Portfolio Selection with Narrow Framing: Probability Weighting Matters

Enrico G. De Giorgi$^{1,2}$  Shane Legg$^{1,3}$

$^1$ Swiss Finance Institute, University of Lugano, Switzerland
$^2$ Group for Mathematics and Statistics, University of St. Gallen, Switzerland
$^3$ Gatsby Computational Neuroscience Unit, University College London

Computing in Economics and Finance, 2009, Sydney, Australia
In portfolio choice and asset pricing models various well known “puzzles” tend to arise, e.g. the models predict that people should own far more stock than they typically do.

Models are usually based on *expected utility theory* (EUT) which is known to be a poor model of human decision making under risk. A more accurate model due to Tversky and Kahneman is *cumulative prospect theory* (CPT).

If we use CPT as a basis for our models, do some of the puzzles go away?
1) **Narrow framing** When deciding between risky alternatives, people tend to analyse the problem as though it exists in isolation. This is probably because optimising over all risks being faced in life would be complex.

2) **Reference return** The utility that you get from an investment is not simply a function of the actual return, but also depends on how that return compares to some mental point of reference.

E.g. if the market went up 20% and you made a 10% you wouldn’t be happy with that. But you would be happy with this return if the market had fallen.
3) **Loss aversion** The pain of losing $100 is typically about twice the magnitude of the pleasure of gaining $100. This may lead people to pass up opportunities to make a gain in order to avoid a loss that is comparatively small.

4) **Probability weighting** People tend to act as though very low probability events, say winning the lottery, are much more likely than they really are. Conversely, moderate probability events are slightly unweighted when making decisions.
5) *Curved value function* People tend to be risk seeking over losses, but risk averse over gains.

E.g. if you have a guaranteed gain of $10,000 or a very likely gain of $11,000, which would you take? Most take the first option.

If you had a guaranteed loss of $10,000 or a likely but not for sure loss of $11,000 what would you take? Most take the second option.
Barberis and Huang (2009) managed to construct a tractable model of investor behaviour that included narrow framing, reference returns and loss aversion. This is the basis for our work.

**Drawbacks of their model:**

1) Does not include probability weighting.

2) Does not include a concave-convex value function.

3) Under a realistic Sharpe ratio the stock participation problem becomes worse.
At time $t$ the investor chooses a consumption level $C_t$ and decides how to invest her remaining wealth $W_t - C_t$ to $n$ assets with gross returns $R_{1,t+1}, \ldots, R_{n,t+1}$ between time $t$ and $t+1$. Time $t+1$ wealth is therefore given by

$$W_{t+1} = (W_t - C_t) \sum_{i=1}^{n} \theta_{i,t} R_{i,t+1}$$

where $\theta_{i,t} \geq 0$ is the proportion of post-consumption wealth invested in asset $i$. We assume that short-sale is not allowed.
A quick walk through the CPT investor model

The investor frames assets $m + 1, \ldots, n$ narrowly. Investor’s utility at time $t$ is then given as follows:

$$V_t = H\left(C_t, \mu(V_{t+1}|I_t) + b_0 \sum_{i=m+1}^{n} U_t(G_{i,t+1})\right)$$

where

$$H(C, x) = \left((1 - \beta) C^\rho + \beta x^\rho\right)^{1/\rho} \quad 0 < \beta < 1, \ 0 \neq \rho < 1$$

$$\mu(kx) = k \mu(x) \quad k > 0$$

$$G_{i,t+1} = \theta_{i,t}(W_t - C_t)(R_{i,t+1} - R_{i,z}) \quad i = m + 1, \ldots, n$$
A quick walk through the CPT investor model

The narrowly framed utility function $U_t$ is defined as follows. For a
random variable $x$ with cumulative distribution $F_t$ at time $t$

$$U_t(x) = \int_{-\infty}^{0} \bar{v}(x) \frac{d}{dx} [w^-(F_t(x))] \, dx + \int_{0}^{\infty} \bar{v}(x) \frac{d}{dx} [-w^+(1-F_t(x))] \, dx$$

with

$$\bar{v}(x) = \begin{cases} 
 x^\zeta & x \geq 0 \\
 -\lambda (-x)^\zeta & x < 0 
\end{cases} \quad \zeta \in (0, 1]$$

$$w^+(p) = \frac{p^{\delta^+}}{(p^{\delta^+} + (1-p)^{\delta^+})^{1/\delta^+}} \quad \delta^+ \in (0.3, 1]$$

$$w^-(p) = \frac{p^{\delta^-}}{(p^{\delta^-} + (1-p)^{\delta^-})^{1/\delta^-}} \quad \delta^- \in (0.3, 1]$$
This model describes 3 types of CPT investor

1. Original Barberis-Huang investor has narrow framing, reference returns and loss aversion. Barberis-Huang derive an efficient method for computing portfolio decisions for this investor. To get this case in our model set $\zeta = \delta^+ = \delta^- = 1$

2. Probability weighting added to the above model: $\zeta = 1$, $\delta^- = 0.69$, $\delta^+ = 0.61$. In our paper we show how the Barberis-Huang solution method can be extended to include this case.

3. Full CPT model, that is, the previous model but with a concave-convex value function added: $\zeta = 0.88$, $\delta^- = 0.69$, $\delta^+ = 0.61$. For this we must rely on a much slower but more general method of computation.
A simple consumption and portfolio choice problem

Our investor makes decisions on a yearly basis with an unlimited lifespan and a temporal discount rate of $\beta = 0.98$. Each year she allocates her wealth across three assets: a risk-free asset with a net return of 2%, and two risky with gross returns:

$$\log R_{i,t+1} = g_i + \sigma_i \epsilon_{i,t+1}, \quad i = 2, 3$$

where

$$\begin{pmatrix} \epsilon_{2,t} \\ \epsilon_{3,t} \end{pmatrix} \sim N \begin{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 1 & \omega \\ \omega & 1 \end{pmatrix} \end{pmatrix}, \quad \text{i.i.d. over time.}$$

Thus

$$W_{t+1} = (W_t - C_t)((1 - \theta_{2,t} - \theta_{3,t})R_f + \theta_{2,t}R_{2,t+1} + \theta_{3,t}R_{3,t+1}.$$
A simple consumption and portfolio choice problem

We set drift and volatility of the two risky assets based on annual returns for the S&P 500 index from January 1946 to January 2009: $g_1 = g_2 = 6.15\%$ and $\sigma_1 = \sigma_2 = 15.49\%$.

We also set

- $\omega = 0$, though our models don’t require this
- $\lambda = 2.25$ (a standard empirical value for loss aversion)
- $\theta_{2,t} = 0.5$ (asset 2 could be a non-financial asset like housing)
- only the third asset is narrowly framed

We will focus on the proportion $\theta_3/(1 - \theta_2)$ of wealth that is allocated to the narrowly framed asset.
Barberis-Huang investor with log-normal returns
Probability weighting investor with log-normal returns
Barberis-Huang investor with log-normal returns
Probability weighting investor with log-normal returns
Log-normal and skew-normal fitted to S&P 500
Barberis-Huang investor with skew-normal returns
Probability weighting investor with skew-normal returns
Again following the work of Barberis and Huang we performed a simple equilibrium analysis assuming homogeneous investors facing skew-normal returns.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>γ = 2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.005</td>
<td>5.23</td>
<td>0.55</td>
<td>5.08</td>
<td>0.79</td>
</tr>
<tr>
<td>0.015</td>
<td>4.66</td>
<td>1.50</td>
<td>4.10</td>
<td>2.42</td>
</tr>
<tr>
<td>0.025</td>
<td>4.20</td>
<td>2.26</td>
<td>3.16</td>
<td>3.98</td>
</tr>
<tr>
<td>0.035</td>
<td>3.87</td>
<td>2.82</td>
<td>2.41</td>
<td>5.21</td>
</tr>
<tr>
<td>γ = 4</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.005</td>
<td>8.12</td>
<td>0.38</td>
<td>7.91</td>
<td>0.55</td>
</tr>
<tr>
<td>0.015</td>
<td>7.25</td>
<td>1.10</td>
<td>6.49</td>
<td>1.73</td>
</tr>
<tr>
<td>0.025</td>
<td>6.47</td>
<td>1.74</td>
<td>4.81</td>
<td>3.10</td>
</tr>
<tr>
<td>0.035</td>
<td>5.80</td>
<td>2.30</td>
<td>2.17</td>
<td>5.29</td>
</tr>
</tbody>
</table>
Mathematical analysis of a full CPT model shows that an interesting wealth effect occurs: with higher wealth the investor acts as though she is not narrow framing as much.

⇒ wealthy investors will have a much higher proportion of their wealth in stock than poor investors.

This is consistent with empirical studies that show that stock market participation increases rapidly as a function of wealth.

Unfortunately, we have not found a fast way to compute the behaviour of the full CPT investor and so rely on a slower but more flexible method that computes directly from the investor’s Bellman equation.
Full CPT investor $100,000 wealth & skew-normal returns
We have extended the model of Barberis and Huang to include probability weighting and a concave-convex value function. Under realistic parameters the probability weighting model does not imply a stock market participation puzzle. A simple equilibrium analysis of the probability weighting investor can give a realistic risk free rate and equity premium. A concave-convex value function introduces an interesting wealth effect to the stock market participation. Perhaps the least desirable aspect of these models is we don’t know what a realistic level of narrow framing is, thus \( b_0 \) is essentially a free parameter. Can it be empirically estimated?