Follow the Leader
Steady State Analysis of a Dynamic Social Network

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Objectives

Understand the emergence of an opinion leader of a product for which preference is subjective and requiring public confirmation.

Consider a dynamic social network in which members receive a reward for being the early adopter of a subsequently popular trend.
Approach

Theory component
• Find the equilibrium structure of the social network
• Determine the parameters that impact the equilibrium
• Understand the social aspects that affect the equilibrium

Simulation component
• Consider the reward structure, social interactions and adjustment mechanisms that impact on the emergence of a hierarchy
Background

Network Externalities: Katz and Shapiro (1985)
Social Interaction: Brock and Durlauf (2001)
Minority Games: Arthur (1994)
Symbiotic relationship in an endogenous social network: Chang and Harrington (2005)
Model

Social Network
• $L$ agents (nodes); $i \in \{1, 2, 3, \ldots, L\}$
• Each agent maintains $d$ one way (directed) links to other agents; “friends” who they can potentially imitate
• **Fully Connected**: For every $i \neq j$ there exists a pathway of links connecting agent $j$ to agent $i$

Choice setting
• $K$ discrete options; agent $i$ chooses $k_i \in \{"a", "b", "c", \ldots, K\}$

Information
Simulations: limited to experiences and observations
Analytical examination: full information
Model

Process of selecting $k_i$

- **Decision 1**: Decide whether to choose independently or to imitate
- **Decision 2**: If imitate, choose which one of $d$ friends
Model

Discrete time

- **Total of** $R$ **rounds**
  - **Round 1**: those acting independently directly and independently select from $K$ options
  - **Round 2**: those imitating those who acted independently select
  - **Round 3**: those imitating those who acted in Round 2 select
  - continue…
Model

Payoff
\[ \pi_i = u(k_i) + J(N_{ki}^J) + T(N_{ki}^T) \]

Conformity reward (Reward based on number of other adopters)
\[ J(N_{ki}^J) = a_J(N_{ki}^J - 1) \]

Timing reward (Reward based on subsequent adoption)
\[ T(N_{ki}^T) = a_T N_{ki}^T \]

\[ 0 \leq a_J \leq a_T = 1 \]
Model

Payoff

\[ \pi_i = u(k_i) + J(N_{ki}^J) + T(N_{ki}^T) \]

Conformity reward (Reward based on number of other adopters)

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\[ T(N_{ki}^T) = a_T N_{ki}^T \]

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Analysis

Equilibrium Analysis
1) Establish the existence of hierarchies
2) Establish the hierarchy as an equilibrium
3) Consider the alternatives when equilibrium conditions do not hold
4) Eliminate possibility of multiple hierarchies in equilibrium

Simulation Analysis
1) Does a leader emerge (and how)?
2) Is the emergent hierarchy the equilibrium hierarchy?
3) Who emerges as the leader?
Analysis

Existence
Proposition 1: For every agent $i$ there exists a structured hierarchy based on agent $i$ as the leader and every agent $j$, $j \neq i$ as a follower.

Proof: True "by construction"
Tier 0, the leadership position, is occupied by agent $i$ from the population.
Tier 1 be populated by every agent with a direct link to agent $i$
Tier $\delta$ is populated by agents possessing a direct link to a member of tier $\delta - 1$ and without links to members of tiers 0 through $\delta - 2$.

Efficiency: The hierarchy under leader $i$ is efficient if each follower links to $i$ via the shortest route available, indicated by $H_i$
Example: \( L = 100, K = 12, H_i, \) agent #2 as leader
Analysis

A more stylized hierarchy
Analysis

Participation

Condition A Participation criterion for agent \( j \) in \( H_i \):

\[
T(N_{ij}^T) + J(N_i^J) \geq \frac{1}{K} (T(N_i^T - 1) + J(N_i^J)) + \frac{K - 1}{K} (T(N_j^T) + J(N_j^J))
\]

or

\[
(n_{ij} + 1)(a_T - (K - 1)a_J) - m_{ij}(a_T + a_J) \leq a_T
\]

Sufficient condition: \( a_T \leq (K - 1)a_J \)
Analysis

Participation
Condition A Participation criterion for agent $j$ in $H_i$:

$$T(N_{ij}^T) + J(N_{i}^J) \geq \frac{1}{K} (T(N_{i}^T - 1) + J(N_{i}^J)) + \frac{K - 1}{K} (T(N_{j}^T) + J(N_{j}^J))$$

or

$$(n_{ij} + 1)(a_T - (K - 1)a_J) - m_{ij}(a_T + a_J) \leq a_T$$

Sufficient condition: $a_T \leq (K - 1)a_J$

- For small $K$: There is a higher probability an independent non-leader to choosing $k$ that coincides with the leader
- For small $a_J$: There is little social benefit to conform “last”
- For either: those at the bottom abandon the hierarchy
Analysis

Participation
Condition A Participation criterion for agent $j$ in $H_i$:

$$T(N_{ij}^T) + J(N_i^J) \geq \frac{1}{K} (T(N_i^T - 1) + J(N_i^J)) + \frac{K - 1}{K} (T(N_j^T) + J(N_j^J))$$

or

$$(n_{ij} + 1)(a_T - (K - 1)a_J) - m_{ij}(a_T + a_J) \leq a_T$$

Sufficient condition: $a_T \leq (K - 1)a_J$

For the agent at the bottom…
Condition B Necessary and Sufficient condition:

$$a_T \frac{(L - 2)}{L - 1} - (K - 1)a_J \leq 0$$
Analysis

Proposition 6: If Condition B holds, then given leader $i$, $H_i$ is a Nash equilibrium for all $i$.

With a few exceptions, $H_i$ is also a subgame perfect equilibrium.
Analysis

Cascades?
Will the abandonment of the hierarchical social structure by those at the bottom cause a dissolution of the hierarchy?

• Reduces the payoff to those who remain
• May increases the expected payoff independent action
Example: $L = 100$, $K = 6$, efficient $H^\lambda_i$, agent #40 as leader
Analysis

\( \lambda \) the proportion of the population participating in the hierarchy

**Condition C** The condition for agent \( j \)'s participation:

\[
(n_{ij}^\lambda + 1)(a_T - (K - 1)a_J) - m_{ij}^\lambda (a_T + a_J) \leq a_T
\]

Necessary and Sufficient condition based on agent in bottom tier:

\[
(\lambda L - 1)(a_T - a_J(K - 1)) \leq a_T
\]

Equilibrium hierarchy size:

\[
\lambda^* = \frac{1}{L} \left( \frac{a_T}{a_T - a_J(K - 1)} + 1 \right)
\]
Analysis

$$\lambda^* = \frac{1}{L} \left( \frac{a_T}{a_T - a_J(K - 1)} + 1 \right)$$

$\lambda^* \in [\frac{2}{L}, 1]$ for $0 \leq a_J(K - 1) \leq \frac{L-2}{L-1} a_T$

$\lambda^*$ is increasing in $a_J(K - 1)$

More social payoff or more options increases the size of the hierarchy

$L^* = \lambda^* L$ is independent of $L$

For an interior $\lambda^*$, the hierarchy supports a constant number of participants rather than a constant proportion of the population.
Analysis

Single hierarchy in Equilibrium

Proposition 8: For two jointly efficient hierarchies, $A$ and $B$, with $N_A^J > N_B^J$. There exists $j_B \in B$ with $d_j^B < d$ for whom $\pi_j^{A^*} > \pi_j^B$.

Proof: Consider efficient hierarchies, $A$ larger than $B$.
- $J(N_A^J) > J(N_B^J)$
- 1, 2, or 4 improve if switch to $A$
- A switch by 3 to $A$ may be worse off, but will induce other’s to switch to 3’s advantage
- $B$ is not subgame perfect, and often not a Nash Eq
Conclusion (Analysis)

With

• sufficient desire for conformity (large enough $a_j$)

• a discouraging number of options (large enough $K$)

Then

• the Nash equilibrium supports a single hierarchy under any individual $i$ in the population

• the subgame perfect equilibrium supports a single hierarchy under the subset of individuals who are sufficiently “fit”
Dynamic Model

The agent settles on a strategy for period $t$ through randomization.

Decision parameters for period $t$

- **Decision 1**: $\Pr(\text{agent } i \text{ acts independently}) = \theta_{i,t}$
- **Decision 2**: $\Pr(\text{agent } i \text{ imitates friend } j|\text{ Imitate}) = w_{i,t}^j$

Adjustments between periods (evolution of parameters)

- **Update** $\theta_{i,t}$ for next period
- **Update** $\{w_{i,t}^j\}_i$ for next period
- [Drop links with $w_{i,t}^j < \bar{w}$ and replace with new friend]
Dynamic Model

Update parameters according to Experience-weighted Attraction Learning: Camerer and Ho (1999)

Through manipulating parameters, can…
  • set memory length
  • down-weight speculative payoffs (of untried strategies)

Original (EWA): Places greater weight on superior strategy(ies)
Modified (RD): Drives weight on dominated strategies to zero
Simulations

Issues to address with simulation

1. Will a randomly linked population organize into a structured hierarchy (when the hierarchy is an equilibrium)?

2. Does a single leader emerge?

3. Is the hierarchy efficient?

4. Is the leader fit?
Simulations

Issues to address with simulation

1. Will a randomly linked population organize into a structured hierarchy (when the hierarchy is an equilibrium)?
   Yes for appropriately large $a_J(K - 1)$ (caveats apply)

2. Does a single leader emerge?

3. Is the hierarchy efficient?

4. Is the hierarchy optimal?
Simulations
Simulations
Simulations

Issues to address with simulation

1. Will a randomly linked population organize into a structured hierarchy?
   Yes for appropriately large $a_J(K - 1)$

2. Does a single leader emerge?

3. Is the hierarchy efficient?

4. Is the leader fit?
Formation, then disintegration, of hierarchy; $K = 6$ (or $a_j = 0$)
Formation, then disintegration, of hierarchy; $K = 6$ (or $a_j = 0$)
Simulations

Issues to address with simulation

1. Will a randomly linked population organize into a structured hierarchy? 
   Yes for appropriately large $a_j(K - 1)$
   But if not, then affected by whether small $K$ or small $a_j$

2. Does a single leader emerge?

3. Is the hierarchy efficient?

4. Is the leader fit?
…or, no formation of hierarchy; $K = 4$
Simulations

Issues to address with simulation

1. Will a randomly linked population organize into a structured hierarchy?
   Yes for appropriately large $a_j(K - 1)$
   But if not, affected by whether small $K$ or small $a_j$

2. Does a single leader emerge?

3. Is the hierarchy efficient?

4. Is the leader fit?
Simulations

Issues to address with simulation

1. Will a randomly linked population organize into a structured hierarchy?  
   Yes for appropriately large $a_J(K - 1)$  
   But if not, affected by whether small $K$ or small $a_J$

2. Does a single leader emerge?  
   Yes (with caveat)

3. Is the hierarchy efficient?

4. Is the leader fit?
Multiple hierarchies when forced into long chains
Simulations

Issues to address with simulation

1. Will a randomly linked population organize into a structured hierarchy?
   Yes for appropriately large \( a_j (K - 1) \)
   But if not, affected by whether small \( K \) or small \( a_j \)

2. Does a single leader emerge?
   Yes

3. Is the hierarchy efficient?
   Yes when population uses RD.
   No if noise or randomness persists (as with EWA).

4. Is the leader fit?
EWA produces fluidity in hierarchy
Simulations

Issues to address with simulation

1. Will a randomly linked population organize into a structured hierarchy?
   Yes for appropriately large $a_j(K - 1)$
   But if not, affected by whether small $K$ or small $a_j$

2. Does a single leader emerge?
   Yes

3. Is the hierarchy efficient?
   Yes when population uses RD.
   No if noise or randomness persists.

4. Is the leader fit?
Simulations

Issues to address with simulation

1. Will a randomly linked population organize into a structured hierarchy?
   Yes for appropriately large $a_J(K - 1)$
   But if not, affected by whether small $K$ or small $a_J$

2. Does a single leader emerge?
   Yes

3. Is the hierarchy efficient?
   Yes when population uses RD.
   No if noise or randomness persists.

4. Is the leader fit?
   No under RD, early transitory events determine the initial leader
   Yes under EWA where short memory and noise enable deposition
Example: Efficient, but not optimal (#78 had 12 potential fans)
EWA Replacement of leader at $t=116$ and $t=239$, increased fitness
Conclusion

- the Nash equilibrium supports a single hierarchy under any individual $i$ in the population

- the subgame perfect equilibrium supports a single hierarchy under the subset of individuals who are sufficiently “fit”

The emergence of a leader and hierarchy is made possible from adjustments to the social network that perpetuates transitory advantage.

- Leadership affirms success

- Hierarchy emerges despite individual agents’ ignorance of greater social structure
Extensions

Modeling

• Introduce non-zero $u(k_i)$ for possible feedback between innate preferences and leader choice

• Introduce layered networks with consumers seeking information from trusted sites and marketers pushing information through traditional channels

Behavioral Laboratory Experiments

• Simulated agents are reactionary. Forward looking strategies would likely affect the social process producing emergence
Simulations

Other Exceptions

Unobservant individuals and reluctance to change
\( \lambda_{RD} = 0 \rightarrow \text{stable leader, transitory hierarchy} \)
\( \delta_A \rightarrow 0 \rightarrow \text{stable leader, transitory hierarchy} \)
\( \delta_B \rightarrow 0 \rightarrow \text{stable leader, stable hierarchy but a suboptimal pool of independent actors} \)
$\delta_B = 0.01$ suboptimal pool of independent actors
Simulations

Allow friend replacement

Drop and replace friends with $w_{j}^{i} < w = 0.2 / d$

- Develop a large population of agents with direct link to leader, but not everyone
Drop low performing friends