Applying Stress Tests to Market Risk Modeling

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Abstract
Stress testing has been proposed by many as a tool to describe how the portfolio would perform under extreme market moves. Although these extreme market moves typically have a very low probability under most Value-at-Risk (VaR) models, they happen once in a while. In this paper, we use Extreme Value Theory (EVT) to undergo stress testing on our sample. Risk managers have found that one of the problems with traditional approaches to risk modeling is that it does not take proper account of the more extreme observations in the data set. This weakness can be rectified by using Extreme Value Theory (EVT), which provides one approach to the estimation of low frequency events with limited data. Moreover, we will apply a Generalized Pareto Distribution (GPD) for our data, and a framework which incorporates stress scenarios into Value-at-Risk and Expected Shortfall models. GPDs are generally considered the most useful for practical applications, due to their more efficient use of data on extreme outcomes.

In this paper, we use historical scenarios of S&P prices, and choose the big losses between the years 2008 and 2009 as extreme events. We shall present a comprehensive review of stress testing for historical simulation and GARCH model with special focus on modeling aspects. Among the original contribution, we propose stress testing for two scenarios and the effect of scenarios on Value at Risk (VaR) and Expected Shortfall (ES) values. Finally, we will compare the change in Value-at-Risk and Expected Shortfall when we add different scenarios for both historical simulation and GARCH models.
I. Introduction

In financial risk management, Value at Risk (VaR) is a widely used measure of the risk of loss on a specific portfolio of financial assets. For a given portfolio, probability and time horizon, VaR is defined as a threshold value such that the probability that the market-to-market loss on the portfolio over the given time horizon exceeds this value (assuming normal markets and no trading in the portfolio) is the given probability level.\(^1\) VaR is an attractive measure because it is simple to understand.

VaR summarizes the total risk in a portfolio of financial assets in a single number. It was established by J.P. Morgan and it has become extensively used by corporate treasuries and financial institutions. Also, the VaR measure is used by the Basel Committee in setting capital requirements for banks.\(^ii\)

Although VaR has many attractions as a risk management measure, “An Academic Response to Basel II” argued that it is insufficient to calculate capital charges rely on VaR. VaR regulation can threaten an economy and induce crashes when they would not otherwise occur.\(^iii\)

Value-at-Risk has a number of problems with how VaR information is usually dispersed and understood. For example, one of the problems in interpreting VaR numbers is that they are just one outcome from an estimated distribution. When there is uncertainty a point which estimates risk can lead risk managers to rely on that estimate and fail to sufficiently consider the precision or accuracy of that estimate or other outcomes from the distribution. On the other hand, Hodder et al (2001) argues that where large amounts of information are provided in VaR, users may not be able to rationally or consistently interpret that given large amount of information.

Another problem with VaR is the failure of subadditivity-the risk measure for two portfolios after they have been merged should be no greater than the sum of their risk measures before they were merged—which is a property that would normally be regarded as absolutely basic to any respectable measure of financial risk. The VaR, in general, does not satisfy the coherent risk measure.\(^iv\)

On the other hand, VaR is a method for measuring the loss magnitudes associated with rare tail events in financial market. Occasionally, either to satisfy management interest or for regulatory compliance, it becomes necessary to quantify the magnitude of the losses that might accrue under events less likely than those analyzed in a standard VaR calculation.\(^v\) The procedure used to calculate potential loss exposures under such special circumstances is often called a stress test.

Stress testing and Value-at-Risk (VaR) have much in common. For example, they both assess market risk, and consider the change in market risk over a fixed horizon due to changes in specific risk factors. However, there was an insight that stress testing allowed for the analysis of extreme events that VaR didn't address. For example, if a firm was using one-day 90% VaR, results would reflect losses that might be experienced one day out of ten, but it would not give any information for the losses that might be experienced one day out of 100 or one day out of 1000. Stress testing, with its ability to assess subjectively extreme events, could give this information. If we consider an extreme enough scenario, then; it is possible to predict the loss for any portfolio.
Stress testing can provide useful information about a firm’s risk exposure that VaR methods can easily miss, potentially when VaR methods focus on normal market risks rather than the risks associated with rare or extreme events. Stress testing has been proposed by many as a tool to describe how the portfolio would have performed under extreme market moves. Although these extreme market moves typically have a very low probability under most Value-at-Risk (VaR) models, they happen time to time.

In all areas of risk management the potential extreme event risk exists. In this paper, we focus on the market risk. In market risk, we are probably concerned with day-to-day determination of Value-at-Risk (VaR) for the losses due to adverse market movements. Here, we consider a model to address the possibility of an extreme outcome. In addition, we need to model the risk to give information about extreme event outcomes. One method which provides this information is extreme value theory (EVT).

In the following sections, we will discuss different approaches of risk managers to stress testing (the second section). Then, we will provide an overview of the role of extreme value theory (EVT) in risk management as a method for modeling and measuring extreme risks (the third section). We will discuss EVT theory and the related Value at Risk (VaR) and Expected Shortfall (ES) models for Historical Simulation method and GARCH model (the fourth section). This will lead us to the main part of our studies, in which we shall apply real data and compare the HS and GARCH methods in terms of VaR and Expected Shortfall (the fifth section). And, eventually, we will examine the empirical analysis of the risk modeling with some conclusion remarks (the sixth section).

II. Literature Review on Stress Testing

In April 1995, the Basel Committee proposed the Internal Model approach in which banks have to perform stress tests which may subject them to extra capital charges. These stress tests examine the effect of simulated large movements in important financial variables on the bank’s portfolio. The advantage of this stress testing, or scenario analysis, is that it may cover situations completely absent from historical data, whereas VaR models are typically solely based on the historical data; stress testing forces management to consider events that they might otherwise ignore.

In recent years risk managers have embraced the idea of supplementing VaR estimates with stress testing. Thus, they inspire to apply resources on developing more and better stress-tests. However, there are some problems with stress testing that leads various practitioners and regulators to bring different ideas.

Berkowitz (1999) believes that using stress-test outside the basic risk model yields two sets of forecasts, one from the stress tests and one from the basic model (e.g. Value-at-Risk (VaR)). If the stress scenarios are conducted outside the model, they are not assigned probabilities. Therefore, there is no guidance about the relevance of stress scenarios and how to incorporate the
stress forecasts into the underlying risk model. So, he suggests folding stress-tests into the risk model, and requires all scenarios to be assigned probabilities. He defines a new factor distribution, \( f_{\text{combined}}(x) \) as a mixture distribution, such that

\[
\begin{align*}
X &\sim f(x) \quad \text{with probability } (1-\alpha) \\
X &\sim f_{\text{stress}}(x) \quad \text{with probability } (\alpha),
\end{align*}
\]

where \( \alpha \) is the probability assigned to a particular scenario. The distribution \( f \) and \( f_{\text{stress}} \) are \( k \)-variate distributions describing the joint behaviour of the \( k \) factors. Using Monte Carlo techniques, we can simulate the factor realizations for both distributions. Stress tests are usually separated from the basic model, so they are not subject to back testing. However, according to Berkowitz (1999), incorporating stress tests into the basic model, we get a single forecast loss distribution where standard back testing tools are applicable. So, through a unified model where stress-test scenarios are incorporated into the basic risk model and assigned probabilities, stress-testing can validate a model. Therefore, the model produces a single forecasted loss distribution which is internally consistent and agreeable to back testing.

Aragones, Blanco, and Dowd (2001) examine some of the weaknesses in current stress-testing practice and suggest some possible improvements.

The most obvious weakness is that stress tests are inevitably subjective. Stress testers choose scenarios; hence the value of stress testing critically depends on the skill of modellers.

The other disadvantage of current stress test happens when we do not apply the probabilities of the events that are concerned. Thus it is difficult to interpret the stress test results. As a consequence, Berkowitz (1999) argues, that probability absence makes statistical issues arise. Aragones, Blanco, and Dowd (2001) follow his argues and suggest a framework to solve these weaknesses.

Incorporating stress tests into market risk modeling is the suggested model by Aragones, Blanco, and Dowd. This framework has some advantages. One is transparency. If risk managers assign a probability to each scenario and are aware of the scenarios being used in the VaR calculation, they are using a transparent model, where the scenarios have a direct input, instead of being integrated in an ad hoc

The other advantage of incorporating stress tests into market risk modeling is applying user-defined scenarios. This framework is independent of choosing scenarios by the risk modeller or group of risk modellers. Thus, this idea makes it simple to assess the usefulness of any stress test and related procedures.

As mentioned above, stress testing is a method to complement VaR estimates but it suffers from some limitations such as lack of coherence between a statistical risk measure and a subjective one. This problem arises when we use stress tests outside the risk estimation framework which can lead to certain inconsistencies between statistical risk measures and stress tests results. Alternatively, there are several agreements that using the same correlation matrix to design various stress tests is not an appropriate depiction of the relationship among risk factors in periods of market stress. Aragones and Blanco (2007) suggest a solution to these problems by considering different correlation regimes and incorporating the result of stress test within the
traditional market risk measurement models. They propose an extension of the framework suggested by Berkowitz (2000) and Aragones (2001). They suggest a process to design stress tests for different market regimes using regime-dependent covariance matrices of market risk factors. Moreover, they use particular risk factors that act as drivers for the rest of the risk factor changes, which are calculated as a function of original shocks on the driver for each scenario and the respective correlations against that driver under each regime.

The impact of the stress events can be viewed through the change in Expected Shortfall (ES), Value-at-Risk (VaR), Probability of Default (PD), and Loss Given Default (LGD). Daniel Rosh and Harald Scheule (2009) discuss (PD) and LGD in their framework.

Their contribution explains the requirement by the Basel Committee on Banking Supervision (2006) and the critique by Berkowitz (2000) by assigning probabilities to the stressed scenarios which include macroeconomic factors and model risk. The authors show that these various stresses can be individually and jointly incorporated into a framework that was first introduced for probability of default and asset correlation estimates by Rosh and Scheule (2007).

Therefore, they extend this framework to Loss Given Default (LGD) and correlations between default events and LGD. These authors have developed a framework for stress testing based on an unobservable systematic risk factor, which is known as VaR. This framework bases integrated, observable macroeconomic risk factors, model risk and a combination of these elements. Via stress testing, they find that there is a large gap between Basel II and the economic VaR, expressing that financial institution may have to provide a significant amount of buffer capital. The regulatory VaR, according to the Basel Committee on Banking Supervision (2006), does not consider the stochastic properties of LGDs and values that are much lower than the economic VaR. By excluding the years of Asian financial crisis, economic and regulatory capital decrease which means that economic and regulatory VaR may be higher for credit portfolios that have experienced severe financial stresses in the past. The models presented are based on history and can only be fitted to observable information and include the uncertainty in relation to history which is a concern. Hence, stress testing may involve research on the underlying default and LGD processes, which can improve understanding of the forecast uncertainty.

Rosch and Harald Scheule (2009) apply Aragones, Blanco and Dowd’s (2001) thesis and incorporate scenario analysis into probabilistic market risk modeling process. They employ the approach initially elaborated by Berkowitz (2000) and then extended by Aragones (2001). They assign probabilities to scenario-based losses, treating them as if they had actually occurred in the historical dataset and incorporate them into a probabilistic risk approach. The resulting risk estimates include traditional market risk scenarios and the outcomes of the different stress tests conducted, like the probabilities assigned to each scenario or stress event. Risk managers who follow this approach can work with a single, integrated set of risk estimates, which are now dependent on the judgmental factors that go into stress testing and into the evaluation of the probabilities of scenario.

Finally, in January 2009, the Basel Committee issued a consultative document on stress testing. The document emphasizes the importance of stress testing in determining how much capital is necessary to absorb losses should large shocks occur. It shows that stress testing is particularly important after long periods of benevolent conditions because such conditions tend
to lead to complacency. In addition, the Basel Committee requires market risk calculations that are based on a bank’s internal VaR models to be accompanied by rigorous and comprehensive stress testing.

In practice, there are a number of competing methodologies in place. This paper, however, tends to follow the work by Berkowitz (2000), Aragones, Blanco, Dowd (2001), and Aragones and Blanco (2007). This methodology bases a unified model in which stress test scenarios are incorporated into the basic risk model and assigned probabilities. Also, in contrast with Rosch and Harald Scheule (2009) framework, we examine the impact of the stress events through expected shortfall and value-at-risk.

III. Generalized Pareto Distribution

Risk managers found that one of the problems with traditional approaches to risk modeling is that it does not take proper account of the more extreme observations in our data set. This weakness can be recertified by using Extreme Value Theory (EVT), which provides one approach to the estimation of low frequency events with limited data.

EVT is very useful because it guides us in the selection of the distribution to use when modeling our risks. It is useful because we often do not know the distributions from which our observations are drawn.

One way and the simplest way to calculate measure of risk is to calculate the mean or variance of a risk. However, these measures do not provide much information about the extreme risk. In this paper, we use two measures VaR and Expected shortfall, which describe the tail of a loss distribution.

Extreme Value Theory (EVT) tells us what the distribution of extreme values should look like in the limit while our sample size increases. There are some recent texts on EVT, including Reiss & Thomas (1997) and Beilant, Teugels & Vynckier (1996). But in this paper we follow the approach to EVT given by Embrechts, Klupelberg & Mikosch (1997). In this approach extreme value is modeled using the Generalized Pareto distribution (GPD) (Embrechts, Resnick & Samorodnitsky 1998). For example, if we have Profit and losses (P/L) observations x from an unknown distribution, then EVT tells us that the distribution of excess returns x beyond a threshold u, converges asymptotically to a Generalized Pareto Distribution. The distribution function of GPD is given by;

\[ G(\xi, \beta) = \begin{cases} 
1 - \left(1 + \frac{\xi x}{\beta}\right)^{-1/\xi}, & \xi \neq 0 \\
1 - \exp\left(-\frac{x}{\beta}\right), & \xi = 0 
\end{cases} \]

where \( \beta \) and \( \xi \) are scale and shape parameters. The parameter \( \xi \) controls the heaviness of the tails; the larger \( \xi \), the heavier the tail. The range of x is \( x > 0 \) for \( \xi \leq 0 \) and \( 0 < x < \beta/\xi \) for \( \xi > 0 \). Choosing the threshold is a significant issue in EVT methodology. If u is set too large, then only very few observations are left in the tail and the estimate of the tail parameter, \( \xi \), will be very uncertain. On the other hand, if u is set too small, then the EVT theory may not hold that means...
the data to the right of the threshold do not conform sufficiently well to the generalized Pareto distribution to generate unbiased estimates of $\xi$.

One particularly interesting and useful property is that the GPD is steady with respect to excess over threshold operations. In fact we have

$$\text{Prob} \left[ X - u \leq x \mid X > u \right] = \frac{\text{Prob} \left[ u < X \leq x + u \right]}{\text{Prob} \left[ X > u \right]} = \frac{F(x + u; \xi, \beta) - F(u; \xi, \beta)}{1 - F(u; \xi, \beta)} = \frac{(1 - \frac{\xi}{\beta})^{1/\xi} - (1 - \frac{\xi(x + u)}{\beta})^{1/\xi}}{(1 - \frac{\xi}{\beta})^{1/\xi}}$$

$$= - \left( 1 - \frac{\xi x}{(\beta - \xi u)} \right)^{1/\xi}.$$

The equation (2) means that if $x$ is a GPD, then $x - u$, given that $x > u$ for any $u$ is a GPD $(\xi, \beta - \xi u)$. For a given threshold, if the model is consistent with a set of data, then it must be consistent with the data for all higher thresholds.

The GPD is generalized in the sense that it contains a number of special cases. The cases follows:

- When $\xi = 0$, $G(0, \beta) = 1 - \exp \left( - \frac{x}{\beta} \right)$, the GPD reduces to the exponential distribution with mean $\beta$.
- When $\xi = 1$, $G(1, \beta) = 1 - \left( (1 + \frac{x}{\beta})^{-1} \right) = \frac{x}{x + \beta}$. The GPD becomes a uniform $U[0, \beta]$ distribution.
- When $\xi < 0$, the GPD has a type II Pareto distribution.
- When $\xi > 0$, the GPD is an ordinary Pareto distribution.

The mean of GPD is defined provided $\xi < 1$, and is

$$E(X) = \frac{\beta}{1 - \xi}.$$

To summerize, EVT tells us that the limiting distribution of extreme returns always has the same form. It is important due to the fact that it allows us to estimate extreme probabilities and extreme quaitities, including VaR and ES, without having to make strong assumptions about an unknown pareto distribution.

A natural model for the excess distribution over a high threshold is the role of the GPD in EVT. We define this concept along mean excess function which will also play an important role in the theory.
III.1 Excess Distribution Over Threshold U

Consider the probability of standardized returns $z$ less a threshold $u$ being below a value $x$ given that the standardized return itself is beyond the threshold, $u$;

$$F_u(x) = P \{ z-u \leq x \mid z > u \}$$

where $x > u$. Let $X$ be a random variable with distribution function $F$. The excess distribution over the threshold $u$ has the distribution function

$$F_u(x) = P ( X-u \leq x \mid X > u ) = \frac{F(x+u)-F(u)}{1-F(u)}$$

(4)

For $0 \leq x < x_F - u$ where $x_F \leq \infty$ is the right endpoint of $F$.

III.2 Mean Excess Function

We can define the mean excess function of a random variable $X$ with finite mean by

$$e(u) = E \{ X-u \mid X > u \}.$$  

(5)

The mean excess function $e(u)$ expresses the mean of $F_u$ as a function of $u$. The excess distribution function of $F_u$ describes the distribution of the excess loss over the threshold $u$, given that $u$ is exceeded. The interpretation of (5) explains in Embrechts et al. (1997) and Hogg and Klugman (1984).

III.3 Excess Distribution of GPD

If $F$ has the distribution function of an exponential random variabels, for all $x$, we can verify that $F_u = F(x)$. According to McNeil, Frey, and Embrechts (2005), if $X$ has distribution function $F = G_x, \beta$, the excess distribution function is easily calculated. By using (4) we can have

$$F_u(x) = G_x, \beta(u)(x), \ \beta(u) = \beta + \xi u$$

(6)

where $0 \leq x < \infty$ if $\xi \geq 0$ and $0 \leq x \leq -\frac{\beta}{\xi} - u$ if $\xi < 0$. This distribution has $\xi$ same as before but $\beta$ grows linearly with the threshold $u$. The mean excess function of GPD is easily calculated from (3) and (6) to be

$$e(u) = \frac{\beta(u)}{1-\xi} = \frac{\beta + \xi u}{1-\xi}$$

(7)

where $0 \leq u < \infty$, if $0 \leq \xi < 1$ and $0 \leq u \leq -\frac{\beta}{\xi}$ if $\xi < 0$. The mean excess function is linear in the threshold $u$, which is a characterizing property of the GPD.
We can find $\beta(u)$ such that

$$\text{Lim sup } |F_u(x) - G_{\xi, \beta(u)}(x)| = 0. \quad (8)$$

That is for a large class of underlying distribution $F$, as the threshold $u$ is progressively raised, the excess distribution $F_u$ converges to a generalized Pareto.

Our model for a risk $X_i$ has distribution $F$ for a certain $u$, the excess distribution above this threshold may be taken to be exactly GPD

$$F_u(x) = G_{\xi, \beta(u)}(x) \quad (9)$$

for $0 \leq x < x_f - u$ and some $\xi, \beta$.

Given loss data $X_1, \ldots, X_n$ from $F$, we use statistics to make the model more precise by choosing a sensible $u$ and estimating $\xi$ and $\beta$. There are various ways of fitting the GPD including Maximum Likelihood Estimation (MLE), where the values are chosen to maximize the joint probability density of the observations.

**IV. Value-at-Risk: Different Approaches**

The VaR techniques can be roughly classified in two categories; the conditional methods and unconditional methods. Unconditional methods are those that yield a constant VaR prediction for each trading day in the year under examination. In contrast, conditional methods predict a different VaR for each trading day in the year under examination depending on the previous days’ portfolio returns. Both Historical Simulation and the GARCH related VaR techniques meet this qualification. The fact that conditional VaR methods incorporate new information each day should make their predictive performance superior to the unconditional VaR methods if the distribution of portfolio returns changes through time.

**IV.1 Historical Simulation**

One approach for VaR assessment is called Historical Simulation (HS). This technique is nonparametric and does not require any distributional assumption. This is because the HS uses only the empirical distribution of the portfolio returns.

Mcneil (1996) declares that choosing an optimal threshold is a significant issue in estimating the Value-at-Risk, so we must decide where to set the threshold. If we set the threshold too high we have few data points and we introduce more parameter uncertainty. On the other hand, if we choose the threshold too low, our estimation would be biased. Therefore, we want $u$ to be sufficiently high that we are truly investigating the shape of the tail of the distribution, but sufficiently low that the number of data items included in the maximum likelihood calculation is not too low. More data lead to more accuracy in the assessment of the shape of the tail.
IV.1.1 Extreme Value VaR

Extreme value theory provides an approach to VaR estimation, given that VaR is primarily concerned with the tails of our return distributions. To estimate VaR, we first estimate the distribution’s parameters. Then we can plug them into a number of alternative formulas to obtain VaR estimates.

Under the assumption number (5) we have

\[ F(x) = P(X > u) P(X > x | X > u) = \]
\[ F(u) P(X - u > x - u | X > u) = F(u) F_u(x - u) = \]
\[ F(u) \left( 1 + \frac{x}{\beta} \right)^{-1/\xi}, \quad \text{for } x > u. \]  

(10)

If \( F(u) \), gives a formula for tail probabilities, the inverted form of the formula gives us a high quantile of the underlying distribution. In the other word it gives us a Value-at-Risk (VaR);

\[ \text{VaR}_q = F^{-1}(q); \]

(11)

where \( F^{-1} \) is the inverse of \( F \).

For \( q \geq F(u) \) we have VaR that is equal to

\[ \text{VaR}_q = u + \beta \left( 1 - \frac{q}{F(u)} \right)^{-\xi} - 1. \]

(12)

Typically it is also straightforward to estimate the probability of exceeding the threshold \( P(X > u) \), when VaR can be computed from this formula.

IV.1.2 Expected Shortfall

The mean excess is closely related to the popular expected shortfall measure. In particular, the expected shortfall is the expected loss given that the loss is greater than or equal to the Value-at-Risk.

Expected shortfall is related to VaR by.

\[ \text{ES} = \text{VaR}_q + \text{mean excess over VaR} \]
\[ = \text{VaR}_q + E[X - \text{VaR}_q | X > \text{VaR}_q] \]

(13)

where the second term is simply the mean of the excess distribution \( F_{\text{VaR}_q}(y) \) over the threshold \( \text{VaR}_q \).

If excesses over a threshold \( u \) have the Generalized Pareto Distribution, then excesses over \( \text{VaR}_q > u \) have the same distribution function with the scaling parameter \( \beta \) replaced by \( \beta + \xi \) \((\text{VaR}_q - u)\), and the mean excess is given by (7), with \( u \) replaces by \( \text{VaR}_q - u \).
Combining this fact with (13), we can write the expected shortfall such as,

$$ES = \frac{\text{VaR}_\xi}{1-\xi} + \frac{\beta - \xi u}{1-\xi}. \quad (14)$$

**IV.2 The GARCH Model**

In this subsection, the Value-at-Risk of a portfolio relates directly to the variance or standard deviation of the portfolio returns. Intuitively, the larger the variance of a portfolio return, the more likely the occurrence of large swings in the portfolio value and the larger the Value-at-Risk. Here we shall look at a methodology that does consider volatility clustering, namely the Generalized Autoregressive Conditional Heteroskedasticity, or GARCH model. Bollerslev (1986) introduces this model.

We can write the GARCH model:

$$\sigma_{t+1}^2 = \omega + \alpha (R_t - \theta \sigma_t)^2 + \beta \sigma_t^2,$$

with $R_t = \sigma_t z_t$, and $Z_t \sim N(0,1)$ \hspace{1cm} (15)

where $\omega > 0$, $\alpha \geq 0$, $\beta \geq 0$ to ensure positive variances. We use Solver in Excel to estimate the parameters in the equation 15. This formula nicely demonstrates the essence of the volatility clustering feature in the GARCH model. If the market was volatile in the current period, next period’s variance will be high, which is intensified or offset in accordance with the magnitude of the return deviation this period. If, on the other hand, today’s volatility was relatively low, tomorrow’s volatility will be low as well, unless today’s volatility return deviates from its mean considerably.

We can use GARCH model to calculate the Volatility Updating into the historical simulation suggested by Hull and White. We will discuss more about this issue in the next section.

**IV.2.1 Parameter Estimation**

If we consider points, $x$ with $x > u$, in the tail of the distribution, and let $y = x + u$, then we can rearrange equation 4 and get;

$$F(y) = 1 - [1 - F(u)] [1 - F_u(y-u)]. \quad (16)$$

Consider that $T$ denotes the total sample size and $T_u$ denotes the number of observations beyond the threshold, $u$. The term $1 - F(u)$ can then be estimated simply by the proportion of data point beyond the threshold $u$, call it $T_u/T$. $F(u)$ can be estimated by MLE on the standardized observations in excess of the chosen threshold. The new distribution would be:

$$F(y) = 1 - \frac{T_u}{T} \left( 1 + \xi \frac{(y-u)}{\beta} \right)^{-1/\xi} \quad (17)$$

There are several ways to estimate the tail parameter, and Hill estimator is one of them. The Hill (1975) estimator is best known and most often applied, due to its easy implementation and asymptotic unbiased. Huisman et al. (2001) suggests the modified Hill estimator that appears to
perform equally well for GARCH (1,1) processes, which are often used to model financial returns. This approach is rather insensitive to the choice of maximum number of tail observations to include. The Hill estimator suggested by Huisman et al. avoids overestimating probabilities on extreme events and provides reliable tail-index estimates even in small samples. His methodology can have many useful applications in real-world situations for instance, in the risk-management industry, where samples can be relatively short while the likelihood of extreme events tends to be high.

We apply Hill estimator for estimating tail parameters like Christoffersen in 2003. Implicitly, we use Christoffersen steps to estimate Value-at-Risk of a portfolio in this paper.

\[ P(z > y) = 1 - F(y) = L(y) y^{-1/\xi} \sim c y^{-1/\xi}, \text{for } y > u. \]  

(18)

Where \( L(y) \) is a slowly varying function of \( y \) for most distributions and is thus set to a constant, \( c \). Given this approximation and using the definition of a conditional distribution, we can define the likelihood function for all observations \( y_i \) larger that the threshold. Thus,

\[ L = \prod_{i=1}^{T_u} f(y_i) \left( 1 - F(u) \right) = \prod_{i=1}^{T_u} \frac{1}{c u^{-1/\xi}} y_i^{-1/\xi}, \text{for } y_i > u \]  

(19)

so that the log likelihood function is

\[ \ln L = - \sum_{i=1}^{T_u} \ln(\xi) - \left( \frac{1}{\xi} + 1 \right) \ln(y_i) + \frac{1}{\xi} \ln(u). \]  

(20)

To apply Hill estimator, we should take the derivative respect to \( \xi \) and setting it to zero.

\[ \xi = \frac{1}{T_u} \sum_{i=1}^{T_u} \ln \left( \frac{y_i}{u} \right). \]  

(21)

We can estimate the \( c \) parameter by ensuring that the fraction of observations beyond the threshold is accurately captured by the density as in

\[ F(u) = 1 - \frac{T_u}{T}. \]

From the definition of \( F(u) \), we can write

\[ 1 - c u^{-1/\xi} = 1 - \frac{T_u}{T} \]  

(22)

Solving this equation for \( c \) yields the estimate

\[ C = \frac{T_u}{T} u^{1/\xi} \]  

(23)

Our estimate of the cumulative density function for observations beyond \( u \) is therefore,

\[ F(y) = 1 - c u^{-1/\xi} = 1 - \frac{T_u}{T} \left( \frac{y}{u} \right)^{-1/\xi} \]  

(24)

As risk managers we are more interested in extreme negative returns corresponding to large losses. So we do the EVT analysis on the negative of returns instead of returns themselves.
To estimate the other parameter of extreme value theory in GARCH model, we just need to use equations number 17 and 24. So we can write

$$\beta = u \xi.$$

### IV.2.2 Calculating VaR from the EVT Quantile

The previous sections we use the Hill estimator to estimate $\xi$ and $c$ from the losses, $y_i$. Now we want to estimate the Value-at-Risk of a portfolio by using the EVT distribution.

We define $y$ to be a standardized loss, that is,

$$y_i = \frac{R_i}{\sigma_i}.$$

To calculate the quantile, we need to compute the inverse cumulative distribution function. According to (24), we set the estimated cumulative probability function equal to $1-p$ so that there is only $p$ probability of getting a standardized loss worse than the quantile, $F_{1-p}^{-1}$, which is defined by

$$F (F_{1-p}^{-1}) = 1 - p,$$

so we can solve for the loss quantile, $F_{1-p}^{-1}$, to get

$$F_{1-p}^{-1} = u \frac{[\frac{p}{T}]^{-\xi}}{T}.$$

To calculate the extreme Value-at-Risk, we combined it with the variance model. So the extreme VaR is,

$$\text{VaR}_{t+1}^p = \sigma_{pF,t+1} F_{1-p}^{-1} = \sigma_{pF,t+1} u \frac{[\frac{p}{T}]^{-\xi}}{T}.$$

We use $(1-p)^{th}$ quantile from the EVT loss distribution in the VaR with coverage rate $p$. The reason is that the quantile such that $(1-p)^\%$ of losses are smaller than it is the same as minus the quantile such that $p^\%$ of returns are smaller than it.

### IV.2.3 Expected Shortfall

As we discussed is the introduction section that VaR is blind to what might happen in its tail. VaR is considered only with the number of losses that exceed the VaR and not the magnitude of these losses. Large VaR exceedences are much more likely to cause financial distress than are small exceedences. Therefore, we want to consider a risk measure that accounts for the magnitude of large losses as well as their probability of occurring.

The most complete measure of large losses is no doubt the entire shape of the tail of the distribution of losses beyond the VaR. Expected Shortfall (ES), or Expected Tail of Loss (ETL) gives exactly the expected loss.

In particular, the expected shortfall is the expected loss given that the loss is greater than or equal to the Value-at-Risk.
Expected shortfall is related to VaR by,
\[ \text{ES}^p_{t+1} = \text{VaR}^p_{t+1} + \text{mean excess over VaR} \] (28)

The expected shortfall tells us the expected value of tomorrow’s return, conditional on it being worse than the VaR.

The tail shape distribution gives us information on the range of possible losses and the probability associated with each outcome. Therefore, VaR tells us how bad can things get, and ES tells us if things do get bad, what the expected loss is.

So the shape of the tail beyond the VaR measure now is important for determining the risk number.

First, we consider the ES has a normal distribution. The expected value of a normal variable with zero mean return is:
\[ E\text{S}^p_{t+1} = \sigma_{PF,t+1} \frac{\phi(-\text{VaR}^p_{t+1}/\sigma_{PF,t+1})}{\Phi(-\text{VaR}^p_{t+1}/\sigma_{PF,t+1})} \] (29)

where \( \phi(*) \) denotes the density function and \( \Phi(*) \) the cumulative density function of the standard normal distribution.

V. Empirical Application and Results

The data set is the daily values of 349 daily S & P / TSX composite index from January 1st 2008 to June 1st 2009. We choose this period of time due to financial crisis of 2008 and 2009 that has caused bank supervisors to place more emphasis on stress testing. A plot of prices displays the volatility phenomenon of large and small swings, see figure 1. The relative performance of historical simulation method and GARCH model with using Generalized Pareto Distribution are compared in terms of Value-at-Risk and Expected Shortfall basis. Throughout the analysis, a holding period of one day will be used. Also, the different losses will be added to the sample and finally we will compare the results.

As mentioned before in the literature part, the solution for all the weaknesses of stress testing is to incorporate stress testing into market risk modeling. To show that this model is practical, we shall calculate VaR, and, then, the same parameter adding two hypothetical high-loss scenarios to the sample set of prices.

In order to apply stress tests into market risk modeling process and to calculate VaR, one has to add the scenario to the data set and assign probability to that scenario, the cumulative value of all probable scenarios adding up to one.

The first and the second scenarios are losses that we will add to the data sample. The first scenario is 10% loss and the second one is 20% loss. We will add them on April 15th 2009 because it is almost near to the last data and will give a good estimation for the future.
It is important to decide where we want to add scenarios in the GARCH model. Because the Value-at-Risk and Expected Shortfall, in GARCH model, are depend on volatility. If the market was volatile in the current period, next period variance will be high and it will have significant effect on the value of VaR and ES. So if we add scenarios near to the last data, we will face a big effect on the values of VaR and ES. On the other hand, by the definition of historical simulation method, we can add scenarios without any concern.

We will distinguish this section to two approaches; historical simulation and the GARCH model. In each case we assume different assumptions in order to find the effect of those assumptions on the value of Value-at-Risk, Expected Shortfall, the shape, and scale parameters.

### V.1 Historical Simulation

**Case 1:** To estimate Value-at-Risk, this case studies the historical simulation. Then, following Hull and White (1998) suggestions, it incorporates volatility updating into the historical simulation approach.

Historical simulation involves using past data as a guide to what will happen in the future. To calculate a one-day VaR with 99% confidence level, and 349 days of data, there is need to 349 alternative scenarios. This methodology shows what can happen between today and tomorrow. For example, scenario 1 is where the percentage changes in the values of all variables are the same as they were between Day 0 and Day 1. The second scenario is where they are the same as between Day 1 and Day 2, and so on. For each scenario we calculate the dollar change in the value of the portfolio between today and tomorrow.

We use (30) to calculate the $i^{th}$ scenario:

$$ V_T^i \frac{V_i}{V_{i-1}}. $$

$V_i$ is the value of a market variable on Day $i$, and $V_T$ is the value of a market variable on Day T, Day T being today.

To calculate the value of portfolio, first, we need to multiply the S&P on June 1st 2009 by the ratio of the S&P on two consecutive days. Then, we calculate the percentage change in the portfolio with respect to June 1st 2009 value. Finally, we order the change in portfolio by ascending order and choose the VaR as being some amount between the third worst portfolio loss and the fourth worst portfolio loss. The Value-at-Risk is -8.14%. The estimate of the 1-day 99% VaR is therefore $8140000$ million dollars, when the investment is $1$ million dollars.

Hull and White suggest a way of incorporating volatility updating into the historical simulation approach. For volatility updating scheme, we apply GARCH (1,1). The expression in (30) will thus change to

$$ V_T \frac{V_{i-1} + (V_i - V_{i-1}) \sigma_{T+1} / \sigma_i}{V_{i-1}} $$

where $\sigma_i$ is the daily volatility for a particular market variable estimated at the end of day $i-1$, and the current estimate of the volatility of the market variable is $\sigma_{T+1}$. We estimate the simple GARCH (1,1) model on the S&P daily log returns by using the Maximum Likelihood Estimation (MLE) techniques. First, we estimate the parameters in equation (15). While all the optimal parameters can be found by using Solver in Excel.
Then, we use (31) to calculate the daily volatility updating into historical simulation data. The rest of procedure for calculating VaR is the same as in historical simulation. The volatility for our data varies from 0.97% to 47.78% per day. After adjustment, the value between the third and the fourth worst outcome will be −9.60%. Therefore, the VaR estimate will be $9600000 million. The Hull and White approach is superior to traditional historical simulation. The value of VaR in volatility updating is greater than historical losses on the days we consider.

After calculating VaRs for historical simulation and Hull and White approaches, we try to calculate the value of expected shortfall for both approaches. The expected shortfall is the expected loss during three-day period, provided that the loss is greater than 99th percentile of the loss distribution. Expected shortfall is the average amount we lose over a three-day period. The historical simulation is $8820000 million for ES, and $10180000 million for volatility updating. All the values in this section are summarized in Table 1.

The Volatility Updating takes account of volatility changes in a natural and intuitive way and produces VaR estimates that incorporate more current information. The VaR in this approach is greater than historical losses that have occurred for the current portfolio during historical period considered. In our data set because the volatilities were highest at the end of historical period, the effect of the volatility adjustment is to create more variability in the gains and losses for the 349 scenarios.

**Case 2:** Here, presuming that we have fitted Generalized Pareto Distribution (GPD) model to excess daily losses above the threshold, we need to estimate 99% VaR and expected shortfall of underlying daily loss distribution.

During January 1st, 2008 and June 1st, 2009, the total number of observations ranged from -9.32% to +9.82%. We consider the left tail of the distribution of returns. Therefore, in equations given above, the variable x is the negative of the daily return on the S&P. We choose a value for u equal to 0.02. There exist 58 returns less than -2%, or \( T_u = 58 \). To estimate VaR we use maximum likelihood estimation. Again, here we use Solver to find the optimal value for \( \beta \), and \( \xi \) that maximize the log-likelihood function, thus \( \hat{\xi} = 0.01 \), and \( \hat{\beta} = 0.017 \).

To calculate the value of one-day 99% VaR, now we can apply the equation (12). Therefore, VaR will be:

\[
0.02 + \frac{0.017}{0.01} \left[ \left(\frac{349}{58} \right) \left(1 - 0.99\right) \right]^{-0.01} - 1 = 0.0692
\]

As such, the value of the VaR for a $1 million that is invested in S&P shall be $6850000 – 6.85% of the portfolio value.

There is a significant point here that the results depend on the choice of u. Actually, it is one of the advantages of extreme value theory that it focuses on the tails and the ability to study each tail separately. On the other hand, applying EVT properly needs careful attention in choosing the threshold. If u is set too large, then only very few observations are left in the tail and the estimate of the tail parameter, \( \xi \), will be bold. It means that the data to the right of the threshold do not conform sufficiently well to the generalized Pareto distribution to generate unbiased estimates of \( \xi \).

Therefore, if \( u = 0.025 \), the best fit values of \( \xi \) is 0.01, and \( \beta \) is 0.016. However, the value of VaR dramatically changes to 6.80%. So, given that the thresholds are 0.02 and 0.025, the one-day 99% VaR for an investment in the S&P are 6.92% and 6.80%, respectively. Choosing higher threshold u leads to lower value of VaR, however, it does not have a big impact on the value. All the numbers are shown in table 2.
To calculate the value of expected shortfall, we focus on the equation (14):

\[
\frac{0.0692}{1-0.1} + \frac{0.017−0.1(0.02)}{1−0.1} = 0.0872
\]

Therefore, when \( u=0.02 \) the expected shortfall is $8720000 million. If we redo equation (14), when \( u=0.025 \), the ES will be equal to $8530000 million. The related values are shown in 2.

In this section, we find that by changing the value of threshold, the values of VaR and expected shortfall do not change significantly. Although the value of threshold will change the values of VaR and ES, this change is not considerably.

**Case 3:** This case applies the integrated modeling process. It goes through stress testing in the traditional way and defines scenarios. Scenario analyses represent the possible future situations in which portfolio may be subjected, and involve extreme moves of a set of factors. Moreover, scenario analyses assess particular scenarios such as worst-case to achieve better perceptive of current situation. Indeed, useful scenarios should be realistic, and should correspond to the approach and portfolio of exposures. Also, they must be informative and valuable to risk management objectives.

This paper follows Aragones, Blanco, and Dowd (2001) suggestions for choosing scenarios. They use the profit and loss trend, and choose their scenarios based on the big losses. In Figure 2, we draw the profit and loss trend for S&P daily returns between the years 2008. During this period, we apply the loss in April 15\textsuperscript{th}, 2009 because in that time the return before and after was positive, so by this way we give a shock to all samples.

As mentioned in the literature part, stress testing is valuable when probabilities are assigned to scenarios. If each scenario is associated with a particular probability of occurrence, we will have one unified and coherent risk measurement system. Following Berkowitz (2000) analysis, each of our 349 historical observations shall therefore weigh for 0.002\% of probability.

After assigning probabilities to all samples, we can perform integrated risk estimation. We use (14) and (16) to calculate VaR and ES for a 99\% confidence level and 0.02 thresholds.

When we add the first scenario – 15\% loss– to the HS, the best fit values of \( \xi \), and \( \beta \) are 0.13 and 0.0168 respectively, and the maximum likelihood is 174.07. To calculate the value of VaR and ES, we apply these optimal values in equation (14) and (16). As such, VaR will be 7.78\%, and expected shortfall 10.60. To add the second scenario – 20\% loss – to the historical simulation, the optimal values for \( \xi \), \( \beta \), and the value of MLE are 0.19, 0.016, and 172.83. Therefore, the value of VaR for the second scenario will be 8.10\%. The value of expected shortfall will be 11.55.

In Table 3, the first row shows VaRs and ESs obtained after adding the first scenario. The second row shows adding the second one. If we keep adding losses that are significantly greater than the second scenario into the historical simulation, the value of VaR will dramatically increase. Moreover, the expected shortfall value will increase by adding the big losses in historical simulation. Note that HS puts the same weight on all observations, including old data points, which may be an undesirable feature.
In this section, as we increase losses, the value of VaR and ES increase as well. Before we reach the reported results, we had predicted that by adding losses, we shall have striking effect on both values.

V.2 GARCH Model

**Case 4:** In order to calculate VaR, we apply EVT to the standardized returns and then combine EVT with the variance models. All results in extreme value theory assume that returns are independent and identically distributed (i.i.d.). Asset returns appear to approach normality at long horizons, thus EVT is more important at short horizons, such as daily. Unfortunately, the i.i.d. assumption is the least appropriate at short horizons due to the time-varying variance patterns. We therefore need to get rid of the variance dynamics before applying EVT. Therefore, we consider the standardized portfolio returns

\[ Z_{t+1} = \frac{R_{PF,t+1}}{\sigma_{PF,t+1}}. \]  

(32)

To calculate returns \( R_{t+1} \), we just need to take a log of the ratio of two consecutive portfolios, and to calculate the \( \sigma_{t+1} \) we estimate the simple GARCH (1,1) model on the S&P 349 daily log returns by using the Maximum Likelihood Estimation (MLE) techniques. First, we estimate conditional variances using equation 15 while all the optimal parameters can be found by using Solver in Excel.

Here, presuming that we have fitted Generalized Pareto Distribution (GPD) model to excess daily losses above the threshold, we need to estimate 99% VaR and expected shortfall of underlying daily loss distribution. For sample around 349 observations, we set the threshold 17, \( T_u = 17 \), and try to estimate \( \xi \), and \( \beta \). To calculate the value of one-day 99% VaR, now we can apply the equation (30). Therefore, VaR will be:

\[ VaR_{350}^{0.01} = \sigma_{0.01 I_{350}} F_{0.99}^{-1}, \phi(\frac{0.01}{17}) 0.016 * 1.808 \left[ \frac{0.01}{17} \right] ^{-0.18} = 0.0403. \]

As such, the value of the VaR for a $1 million that is invested in S&P shall be $4030000 or – 4.03% of the portfolio value. Also the value of \( \beta \) according to (25) is 0.33.

After calculating VaR for EVT methodology, we try to calculate the value of expected shortfall. The expected shortfall shows the loss that is greater than 99th percentile of the loss distribution. We use equation (29) to calculate ES,

\[ ES_{350}^{0.01} = 0.016 \frac{\phi(-2.43)}{\phi(-2.43)} = 0.0438 \]

so the expected shortfall for the last date is $4380000 million. The results in this section show that the values of VaR and expected shortfall are smaller than historical simulation. This result confirms the result in Goorbergh and Vlaar in 1999.

**Case 5:** In the historical simulation section we argued that how sensitive is the value of VaR on the choice of \( u \). Here, we assume different threshold and want to know the probably change in the values of VaR, and ES is the GARCH model. If the value of threshold increases the value of VaR dramatically decreases. The values of Value-at-Risk with different thresholds are shown in table 4. So, given that the \( T_u \) is 17 or 14, the one-day 99% VaR for an investment in the S&P will change is 4.03% and 3.94%, respectively.
Choosing higher threshold $u$ does not have a big impact on the value of VaR, it just have a little decrease in the value of VaR. Also, by changing the value of threshold the value of expected shortfall will be changed. The value for the new expected shortfall is shown in table 4. Christoffersen (2003) argues that if the tail is fatter, it leads to have a larger ratio of ES to VaR. The results in this section display that as we consider higher threshold, the values of VaR and expected shortfall do not change significantly. That is the value of threshold does not have a big impact on VaR and ES.

**Case 6:** This case applies the integrated modeling process. We want to give a shock into our observations. In the GARCH model it is important to give a shock to which day. If we give a shock to last observations the value of VaR will change a lot comparing to the first days. Like the historical simulation section, we give shock to April $15^{th}$ 2009. First we give 10% loss to the sample, and want to calculate the values of VaR, ES, $\xi$, and $\beta$ base on this shock. According to the VaR formula, if we add 10% loss to the data, the value of VaR will be 5.24%. If we compare this result with the case one that we do not add any loss to data, we will understand that the Value-at-Risk will increase by adding a big loss. We use the equation (26) and the result confirms that we have a bigger VaR than before (10% loss). Also, this increase trend is true for expected shortfall and $\xi$, and $\beta$. As we add the loss to the observation, the values of expected shortfall, the shape and the scale parameters will go up.

The second row in table 5 shows that having the 10% loss expected shortfall will increase from 4.38 to 6.03.

$$ES_{\frac{10}{350}} = 0.0196 \frac{\phi(-2.86)}{\phi(-2.86)} = 0.0196 \frac{0.0113}{0.0036} = 0.0603$$

The value of $\xi$, and $\beta$ are also increasing to 0.29, and 0.49.

As we see in the table 5, by adding losses to the log returns samples the Value-at-Risk and the expected shortfall have increased.

Also, we want to know that if we add the bigger loss than the first time, i.e. 20%, how much the Value-at-Risk will change. We expect that by adding the bigger loss, we will have the bigger Value-at-Risk. So we apply the 20% loss on 15 April to find that what will happen to the results.

The VaR increases to 7.91, and the ES to 8.73. Also the value of $\xi$ increase to 0.36. So as we supposed to add big losses, we will have the big change in VaR, the results in this paper also confirm our content.

In this case, results show that the Value-at-Risk will be higher in tumultuous times than when the financial markets are in smooth conditions. The striking example of extreme instability in our data set was the stock market crash on $15^{th}$ April 2009. Adding losses to the data set does have big impact on values of Value-at-Risk and expected shortfall. As we reported in table 5, the more losses we added to the data set lead to the higher the value of VaR and ES.
VI. Review of the Results and Conclusions

Stress testing involves estimating the performance of the portfolio of a financial institution under extreme market moves. There are two steps in stress testing: developing scenarios involving plausible extreme market moves, and evaluating the portfolio under these scenarios. As such, in this paper we have applied the approach of Aragones, Blanco, and Dowd (2001), who suggest incorporating stress tests to market risk modeling. In order to choose different scenarios into our observations, we have assigned probability to each of them. In the empirical section we have surveyed the change in Value-at-Risk and expected shortfall by adding different losses in different times. Typically, the biggest risks to a portfolio are the abrupt occurrence of a single large negative return. Having precise knowledge of the probabilities of such extremes is, therefore, the essence of financial risk management.

In this section we have applied two models to assess the Value-at-Risk of a portfolio. An empirical evaluation of these models was done by means of an investment in the S&P stock index. For each VaR models an evaluation period of 349 trading days was used so that their performances at different given times can be compared. Here, we reproduce the results for VaR techniques that we have obtained.

The estimation results for the entire sample are presented in tables form. When we add big losses to the sample and apply historical simulation method, the value of VaR and expected shortfall will increase. In other words, in such circumstances when losses occur, Value-at-Risk consequently incurs larger losses. This increase means that when the market is in a bad state and big loss occurs, we will lose much more than before. The results of stress testing we have produced sustain John C. Hull in 2009. Moreover, if we add losses to the sample and use the GARCH model to calculate VaR we will see that the value of VaR increases, and so will the value of expected shortfall. This result confirms Basak and Shapiro in 2001.

In general, the impact of adding losses to the data sample in the both HS and GARCH methods shows that Value-at-Risk and expected shortfall are increasing, as shows in tables.

To compare the two approaches, it is better to take a look at Goorbergh and Vlaar in 1999. They calculated Value-at-Risk for both historical simulation and variance techniques, and concluded that changing volatility over time is the most important characteristic of stock returns when modeling Value-at-Risk. The fact that GARCH model incorporate new information each day should make their predictive performance superior to the historical simulation. They argue that GARCH method is able to deal with changing distribution much better than HS by attaching decaying weights to the past observations, so that past portfolio returns become less and less important as time passes.

The empirical results in their paper show that the historical simulation performs worse that the GARCH models because in GARCH model the Value-at-Risk incorporate new information each day. As a result, estimating VaR base on GARCH is superior to the historical simulation method.

Also, Eksi and Yildirim (2005) support our result in which the value of Value-at-Risk of portfolio in historical simulation is higher than in the GARCH model. Moreover, the value of expected shortfall is also more than in historical simulation rather than in the GARCH model.
Furthermore, they declare that calculating risk based on extreme value are more consistent than the GARCH model, because the estimates are very volatile for the latter. Therefore, applying the extreme value theory, banks shall hold stable amount, whereas in accordance to the GARCH model they shall constantly and regularly experience a change of their capital. Moreover, their empirical studies show that in historical simulation and the GARCH model expected shortfall offers higher risk capital allocation than Value-at-Risk. Their main concern is to determine the level of risk capital, which is a burden, the amount of money that can be invested to higher return assets. Thus, overestimating the risk capital can prove to be as costly as underestimating it. They conclude that the GARCH seem to perform well in VaR estimations. Their results show that HS overestimates the risk in terms of both ES and VaR while the tested period is quite stable and there are no significant movements in losses. Finally, they conclude that the GARCH model is more promising compared to the historical simulation method.

Although our results confirm the findings of the both papers cited above, this does not invariably mean that one should use the GARCH model in each and every situation. The model has indeed performed well in our specific portfolio setting. For further studies, it is advised to apply other distribution for estimating the possible future path of dynamic volatility. Also, it is recommended to add different scenarios in different places in the GARCH model, as the Value-at-Risk and Expected Shortfall in the GARCH model are dependent on volatility. If the market were volatile in the current period, next period variance would be high and it would have significant effect on the value of VaR and ES.
Table 1: Estimating Value-at-Risk and Expected Shortfall from Historical Simulation and Updating Volatility Methods

<table>
<thead>
<tr>
<th></th>
<th>VaR</th>
<th>ES</th>
</tr>
</thead>
<tbody>
<tr>
<td>Historical Simulation</td>
<td>8.14</td>
<td>8.82</td>
</tr>
<tr>
<td>Volatility Updating</td>
<td>9.60</td>
<td>10.18</td>
</tr>
</tbody>
</table>

Note: First row shows the historical simulation method and the second row shows the incorporating volatility updating into the historical simulation approach. A volatility updating; GARCH (1,1); is used parallel with the historical simulation approach for all market variables. We estimate the 1-day VaR for 1% confidence level. Here we have 349 samples, so the value of VaR will be the loss between the third and the forth worst loss. The Volatility Updating takes account of volatility changes in a natural and intuitive way and produces VaR estimates that incorporate more current information. The VaR in this approach is greater than historical losses that have occurred for the current portfolio during historical period considered. In our data set because the volatilities were highest at the end of historical period, the effect of the volatility adjustment is to create more variability in the gains and losses for the 349 scenarios.

Table 2: The impact of threshold VaR and Expected Shortfall in Historical Simulation Method

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>T_u</td>
<td>VaR</td>
<td>ES</td>
</tr>
<tr>
<td>-----</td>
<td>-----</td>
<td>-----</td>
</tr>
<tr>
<td>Threshold 2%</td>
<td>58</td>
<td>6.92</td>
</tr>
<tr>
<td>Threshold 2.5%</td>
<td>45</td>
<td>6.80</td>
</tr>
</tbody>
</table>

Note: Changing in value of threshold does have a little change in Value-at-Risk and expected shortfall values. We want u to be sufficiently high that we are truly investigating the shape of the tail of the distribution, but sufficiently low that the number of data items included in the maximum likelihood calculation is too low. More data lead to more accuracy in the assessment of the shape of the tail. In this table, we increase the value of threshold so it makes changes in the both VaR and expected shortfall value. Both values decrease a little by increasing the value of threshold. The value of threshold does not have a big impact on both values of VaR and ES.
Table 3: Stress-VaR Results in Historical Simulation Method

<table>
<thead>
<tr>
<th>EVT-HS</th>
<th>VaR</th>
<th>ES</th>
<th>Pareto ξ</th>
<th>Pareto β</th>
</tr>
</thead>
<tbody>
<tr>
<td>HS</td>
<td>6.92</td>
<td>8.72</td>
<td>0.01</td>
<td>0.017</td>
</tr>
<tr>
<td>HS-Scenario 1</td>
<td>7.78</td>
<td>10.60</td>
<td>0.13</td>
<td>0.016</td>
</tr>
<tr>
<td>HS-Scenario 2</td>
<td>8.10</td>
<td>11.55</td>
<td>0.19</td>
<td>0.016</td>
</tr>
</tbody>
</table>

Note: Estimated with a 1-day horizon and 99% VaR quantile. The threshold is 2% tail. The first row shows losses using historical simulation alone. The second row shows the historical simulation plus scenario 1 and the third row shows adding the second scenario into historical simulation. Scenario 1 is adding the 10% loss into sample and the scenario 2 is adding the 20% loss into sample. Before calculating the values of VaR and expected shortfall, we need to assign probabilities to the scenarios. To be simple, we assume the added scenario has the same probability as the historical simulation. So each assuming that equal weighting is used, each historical simulation scenario is assigned a probability of 1/350 = 0.0028. As shown in table, by adding the losses into sample, we are facing higher Value-at-Risk of portfolio. Implicitly, when we add big losses to the sample and apply historical simulation method the value of VaR and expected shortfall will increase, or in the other word when losses occur, Value-at-Risk consequently incur larger losses. The values of expected shortfall and ξ are also increasing, but the value of β does not change a lot.

Table 4: The impact of threshold on VaR and Expected Shortfall in GARCH (1,1) Model

<table>
<thead>
<tr>
<th>EVT-GARCH</th>
<th>$T_u$</th>
<th>VaR</th>
<th>ES</th>
</tr>
</thead>
<tbody>
<tr>
<td>Threshold 1.8%</td>
<td>17</td>
<td>4.03</td>
<td>4.38</td>
</tr>
<tr>
<td>Threshold 1.9%</td>
<td>14</td>
<td>3.94</td>
<td>4.21</td>
</tr>
</tbody>
</table>

Note: We want to find how the results of Value-at-Risk and expected shortfall depend on the choice of threshold in GARCH model. The result is the same as historical simulation method. When we choose the higher threshold, the value of VaR and ES will decrease. Note that the change on threshold does not have a big impact on values of VaR and ES.
Table 5: Stress-VaR Results in GARCH (1,1) Method

<table>
<thead>
<tr>
<th>EVT-GARCH</th>
<th>VaR</th>
<th>ES</th>
<th>Pareto $\xi$</th>
<th>Pareto $\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>GARCH</td>
<td>4.03</td>
<td>4.38</td>
<td>0.18</td>
<td>0.33</td>
</tr>
<tr>
<td>GARCH-Scenario 1</td>
<td>5.24</td>
<td>6.03</td>
<td>0.29</td>
<td>0.49</td>
</tr>
<tr>
<td>GARCH-Scenario 2</td>
<td>7.91</td>
<td>8.73</td>
<td>0.36</td>
<td>0.58</td>
</tr>
</tbody>
</table>

Note: Estimated with a 1-day horizon and 99% VaR quantile. For samples of around 349 observations, corresponding to about 1 year of daily data, we set the threshold to keep the largest 5% of the observations. Or the threshold $u$ is the 95th percentile of data set. To estimate the parameters of EVT, we use Hill estimator. The first row shows losses using simple GARCH model alone. The second row shows the GARCH (1,1) plus scenario 1 and the third row shows adding the second scenario into GARCH. Scenario 1 is adding the 10% loss into sample and the scenario 2 is adding the 20% loss into sample. Again we assign same probabilities to sample; like we did in historical simulation stress testing. As we add big loss to the sample the value of VaR and Expected Shortfall will increase. Moreover, the values of parameters of EVT increase by adding losses to the sample.\textsuperscript{xiii}

Figure 1: The S&P Price Trend between 2008 and 2009

Note: We have a sizeable timeseries of 349 daily data from 1 January 2008 to 1 June 2009. We choose this period of time due to financial crisis of 2008 and 2009 that has caused bank supervisors to place more emphasis on stress testing. A plot of prices displays the volatility
phenomenon of large and small swings, see figure 1. As is intuitively important clear, this will turn out to be important for the measurement of risk. The Value-at-Risk will be higher in tumultuous times than when the financial markets are in smooth conditions. The most striking example of extreme instability in our data set was the stock market crash of 28 November 2008.

Figure 2: Histogram of Daily Profits and Losses

Note: This figure displays the distribution of daily profits and losses. We use historical changes in market rates and prices to construct a distribution of potential future portfolio profits and losses. The Value-at-Risk as the loss that is exceeded 8% of the portfolio. The distribution of profits and losses is constructed by taking the current portfolio, and subjecting it to the actual changes in the market factors experienced during each of the last N days.
Figure 3: Profit and Loss for S&P Prices between 2008 and 2009.

Note: We illustrate the distribution of profits and losses in this figure. Based on this figure we can find that how the stock prices are changing during a year. As we see the stock had much fluctuations at the end of 2008 year, and the big loss which is -9.32% occurred in 28\textsuperscript{th} November. Stress testing involves estimating how the portfolio of a financial institution would perform under extreme market moves. To stress test observations through historical simulation approach, there is no need to add it in a specific date due to the Value-at-Risk formula in this approach. However, adding the loss in the GARCH model is significantly important due to the formula of Value-at-Risk. Here, we determined to add the loss in 15 April 2009 which is really near to last data.
Endnotes


iv Dowd. (2002). points out “unlike VaR, the ES satisfies the conditions for a coherent risk measure, and coherent risk measures have a number of attractive features. Users of VaR would be well advised to switch over.

v For example, the qualitative standards of the Basel internal models approach [1995] for market risk capital requirements specify that a bank must conduct periodic stress tests in addition to its VaR calculations. The standards; however, are silent on how banks should conduct these stress tests.

vi Berkowitz (2000) proposes the assignment of probabilities to defined stress tests and incorporate them into the marker risk modeling process and Aragones et al (2001) extended the analysis to incorporate coherent measures of risk and use extreme value theory to estimate large confidence levels for VaR and ES estimates. Once each scenario is associated with a particular probability of occurrence, we will have one unified and coherent risk measurement system rather than two incompatible ones and we will be able to apply back testing procedures to impose some check on our scenarios.


xi Yahoo Finance Canada Website: http://ca.finance.yahoo.com/q?s=^GSPTSE
Bibliography


