Adverse Selection, Liquidity, and Market Breakdown

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Abstract

This paper develops a model that illustrates how even a small amount of adverse selection in the asset market can lead to an increased asset price volatility and possibly to a breakdown of trade during crises. Asymmetric information about asset returns generates the Akerlof’s "lemons" problem when buyers do not know whether an asset is sold because of its low quality or because the seller experienced a sudden need for liquidity. The adverse selection can lead to an equilibrium with no trade, reflecting the buyers’ belief that most assets offered for sale are of low quality. I analyze the role of market liquidity, uncertainty about the assets value, and beliefs about the likelihood of a crisis in amplifying the effect of adverse selection. The policy implications depend on which amplification mechanism causes the market breakdown.

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1 Introduction

In the recent crisis of 2007-2009, the market for securities backed by subprime mortgages was the first to suffer a sudden dry up in liquidity. Among possible explanations for market freezes are increased uncertainty and information asymmetries about the value of assets. In particular, the difficulty in assessing the fundamental value of securities may lead to the adverse selection issue.

Financial institutions held significant amount of mortgage backed securities (MBS). Before the crisis, many of those MBS were rated AAA, which implied a minimal risk of default. These assets were considered liquid: if a financial institution needed cash, it could sell these securities at a fair market price. When in February 2007 subprime mortgage defaults had increased, triggering the liquidity crisis, a large fraction of these securities have been downgraded. The impact of declining housing prices on securities depended on the exact composition of assets and mortgages that backed them. Due to the complexity of structured financial products and heterogeneity of the underlying asset pool, owners had an informational advantage in estimating how much those securities are worth.

The asymmetric information about the assets value leads to the Akerlof (1970) lemons problem: a buyer does not know whether the seller is selling the security because of a sudden need for liquidity, or because the seller is trying to unload the toxic assets. This adverse selection issue can generate the market illiquidity reflecting buyers’ beliefs that most securities have skewed payoffs: they offer high expected return in most states of nature but suffer substantial losses in extremely bad states. When an economy is in a normal state with strong fundamentals, the asymmetric information does not significantly affect the asset value. However, when an economy is subject to a negative shock, the value of securities becomes more sensitive to private information and the adverse selection may influence the trading decisions. (Morris and Shin [23])

2 For example, 27 of the 30 tranches of asset-backed CDOs underwritten by Merrill Lynch in 2007 were downgraded from AAA ratings to “junk” (Coval, Jurek and Stafford [13]).

3 The junior equity tranches (also referred to as "toxic waste") were usually held by the issuing bank; they were traded infrequently and were therefore hard to value. Also, these structured finance products received overly optimistic ratings from the credit rating agencies. One of the reason the underlying securities default risks were underestimated is that the statistical models were based on the historically low mortgage default and delinquency rates. (Brunnermeier [8])
securities offered for sale are of low quality. Krishnamurthy [21] identifies this issue as one of diagnoses of the current crisis: market participants may fear that if they transact they will be left with a "lemon". Drucker and Mayer [14] find that underwriters of prime MBS appeared to exploit access to better information when trading in the secondary market. Elul [17] also finds evidence of adverse selection in the prime mortgage market. Moreover, the extent of asymmetric information was not fully known, e.g., Gorton [18] argues that there was a loss of information due to the complexity of securitization.

Furthermore, as market condition worsened, investors’ value for liquidity had increased which was reflected in the high spreads of MBS relative to Treasury bills (Krishnamurthy [21]). The deleveraging that accompanies the initial shock can further aggravate the adverse selection problem. Because of the losses on their MBS, some banks became undercapitalized; however, their attempts to recapitalize pushed their market price further down. This reflects the investors’ fear that any bank that issues new equity or debt may be overvalued, leading to the liquidity crunch. As market liquidity falls, it becomes difficult to find trading partners which leads to the fire-sale pricing.

The demand for ABS collapsed from over $500 billion in 2007 to $20 billion in 2009 as illustrated in Figure 1 which is taken from Adrian, Ashcraft, and Pozsar (2010).

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4 For example, the repo market in 2007-2009, as described by Gorton and Metrick [19].
5 "The large haircuts on some securities could be seen as a response by leveraged entities to the potential drying up of trading possibilities in the asset-backed securities (ABS) market. The equity market, in contrast, is populated mainly with non-leveraged entities such as mutual funds, pension funds, insurance companies, and households, and hence is less vulnerable to the drying up of trading partners." Morris and Shin [23]
6 Brunnermeier and Pedersen [9] refer to this phenomena as a "loss spiral" and a "margin spiral". Adrian and Shin [1] documented evidence of these phenomena for investments banks.
7 The haircut on ABSs increased from 3-5% in August 2007 to 50-60% in August 2008. The haircut on equities increased from 15% to 20% for the same period (Gorton and Metrick [19]).
In this paper, I develop a model that illustrates how adverse selection in an asset market can lead to an equilibrium with no trade during the crisis. Also, I analyze the role of market liquidity, crisis expectations, and uncertainty about assets value, in amplifying the effect of adverse selection.

In my model, agents have the Diamond-Dybvig\(^8\) type of preferences: they consume in period one or in period two, depending on whether they receive a liquidity shock in period one. In period zero, investors choose how much to invest into risky long-term assets which have idiosyncratic payoffs. In period one, liquidity shocks are realized and, subsequently, risky investments are traded in the financial market. The late consumers (who have not experienced a liquidity shock) are the buyers in the financial market.

Market liquidity, defined as the demand for risky investments in the interim period, depends on the amount of the safe asset held by the investors that is available to buy risky assets from liquidity traders. Also, market liquidity can be characterized by the cost (in terms of foregone payoff) of selling long-term asset prematurely. The equilibrium market price is determined by investors preference for liquidity as well as the quality of assets sold. Similarly to the Allen and Gale [3] "cash-in-the-market" framework, the higher is the average liquidity preference of investors in the market, the greater is the average holdings of the safe asset in investors’ portfolios, and therefore, the greater is the market ability to absorb liquidity trading without large price changes. If the preference for liquidity is

\(^8\)Diamond and Dybvig (1983)
high, the "cash-in-the-market" pricing\(^9\) leads to the market prices below fundamentals. I demonstrate that the presence of adverse selection in the market can further depress the asset prices exacerbating price volatility.

I begin by examining the portfolio choice when investors have private information about their investment payoff and it is public information which investors have received a liquidity shock. Then I analyze the situation when the identity of investors hit by a liquidity shock is private information. In the latter case, investors can take advantage of their private information by selling the low-payoff investments and keeping the ones with high payoffs. This generates the lemons problem.\(^10\)

When economy is in a normal state with a small fraction of low quality assets, adverse selection does not significantly affect market liquidity. If the market is liquid then informed investors can gain from trading on private information by pretending to be liquidity traders. However, during a crisis (when the fraction of low quality assets offered for sale is large) the adverse selection can lead to no-trade outcome. This result is consistent with the fact that liquidity crises tend to be associated with economic downturns.\(^11\)

Furthermore, I show that even a small amount of adverse selection can lead to a market breakdown during the crisis if it is accompanied by any the following amplification mechanisms: increase in the liquidity preference during the crisis, beliefs about the likelihood of a crisis, or uncertainty about assets returns. On one hand, higher preference for liquidity alleviates the adverse selection since assets are more likely to be sold due to seller’s liquidity needs than due to their low quality. On the other hand, higher liquidity preference implies lower demand for risky assets and therefore leads to lower prices. I show that if a crisis is accompanied by the flight to liquidity, the effect of adverse selection can be amplified, leading to the fire-sale pricing (assets are priced significantly below their expected payoffs) and possibly to a complete breakdown of trade. Underestimating the likelihood of a crisis

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\(^9\)The equilibrium price of the risky asset is equal to the lesser of two amounts: the discounted value of future dividends and the amount of cash available from buyers divided by the number of shares sold. (Allen and Gale [5])

\(^10\)This setting is different from models where investors have private information about aggregate (common) payoff and information can be revealed through trading.

can also aggravate the effect of adverse selection, leading to increased asset price volatility or market breakdown during the crisis. A Knightian uncertainty (ambiguity) about the fraction of low quality assets in the market can cause market illiquidity as well. The private information about asset quality may be welfare beneficial if adverse selection does not lead to the market breakdown. The ability to trade based on private information smoothes the ex-ante consumption and consequently may lead to an increase in welfare.

These amplification mechanisms lead to different policy implications. If the market breakdown is due to an increase in the liquidity preference or underestimating the likelihood of a crisis then injecting liquidity into the market can restore the trading. However, if the no-trade outcome is caused by a large fraction of lemons or by the Knightian uncertainty about it, then it is more effective to remove these low quality assets from the market. The requirement of larger liquidity holdings prevents the market breakdown during the crisis.

The central planner can reduce the adverse selection problem by increasing holdings of safe asset. Since adverse selection leads to the larger supply of low quality assets, more market liquidity is needed to absorb these trades. The central planner allocation reduces consumption volatility by improving consumption of liquidity investors and investors with low quality assets. As a result, it achieves a higher welfare than any of the market equilibria. The welfare improvement is more significant relative to an equilibrium with the market breakdown.

This paper is organized as follows. In the next section I discuss the related literature. Section 3 describes the model environment, and Section 4 characterizes the equilibrium. Section 5 analyzes the model from the central planner perspective and provides the welfare comparison. Section 5 concludes the paper. All results are proved in the Appendix.

2 Related Literature

As has been demonstrated in line of work started by Akerlof [2], asymmetric information between buyers and sellers can lead to a complete breakdown of trade. Morris and Shin [23] show that adverse selection may lead to the failure of trade in a coordination game among differently informed traders. If the condition of approximate common knowledge of
an upper bound on expected losses fails then traders withdraw from trade because they fear their uninformed partners may refrain from trade. Heider, Hoerova, and Holthausen [20] study the interbank market in the presence of counterparty risk. They show that private information about the risk of banks’ assets and heterogeneous liquidity needs can result in a market breakdown and liquidity hoarding. Bolton, Santos, and Scheinkman [7] analyze the efficiency of trading equilibria in the presence of asymmetric information about asset values. The delay in trading increases the adverse selection problem and may inefficiently accelerate asset liquidation. Eisfeldt [16] shows that higher investment productivity leads to the increased liquidity in a model where long-term risky assets are illiquid due to the adverse selection.

The importance of Knightian uncertainty has been emphasized by Easley and O’Hara [15], Caballero [10], Caballero and Krishnamurthy [11], Krishnamurthy [22], and Uhlig [24].

Uhlig [24] develops a model of a systemic bank run. He considers two variants, uncertainty aversion and adverse selection, and illustrates that the former generates the following feature of financial crisis: a larger share of troubled financial institutions results in a steeper asset price discount. However, in my model it is possible that the adverse selection can lead to a larger price discount even if there is no uncertainty about assets value.

Caballero [10] argues that complexity and Knightian uncertainty are key multipliers that can greatly increase the impact of an initial shock. Krishnamurthy [22] examines two amplification mechanisms that operate during liquidity crises. The first mechanism involves asset prices and balance sheets: a negative shock to agents’ balance sheets causes them to liquidate assets, lowering prices, further deteriorating balance sheets and amplifying the shock. The second mechanism involves investors’ Knightian uncertainty: shocks to financial innovations increase agents’ uncertainty about their investments, causing them to disengage from risk and seek liquid investments, which amplifies the crisis. Caballero and Simsek [12] developed a model of fire sales due to the endogenously increased complexity of financial network during crises. Easley and O’Hara [15] show that uncertainty about the true value of an asset can lead to a no-trade equilibrium when investors have incomplete preferences over portfolios.

Allen and Gale ([3], [4], [5], [6]) developed a liquidity-based approach to study financial
crises. When supply and demand for liquidity are inelastic in the short run, a small degree of aggregate uncertainty can have a large effect on asset prices and lead to financial instability.

My paper contributes to the literature by analyzing the interaction between adverse selection and liquidity, and their role during the crisis. I show that the adverse selection leads to lower market liquidity and asset price volatility even if there is no aggregate uncertainty about liquidity preferences. The aggregate uncertainty about preference for liquidity, underestimating probability of a crisis and Knightian uncertainty about assets value can further amplify the effect of adverse selection, potentially resulting in a market breakdown.

3 Model

I consider a model with three dates indexed by $t = 0, 1, 2$. There is a continuum of ex-ante identical financial institutions (investors, for short) with an aggregate Lebesgue measure of unity. There is only one good in the economy that can be used for consumption and investment. All investors are endowed with $\omega$ units of good at date $t = 0$, and nothing at the later dates.

3.1 Preferences

Investors consume at date one or two, depending on whether they receive a liquidity shock at date one. The probability of receiving a liquidity shock in period one is denoted by $\lambda$. So $\lambda$ is also a fraction of investors hit by a liquidity shock. Investors who receive a liquidity shock have to liquidate their risky long-term asset holdings and consume all their wealth in period one. So they are effectively early consumers who value consumption only at date $t = 1$. I will also refer to them as liquidity traders. The rest are the late consumers who value the consumption only at date $t = 2$.

Investors have Diamond-Dybvig type of preferences:

$$U(c_1, c_2) = \lambda u(c_1) + (1 - \lambda)u(c_2)$$

(1)

where $c_t$ is the consumption at dates $t = 1, 2$. In each period, investors have logarithmic utility: $u(c_t) = \log c_t$. 

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3.2 Investment technology

Investors have access to two types of constant returns investment technologies. One is a storage technology (also called the safe asset or cash), which has zero net return: one unit of safe asset pays out one unit of safe asset in the next period. Another type of technology is a long-term risky investment project (also called a risky asset). The risky assets pay off \( \tilde{R} \in \{R_H, R_L\} \) per unit of investment at date two that represents an idiosyncratic (investment specific) productivity. The risky investment with payoff \( R_H \) is a high-quality asset while an investment with payoff \( R_L \) is a low-quality asset (lemon).

There are two states of nature \( s = 1 \) and \( s = 2 \) that are revealed at date \( t = 1 \). The state 1 is a normal state and the state 2 is a crisis state. These states are realized with ex-ante probabilities \((1 - q)\) and \( q \). I will also use the notation \( q_1 = 1 - q \) and \( q_2 = q \). The states differ with respect to aggregate (market) productivity. There are more high-quality investments in the normal state \( s = 1 \) relative to the crisis state \( s = 2 \).

The quality of assets are independent across investors. Each investor \( i \) has a choice of starting his own investment project \( i \) by investing a fraction of his endowment. The investor can start only one project, and each project has only one owner. The idiosyncratic payoff of each investment \( i \) is an independent realization of a random variable \( \tilde{R}^i \) that takes two values: a low value \( R_L \) with probability \( \pi_s \) and a high value \( R_H \) with probability \( (1 - \pi_s) \) where \( s \in \{1, 2\} \). The state 1 is a normal state where the fraction of low quality assets is small: \( \pi_s = \pi_1 \). The state 2 is a crisis state with a larger fraction of low quality assets: \( \pi_s = \pi_2 > \pi_1 \).

Alternative specification\(^{12}\) is that the payoff of each investment \( i \) consists of two components: \( \tilde{R}^i(s) = \alpha_i(s)R_L + (1 - \alpha_i(s))R_H \). The fraction \( \alpha_i(s) \) represents the investment’s exposure to an asset with a low payoff \( R_L \). The individual exposure \( \alpha_i(s) \) is a random variable that takes two values: a high value \( \alpha_h \) with probability \( \pi_s \) and a low value \( \alpha_l \) with probability \( (1 - \pi_s) \) where \( s \in \{1, 2\} \). So that the market exposure is given by \( \alpha_m(s) = \pi_s\alpha_h + (1 - \pi_s)\alpha_l \) and the market (aggregate) payoff is \( R^m(s) = \alpha_m(s)R_L + (1 - \alpha_m(s))R_H \). As before, the state 1 is a normal state where the fraction of low quality assets is small: \( \pi = \pi_1 \). The state 2 is a crisis state with more low quality assets.

\(^{12}\)This specification is equivalent to the above but it makes the model more applicable to the MBS market.
sets: $\pi = \pi_2 > \pi_1$, so that $R^m(1) > R^m(2)$. Denote the payoff of low-quality investment as $R_L$, i.e., $R_L = \alpha_h R^L + (1 - \alpha_h) R^H$. Similarly, the high-quality investment payoff is denoted by $R_H$ such that $R_H = \alpha_l R^L + (1 - \alpha_l) R^H$.

The expected payoff of each individual risky project in state $s$ is denoted by $\bar{R}_s = \pi_s R_L + (1 - \pi_s) R_H$ with $R_L < 1 < R_H$. The expected payoff when an economy is in a normal state is higher than when it is in a crisis state: $\bar{R}_1 > \bar{R}_2$. The expected payoff before states are realized is denoted by $\bar{R} = (1 - q) \bar{R}_1 + q \bar{R}_2$ with $\bar{R}_s > 1$ in each state $s = 1, 2$.

The long-term asset can be liquidated prematurely at date $t = 1$, in this case, one unit of the risky asset $R_k$ yields $r_k$ units of the good, where $k = L, H$ and $R_L < r_L \leq r_H < 1$. The holdings of the two-period risky asset can be traded in financial market at date $t = 1$.

Figure 1 summarizes the payoff structure.

<table>
<thead>
<tr>
<th>time</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>safe asset</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>risky asset</td>
<td>1</td>
<td>$r_k$</td>
<td>$R_k$</td>
</tr>
</tbody>
</table>

3.3 Information

At date $t = 0$, investors make investment choices between the two technologies, safe and risky, in proportion $x$ and $(1 - x)$ respectively. They choose their asset holdings to maximize their expected utility.

At date $t = 1$, the liquidity shocks and the aggregate state are realized, and the financial market opens. Investors privately observe their asset payoffs. The supply of the risky assets comes from the investors who have experienced a liquidity shock. The demand for risky assets comes from investors who have not received a liquidity shock. Any investor can liquidate his investment project at date one, receiving $r_k$ units of the good per unit of investment.

The timeline of the model is summarized in the figure below.
Note the markets are incomplete since there are two frictions in this economy: liquidity shock and asymmetric information about asset quality, which generates four possible types of investors in each state.

I will consider two cases. In the first case, it is public information which investors have experienced a liquidity shock. If an investor gets a liquidity shock, he sells or liquidates his holdings of the risky asset in order to consume as much as possible in period one. In the second case, identity of investors hit by a liquidity shock is private information. Therefore, after observing investment payoffs, agents can take advantage of this private information by selling low quality projects in the market at date $t = 1$. In this case, buyers are not able to distinguish whether an investor is selling his asset holdings because of its low payoff or because of the liquidity needs. This generates adverse selection problem, and leads to a discount on the investments sold in the market at date $t = 1$.

4 Equilibrium

4.1 Equilibrium without Adverse Selection

First, I consider the case where identity of investors hit by a liquidity shock is public information. Therefore, there is no adverse selection. All risky assets at $t = 1$ are sold by liquidity traders who cannot wait for the maturity of their investments at date $t = 2$.

Since all the investments have idiosyncratic productivity, the expected payoff of the risky asset sold in period one is $R_s$ in state $s$. All risky assets sold at $t = 1$ are aggregated in the market, hence, the variance of an asset bought at date $t = 1$ is zero. Therefore, the
return on risky asset bought in period one is \( R_s/p_s \), where \( p_s \) is the market price in state \( s \). The late consumers will be willing to buy risky asset at date \( t = 1 \) if the market price \( p_s \) is less than the expected payoff \( R_s \). The earlier consumers will be willing to sell their projects if the market price \( p_s \) is greater than the liquidation value \( r_k \).\(^{13}\)

At date \( t = 0 \), investors choose the investment allocations between the risky and safe technologies, in proportion \( x \) and \( (1 - x) \) respectively, in order to maximize their expected utility. The consumption of early consumers in state \( s \) is denoted by \( c_{1k}(s) \) and the consumption of late consumers in state \( s \) is denoted by \( c_{2k}(s) \) where \( k = L, H \) refers to payoff of an investment project \( i \).

\[
\sum_{s=1,2} q_s \left[ \lambda \log (\pi_s \log c_{1L}(s) + (1 - \pi_s) \log c_{1H}(s)) + (1 - \lambda) (\pi_s \log c_{2L}(s) + (1 - \pi_s) \log c_{2H}(s)) \right] 
\]

\[ s.t. \quad (i) \quad c_{1k}(s) = \begin{cases} 1 - x + p_s x & \text{if } p_s > r_k \\ 1 - x + r_k x & \text{if } p_s \leq r_k \end{cases} \]

\[ (ii) \quad c_{2H}(s) = \begin{cases} xR_H + y_s R_s & \text{if } p_s > r_k \\ xR_H + (1 - x) & \text{if } p_s \leq r_k \end{cases} \]

\[ (iii) \quad c_{2L}(s) = \begin{cases} xR_L + y_s R_s & \text{if } p_s > r_k \\ xR_L + (1 - x) & \text{if } p_s \leq r_k \end{cases} \]

The late consumers are willing to buy risky assets at \( t = 1 \) if the market price \( p_s \) is less than or equal to the expected payoff \( R_s \). Therefore, the demand for risky asset at \( t = 1 \) in state \( s \) is given by

\[
y(s) = \begin{cases} \frac{1-x}{p_s} & \text{if } p_s \leq R_s \\ 0 & \text{if } p_s > R_s \end{cases}
\]

Therefore, the aggregate demand at \( t = 1 \) in state \( s \) is given by

\[
D(s) = \begin{cases} (1 - \lambda) \frac{1-x}{p_s} & \text{if } p_s \leq R_s \\ 0 & \text{if } p_s > R_s \end{cases}
\]

The early consumers are willing to sell their investments if the market price \( p_s \) is greater than the liquidation value \( r_k \). Therefore, the aggregate supply at \( t = 1 \) in state \( s \) is given

\(^{13}\)For simplicity, I assume that if the asset price is equal to the liquidation value, investors choose to liquidate their assets rather than to sell.
by

\[ S(s) = \begin{cases} 
\lambda x & \text{if } p_s > r_H \\
\lambda \pi_s x & \text{if } r_L < p_s \leq r_H \\
0 & \text{if } p_s \leq r_L
\end{cases} \]  \quad (5)

Therefore, the price in state \( s \) is determined by the market clearing conditions:

\[ p_s = \min \left\{ \frac{(1 - \lambda)(1 - x)}{x} \lambda \bar{R}_s, \frac{(1 - \lambda)(1 - x)}{x} R_s \right\} \]  \quad (6)

Since the investment allocations are determined at \( t = 0 \) and there are no aggregate uncertainty about the probability of a liquidity shock \( \lambda \), the price is the same in both states: \( p_1 = p_2 \equiv p \).

**Proposition 1.** The market equilibrium achieves the first-best investment allocation \( x = (1 - \lambda) \) with market prices \( p_s = 1 \). The consumption allocations are given by

\[
\begin{align*}
   c_{1k}(s) &= 1 \\
   c_{2H}(s) &= (1 - \lambda) R_H + \lambda \bar{R}_s, \\
   c_{2L}(s) &= (1 - \lambda) r_L + \lambda \bar{R}_s.
\end{align*}
\]  \quad (7)

In the absence of adverse selection, the market price is the same across states. The investment and consumption allocations of early allocation are efficient in the sense that they coincide the central planner solution under full information. The consumption allocations of late consumers differ from the central planner solution under full information: \( c_{2K}(s) = \bar{R}_s \) because the investment quality is not observable.

### 4.2 Equilibrium with Adverse Selection

Now suppose identity of investors who have received a liquidity shock is private information. Therefore, after observing investment payoff, agents can take advantage of this private information by selling low productive investments in the market at date \( t = 1 \). This generates the adverse selection problem and therefore, leads to the discount on the price of risky assets sold at \( t = 1 \). Investors always can choose to liquidate their asset holdings if the investment payoff is low.

An investor who buys a risky asset at date \( t = 1 \), does not know whether it is sold due to a liquidity shock or because of its low payoff (low quality). The buyers believe
that with probability \( \lambda \) investment is sold due to a liquidity shock, and with probability 
\( (1 - \lambda)(1 - \pi_s) \) it sold because of its low payoff. Hence, buyers believe that the payoff of 
risky assets sold in state \( s \) is \( \hat{R}_s \) such that

\[
\hat{R}_s = \frac{\lambda}{\lambda + (1 - \lambda)\pi_s} R_s + \frac{(1 - \lambda)\pi_s}{\lambda + (1 - \lambda)\pi_s} r_L
\]

(8)

The late consumers are willing to buy risky asset at \( t = 1 \) if the market price \( p_s \) is less 
than the expected payoff \( \hat{R}_s \). Therefore, the demand for risky asset at \( t = 1 \) is given by

\[
y_s = \begin{cases} 
\frac{1 - x}{p_s} & \text{if } p_s \leq \hat{R}_s \\
0 & \text{if } p_s > \hat{R}_s 
\end{cases}
\]

(9)

The earlier consumers are willing to sell their investment if the market price \( p_s \) is greater 
than or equal to the liquidation value \( r_k \).

As before, the price in state \( s \) is determined by market clearing conditions.

\[
(\lambda + (1 - \lambda)\pi_s)xp_s = (1 - \lambda)(1 - x)
\]

(10)

However, the supply of risky assets is larger because of the adverse selection.

Therefore, the market price in state \( s \) can be expressed as the lesser of the two: market 
clearing and expected payoff \( \hat{R}_2 \):

\[
p_s = \min \left\{ \frac{(1 - \lambda)}{\lambda + (1 - \lambda)\pi_s} \frac{(1 - x)\hat{R}_2}{x} \right\}
\]

(11)

Note, that the price is no longer the same in both states since the fraction of low 
productive investments is larger in a crisis state: \( \pi_2 > \pi_1 \). Therefore, the price in a crisis 
state is lower than the price in a normal state: \( p_2 < p_1 \).

If the market price \( p_s \) is such that \( r_L < p_s \leq r_H \) then all asset with high payoffs will 
be liquidated so that only lemons (assets with low payoffs) are traded in the market. Therefore, 
the expected payoff of a risky asset is \( r_L \) since a buyer gets an asset with payoff \( R_L \) at date 
\( t = 2 \) which can be liquidated at date \( t = 1 \) yielding \( r_L \). In this case, there is no trading as 
no one would be willing to buy these low quality assets. If the fraction of low quality assets 
\( \pi_2 \) is sufficiently large so that the expected payoff is less than or equal to the liquidation 
value: \( \hat{R}_2 \leq r_H \), then there is no trading as well.

Investors choose their asset holdings, \( (x, 1 - x) \) to maximize their expected utility:
\[
\max \left\{ \sum_{s=1,2} q_s \left( \lambda \log (\pi_s \log c_{1L} (s) + (1 - \pi_s) \log c_{1H} (s)) + \right. \right.
\]
\[
\left. \left. \left. + (1 - \lambda) \left( \pi_s \log c_{2L} (s) + (1 - \pi_s) \log c_{2H} (s) \right) \right) \right\} \quad (12)
\]

s.t. 

(i) \[c_{1k} (s) = \begin{cases} 
1 - x + p_s x & \text{if } p_s > r_k \\
1 - x + r_k x & \text{if } p_s \leq r_k 
\end{cases} \]

(ii) \[c_{2H} (s) = \begin{cases} 
xR_H + (1 - x) \tilde{R}_s/p_s & \text{if } p_s > r_k \\
xR_H + (1 - x) & \text{if } p_s \leq r_k 
\end{cases} \]

(iii) \[c_{2L} (s) = \begin{cases} 
xp_s + (1 - x) \tilde{R}_s/p_s & \text{if } p_s > r_k \\
xR_L + (1 - x) & \text{if } p_s \leq r_k 
\end{cases} \]

Proposition 2. There exists a unique equilibrium, and there are two possible equilibrium types:

I. equilibrium with market trading in both states with the market price in a crisis state \( p_2 \) being lower than the market price in a normal state \( p_1 \);

II. equilibrium with market trading in normal state \( s = 1 \) and no trade in a crisis state \( s = 2 \).

Furthermore, the presence of adverse selection leads to a lower level of investments \( x \) relative to an equilibrium without adverse selection.

Type I is a pooling equilibrium where both high and low quality assets are sold. Type II is a separating equilibrium where in a crisis state investors choose to liquidate high quality assets rather than to sell them, which leads to a no-trade outcome.

Consider an equilibrium of type I with market trading in both states. The presence of adverse selection leads to the price volatility across states. The market price of risky asset in a crisis state is lower relative to a normal state since there more lemons in the market. Because of the adverse selection, assets offered for sale at \( t = 1 \) have lower expected return. As a result, the optimal investment allocation is lower than in an equilibrium without adverse selection. This leads to a loss in aggregate welfare. However, investors with low quality assets benefit from the private information at the expense of liquidity traders.

In a crisis state, the price of risky asset can be determined either by market clearing or the expected payoff \( \tilde{R}_2 \). If the price of risky asset at date \( t = 1 \) is below the liquidation
value $r_H$ then all asset with high payoffs will be liquidated. Therefore, there is no trade since only low quality assets are available in the market. Also, if there too many lemons 

$$\left( \pi_2 \geq \frac{\lambda(R_H - r_H)}{\lambda R_H + (1 - \lambda)r_H - r_L} \right)$$

so that the expected payoff $\hat{R}_2$ is below the liquidation value $r_H$, then there is no trading in a crisis state.

4.2.1 Properties of Equilibrium

Probability of a crisis state. The probability of a crisis state $q$ reflects the investors’ beliefs about the likelihood of a crisis. In this section, I examine how changes in $q$ affect the equilibrium values.

**Corollary 1.** If investors believe a crisis state is more likely to occur ($q$ is larger) then (i) investment allocation is smaller; (ii) market prices are higher; (iii) expected utility is lower. If the economy is in a type II equilibrium with market trading in a normal state and no trade in a crisis state then an increase in $q$ may lead to a type I equilibrium with trading in both states.

The higher probability of a crisis state $q$ implies that an asset is more likely to become a lemon, which makes it ex-ante less profitable. Therefore, an increase in $q$ leads to a lower level of investment and to a lower expected utility. The smaller investment at date $t = 0$ implies less supply and more demand for risky assets at date $t = 1$. As a result, market prices are higher, both in a type I and a type II equilibria.

Let us compare equilibria sequentially. The fact that the market price is increasing in the probability of a crisis makes it is possible to move from one equilibrium type to another. Suppose an economy is in a type II equilibrium with no market in a crisis state, and the probability of a crisis $q$ increases. Then it is possible that the price in a crisis state will increase sufficiently to switch to a type I equilibrium with market trading in both states. (If an economy is initially in a type I equilibrium then the equilibrium type does not change if $q$ is increased. If an economy is in a type II equilibrium and the probability $q$ is decreased then the equilibrium type does not change either.)

Consider the following numerical example. The asset return parameters are given $R_H = 1.3$, $r_H = 0.5$, $r_L = 0.3$, the fraction of low quality investments in a normal state: $\pi_1 = 0.03$ and in a crisis state: $\pi_2 = 0.3$, and probability of a liquidity shock: $\lambda = 0.2$. In this
example, 3% of assets become lemons in a normal state, and in a crisis state, the 30% of all assets are lemons. Figure 5 depicts the equilibrium values of investment, prices and expected utility as a function of probability of a crisis state $q$. At $q = 0.13$, there is a switch from an equilibrium with no trade in a crisis state to an equilibrium with trading in both states.

Figure 5. Equilibrium values of investment, prices and expected utility as a function of $q$.

Therefore, an equilibrium type depends on the probability of a crisis. If a crisis is considered to be a rare event (probability is small) then there is no market trading during the crisis.

Let the probability of a crisis $q$ depend on the previously realized state. So that conditional probability of transition from a normal state to a crisis state is smaller than the conditional probability of remaining in a crisis state. The transition matrix is given by

$$
\begin{bmatrix}
1 - q_{12} & q_{12} \\
1 - q_{22} & q_{22}
\end{bmatrix}
$$

where $q_{22} > q_{12}$ and $q_{jk} = \Pr(s = s_k|s = s_j), k, j \in \{1, 2\}$, so it is more likely that an economy continues to stay in a crisis state if it is realized. Let us look again at the numerical example. Suppose $q_{11} = 0.05$ and $q_{22} = 0.5$. If an economy is in a normal state then it is in a type II equilibrium with no trading during the crisis. Once an economy is in a crisis state, beliefs are revised and investment allocations are adjusted, and an economy moves to a type I equilibrium. So, the market trading is resumed next period even if the crisis persists.

Next, examine the role of beliefs about the likelihood of a crisis. Suppose the (true) probability of a crisis is $q_o$, however, investors believe that the probability is $q$ which can be less or greater than $q_o$. Let us look again at the numerical example considered before.
Suppose the (true) probability of a crisis is \( q_o = 0.1 \). Figures 6 depicts the equilibrium values of investment, prices and expected utility as a function of beliefs about probability of a crisis state \( q \in (0, 0.2) \). At \( q = 0.052 \), there is a switch from an equilibrium with no trade in a crisis state to an equilibrium with trading in both states.

Therefore, the initial expectation can affect an equilibrium type. Underestimating the probability of a crisis is more costly in term of welfare than overestimating as it may result in a no-trade outcome during the crisis.

**Liquidity preference** Now consider the situation when a crisis is accompanied by an exogenous increase in liquidity preference \( \lambda \) in addition to a larger fraction of low quality assets.

**Corollary 2.** Suppose the economy is in a type I equilibrium with market trading in both states. The increase in liquidity preference \( \lambda \) in a crisis state may lead to shift to a type II equilibrium with market trading in a normal state and no trade in a crisis state.

The price \( p_s \) is a decreasing function of the liquidity preference \( \lambda_s \). Therefore, the higher preference for liquidity in a crisis state results in the further price decrease relative to a normal state. Hence, a lack of liquidity during the crisis may amplify the adverse selection problem pushing the asset prices further down and possibly leading to a complete breakdown of trade. This reflects the fire-sale phenomenon when depressed prices reflect the difficulty of finding buyers during the crisis.

Again consider the numerical example: asset returns are given by \( R_H = 1.3, r_H = \)
0.5, \( r_L = 0.3 \), the fraction of low quality investments in a normal state: \( \pi_1 = 0.03 \) and in a crisis state: \( \pi_2 = 0.25 \), the probability of a crisis: \( q = 0.1 \). The probability of a liquidity shock in a normal state is \( \lambda_1 = 0.2 \). The figure below illustrates the effect of an increase in the liquidity preference in a crisis state \( \lambda_2 \) from 0.2 to 0.25 on the equilibrium investment and prices. When preference for liquidity is the same in both states \( \lambda_1 = \lambda_2 = 0.2 \), there is market trading in both states. However, if \( \lambda_2 > 0.224 \) then there is no trade during the crisis.

Figure 7. Equilibrium values of investment, prices and expected utility as a function of \( \lambda_2 \).

If a crisis is accompanied by the flight to liquidity, the adverse selection effect is magnified which exacerbates the asset price volatility. If the expected payoff \( \tilde{R} \) is higher than the liquidation value \( r_H \) and there is no trade then the trading can potentially be restored by providing liquidity (safe asset) to the market.

**Equilibrium types** Next I examine how equilibrium types depends on the interaction between liquidity preference (\( \lambda_2 \)), fraction of lemons (\( \pi_2 \)), and probability of a crisis (\( q \)).

Figure 8 depicts the possible equilibria regions for different values of \( \pi_2 \) and \( \lambda_2 \). Each point in the \((\pi_2, \lambda_2)\) plane corresponds to a particular type of equilibria: type I or type II. Thus, each type corresponds to a region in the plane which is marked accordingly. I consider three examples with the same values of \( R_H = 1.3, r_H = 0.5, r_L = 0.3 \) and different
values of probability of a crisis: $q = 0.05, q = 0.1,$ and $q = 0.5$.

As can be seen from the figure, even small amount of adverse selection (small $\pi_2$) can lead to the no-trade outcome if a crisis is considered to be a rare event (small $q$) and preference for liquidity is high (large $\lambda_2$). However, if both states are equally expected than there is trading even if there are many lemons in the market. The increase in liquidity preferences during the crisis is more likely to lead to the market breakdown (type II equilibrium) if the probability of a crisis is smaller.

Next figure illustrates the possible equilibria regions for different values of $q$ and $\lambda_2$. Again, I consider three examples with the same values of $R_H = 1.3, r_H = 0.5, r_L = 0.3$ and different values of $\pi_2$: $\pi_2 = 0.05, \pi_2 = 0.15, \pi_2 = 0.25$.

Even if the fraction of lemons in a crisis state is not significantly larger ($\pi_2 = 5\%$ and $\pi_1 = 3\%$), an increase in the liquidity preferences can lead to the no-trade equilibrium during the crisis. As before, the market breakdown is more likely to happen if the probability of a crisis is small. The threshold value of the crisis probability when economy switches from
4.3 Equilibrium with Adverse Selection and Knightian Uncertainty

Now consider the case when a crisis state is accompanied by an unanticipated shock in period one. The shock can be viewed as an "unforeseen contingency", an event that investors are not aware about so they do not plan for it. As a result of this shock, investors face Knightian uncertainty (ambiguity) about the fraction of low quality assets in a crisis state, i.e., $\hat{\pi}_2 \in [\pi_2, \bar{\pi}_2]$ where $\pi_1 \leq \pi_2 < \bar{\pi}$. Investors do not know the actual probability of an asset being a lemon, instead they believe the probability $\hat{\pi}_2$ belongs to the set: $[\pi, \bar{\pi}]$. Investors are assumed to have Gilboa-Schmeidler maxmin utility: $U(c) = \min_{\hat{\pi}_2} E_{\hat{\pi}_2} [\log(c)]$. This assumption does not change the investment decision made at date $t = 0$ since there are no ambiguity at date $t = 0$. The investment allocation $x$ depends on the initial beliefs $\pi_2$ (before the unanticipated shock is realized).

The investment decisions of liquidity traders are unaffected by this uncertainty about $\hat{\pi}_2$. The late consumers make decision about buying assets at date $t = 1$ based on the worst among possible priors: $\bar{\pi}$. Therefore, investors are willing to buy risky asset at $t = 1$ during the crisis if the market price $p_2$ is less than the (worst) expected payoff $\hat{R}(\pi_2)$ which is given by

$$\hat{R}(\pi_2) = \frac{\lambda(1 - \pi_2)}{\lambda + (1 - \lambda)\pi_2} R_H + \frac{\pi_2}{\lambda + (1 - \lambda)\pi_2} r_L$$

(13)

Suppose $\hat{\pi}_2$ is the actual (true) fraction of low quality assets such that $\pi_2 \leq \hat{\pi}_2 < \bar{\pi}_2$. Therefore, the price $p_2$ is given by

$$p_2 = \frac{(1 - \lambda)}{(\lambda + (1 - \lambda)\pi_2)} \frac{(1 - x(\pi_2))}{x(\pi_2)}$$

(14)

Unforeseen contingencies are defined as "possibilities that the agent does not think about or recognize as possibilities at the time he makes a decision" (Lipman, The New Plaggrave Dictionary of Economics 2008). In modeling unanticipated uncertainty about the asset value, I am following Easley and O’Hara (2008) and Uhlig (2009).
Consider the case when \( p_2 > r_H \). This implies that \( p_1 > r_H \). So, if there are no ambiguity about \( \hat{\pi}_2 \) then there is market trading in each state. However, in the presence of ambiguity about \( \hat{\pi}_2 \) there is no trade equilibrium if \( \pi_2 \) is sufficiently large so that \( \tilde{R}_2(\pi_2) \leq r_H \), i.e.,

\[
\pi \geq \frac{\lambda (R_H - r_H)}{\lambda R_H + (1 - \lambda) r_H - r_L}
\]

(15)

Again consider the numerical example: asset returns are given by \( R_H = 1.3 \), \( r_H = 0.5 \), \( r_L = 0.3 \), \( \pi_1 = 0.03 \), \( q = 0.1 \), and \( \lambda = 0.2 \). The figure below illustrates the effect of an increase in the fraction of lemons in a crisis state \( \pi_2 \) from 0.05 to 0.5 on the equilibrium values of investment and prices. If the fraction of lemons during a crisis exceeds 37% then the market breaks down.

![Figure 10. Equilibrium values of investment, prices and expected utility as a function of beliefs \( \pi_2 \).](image)

Therefore, the uncertainty about fraction of low quality assets can amplify the affect of adverse selection and result in a breakdown of trade. If the market breakdown of trade is caused by large fraction of low quality assets then extra liquidity is not helpful since it does not affect the expected payoff, and therefore, leads to the hoarding of liquidity. In this case, it is more effective to liquidate some of low quality assets. This reduces adverse selection, and therefore can restore the market trading.

5 Welfare Analysis

In this section, I analyze this model from the central planner perspective, and compare it with the market equilibrium.
Under full information (when it is known who receives a liquidity shock and the quality of asset is observable) the optimal investment allocation is \( x = (1 - \lambda), \) consumption allocation of liquidity investors \( c_1(s) = 1, \) and late consumers receive \( c_2(s) = R_s. \) This is a first-best allocation.

With asymmetric information about the quality of assets and identity of liquidity traders, the first-best allocation is not incentive compatible because investors with low quality assets have an incentive to pretend to be liquidity traders to get one unit of \( \text{good} \) per unit of low quality asset instead of liquidating it for \( r_L \) units of \( \text{good} \) since \( r_L < 1. \)

Therefore, the incentive-compatible maximization problem becomes:

\[
\max_x \sum_{s=1,2} \sum_{k=L,H} q_s \left( \lambda \pi_{sk} \log c_{1k}(s) + (1 - \lambda) \pi_{sk} \log c_{2k}(s) \right) \quad \text{(16)}
\]

\[
s.t. \quad (i) \quad \lambda c_1(s) \leq 1 - x \\
(ii) \quad (1 - \lambda) \sum_{k=L,H} \pi_{sk} c_{2k}(s) = x (1 - \pi_s) R_h + 1 - x - \lambda \sum_{k=L,H} \pi_{sk} c_{1k}(s) \\
(iii) \quad c_1(s) \geq x r_H + 1 - x \\
(iv) \quad c_{2L}(s) \geq x r_L + 1 - x \\
(v) \quad c_{2H}(s) \geq x R_H + 1 - x \\
(vi) \quad c_1(s) \leq c_{2k}(s) \ \forall k, s
\]

Since the quality of assets is not observable, all liquidity investors consume the same amount: \( c_{1k}(s) \equiv c_1(s) \) for each \( k, s. \) The constraints (i) and (ii) are resource constraints for period one and two, respectively. The constraints (iii), (iv) and (v) are participation constraints for each type. The constraints (vi) are incentive compatibility constraints. In equilibrium, these constraints bind for investors with low quality assets: \( c_1(s) = c_{2L}(s) \) in each state \( s. \)

**Proposition 3.** The optimal holdings of safe asset \( (1 - x) \) in the central planner solution are larger than in the first-best allocation, and larger than in the market equilibrium. The central planner achieves higher welfare relative to the market equilibrium.

The central planner can reduce the adverse selection problem but cannot completely eliminate it. Due to the adverse selection there are more assets traded in the market at date \( t = 1, \) in particular, more assets of low quality. To absorb this trading, more market
liquidity is required. In the market equilibrium, investors do not take into account the
effect of their investment choice on prices. The central planner problem is equivalent to
the investor maximization when they take this effect into account, which results in a larger
fraction of endowment allocated to the safe asset at date $t = 0$. This allocation smooths the
ex-ante consumption by improving consumption of liquidity investors and investors with
bad assets. As a result, it achieves higher welfare.

Consider the numerical example discussed in Section 4: asset returns are given by
$R_H = 1.3$, $r_H = 0.5$, $r_L = 0.3$, $\pi_1 = 0.03$, $\pi_2 = 0.3$, and $\lambda = 0.2$. Figure 11 depicts
the equilibrium values of investment and expected utility as a function of a crisis state
probability $q$ for the following scenarios:

A) market equilibrium without adverse selection,
B1) type I market equilibrium with adverse selection (with trading in both states),
B2) type II market equilibrium with adverse selection (with no-trade in the crisis state),
C) incentive compatible central planner solution, and
D) central planner solution under full information (first-best).

The central planner can increase welfare relative to the market equilibrium. The welfare
improvement is more significant relative to an equilibrium with the market breakdown. Also,
there are welfare gains from allowing informed traders to benefit from private information

![Figure 11. Central Planner vs Market Equilibrium with and without Adverse Selection](image-url)
on their asset quality. This ability to hide behind liquidity trades provides some protection in case an investor receives a low quality asset, especially in the crisis state. It leads to the consumption smoothening across different types of investors, and therefore, improves the welfare. However, if there is no trade during the crisis, then investors are left with their bad assets, which increases consumption volatility and leads to a lower welfare relative to an equilibrium where informed trading is not possible. Therefore, the ability to take advantage of private information is beneficial if there is market trading in both states and harmful if the market breaks down during the crisis.\footnote{This result holds for numerous numerical examples. I conjecture that this result holds in general, it remains to be proved (work in progress).}

The central planner solution suggests another policy implication: requiring ex-ante a larger holdings of safe asset (liquidity) would alleviate the adverse selection problem and prevent market breakdown during the crisis.

6 Conclusion

I analyze the effect of adverse selection in the asset market. The asymmetric information about asset returns generates the lemons problem when buyers do not know whether the asset is sold because of its low quality or because the seller’s sudden need for liquidity. This adverse selection can lead to market breakdown reflecting the buyers’ belief that most assets that are offered for sale are of low quality.

Further, I examine the following amplification mechanisms: increase in the liquidity preference during the crisis, underestimating the likelihood of a crisis, and uncertainty about the fraction of low quality assets. Any of these phenomena can amplify the effect of adverse selection leading to the increased asset price volatility and possibly to the breakdown of trade during the crisis.

The policy implication depends on which amplification mechanism causes the market breakdown. If it is due to the higher liquidity preference or to underestimating the likelihood of a crisis, then injecting liquidity into the market can restore the trading. However, if the no-trade outcome is a result of large fraction of lemons in the market or Knightian uncertainty about it then it is more effective to liquidate these assets. Removing such
assets from the market reduces adverse selection and uncertainty problems. In this case, the liquidity injection is not useful since it does not affect the expected value of assets, and therefore, leads to the liquidity hoarding.

The ability to trade based on private information may be welfare improving if adverse selection does not lead to the market breakdown. The central planner can reduce the adverse selection problem by requiring larger liquidity holdings, which prevents market breakdown during the crisis and increases the aggregate welfare.

17 This is consistent with arguments about effectiveness of the TARP proposal. However, as has been extensively noted, there are various implementation issues associated with it.
References


7 Appendix

7.1 Equilibrium without Adverse Selection

7.1.1 Proof of Proposition 1

Proof. First let’s establish that \( p_s \leq 1 \). Suppose \( p_s > 1 \) at least in one state \( s = 1, 2 \). If a price in the crisis state is such that \( 1 < p_2 \leq \overline{p}_2 \) then \( p_1 = p_2 \leq \overline{p}_2 \leq \overline{p}_1 \). Therefore, prices in both states \( p_s > 1 \) which means that the risky asset dominates safe asset at \( t = 0 \). Hence, no one will choose to hold the safe asset at \( t = 0 \), so it is not an equilibrium. The market prices in both states are not the same only if \( p_2 = \overline{p}_2 \) and \( p_1 > \overline{p}_2 > 1 \). These prices again cannot be the equilibrium prices since there is no safe asset holdings at \( t = 0 \). Therefore, we should have \( p_1 = p_2 \leq 1 \).

Denote \( p = p_1 = p_2 \). Then an investor’s maximization problem becomes

\[
\max_x \left\{ \lambda \log (1 - x + px) + (1 - \lambda) \sum_{s=1,2} q_s \left( \pi_s \log \left( x R_L + (1 - x) \overline{R}_s / p \right) + (1 - \pi_s) \log \left( x R_H + (1 - x) \overline{R}_s / p \right) \right) \right\}
\]

The equilibrium price and investment allocation \((x, p)\) are determined by the following system of equations:

\[
\lambda \frac{p - 1}{x(p - 1)} + (1 - \lambda) \sum_{s=1,2} q_s \left( \pi_s \frac{x - \overline{R}_s / p}{x(R_H - \overline{R}_s / p) + \overline{R}_s / p} + (1 - \pi_s) \frac{R_H - \overline{R}_s / p}{x(R_H - \overline{R}_s / p) + \overline{R}_s / p} \right) = 0
\]

\[
\lambda xp - (1 - \lambda) (1 - x) = 0
\]

Therefore, the equilibrium price is given by

\[
p = \lambda + \sum_{s=1,2} q_s \left( \pi_s \frac{x - \overline{R}_s / p}{x(R_H - \overline{R}_s / p) + \overline{R}_s / p} + (1 - \pi_s) \frac{R_H - \overline{R}_s / p}{x(R_H - \overline{R}_s / p) + \overline{R}_s / p} \right)
\]

It can be shown that \( p \geq 1 \). Since both prices must be less than one, we have \( p = 1 \). Denote equilibrium prices and investment allocation by \((p^o, x^o)\): \( p_s^o = 1 \) and \( x^o = (1 - \lambda) \).

Consider the maximization problem from the central planner prospective:

\[
\max_x \left\{ \lambda \log c_1 + (1 - \lambda) \sum_{s=1,2} q_s \left( \pi_s \log c_2 L (s) + (1 - \pi_s) \log c_2 H (s) \right) \right\}
\]

s.t. \( i \) \( \frac{c_1}{\lambda} = \frac{x - s}{x} \)

\( ii \) \( c_{2k} (s) = \frac{s \overline{R}_s}{(1 - \lambda)} \)

The solution is \( x = (1 - \lambda) \) which is the same as the market equilibrium allocation. Therefore, consumption allocations of early consumers are also the same. However, the consumption allocation of late consumers is different due to the private information about investment payoffs.
7.1.2 Equilibrium without Adverse Selection and Aggregate Liquidity Risk

Now the crisis state is parameterized by high demand for liquidity ($\lambda_2 > \lambda_1$) in addition to larger fraction of low quality assets ($\pi_2 > \pi_1$). This increase in $\lambda$ captures the flight-to-liquidity during the crisis when more investors are subject to liquidity shocks.

Impose additional assumption on $r_h$ such that $r_h \leq \frac{\lambda_2}{1-\lambda_2} \max_{s=1,2} \left\{ \sum_{x=1,2} q_x \lambda_s \left( \frac{\lambda_s + \pi_s \left( \frac{r_l}{r_h + \pi_x \frac{r_l}{r_h}} \right) + (1-\pi_s)}{R_h + \pi_x \frac{r_l}{r_h}} \right) \right\}$. This assumption ensures that the increase in liquidity preference by itself does not lead to the market breakdown. Of course it is possible to have $\lambda_2$ or $r_h$ sufficiently high to cause a breakdown. In this case adverse selection will only amplify this effect.

Then an investor’s maximization problem is given by

$$\max \left\{ \sum_{s=1,2} \left[ \lambda_s \log (1-x + p_s x) + (1 - \lambda_s) \left( \pi_s \log \left( xR_l + (1-x)R_h/p_s \right) \right) \right] \right\}$$

Market clearing conditions imply

$$p_s = \min \left\{ \frac{(1 - \lambda_s) (1-x)}{\lambda_s} \right\}$$

Condition 1: $\frac{\sum_{x=1,2} q_x \lambda_s \left( \frac{\lambda_s + \pi_s \left( \frac{r_l}{r_h + \pi_x \frac{r_l}{r_h}} \right) + (1-\pi_s)}{R_h + \pi_x \frac{r_l}{r_h}} \right)}{\sum_{x=1,2} q_x \lambda_s \left( \frac{\lambda_s + \pi_s \left( \frac{r_l}{r_h + \pi_x \frac{r_l}{r_h}} \right) + (1-\pi_s)}{R_h + \pi_x \frac{r_l}{r_h}} \right)} \leq \min_s \left( \frac{\lambda_s}{1-\lambda_s} \right)$

Condition 2: $R_h^1 < \frac{\lambda_2}{\lambda_1} \frac{(1 - \lambda_1)}{(1 - \lambda_2)}$

If condition 1 is satisfied then the equilibrium prices and investment allocation are given by

$$x = \sum_{s=1,2} q_s (1 - \lambda_s) \left( \lambda_s + \pi_s \left( \frac{r_l}{r_h + \pi_s \frac{r_l}{r_h}} \right) + (1-\pi_s) \right) \frac{R_h}{R_h + \pi_s \frac{r_l}{r_h}}$$

$$p_s = \frac{(1 - \lambda_s)}{\lambda_s} \sum_{s=1,2} q_s \lambda_s \left( \lambda_s + \pi_s \left( \frac{r_l}{r_h + \pi_s \frac{r_l}{r_h}} \right) + (1-\pi_s) \right) \frac{R_h}{R_h + \pi_s \frac{r_l}{r_h}}$$

If condition 1 fails but condition 2 is satisfied then the equilibrium prices and investment allocation are given by
\[ x = \frac{(1 - \lambda_1)}{(1 - \lambda_1) + \lambda_1 \bar{R}_1} \]
\[ p_1 = \bar{R}_1 \]
\[ p_2 = \frac{(1 - \lambda_2) - \lambda_1}{\lambda_2} \frac{1}{(1 - \lambda_1) \bar{R}_1} \]

Otherwise, the equilibrium prices and investment allocation are the same as in case with no aggregate liquidity risk.

It can verified that both dynamic consistency conditions are satisfied, i.e., given the equilibrium prices \((p_1, p_2)\) and investment \(x\),
\[ EU(x, p) \geq \max \{ EU(0, p), EU(1, p) \} \]

7.1.3 Equilibrium with Adverse Selection

7.1.4 Proof of Proposition 2

**Proof.** Let us start with a type I equilibrium with market trading in both states. The investors’ maximization problem is given by

\[
\max_x \quad \lambda \log (1 - x + p_s x) + (1 - \lambda) \sum_{s=1,2} q_s \left( \pi_s \log \left( x p_s + (1 - x) \bar{R}_s / p_s \right) + (1 - \pi_s) \log \left( x R_H + (1 - x) \bar{R}_s / p_s \right) \right) \\
\text{s.t} \quad (i) \quad 0 \leq x \leq 1 \\
\quad (ii) \quad p_s > r_h \quad \forall s
\]

Therefore, an investment allocation \(x\) and market prices \(p_s\) are determined by the following equations:

\[
\sum_{s=1,2} q_s \left( \lambda \frac{p_s - 1}{1 - x + p_s x} + (1 - \lambda) \left( \pi_s \frac{p_s - \bar{R}_s / p_s}{x p_s + (1 - x) \bar{R}_s / p_s} + (1 - \pi_s) \frac{R_H - \bar{R}_s / p_s}{x R_H + (1 - x) \bar{R}_s / p_s} \right) \right) = 0 \\
(\lambda + (1 - \lambda) \pi_s) p_s x = (1 - \lambda) (1 - x) \quad \forall s
\]

Substituting prices \(p_s\), we can get
\[ F(x) \equiv \sum_{s=1,2} q_s F_s(x) = 0 \]

where
\[
F_s(x) \equiv \left( \lambda \frac{1}{(1 - \lambda) + \pi_s} + (1 - \lambda) \pi_s \frac{(1 - x)}{(1 - x) + \bar{R}_s \left( \frac{\lambda}{(1 - \lambda)} + \pi_s \right)^2 x} + (1 - \lambda) (1 - \pi_s) \frac{R_H - \bar{R}_s / p_s}{x R_H + \bar{R}_1 \left( \frac{\lambda}{(1 - \lambda)} + \pi_s \right)} - x \right)
\]
This is a monotonically decreasing function of \( x \). At \( x = 0 \), \( F \) is greater than 0 and at \( x = 1 \), \( F \) is less than zero. Therefore, by Intermediate Function Theorem, there exist a unique \( x^* \) such that at \( F(x^*) = 0 \). The \( x^* \) can be derived as a root to a cubic equation: \( a_1x^3 + a_2x^2 + a_3x + a_4 = 0 \), where

\[
\begin{align*}
a_1 & = -d_1d_2 \\
a_2 & = d_1d_2d_3 - ((1 - \lambda)q_1\pi_1 + 1)d_2 - ((1 - \lambda)q_2\pi_2 + 1)d_1 \\
a_3 & = ((d_1 + d_2)d_3 - 1) + ((1 - \lambda)q_1(d_2 - 1) + (1 - \lambda)q_2(d_1 - 1)) \\
a_4 & = d_3 + (1 - \lambda)q_1\pi_1 + (1 - \lambda)q_2\pi_2 \\
d_1 & = \left( \hat{R}_1 \left( \frac{\lambda}{(1 - \lambda)} + \pi_1 \right)^2 - 1 \right) \\
d_2 & = \left( \hat{R}_2 \left( \frac{\lambda}{(1 - \lambda)} + \pi_2 \right)^2 - 1 \right) \\
d_3 & = \lambda \sum_{s=1,2} q_s \frac{1}{(1 - \lambda) + \pi_s} + (1 - \lambda) \sum_{s=1,2} q_s \frac{(1 - \pi_s)R_H}{R_H + R_s \left( \frac{\lambda}{(1 - \lambda)} + \pi_s \right)} 
\end{align*}
\]

Denote the solution as \( x^* \), then the prices are given by

\[
p_s^* = \frac{(1 - \lambda)}{(\lambda + (1 - \lambda)\pi_s)} \frac{(1 - x^*)}{x^*}
\]

If \( p_s^* > r_h \) and \( p_s^* \leq \hat{R}_2 \) then \( (x^*, p_s^*) \) are an equilibrium investment and prices. Furthermore, an investment allocation in an equilibrium without adverse selection \( (x^o) \) is larger than an investment allocation in a market equilibrium with adverse selection: \( x^* < x^o \).

Consider a solution to maximization problem in Proposition 1 but with prices \( p'_s = \frac{(1 - \lambda)}{(\lambda + (1 - \lambda)\pi_s)} \frac{1 - x}{x} \) instead of \( p_s = \frac{(1 - \lambda)}{(\lambda + (1 - \lambda)\pi_s)} \frac{1 - x}{x} \). Denote the solution as \( (x', p'_s) \), then

\[
\begin{align*}
x' & = \sum_{s=1,2} q_s \left( \frac{\lambda}{(\lambda + \pi_s)} + 1 \right) + (1 - \lambda) \left( \frac{\pi_s}{r_L + R_s \left( \frac{\lambda}{(1 - \lambda)} + \pi_s \right)} + (1 - \pi_s) \frac{R_H}{R_H + R_s \left( \frac{\lambda}{(1 - \lambda)} + \pi_s \right)} \right) < \\
& < (1 - \lambda) = x^o
\end{align*}
\]

It can be shown that \( F(x') < 0 \). Therefore, \( x^* < x' \) such that \( F(x^*) = 0 \). Hence, \( x^* < x' < x^o \).

If \( p_s^* > \hat{R}_2 \) then \( p_s^* = \hat{R}_2 \) and \( p_s^* = \frac{(1 - \lambda)}{(\lambda + (1 - \lambda)\pi_s)} \frac{1 - x}{x} \). The equilibrium investment is given by \( x^* = \frac{1}{(\lambda + \pi_1 + \pi_2) \hat{R}_2 + 1} \).

If \( p_s^* > \hat{R}_1 \) then \( p_s^* = \hat{R}_1 \) and \( p_s^* = \frac{(1 - \lambda)}{(\lambda + (1 - \lambda)\pi_s)} \frac{1 - x}{x} \). The equilibrium investment is given by \( x^* = \frac{1}{(\lambda + \pi_1 + \pi_2) \hat{R}_1 + 1} \).
If $p_2^* \leq r_h$ then in a crisis state liquidity traders with high quality investment choose to liquidate their investment rather than selling it at $t = 1$. Therefore, the expected return $\hat{R}_2 = r_1$, so there no demand for risky assets. Hence, $(x^*, p_2^*)$ cannot be an equilibrium investment and prices if $p_2^* \leq r_h$.

Consider a type II equilibrium with no trading in a crisis state. The investors maximization problem becomes

$$\max_{x} \left\{ \lambda \log (1 - x + p_1 x) + (1 - \lambda) (1 - q) \left( \pi_1 \log \left( x p_1 + (1 - x) \frac{\hat{R}_1}{p_1} \right) + (1 - \pi_1) \log \left( x R_H + (1 - x) \frac{\hat{R}_1}{p_1} \right) \right) + \right. \right. \right. \right. \right.$$ \[ \left. \left. \left. \left. + \lambda \log (1 - x + r_k x) + (1 - \lambda) + q (\pi_2 \log (x r + (1 - x)) + (1 - \pi_2) \log (x R_H + (1 - x)) \right) \right\} \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \righth
The equilibrium consumption of early and late consumers in a type I equilibrium are given by

\[
c_{1h}(s) = \begin{cases} 
  1 - x + p_s x & \text{if } p_s > r_h \\
  1 - x + r_h x & \text{if } p_s \leq r_h 
\end{cases}
\]
\[
c_{2h}(s) = \begin{cases} 
  xR_H + (1 - x) \tilde{R}_s/p_s & \text{if } p_s > r_h \\
  xR_H + (1 - x) & \text{if } p_s \leq r_h
\end{cases}
\]
\[
c_{2L}(s) = \begin{cases} 
  xp_s + (1 - x) \tilde{R}_s/p_s & \text{if } p_s > r_h \\
  xR_L + (1 - x) & \text{if } p_s \leq r_h
\end{cases}
\]

In the presence of adverse selection, the expected utility is higher when there is a market trading in both states. \(\blacksquare\)

7.2 Comparative Statics

7.2.1 Proof of Corollary 1

Proof. First consider an equilibrium with trade in both states.

The equilibrium investment allocation is determined from the following equation: \(F(x) = 0\) \((F(x)\)

is defined in the proof of Proposition 2, it is derived by substituting market clearing conditions into the FOC condition.) \(F_s(x) = 0\) provides the solution for the problem with one state \(s\). \(F_s(x)\) is decreasing in \(\pi_s\). Therefore, \(F(x)\) is decreasing in \(q\). Also, \(F(x)\) is decreasing in \(x\). Hence, the solution \(x^*\) is decreasing in \(q\). The prices are determined by \(p_s^* = \frac{(1-\lambda)}{(1+(1-\lambda)s_x)}\frac{(1-x^*)}{x^*}\). Therefore, the market prices \(p^*_s\) are increasing in \(q\). The one-state expected utility is decreasing in \(\pi_s\). Therefore, as \(q\) becomes larger the expected utility decreases.

Now consider an equilibrium with no trade in a crisis state. If we compute \(xt\) such that \(G_2(x') = 0\) and \(x''\) such that \(F_1(x'') = 0\) then \(x'' > x'\). The equilibrium \(x\) in a two-state problem is determined by \(G(x) = (1-q)F_1(x) + qG_2(x) = 0\). Since \(G(x)\) is decreasing in \(x\) then the optimal \(x\) is decreasing in \(q\). Therefore, \(p_1\) is increasing in \(q\) since it negatively depends on \(x\). The one-state expected utility is lower in a no-trade state vs the one with a trade. Therefore, as \(q\) becomes larger the expected utility decreases. The no-trade outcome occurs when the price in a crisis state falls below the liquidation value \(r_H\). The increase in \(q\) may increase the price in a crisis state sufficiently to restore the trading.

Consider some \(q\) such that \(p_2 = r_H - \epsilon\) with \(\epsilon > 0\). Then there is no trading in state 2,

\[
F_1(x) = \lambda \frac{1}{(\tau - x) + \pi_1} + (1 - \lambda) \pi_1 \frac{1}{1 + R_1(\frac{1}{(\tau - x) + \pi_1})(\tau - \epsilon)/(\tau - x) + \pi_2)} + (1 - \lambda) (1 - \pi_1) \frac{R_H}{R_H + R_2(\frac{1}{(\tau - x) + \pi_2})(\tau - \epsilon)/(\tau - x) + \pi_2)}
\]
\[
F_2(x) = \lambda \frac{1}{(\tau - x) + \pi_2)} + (1 - \lambda) \pi_1 \frac{1}{1 + R_2(\frac{1}{(\tau - x) + \pi_2})(\tau - \epsilon)/(\tau - x) + \pi_2)} + (1 - \lambda) (1 - \pi_1) \frac{R_H}{R_H + R_2(\frac{1}{(\tau - x) + \pi_2})(\tau - \epsilon)/(\tau - x) + \pi_2)}
\]

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Therefore, $F_2(x) > G_2(x)$. If $q$ increases sufficiently so that $x$ goes down by more than
$$\frac{\lambda_1^2}{(1 - \lambda_2)^2} \frac{1}{x} \frac{1}{(1 + \lambda_1 \pi_2)(1 + \lambda_1 \pi_2 + \pi_2)}$$
then the trading in a crisis state restores. ■

### 7.2.2 Proof of Corollary 2

**Proof.** Suppose now the economy is parametrized by state 1: $(\lambda_1, \pi_1)$ and state 2: $(\lambda_2, \pi_2)$ such that $\lambda_1 < \lambda_2$ and $\pi_1 < \pi_2$.

First consider an equilibrium with trade in both states. The equilibrium investment allocation is determined from the following equation: $F(x) = 0$

$F_s(x) = 0$ provides the solution for the problem with one state $s$. $F_s(x)$ is decreasing in $\lambda_s$. Also, $F_s(x)$ is decreasing in $x$. Hence, $x$ is decreasing in $\lambda_s$.

The effect of increase in $\lambda_2$ on the price in state 2 is determined by

$$\frac{\partial p_2}{\partial \lambda_2} = -\frac{1}{(1 - \lambda_2)^2} \frac{(1 - x)}{x} - \frac{1}{(1 - \lambda_2)^2 + \pi_2} \frac{1}{x^2} \frac{\partial x}{\partial \lambda_2}$$

Therefore, increase in $\lambda_2$ can lead to the decrease in $p_2$, potentially resulting in $p_2 \leq r_H$. ■

### 7.3 Central Planner problem

#### 7.3.1 Proof of Proposition 3

**Proof.** The central planner maximization problem can be written as following,

$$\max_x \{ \sum_{s=1,2} q_s (\lambda \log (1 - x) + (1 - \lambda) (\pi_s \log c_{2L} + (1 - \pi_s) \log c_{2H})) \}

s.t. (i) \quad (\lambda + (1 - \lambda) \pi_s) \Delta c_1(s) = (1 - \lambda) (1 - x) 

(ii) \quad (1 - \lambda) \Delta c_2(s) = (\lambda + (1 - \lambda) \pi_s) x R 

(iii) \quad c_1(s) = \Delta c_1(s) + (1 - x) 

(iv) \quad c_{2L}(s) = \Delta c_1(s) + \Delta c_2(s) 

(v) \quad c_{2H}(s) = x R H + \Delta c_2(s) 

(vi) \quad c_1 \geq x R H + (1 - x) 

(vii) \quad c_{2H} \geq x R H + (1 - x) 

(viii) \quad c_{2L} \geq c_1$$

where $\Delta c_1(s)$ is a transfer of cash holdings to liquidity investors in exchange of their risky asset holdings at date $t = 1$ and $\Delta c_2(s)$ is a transfer of risky asset holdings in exchange for cash holding to non-liquidity traders.
The maximization problem can be reduced to the following,

\[
\max_x \left\{ \sum_{s=1,2} q_s \left[ \lambda \log (1-x) \left( \frac{1+(1-\lambda)\pi_s}{\lambda+(1-\lambda)\pi_s} \right) + (1-\lambda) \left( \pi_s \log \left( \frac{(1-x)}{(1-\lambda)\pi_s} \frac{(1-\lambda)}{(x+(1-\lambda)\pi_s)} + x \frac{(\lambda+(1-\lambda)\pi_s)}{(1-\lambda)} \vec{R}_s \right) + (1-\pi_s) \log x \left( R_h + \frac{(\lambda+(1-\lambda)\pi_s)}{(1-\lambda)} \vec{R}_s \right) \right) \right\}
\]

s.t. \quad 0 \leq x \leq 1

The optimal investment \( x \) is a solution to the following equation \( G(x) \equiv \sum_{s=1,2} q_s G_s(x) = 0 \), where

\[
G_s(x) \equiv \left( -\frac{1}{(1-x)} + (1-\lambda) \pi_s \right) \frac{\left( \frac{(\lambda+(1-\lambda)\pi_s)}{(1-\lambda)} \frac{2}{\vec{R}_s} - 1 \right)}{(1-x) + x \left( \frac{(\lambda+(1-\lambda)\pi_s)}{(1-\lambda)} \frac{2}{\vec{R}_s} \right) + (1-\lambda) \left( 1-\pi_s \right) \frac{1}{x}}
\]

This is a monotonically decreasing function of \( x \). At \( x = 0 \), \( G \) is greater than 0 and at \( x = 1 \), \( G \) is less than zero. Therefore, by Intermediate Function Theorem, there exist a unique \( x^o \) such that at \( G(x^o) = 0 \) the \( x^o \) can be derived as a root to a cubic equation.

Furthermore at \( x = 1-\lambda \), we have \( G(x) < 0 \) which implies that \( x^o < 1-\lambda \), i.e., the investment allocation in the incentive compatible equilibrium is smaller than the first-best investment allocation.

Denote the solution as \( V^o \equiv EU(x^o) \). To compare welfare achieved by central planner with a market equilibrium, let’s denote the expected utility in the market equilibrium by \( V^* \equiv EU(x^*, p_s^*) \).

If \( p_s^* = \frac{(1-\lambda)}{(\lambda+(1-\lambda)\pi_s)} \frac{(1-x)}{x} \) for all \( s \), then

\[
V^* = \left\{ \sum_{s=1,2} q_s \left[ \lambda \log (1-x^*) \left( \frac{1+(1-\lambda)\pi_s}{\lambda+(1-\lambda)\pi_s} \right) + (1-\lambda) \left( \pi_s \log \left( \frac{(1-x^*)}{(1-\lambda)\pi_s} \frac{(1-\lambda)}{(x^*+(1-\lambda)\pi_s)} + x^* \frac{(\lambda+(1-\lambda)\pi_s)}{(1-\lambda)} \vec{R}_s \right) + (1-\pi_s) \log x^* \left( R_h + \frac{(\lambda+(1-\lambda)\pi_s)}{(1-\lambda)} \vec{R}_s \right) \right) \right\}
\]

Therefore, \( V^* \leq V^o \) since \( x^o = \arg \max V(x) \), i.e. the central planner always achieves a higher welfare level. Furthermore, comparing \( F(x) \) defined in the proof of Proposition 2 and \( G(x) \), it can be shown that \( F(x^o) \geq 0 \). It implies that \( x^* \) such that \( F(x^*) = 0 \), \( x^* \geq x^o \), the investment allocation in a market equilibrium is larger than the central planner investment allocation. ■