1}{\gamma_c}} \int_t^T e^{-\int_t^s (\lambda y + u + K_c) du} ds, \]

where $Y^*_t$ satisfies the equation (13).
Lemma 4.1

If $\gamma_p = \gamma_c > 1$, the optimal life insurance premium at time $t$ is not affected by wealth level of family at time $t$ if $\omega_1 = 0$ or $\omega_1 = 1$ and the optimal life insurance premium decreases as wealth of family increases if $\omega_1 \in (0, 1)$.

If $0 < \gamma_p = \gamma_c < 1$, the optimal life insurance premium at time $t$ is not affected by wealth level of family at time $t$ if $\omega_1 = 0$ or $\omega_1 = 1$ and the optimal life insurance premium increases as wealth of family increases if $\omega_1 \in (0, 1)$.

\[
\frac{\partial (I^*(t)/\lambda_{y+t})}{\partial X_t} = 0, \quad \text{if } \omega_1 = 0 \text{ or } \omega_1 = 1,
\]
\[
\frac{\partial (I^*(t)/\lambda_{y+t})}{\partial X_t} < 0, \quad \text{for } \omega_1 \in (0, 1) \text{ and } \gamma_p = \gamma_c > 1,
\]
\[
\frac{\partial (I^*(t)/\lambda_{y+t})}{\partial X_t} > 0, \quad \text{for } \omega_1 \in (0, 1) \text{ and } 0 < \gamma_p = \gamma_c < 1.
\]
Lemma 4.2

\[
\frac{\partial I^*(t)/\lambda y + t}{\partial \omega_1} < 0 \quad \text{for } \gamma_p = \gamma_c > 1 \text{ and } \omega_1 \in (0, 0.5),
\]
\[
\quad \text{or } \gamma_p = \gamma_c < 1 \text{ and } \omega_1 \in (0.5, 1).
\]
\[
\frac{\partial I^*(t)/\lambda y + t}{\partial \omega_1} > 0 \quad \text{for } \gamma_p = \gamma_c > 1 \text{ and } \omega_1 \in (0.5, 1),
\]
\[
\quad \text{or } \gamma_p = \gamma_c < 1 \text{ and } \omega_1 \in (0, 0.5).
\]
\[
\frac{\partial I^*(t)/\lambda y + t}{\partial \omega_1} = 0 \quad \text{if } \omega_1 = 0.5.
\]
Lemma 4.3

The optimal investment in risky asset decreases as $\omega_1$ increases if $\gamma_p > \gamma_c$ and increases as $\omega_1$ increases if $\gamma_p < \gamma_c$. If $\gamma_p = \gamma_c$, the optimal investment in risky asset is not affected by $\omega_1$.

$$\frac{\partial \pi^*(t)}{\partial \omega_1} < 0, \quad \text{if } \gamma_p > \gamma_c,$$

$$\frac{\partial \pi^*(t)}{\partial \omega_1} > 0, \quad \text{if } \gamma_p < \gamma_c,$$

$$\frac{\partial \pi^*(t)}{\partial \omega_1} = 0, \quad \text{if } \gamma_p = \gamma_c.$$
Assumptions on labor income and hazard rate for numerical examples

Assumption 5.1

*Throughout this section, we assume that*

\[ w_t = C_w e^{k_w t}, \]
\[ \lambda_{y+t} = A_L + B_L t, \]

*for some positive constants \( C_w, k_w, A_L, B_L \). (See Pliska and Ye (2007).)*
Examples for Lemma 4.1: Effects of wealth on the optimal life insurance premium when $\gamma_p = \gamma_c$

Remark 1 on Figure 1: $\gamma_p = \gamma_c = 2 > 1$

Figure 1 illustrates the optimal policies at time $t = 0$ with $\gamma_p = \gamma_c = 2$. As we have shown in Lemma 4.1, we can observe that

- the optimal life insurance premium decreases as wealth increases when $\omega_1 = 0.2, 0.5, 0.8$,
- constant when $\omega_1 = 0$ or $\omega_1 = 1$.

Remark 1 on Figure 2: $0 < \gamma_p = \gamma_c = 1/2 < 1$

Figure 2 illustrates the optimal policies at time $t = 0$ with $\gamma_p = \gamma_c = 1/2$. We can check that

- the optimal life insurance premium increases as wealth increases when $\omega_1 = 0.2, 0.5, 0.8$.
- constant when $\omega_1 = 0$ or $\omega_1 = 1$. 
Numerical Examples

Examples for Lemma 4.2: Effects of weight \((\omega_1, \omega_2)\) on the optimal life insurance premium when \(\gamma_p = \gamma_c\)

Remark 2 on Figure 1: \(\gamma_p = \gamma_c = 2 > 1\)

The optimal life insurance premium in Figure 1 is
- **decreases** as \(\omega_1\) increases from 0 to 0.5,
- **increases** as \(\omega_1\) increases from 0.5 to 1.
- Minimized when \(\omega_1 = \omega_2 = 0.5\),
- maximized when \(\omega_1 = 1\) or \(\omega_1 = 0\).

Remark 2 on Figure 2: \(0 < \gamma_p = \gamma_c = 1/2 < 1\)

The optimal life insurance premium in Figure 2 is
- **increases** as \(\omega_1\) increases from 0 to 0.5,
- **decreases** as \(\omega_1\) increases from 0.5 to 1,
- Maximized when \(\omega_1 = \omega_2 = 0.5\),
- Minimized when \(\omega_1 = 1\) or \(\omega_1 = 0\).
Figure 1: Consumption of parents, Consumption of children, Portfolio, Life insurance for different $\omega_1$ at $t = 0$ with $\gamma_p = 2$, $\gamma_c = 2$.

($R_p = 0.12$, $R_C = 0.10$, $\delta = 0.03$, $r = 0.03$, $\mu = 0.07$, $\sigma = 0.2$, $C_W = 5.0$, $k_W = 0.03$, $A_L = 0.005$, $B_L = 0.001125$, $T = 30$)
Figure 2: Consumption of parents, Consumption of children, Portfolio, Life Insurance for different $\omega_1$ at $t = 0$ with $\gamma_p = 1/2, \gamma_c = 1/2$.
($R_p = 0.12, R_c = 0.10, \delta = 0.03, r = 0.03, \mu = 0.07, \sigma = 0.2, C_w = 5.0, k_w = 0.03, A_L = 0.005, B_L = 0.001125, T = 30$)
Examples for Lemma 4.3: Effects of weight \((\omega_1, \omega_2)\) on the optimal portfolio

In Lemma 4.3 we have shown that

- if \(\gamma_p = \gamma_c\), the optimal investment in risky asset is not affected by \(\omega_1\): See Figure 1 and 2
- if \(\gamma_p < \gamma_c\), the optimal investment in risky asset increases as \(\omega_1\) increases: See Figure 3
- if \(\gamma_p > \gamma_c\), the optimal investment in risky asset decreases as \(\omega_1\) increases: See Figure 4
Figure 3: Consumption of parents, Consumption of children, Portfolio, Life insurance for different $\omega_1$ at time $t = 0$ with $\gamma_p = 2$, $\gamma_c = 3$.

$(R_p = 0.12, R_c = 0.10, \delta = 0.03, r = 0.03, \mu = 0.07, \sigma = 0.2, C_w = 5.0, k_w = 0.03, A_L = 0.005, B_L = 0.001125, T = 30)$
Figure 4: Consumption of parents, Consumption of children, Portfolio, Life insurance for different $\omega_1$ at time $t = 0$ with $\gamma_p = 3$, $\gamma_c = 2$.
($R_p = 0.12$, $R_c = 0.10$, $\delta = 0.03$, $r = 0.03$, $\mu = 0.07$, $\sigma = 0.2$, $C_W = 5.0$, $k_W = 0.03$, $A_L = 0.005$, $B_L = 0.001125$, $T = 30$)
Effects of weight \((ω_1, ω_2)\) on the optimal life insurance premium when \(γ_p ≠ γ_c\)

Remark on Figure 3: \(γ_p = 2 < γ_c = 3\)

Figure 3 represents the optimal policies at time \(t = 0\) with \(γ_p = 2, γ_c = 3\)

- Optimal life insurance premium decreases as wealth increases when \(ω_1 = 0.2, 0.5, 0.8\) and also \(ω_1 = 1\).
- If \(ω_1 = 0\), life insurance premium is not affected by wealth level of family as previous two cases.

Remark on Figure 4: \(γ_p = 3 > γ_c = 2\)

Figure 4 represents the optimal policies at time \(t = 0\) with \(γ_p = 3, γ_c = 2\).

- If \(ω_1 = 0\), life insurance premium is not affected by wealth level of family as before.
- If \(w ∈ [0, 1)\), as wealth of family increases, optimal life insurance premium decreases or increases depending on \(ω_1\).
Effects of risk aversion on the optimal policies

Remark on Figure 5

For fixed $\gamma_p$, as $\gamma_c$ increases,
- consumption of parents increases,
- consumption of children decreases,
- life insurance premium decreases.

For fixed $\gamma_c$, as $\gamma_p$ increases,
- consumption of parents decreases,
- consumption of children increases,
- life insurance premium increase.

Both $\gamma_p$ and $\gamma_c$ have negative effects on portfolio.
Numerical Examples

Figure 5: Consumption of parents, Consumption of children, Portfolio, Life insurance at time $t=0$ for different pairs of $\gamma_p$ and $\gamma_c$.

$(R_p = 0.12, R_c = 0.10, \delta = 0.03, r = 0.03, \mu = 0.07, \sigma = 0.2, C_w = 5.0, k_w = 0.03, A_L = 0.005, B_L = 0.001125, T = 30)$
Effects of consumption floor on the optimal policies

Remark on Figure 6

For fixed $R_p$, as $R_c$ increases,
- consumption of parents decreases,
- consumption of children increases,
- life insurance premium increases.

For fixed $R_c$, as $R_p$ increases,
- consumption of parents increases,
- consumption of children decreases,
- life insurance premium decreases.

Both $R_p$ and $R_c$ have negative effects on portfolio.
Figure 6: Consumption of parents, Consumption of children, Portfolio, Life insurance at time $t = 0$ for different pairs of $R_p$ and $R_c$.

$(\gamma_p = 2, \gamma_c = 2, \delta = 0.03, r = 0.03, \mu = 0.07, \sigma = 0.2, C_W = 5.0, k_W = 0.03, A_L = 0.005, B_L = 0.001125, T = 30)$
Effects of labor income on the optimal policies

Remark on Figure 7

- Figure 7 illustrates the optimal policies for different labor income.
- Labor income has **positive effect on all policies**: consumption of parents, consumption of children, portfolio, life insurance premium.
Numerical Examples

Figure 7: Consumption of parents, Consumption of children, Portfolio, Life insurance at time $t = 0$ for different labor income.

($\gamma_p = 2$, $\gamma_c = 2$, $R_p = 0.12$, $R_c = 0.10$, $\delta = 0.03$, $r = 0.03$, $\mu = 0.07$, $\sigma = 0.2$, $A_L = 0.005$, $B_L = 0.001125$, $T = 30$)
Numerical Examples

Effects of hazard rate on the optimal policies

Remark on Figure 8

In figure 8, we can compare the optimal policies for different $A_L$ and $B_L$ for $\lambda_{y+t}$ at time $t = 0$. Figure 8 indicates that hazard rate has

- **positive effect** on the optimal life insurance premium,
- **negative effects** on other policies: consumption of parents, consumption of children, portfolio.
Figure 8: Consumption of parents, Consumption of children, Portfolio, Life insurance at time $t = 0$ for different hazard rate.
($\gamma_p = 2, \gamma_c = 2, R_p = 0.12, R_c = 0.10, \delta = 0.03, r = 0.03, \mu = 0.07, \sigma = 0.2, C_w = 5.0, k_w = 0.03, T = 30$)
Conclusion

- We investigated an optimal portfolio, consumption and life insurance choice problem of family.
- We obtained analytic solution for value function and the optimal policies.
- We figured out that if $\gamma_p = \gamma_c > 1$, wealth level of family has negative effect on the optimal life insurance premium except the cases with $\omega_1 = 0$ or $\omega_1 = 1$, and if $\gamma_p = \gamma_c < 1$, wealth level of family has positive effect on the optimal life insurance premium except the cases with $\omega_1 = 0$ or $\omega_1 = 1$.
- We observed that if $\gamma_p = \gamma_c > 1$, the optimal life insurance premium is minimized when $\omega_1 = \omega_2 = 0.5$ and maximized when $\omega_1 = 0$ or $\omega_1 = 1$. If $\gamma_p = \gamma_c < 1$, the optimal life insurance premium is maximized when $\omega_1 = \omega_2 = 0.5$ and minimized when $\omega_1 = 0$ or $\omega_1 = 1$.
- $\omega_1$ has negative effect for $\gamma_p > \gamma_c$ and positive effect for $\gamma_p < \gamma_c$ on the optimal portfolio. If $\gamma_p = \gamma_c$, $\omega_1$ does not affect the optimal portfolio.
- We examined effects of various parameters on the optimal policies by using numerical examples.
Further research

- Note that among examples, there are some cases with negative life insurance premium because we didn’t impose nonnegative life insurance premium constraint.
- For future study, we will investigate an optimal portfolio, consumption and life insurance choice problem of family with nonnegative life insurance premium constraint.