Solving Dynamic Models with Heterogeneous Agents and Aggregate Uncertainty with Dynare or Dynare++

Wouter J. DEN HAAN\textsuperscript{1} Tarik OCAKTAN\textsuperscript{2}

\textsuperscript{1}Department of Economics
University of Amsterdam

\textsuperscript{2}Paris School of Economics

Society for Economic Dynamics (Istanbul, 2009)
Outline

Motivation
Heterogeneous agent models
Previous Work

Our Results/Contribution
Main Results
Basic Ideas for Implementation
Reasons to study heterogeneous agents framework.

- **Representative agent framework:**
  - New Keynesian models rely either on the representative agent paradigm or on a very **limited degree of heterogeneity** (i.e., patient and impatient agents).
  - Assumption needed to justify the representative agent model are **not realistic** (i.e., perfect insurance of idiosyncratic risk).
  - Popularity due to the availability of **solution toolboxes** (i.e., Uhlig’s toolbox, Schmitt-Grohe and Uribe codes, Dynare, ...).

- **Issues** that require heterogeneous agent framework
  - **Distribution** of wealth and income.
  - Inequality.

- **Macroeconomic fluctuations** are likely to affect agents differently and **economic policy** should take this into account.
Solving Heterogeneous Agent Models: Challenges.

- Combining heterogeneity and aggregate risk makes it difficult to obtain a numerical solution.
- Time varying distribution of capital holdings across agents.
- Knowledge about this distribution is required for forecasting future prices
- Time-varying distribution means high-dimensional state space
Solving Heterogeneous Agent Models

Previous Work.

- Krusell and Smith (JPE, 1998):
  1. Set an arbitrary law of motion for selected moments.
  2. Solve the individual problem.
  3. Simulate the economy for $N$ agents and $T$ time periods.
  4. Regress and update the aggregate law of motion.

- Algan, Allais, and den Haan (JEDC, 2008):
  - Parameterized approximation of the distribution function.
  - Avoids simulation step.

- Preston and Roca (NBER, 2007):
  - Perturbation method.
  - Choice of the moments is determined by the order of the approximation.

- den Haan and Rendahl (JEDC, forthcoming):
  - Explicit aggregation (XPA).
Our Contribution.


2. Describe (simple) steps that allow one to solve this program using Dynare and a simple "mother" program.

3. Compare two algorithms: KS & XPA.
   3.1 KS very popular.
   3.2 XPA new algorithm but makes "mother" program very simple if individual policy rules are solved with Dynare.

4. Compare global solution method with local solution method

Individual Optimization Problem.

The model is characterized by three modifications:

1. Continuum of individuals indexed by $i \in [0, 1]$.
2. Replace the borrowing constraint by a penalty term.
3. Shocks have continuous support.

$$\max \{c_{it}, a_{i,t+1}\} \quad E_t \sum_{t=0}^{\infty} \beta^t \frac{c_{it}^{1-\gamma} - 1}{1 - \gamma} - \phi P(a_{i,t+1})$$

s.t. $c_{it} + a_{i,t+1} = r(k_t, l_t, z_t) a_{it} + w(k_t, l_t, z_t) e_{it} \bar{l} + (1 - \delta) a_{it}$

with $z_t \sim AR(1)$ and $e_{it} \sim AR(1)$. The law of motion for the distribution is given by $\Gamma_{t+1} = H(\Gamma_t)$

- $k_t = \int a_{it} di$
- $r_t = F_k$
- $w_t = F_l$
- $l_t = \bar{l}$
Explicit Aggregation.

The $j$th iteration.

1. $k' = \zeta_0^j + \zeta_1^j k + \zeta_2^j z$

2. Solve for the individual policy functions (using projection or perturbation):

$$a'_i = \theta_0^j + \theta_1^j a_i + \theta_2^j e_i + \theta_3^j z + \theta_4^j k$$

3. Update the aggregate law of motion, using $k' = \int a'_i \, di$:

$$k' = \theta_0^j + \theta_1^j \int a_i \, di + \theta_2^j \int e_i \, di + \theta_3^j z + \theta_4^j k$$

Using $k \equiv \int a_i \, di$ and $\int e_i \, di = 1$, we get:

$$\begin{align*}
\zeta_0^{j+1} &= \theta_0^j + \theta_2^j \\
\zeta_1^{j+1} &= \theta_1^j + \theta_4^j \\
\zeta_2^{j+1} &= \theta_3^j
\end{align*}$$
Solving the Model.

Convergence

The coefficients of the aggregate law of motion stabilize after several iterations ($||\zeta^{j+1} - \zeta^j|| < 10^{-6}$).
First Order Approximation.

Initialize aggregate law of motion:

\[ k' = \zeta_0^0 + \zeta_1^0 k + \zeta_2^0 z \]

1. Solve for individual policy rule using perturbation or projection method

\[ a'_i = \theta_0 + \theta_1 a_i + \theta_2 e_i + \theta_3 z + \theta_4 k \]

2. Update aggregate law of motion

A. Simulation and regression
   2.1. **Simulate economy** using the policy rules obtained in step 1. and **construct** a time series for \( a_i \) with \( i = 1, \ldots, N \) and aggregate \( k_t = \frac{1}{N} \sum_i a_{it} \)
   2.2. Use a **regression analysis** to get a new estimate for the law of motion and **update** \( \zeta^i \) to \( \zeta^{i+1} \)

B. Explicit Aggregation
   2.1. Compute \( \int a_{it} di \) explicitly
       \[ k' = \theta_0 + \theta_1 \int a_{it} di + \theta_2 \int e_{it} di + \theta_3 z + \theta_4 k \]
   2.2. Reordering
       \[ \zeta_0^{i+1} = \theta_0 + \theta_2 \]
       \[ \zeta_1^{i+1} = \theta_1 + \theta_4 \]
       \[ \zeta_2^{i+1} = \theta_3 \]

3. If \( \zeta^i - \zeta^{i+1} < tol \)

   Yes

   End

No
Initialize aggregate law of motion and 2nd order moments:
\[ k' = g_2(k, M_{ae}, M_{a^2}, z; \zeta^0) \]
\[ M'_{a^2} = g_2(k, M_{ae}, M_{a^2}, z; \bar{\zeta}^0) \]
\[ M'_{ae} = g_2(k, M_{ae}, M_{a^2}, k', z; \bar{\zeta}^0) \]

1. **Solve** for individual policy rule using perturbation or projection method
\[ a' = P_2(a, e, z, k, M_{ae}, M_{a^2}; \theta) \]
\[ a'^2 = P_2(a, e, z, k, M_{ae}, M_{a^2}; \bar{\theta}) \]
\[ a'e = P_2(a, e, z, k, M_{ae}, M_{a^2}; \tilde{\theta}) \]

2. **Update** aggregate law of motion and 2nd order moments
\[ k' = \int P_2(a, e, z, k, M_{ae}, M_{a^2}; \theta) \, di \]
\[ M'_{a^2} = \int P_2(a, e, z, k, M_{ae}, M_{a^2}; \bar{\theta}) \, di \]
\[ M'_{ae} = (1 - \rho_e)k' + \rho_e \int P_2(a, e, z, k, M_{ae}, M_{a^2}; \tilde{\theta}) \, di \]

3. If \( \zeta^i - \zeta^{i+1} < tol \)

   - **Yes**
   - **End**
Comparing Solution Methods.

Coefficients of the aggregate law of motion ($\gamma = 1$).

The results obtained through **explicit aggregation** are similar to the Krusell and Smith (1998) approach.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$\sigma_e$</th>
<th>Method for ALM</th>
<th>Coefficients for the ALM</th>
<th>$\zeta_0$</th>
<th>$\zeta_1$</th>
<th>$\zeta_2$</th>
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</table>
Comparing Solution Methods.

Coefficients of the aggregate law of motion ($\gamma = 2$).

Differences occur between the approximations obtained from projection and perturbation for high idiosyncratic uncertainty.

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<tr>
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<th>Method for ALM</th>
<th>Coefficients for the ALM</th>
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<tr>
<td></td>
<td></td>
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<td>Projection</td>
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</table>
Approximation Accuracy.
log Euler equation errors

**Perturbation** is less accurate than projection if idiosyncratic risk is high.

| $\gamma$ | $\sigma_e$ | $||E||_\infty$ | $||E||_1$ | $||E||_\infty$ | $||E||_1$ |
|----------|------------|----------------|----------|----------------|----------|
| 1.0      | 0.005      | -5.2119        | -7.1647  | -5.2734        | -7.5850  |
|          | 0.050      | -5.1441        | -7.1013  | -4.9779        | -6.6844  |
|          | 0.100      | -4.8109        | -6.0265  | -4.4006        | -5.1563  |
| 1.5      | 0.005      | -5.2679        | -7.4768  | -5.3530        | -7.6463  |
|          | 0.050      | -5.3538        | -7.1730  | -4.9949        | -6.5249  |
|          | 0.100      | -4.7907        | -5.7823  | -4.2882        | -4.9477  |
| 2.0      | 0.005      | -5.2555        | -7.5171  | -5.3215        | -7.6430  |
|          | 0.050      | -5.1424        | -7.0488  | -4.9148        | -6.3372  |
|          | 0.100      | -4.5529        | -5.6149  | -4.0674        | -4.7447  |
Aggregate Capital Dynamics.

Idiosyncratic Risk

Increase $\sigma_e$ from 0.005 to 0.1, for given $\gamma = 2$:

- **First order perturbation** does not capture the effect of rising idiosyncratic risk.
- Using **second order perturbation**, higher idiosyncratic uncertainty lowers the mean of the distribution.
Aggregate Capital Dynamics.

Risk Aversion

Higher risk aversion yields:

- Higher volatility of $k$.
- Lower mean of $k$.

Sample mean and standard deviation for aggregate capital ($T = 400$ and $\sigma_e = 0.005$):

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>Method Mean</th>
<th>PerturbationStd.dev.</th>
<th>Mean Std.dev.</th>
<th>Projection Mean Std.dev.</th>
<th>Mean Std.dev.</th>
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<tr>
<td>1.00</td>
<td>5.1283</td>
<td>0.1330</td>
<td>5.1284</td>
<td>0.1337</td>
<td>5.1244</td>
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<td>1.50</td>
<td>5.1255</td>
<td>0.1496</td>
<td>5.1256</td>
<td>0.1506</td>
<td>5.1154</td>
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<td>2.00</td>
<td>5.1231</td>
<td>0.1638</td>
<td>5.1236</td>
<td>0.1652</td>
<td>5.1062</td>
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<td>2.50</td>
<td>5.1209</td>
<td>0.1759</td>
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<td>3.00</td>
<td>5.1190</td>
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<td>3.50</td>
<td>5.1174</td>
<td>0.1950</td>
<td>5.1191</td>
<td>0.1983</td>
<td>5.0811</td>
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</table>
Aggregate Capital Dynamics.

Risk Aversion

(a)-(c): perturbation and (d)-(f): projection. Dotted line: second order approximation. $T = 400$ and $\sigma_e = 0.005$. 

(a) $\gamma = 1.00$

(b) $\gamma = 2.00$

(c) $\gamma = 3.50$

(d) $\gamma = 1.00$

(e) $\gamma = 2.00$

(f) $\gamma = 3.50$
for idxIter = 1:iIter

    % Solve the model for given coeffs. of the aggregate law of motion (vZetaOld)
dynare Dimension4PF noclearall;

    vZetaNew(1) = vTheta(1) + vTheta(3);
vZetaNew(2) = vTheta(2) + vTheta(5);
vZetaNew(3) = vTheta(4);

    % Check convergence of coefficients
dConv = fnConvergence(vZetaNew,vZetaOld,iTol);
if dConv == 1
    break;
end

    vZetaOld = dLambda * vZetaNew + (1-dLambda) * vZetaOld;

    pZeta0 = vZetaOld(1);
pZeta1 = vZetaOld(2);
pZeta2 = vZetaOld(3);
delete InitParams.mat;
    save InitParams.mat pZeta0 pZeta1 pZeta2;

end
The Dynare code:

```matlab
var ... ;       // declare endogeneous variables
varexo ...;     // declare exogeneous variables
parameters ... ; // declare parameters

load InitParams;   // load coefficients for ALM
    set_param_value('pZeta0',pZeta0); set_param_value('pZeta1',pZeta1);
    set_param_value('pZeta2',pZeta2);
load StructParams;  // load structural parameters (sensitivity analysis)
    set_param_value('pGamma',pGamma); set_param_value('pSigmae',pSigmae);

model; ... end;     // model block
initval; ... end;    // initial values for solver
shocks; ... end;     // declare shocks
stoch_simul(order=1,nocorr,noprint,nomoments,IRF=0);

// Collecting parameters
mPolicy = [oo_.dr.ys'; oo_.dr.ghx'; oo_.dr.ghu']; // read coefficients of policy functions
mPolA = mPolicy(:,2);
// Rearrange parameters
dTheta0  = mPolA(1)-mPolA(2)*mPolA(1)-mPolA(6)-mPolA(7)-mPolA(5)*mPolicy(1,5);
dTheta1  = mPolA(2);
dTheta2  = mPolA(6);
dTheta3  = mPolA(7);
dTheta4  = mPolA(5);

vTheta = [dTheta0 dTheta1 dTheta2 dTheta3 dTheta4];
```
Summary

- We showed how a heterogeneous agent model with aggregate uncertainty can be easily solved with Dynare and explicit aggregation.
- The present algorithm yields similar results to the ones obtained by the Krusell and Smith (1998) approach, but it is faster and requires less memory space.
- We have illustrated approximations up to the second order. Using Dynare++, it can easily be extended to the $n$th order case.

Outlook

- Problem: Convergence problems when $\sigma_e$ and $\gamma$ are high when solving with second order projection method.
- Third order approximation necessary for studying skewness of the distribution.
For Further Reading


The Model.

\[
V(a_i, e_i; z, \Gamma) = \max_{\{c_i, a'_i\}} \left\{ \frac{c_i^{1-\gamma} - 1}{1-\gamma} + \beta E\left[ V(a'_i, e'_i; z', \Gamma') - \phi P(a'_i) \right] \right\}
\]

s.t. \[ (1 - \delta)a_i + r(k, l, z)a_i + w(k, l, z)e_i \bar{l} - c_i - a'_i \geq 0 \]
\[
z' = (1 - \rho_z)\mu_z + \rho_z z + \varepsilon'^z \\
e'_i = (1 - \rho_e)\mu_e + \rho_e e_i + \varepsilon'_i \\
\Gamma' = H(\Gamma, z)\]
The Model.
First Order Approximation.

\[ V(a_i, e_i; z, k) = \max_{\{c_i, a'_i\}} \left\{ \frac{c_i^{1-\gamma} - 1}{1 - \gamma} + \beta E[V(a'_i, e'_i; z', k') - \phi P(a'_i)] \right\} \]

s.t \( (1 - \delta)a_i + r(k, l, z)a_i + w(k, l, z)e_i\bar{l} - c_i - a'_i \geq 0 \)

\[ z' = (1 - \rho_z)\mu_z + \rho_z z + \varepsilon' z \quad (2) \]

\[ e'_i = (1 - \rho_e)\mu_e + \rho_e e_i + \varepsilon'_e \]

\[ k' = \zeta_0 + \zeta_1 k + \zeta_2 z \]
The Model.

Second Order Approximation.

\[ V(a_i, e_i; z, k, M_{ae}, M_{a^2}) = \max_{\{c_i, a_i'\}} \left\{ \frac{c_i^{1-\gamma} - 1}{1 - \gamma} + \beta E[V(a_i', e_i'; z', k', M_{ae}', M_{a^2}') - \phi P(a_i')] \right\} \]

s.t. \((1 - \delta)a_i + r(k, l, z)a_i + w(k, l, z)e_i - c_i - a_i' \geq 0\)

\[
\begin{align*}
z' &= (1 - \rho_z)\mu_z + \rho_z z + \varepsilon_i' z \\
e_i' &= (1 - \rho_{e})\mu_e + \rho_{e} e_i + \varepsilon_i' e \\
k' &= \xi_0 + \xi_1 k + \xi_2 z + \xi_3 M_{ae} + \xi_4 M_{a^2} + \xi_5 k^2 + \xi_6 z^2 + \xi_7 k z \\
M_{a^2}' &= \tilde{\xi}_0 + \tilde{\xi}_1 k + \tilde{\xi}_2 z + \tilde{\xi}_3 M_{ae} + \tilde{\xi}_4 M_{a^2} + \tilde{\xi}_5 k^2 + \tilde{\xi}_6 z^2 + \tilde{\xi}_7 k z \\
M_{ae}' &= \ddot{\xi}_0 + \ddot{\xi}_1 k + \ddot{\xi}_2 z + \ddot{\xi}_3 M_{ae} + \ddot{\xi}_4 M_{a^2} + \ddot{\xi}_5 k^2 + \ddot{\xi}_6 z^2 + \ddot{\xi}_7 k z
\end{align*}
\]
Approximating Functions.
Family of Monomials (2nd order projection method).

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<th>e</th>
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### Approximating Functions.

**Family of Monomials (2nd order perturbation method).**

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