Can taxes stabilize the economy in the presence of consumption externalities?

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Abstract

We extend the Woodford (1986)/Grandmont et al. (1998) framework introducing consumption externalities of the ‘keeping up with the Joneses’ type and fiscal policy rules a la Lloyd-Braga et al (2008) that are characterized by two parameters: the tax rate and the elasticity of the tax rate with respect to the tax base. In the Grandmont et al. (1998) setup indeterminacy can only occur if the elasticity of substitution between capital and labor takes implausibly small values lower than the capital share of output. In contrast, with consumption externalities indeterminacy may occur with higher values of this elasticity, including the case of a Cobb-Douglas technology. We then analyze the power of the fiscal policy rules considered to eliminate indeterminacy, comparing the required values, both of the tax rate and of the elasticity of the tax rate, when two different tax bases are considered: labor income and capital income.

JEL classification: E32, E62
Keywords: Indeterminacy, consumption externalities, capital and labor income taxation.

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1 Introduction

2 The Model

The model here considered extends the Woodford (1986)/Grandmont et al. (1998) framework introducing consumption externalities and taxation. We consider a perfectly competitive monetary economy with discrete time \( t = 1, 2, ..., \infty \) and heterogeneous infinite lived agents of two types: workers and capitalists. Both consume the final good, but only workers supply labor. There is a financial market imperfection that prevents workers from borrowing against their wage income and workers are more impatient than capitalists, i.e. they discount the future more than the latter. So, in a neighborhood of a monetary steady state, capitalists hold the whole capital stock and no money, whereas workers save their wage earnings through money balances. The final good is produced by firms under a representative technology characterized by constant returns to scale. We introduce consumption externalities in this framework, i.e., we assume that the individual utility of consumption is affected by the current consumption of others. Finally, we consider "wasteful" public spending, that is financed by labor and/or capital income taxes. The detailed description of the model is provided below.

2.1 Production

In each period \( t = 1, 2, ..., \infty \), output is produced under a representative technology \( AF(k_{t-1}, l_t) \), where \( A > 0 \) is a scaling parameter, \( F(k, l) \) is a strictly increasing concave function, homogeneous of degree one in capital, \( k > 0 \), and labor, \( l > 0 \). From profit maximization, the real interest rate \( \rho_t \) and the real wage \( \omega_t \) are respectively equal to the marginal productivities of capital and labor, i.e. \( \rho_t = AF_k(k_{t-1}, l_t) \) and \( \omega_t = AF_l(k_{t-1}, l_t) \).

2.2 The Government

The government chooses the tax policy and balances its budget at each period in time. Therefore, real public spending in goods and services in period \( t, G_t \geq 0 \) is given by \( G_t = \tau_L (\omega_t l_t) \omega_t l_t + \tau_K (\rho_t K_{t-1}) \rho_t K_{t-1} \), where \( \tau_L (\omega_t l_t) \) represents the labor tax rate determined as a function of aggregate labor income in the economy and \( \tau_K (\rho_t K_{t-1}) \) represents the capital income tax rate determined as a function of aggregate capital income in the economy. We assume tax rules \( a \ la \) Lloyd-Braga, Modesto and Seegmuller (2008), i.e., tax rates on labor and capital incomes are determined respectively by the
fiscal policy rules:

$$
\tau_i(\omega_l l_t) = \alpha_i \left( \frac{\omega_l l_t}{\omega l} \right)^{\phi_i}
$$

(1)

$$
\tau_k (\rho k_{t-1}) = \alpha_k \left( \frac{\rho k_{t-1}}{\rho k} \right)^{\phi_k}
$$

(2)

with parameters $\alpha_i \in (0, 1)$ and $\phi_i \in R$ for $i = l, k$, and where $\omega l$ and $\rho k$ are respectively the wage bill and capital income, both evaluated at the steady state. Note that $\alpha_i$ represents the tax rate at the steady state and that $\phi_i$ denotes the elasticity of the tax rate with respect to the tax base. When $\phi_i < 0$ the tax rate decreases when the tax base expands, $\phi_i > 0$ corresponds to the cases where the tax rate increases with the tax base, and for $\phi_i = 0$ the tax rate is constant at the level $\alpha_i$.\(^1\)

### 2.3 Workers

We introduce externalities in the consumption of workers. Consumption externalities correspond to the idea that the individual utility of consumption is affected by the current consumption of others (envy or altruism), so that aggregate or average consumption becomes an argument of the utility function (Alonso-Carrera et al. (2008), Gali (1994), Ljungqvist and Uhlig (2000), Weder (2000b)). Here we assume that individual workers compare their own consumption $C_{w,t}$ to that of the average worker, $\overline{C}_{w,t}$:

The behavior of the representative worker can then be summarized by the maximization of $U(c_{w,t+1}^{w} \varphi(\overline{C}_{w,t+1})/B) - V(l_t)$, subject to the budget constraint $p_{t+1}c_{t+1}^{w} = (1 - \tau_l(\omega_l l_t))w_l l_t = m_t$, where $p_t$ is the price of the final good and $w_t$ the nominal wage at period $t$, $c_{t+1}^{w} \geq 0$ the worker’s consumption at period $t + 1$, $B > 0$ a scaling parameter, $V(l)$ the desutility of labor in $l \in [0, l^*]$, where $l^*$ is the worker’s time endowment, $U(c_{t+1}^{w} \varphi(\overline{C}_{t+1}^{w})/B)$ the utility of consumption and $\varphi(\overline{C}_{t+1})$ is the externality function. Workers take taxes as given when solving their maximization problem.\(^2\) The solution of this problem is given by the intertemporal trade-off between future consumption and leisure:

$$
c_{t+1}^{w} \varphi(\overline{C}_{t+1}^{w})/B = \gamma(l_t)
$$

(3)

\(^1\)This specification nests most cases considered in the literature. For example the case considered in Gokan (2006), Pintus (2004) and Schmitt-Grohé and Uribe (1997) where a constant amount of public expenditures is financed by taxes corresponds to the case where $\phi_i = -1$.

\(^2\)Since in our framework the tax rates depend on aggregate variables (see (1) and (2)) individuals, being atomistic, take tax rates as given.
where \( \gamma(l_t) \) is the usual offer curve with \( \varepsilon, \gamma(l) = \gamma'(l)l/\gamma(l) \geq 1. \)

### 2.4 Capitalists

The representative capitalist maximizes the log-linear lifetime utility function \( \sum_{t=1}^{\infty} \beta^t \ln c_t^x \) subject to the budget constraint \( c_t^x + k_t = (1 - \delta + (r_t/p_t)(1 - \tau_k(k_{t-1})))k_{t-1} \), where \( c_t^x \) represents his consumption at period \( t \), \( \beta \in (0, 1) \) his subjective discount factor, \( r_t \) the nominal interest rate and \( \delta \in (0, 1) \) the depreciation rate of capital.\(^4\) Capitalists take also taxes as given. Solving the capitalist’s problem we obtain the capital accumulation equation:

\[
k_t = \beta \left[ 1 - \delta + (r_t/p_t)(1 - \tau_k(k_{t-1})) \right] k_{t-1}.
\]

### 2.5 Equilibrium

Equilibrium on labor and capital markets requires \( \omega_t = w_t/p_t, \rho_t = r_t/p_t \). Considering that \( m > 0 \) is the constant money supply, at the monetary equilibrium, where \( w_t l_t = m \) in every period \( t \), we have \( c_t^{w} = \omega_{t+1}(1 - \tau_l(\omega_{t+1} l_{t+1}))/l_{t+1} \). Therefore:

**Definition 1** A perfect foresight intertemporal equilibrium is a sequence \( (k_{t-1}, l_t) \in \mathbb{R}_{t+1}^2, t = 1, 2, \ldots, \infty \), that, for a given \( k_0 > 0 \), satisfies

\[
\begin{align*}
\omega_{t+1}(1 - \tau_l(\omega_{t+1} l_{t+1}))/l_{t+1} \varphi(c_{t+1}^w)/B &= \gamma(l_t) \\
k_t &= \beta \left[ 1 - \delta + \rho_t(1 - \tau_k(k_{t-1})) \right] k_{t-1}.
\end{align*}
\]

Equations (5) and (6) determine the dynamics of this economy through a two-dimensional dynamic system with one predetermined variable, the capital stock \( k_t \). Indeed \( k_t \) is a variable determined by past actions. The value of \( l_t \), on the contrary, is affected by expectations of future events.

Finally, note that in the absence of government intervention or externalities in consumption (5) and (6) describe the dynamics of the standard Woodford (1986) model studied in Grandmont et al. (1998).

\(^3\)It is assumed that \( V(L) \) is a continuous function for \([0, L^*]\), and \( C^r \), with \( r \) high enough, \( V' > 0, V'' \geq 0 \) for \((0, L^*)\) and \( \lim_{L \to L^*} V'(L) = +\infty \), with \( L^* \) possibly infinite. Also, \( U(C_{t+1}^w/B) \) is a continuous function of \( C_{t+1}^w \geq 0 \), and \( C^r \), with \( r \) high enough, \( U' > 0, U'' \leq 0 \) for \( C_{t+1}^w \geq 0 \), and \(-xU''(x)/U'(x) < 1\). All this implies that \( \varepsilon, (L) \geq 1. \)

\(^4\)We do not introduce consumption externalities into capitalists’ preferences because, since they have a log-linear utility function, such externalities would not affect the dynamics.
3 Steady State Analysis

In this section, we establish conditions for the existence of the normalized steady state \((k, l) = (1, 1)\) of the dynamic system (5) and (6).

**Proposition 1** \((k, l) = (1, 1)\) is a stationary solution of the dynamic system (5)-(6) if and only if
\[
A = \frac{\theta}{\beta p(1,1)(1-\alpha_k)} > 0 \quad \text{and} \quad B = \frac{\theta \omega(1,1)(1-\alpha_l)\gamma(1-\alpha_l)}{\beta p(1,1)(1-\alpha_k)\gamma(1)} > 0.
\]

Proof: See Appendix

4 Local Stability properties

We first log-linearize the system (5)-(6) around the steady state, obtaining:
\[
\begin{bmatrix}
\hat{k}_{t+1} \\
\hat{l}_{t+1}
\end{bmatrix} = [J] \begin{bmatrix}
\hat{k}_{t-1} \\
\hat{l}_t
\end{bmatrix}
\]

where hat-variables denote deviation rates from their steady-state and \(J\) is the Jacobian matrix of the system (5) and (6) evaluated at the steady state.

The local stability properties of the model are determined by the eigenvalues of the Jacobian matrix \(J\) or, equivalently, by its trace, \(T\), and determinant, \(D\), which correspond respectively to the product and sum of the two roots (eigenvalues) of the associated characteristic polynomial \(Q(\lambda) \equiv \lambda^2 - \lambda T + D\), and that can be written as:

\[
T = \frac{\sigma}{(s-\varepsilon)(1+\chi)(1-a_L)} (\varepsilon-1) + T_1 \quad \text{with}
\]

\[
T_1 = 1 + \frac{\sigma [1-\theta a_K (1+\chi) (1-a_L)]}{(s-\varepsilon)(1+\chi)(1-a_L)} - \frac{\theta [1-s-a_K] (1+\chi) (1-a_L)}{(s-\varepsilon)(1+\chi)(1-a_L)}
\]

\[
D = D_1 (\varepsilon-1) + D_1 \quad \text{with}
\]

\[
D_1 = \frac{\sigma [1-\theta a_K - \theta (1-s) (1-a_K)]}{(s-\varepsilon)(1+\chi)(1-a_L)}
\]

where \(a_L = \frac{\phi_L}{1-\alpha_L}\), \(a_K = \frac{\phi_K}{1-\alpha_K}\), \(\theta \equiv 1 - \beta (1-\delta) \in (0, 1)\), \(\varepsilon - 1 \geq 0\) represents the inverse of the elasticity of labor supply of the representative worker, \(s \in (0, 1)\) is the elasticity of output with respect to capital, \(\sigma > 0\) is the elasticity of capital-labor substitution in production, and \(\chi > 0\) is the elasticity of the externality function all evaluated at the steady state.
4.1 Assumptions on the parameters

In accordance with empirical studies we assume that inputs are not weak substitutes i.e., that $\sigma > s$, which in the competitive case implies that the wage bill, which coincides with the consumption of workers, is increasing in labor. Following Lloyd-Braga, Modesto and Seegmuller (2010), and as suggested by empirical works, we extend this last assumption assuming that effective consumption, $c_{t+1}^w \varphi(c_{t+1}^w)$, is increasing in labor, which for $\sigma > s$ implies that $(1 + \chi) (1 - a_L) > 0$. For this last inequality to hold without consumption externalities, i.e., for the wage bill to be increasing in labor, we need to have $(1 - a_L) > 0$. Since this seems an empirically plausible assumption we will impose it, which in turn means that $(1 + \chi) > 0$. Also, as typically done in Woodford economies, we assume that $0 < \theta(1 - s) < s < 1/2$, i.e., that the period is short so that $\theta$ is small, and that $s$ is also small. Moreover, since $\sigma > s$ and $\theta$ is small we will also assume that $1 - \theta a_K > 0$ and that $\sigma > s > \theta(1 - s) (1 - a_K) / (1 - \theta a_K)$. Note that this last inequality that becomes $s > \theta(1 - s)$ in the absence of capital taxation, together with the assumption that $\sigma > s$ implies that capital income, $\rho_t (1 - \tau_k(r_t k_{t-1})) k_t$ is increasing with capital as suggested by empirical works. All these assumptions are summarized below in Assumption 1 and we consider them satisfied in the rest of the paper.

**Assumption 1**

1. $0 < \theta(1 - s) < s < 1/2$
2. $(1 - \theta a_K) > 0$
3. $\sigma > s > \theta(1 - s) (1 - a_K) / (1 - \theta a_K)$
4. $(1 + \chi) (1 - a_L) > 0$
5. $(1 - a_L) > 0$

From 9 we can see that, under Assumption 1, $D$ is an increasing function of $\varepsilon_\gamma$.

Below we present our results on local dynamics considering separately three cases: (i) the case without taxation, (ii) the case with labor income taxation only and (iii) the case with capital income taxation only.
4.2 The case of no taxation (or of constant taxation)

Without taxes (or when tax rates are constant) $\phi_i = 0$, so that $a_i = 0$, for $i = l, k$.

The full characterization of the local stability properties of the model without taxes or with constant tax rates in terms of the relevant parameters is provided in Proposition 2 below. As explained in the Appendix, these results were obtained using the geometrical method developed in Grandmont et al. (1998), i.e. we analyzed how $T$ and $D$ evolve in the space $(T, D)$ as $\varepsilon_\gamma \in [1, \infty)$, the bifurcation parameter, continuously varies in its admissible range.

**Proposition 2** Consider that Assumption 1 is verified, that $\chi > 0$ and that $\phi_i = 0$, so that $a_i = 0$, for $i = l, k$. Let $\chi$ further take admissible values in intervals specified by referring to the critical values $\chi^b = \frac{2[s-\theta(1-s)]}{\theta(1-s)}$ and $\chi^c = \frac{2s-\theta(1-s)+2\sqrt{2[s-\theta(1-s)]}}{\theta(1-s)}$. Let $\sigma \in (s, +\infty)$ take values in intervals specified by referring to the critical values $\sigma_F, \sigma_{H_1}$ and $\sigma_{H_3}$ defined in Appendix 6.3.3., and let $\varepsilon_\gamma \in [1, \infty)$ take values in intervals specified by referring to the critical values $\varepsilon_{\gamma_H}, \varepsilon_{\gamma_F}$ and $\varepsilon_{\gamma_T}$ given in Appendix 6.3.4. Then, the nature of the steady state, whether a saddle, a sink or a source, depends upon the values of the parameters $\chi, \sigma$ and $\varepsilon_\gamma$ belonging to the intervals indicated in Table 1. Also, whenever the critical value $\varepsilon_{\gamma_H}$ (resp. $\varepsilon_{\gamma_F}$ or $\varepsilon_{\gamma_T}$) appears in some row of Table 1 a Hopf bifurcation (resp. a flip or transcritical bifurcation) generically occurs as $\varepsilon_\gamma$ crosses the corresponding value.

Proof. See Appendix 6.2

We start the discussion of the results with two remarks. First, without consumption externalities indeterminacy is not possible. Indeed, indeterminacy only emerges in the presence of not too weak consumption externalities. Second, even when consumption externalities are sufficiently high, indeterminacy requires also a sufficiently high elasticity of substitution in production between capital and labor. For values of consumption externalities sufficiently high to allow for the emergence of indeterminacy, but not too high, i.e. for $\chi^b < \chi < \chi^c$, indeterminacy requires that $\sigma > \sigma_{H_1}$.

4.3 The labor income taxation case

When we only have labor taxation $\alpha_k = \phi_K = 0$, or when when capital income tax rates are constant, $\phi_K = 0$, we have. $a_K = 0$. In Proposition 3
below we provide, for this case, the full characterization of the local stability properties of the model in terms of the relevant parameters. Again, these results were obtained using the geometrical method developed in Grandmont et al. (1998). A detailed explanation of the procedure involved in provided in the Appendix.

**Proposition 3** Consider that Assumptions 1 is verified, that $\chi \geq 0$ and that $a_K = 0$. Let $\chi$ further take admissible values in intervals specified by referring to the critical values $\chi^a = \frac{a_1}{(1-a_L)}$, $\chi^b = \frac{2s - \theta(1-s)[2-a_L]}{\theta(1-s)(1-a_L)}$ and $\chi^c = \frac{2s - \theta(1-s) + 2\sqrt{s[\theta(1-s) - (1-a_L)]}}{\theta(1-s)(1-a_L)}$. Let $\sigma \in (s, +\infty)$ take values in intervals specified by referring to the critical values $\sigma_F$, $\sigma_{H_1}$, and $\sigma_{H_2}$ defined in Appendix 6.3.3., and let $\varepsilon_{\gamma} \in [1, \infty)$ take values in intervals specified by referring to the critical values $\varepsilon_{\gamma_H}$, $\varepsilon_{\gamma_F}$ and $\varepsilon_{\gamma_T}$ given in Appendix 6.3.4. Then, the nature of the steady state, whether a saddle, a sink or a source, depends upon the values of the parameters $\chi$, $\sigma$ and $\varepsilon_{\gamma}$ belonging to the intervals indicated in Table 1. Also, whenever the critical value $\varepsilon_{\gamma_H}$ (resp. $\varepsilon_{\gamma_F}$ or $\varepsilon_{\gamma_T}$) appears in some row of Table 1 a Hopf bifurcation (resp. a flip or transcritical bifurcation) generically occurs as $\varepsilon_{\gamma}$ crosses the corresponding value.

Proof. See Appendix 6.2

Table 1: Local stability properties and bifurcations - The no taxation case

<table>
<thead>
<tr>
<th>$\chi$</th>
<th>$\sigma$</th>
<th>$\varepsilon_{\gamma}$</th>
<th>$\varepsilon_{\gamma}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0 &lt; \chi &lt; \chi^b$</td>
<td>$(s, \sigma_{H_1})$</td>
<td>$[\varepsilon_{\gamma_T}, \infty)$</td>
<td>$[1, \varepsilon_{\gamma_H})$</td>
</tr>
<tr>
<td>$\chi^b &lt; \chi &lt; \chi^c$</td>
<td>$(s, \sigma_F)$</td>
<td>$[\varepsilon_{\gamma_F}, \varepsilon_{\gamma_T})$</td>
<td>$[1, \varepsilon_{\gamma_H})$</td>
</tr>
<tr>
<td>$\chi^c &lt; \chi &lt; +\infty$</td>
<td>$(s, \sigma_{H_2})$</td>
<td>$[\varepsilon_{\gamma_F}, \varepsilon_{\gamma_T})$</td>
<td>$[1, \varepsilon_{\gamma_H})$</td>
</tr>
</tbody>
</table>
4.4 The capital income taxation case

Without labor income taxation, \( \alpha_l = \phi_k = 0 \), or when labor income tax rates are constant, \( \phi_l = 0 \), so that \( a_L = 0 \).

**Proposition 4** Consider that Assumption 1 is verified, that \( \chi \geq 0 \) and that \( a_L = 0 \). Let \( \chi \) further take admissible values in intervals specified by referring to the critical values \( \chi_a = \frac{s a K}{(1-s)(1-a K)} \), \( \chi_b = \frac{2s-\theta(1-s)(1-a K)-\theta s a K}{\theta(1-s)(1-a K)} \), \( \chi_c = \frac{d+\sqrt{d(d+4)}}{2} \) where \( d = \frac{4s-\theta(1-s)(1-a K)-\theta s a K}{\theta(1-s)(1-a K)} \), \( \chi_1 = \frac{\theta s a K}{1-s-a K(1-2s)} \) and \( \chi_2 \) defined in Appendix . Let \( \sigma \in (s, +\infty) \) take values in intervals specified by referring to the critical values \( \sigma_a, \sigma_F, \sigma_T, \sigma_{H_1}, \sigma_{H_2} \) and \( \sigma_{H_3} \) defined in Appendix 6.3.3., and let \( \varepsilon_\gamma \in [1, \infty) \) take values in intervals specified by referring to the critical values \( \varepsilon_{\gamma_H}, \varepsilon_{\gamma_T} \) and \( \varepsilon_{\gamma_T} \) given in Appendix 6.3.4. Then:

(i) For \( 0 < a_K < \min \left\{ \frac{1}{\theta}, \frac{1-s}{1-2s} \right\} \), the nature of the steady state, whether a saddle, a sink or a source, depends upon the values of the parameters \( a_K, \chi, \sigma \) and \( \varepsilon_\gamma \) belonging to the intervals indicated in Table 3. Also, whenever the critical value \( \varepsilon_{\gamma_H} \) (resp. \( \varepsilon_{\gamma_T} \) or \( \varepsilon_{\gamma_T} \)) appears in some row of Table 3 a Hopf bifurcation (resp. a flip or transcritical bifurcation) generically occurs as \( \varepsilon_\gamma \) crosses the corresponding value.

(ii) For \( \frac{s-\theta(1-s)}{\theta(1-2s)} < a_K < 0 \), the nature of the steady state, whether a saddle, a sink or a source, depends upon the values of the parameters \( \chi, \sigma, \) and \( \varepsilon_\gamma \).
\( \sigma \) and \( \varepsilon, \gamma \) belonging to the intervals indicated in Table 4. Also, whenever
the critical value \( \varepsilon_{\gamma H} \) (resp. \( \varepsilon_{\gamma F} \) or \( \varepsilon_{\gamma T} \)) appears in some row of Table 4
a Hopf bifurcation (resp. a flip or transcritical bifurcation) generically
occurs as \( \varepsilon, \gamma \) crosses the corresponding value.

Proof. See Appendix 6.3

We start the discussion of the results with two remarks.

5 Appendix

5.1 Proof of Proposition 1

A stationary equilibrium of the dynamic system (5)-(6) is a solution
\( (k, l) = (k_{l-1}, l) \) for all \( t \), that satisfies
\( A \rho(k/l)(1 - \alpha_k) = \theta/\beta \) and \( (A/B) \omega(k/l)(1 - \alpha_l) \varphi[(\omega(k/l)(1 - \alpha_l)]l = \gamma(l) \). The existence of a steady state can be estab-
lished by choosing appropriately the two scaling parameters \( A > 0 \) and \( B > 0 \)
so as to ensure that one steady state coincides with \( (k, l) = (1, 1) \). From the
first equation, we obtain a unique solution
\( A = \frac{\theta}{[\beta \rho(1, 1)(1 - \alpha_k)]} > 0 \). Sub-
stituting this into the second equation we then obtain the unique solution
for \( B = [\beta \rho(1, 1)(1 - \alpha_k) \gamma(1)]^{-1} \theta \omega(1, 1)(1 - \alpha_l) \varphi(\omega(1, 1)(1 - \alpha_l)) > 0 \).

5.2 Local Dynamics

Our results are obtained using the geometrical method developed in Grand-
mont et al. (1998), which allows us to characterize the occurrence of local
indeterminacy and bifurcations, and ultimately of stochastic and determinis-
tic endogenous fluctuations around the steady state, in terms of the relevant
parameters. We consider that \( s \) and \( \theta \) are fixed throughout the analysis and,
for given different values of \( \chi, \alpha_i \) and \( \phi_i \) for \( i = l, k \), satisfying all the Assump-
tions considered, we study how \( (T, D) \), or equivalently the local eigenvalues,
change as \( \varepsilon, \gamma \) and \( \sigma \) vary in their admissible ranges.

In Figure 2 we have represented in the plane \( (T, D) \) three lines relevant
for our purpose: the line \( AC \) \( (D = T - 1) \) where a local eigenvalue is equal
to 1; the line \( AB \) \( (D = -T - 1) \), where one eigenvalue is equal to -1; and
the segment \( BC \) \( (D = 1 \) and \( |T| < 2) \) where two eigenvalues are complex
conjugates of modulus 1. When \( T \) and \( D \) fall in in the interior of triangle
Table 3: Local stability properties and bifurcations with capital taxation when \( a_k > 0 \)

<table>
<thead>
<tr>
<th>( a_K &lt; \min \left{ \frac{1}{\delta}, \frac{1-a}{1-a} \right} )</th>
<th>( 0 &lt; \chi &lt; \infty )</th>
<th>( \sigma )</th>
<th>( \varepsilon )</th>
<th>Saddle</th>
<th>Sink</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 0 &lt; \chi &lt; \chi_2 )</td>
<td>( (s, \sigma) )</td>
<td>( \varepsilon )</td>
<td>( 1, \infty )</td>
<td>-</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>( (\sigma, \infty) )</td>
<td>( \varepsilon )</td>
<td>( \varepsilon_f, \infty )</td>
<td>( 1, \varepsilon_T )</td>
<td>-</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

| \( \chi_2 < \chi < \chi_1 \) | \( (s, \sigma) \) | \( \varepsilon \) | \( 1, \infty \) | - | - |
| \( (\sigma, \sigma_H) \) | \( \varepsilon \) | \( \varepsilon_f, \infty \) | - | \( 1, \varepsilon_T \) |
| \( (\sigma_H, \sigma_H) \) | \( \varepsilon \) | \( \varepsilon_f, \infty \) | \( 1, \varepsilon_T \) | \( 1, \varepsilon_T \) |
| \( (\sigma_H, \infty) \) | \( \varepsilon \) | \( \varepsilon_f, \infty \) | \( 1, \varepsilon_T \) | \( 1, \varepsilon_T \) |

| \( \chi_1 < \chi < \chi^a \) | \( (s, \sigma) \) | \( \varepsilon \) | \( 1, \infty \) | - | - |
| \( (\sigma, \sigma_H) \) | \( \varepsilon \) | \( \varepsilon_f, \infty \) | \( 1, \varepsilon_T \) | \( 1, \varepsilon_T \) |
| \( (\sigma_H, \sigma_H) \) | \( \varepsilon \) | \( \varepsilon_f, \infty \) | \( 1, \varepsilon_T \) | \( 1, \varepsilon_T \) |
| \( (\sigma_H, \infty) \) | \( \varepsilon \) | \( \varepsilon_f, \infty \) | \( 1, \varepsilon_T \) | \( 1, \varepsilon_T \) |

| \( \chi^a < \chi < \chi^b \) | \( (s, \sigma) \) | \( \varepsilon \) | \( 1, \infty \) | - | - |
| \( (\sigma, \sigma_H) \) | \( \varepsilon \) | \( \varepsilon_f, \infty \) | \( 1, \varepsilon_T \) | \( 1, \varepsilon_T \) |
| \( (\sigma_H, \sigma_H) \) | \( \varepsilon \) | \( \varepsilon_f, \infty \) | \( 1, \varepsilon_T \) | \( 1, \varepsilon_T \) |
| \( (\sigma_H, \infty) \) | \( \varepsilon \) | \( \varepsilon_f, \infty \) | \( 1, \varepsilon_T \) | \( 1, \varepsilon_T \) |

| \( \chi^b < \chi < \chi^c \) | \( (s, \sigma) \) | \( \varepsilon \) | \( 1, \infty \) | - | - |
| \( (\sigma, \sigma_H) \) | \( \varepsilon \) | \( \varepsilon_f, \infty \) | \( 1, \varepsilon_T \) | \( 1, \varepsilon_T \) |
| \( (\sigma_H, \sigma_H) \) | \( \varepsilon \) | \( \varepsilon_f, \infty \) | \( 1, \varepsilon_T \) | \( 1, \varepsilon_T \) |
| \( (\sigma_H, \infty) \) | \( \varepsilon \) | \( \varepsilon_f, \infty \) | \( 1, \varepsilon_T \) | \( 1, \varepsilon_T \) |

<p>| ( \chi^c &lt; \chi &lt; \infty ) | ( (s, \sigma) ) | ( \varepsilon ) | ( 1, \infty ) | - | - |
| ( (\sigma, \sigma_H) ) | ( \varepsilon ) | ( \varepsilon_f, \infty ) | ( 1, \varepsilon_T ) | ( 1, \varepsilon_T ) |
| ( (\sigma_H, \sigma_H) ) | ( \varepsilon ) | ( \varepsilon_f, \infty ) | ( 1, \varepsilon_T ) | ( 1, \varepsilon_T ) |
| ( (\sigma_H, \infty) ) | ( \varepsilon ) | ( \varepsilon_f, \infty ) | ( 1, \varepsilon_T ) | ( 1, \varepsilon_T ) |</p>
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Table 4: Local stability properties and bifurcations with capital taxation for $ak<0$
ABC the steady state is a sink (both eigenvalues with modulus lower than one), i.e., asymptotically stable. In the present context, where only capital is a predetermined variable, the steady state is locally indeterminate in this case, and, as known, there are infinitely many stochastic endogenous fluctuations (sunspots) arbitrarily close to the steady state. In all other cases the steady state is locally determinate: a saddle (one eigenvalue with modulus higher than one and one eigenvalue with modulus lower than one) when $|T| > |D + 1|$ and a source (both eigenvalues with modulus higher than one) in the remaining regions.

We can also use the diagram of Figure 2 to study the occurrence of local bifurcations. When, by slightly changing a (bifurcation) parameter, the values of $T$ and $D$ cross the interior of the segment $BC$ (a pair of complex conjugate eigenvalues crossing the unit circle) a Hopf bifurcation generically occurs, and deterministic cycles (periodic or quasiperiodic orbits) surrounding the steady state in the state space emerge. A flip bifurcation occurs when the values of $T$ and $D$ cross the $AB$ line, and deterministic cycles of period two appear. Finally, when the values of $T$ and $D$ cross the $AC$ line, and if Propositions 1 and 3 are satisfied, a transcritical bifurcation generically occurs, by which two close steady states exchange stability properties. When $T$ and $D$ cross the line $AC$ other types of bifurcations may occur (pitchfork or saddle-node). However, since we have assumed the existence of at least one steady state (Proposition 1) and at most of two (Proposition 3) these other bifurcations are ruled out.

5.2.1 The half-line $\Delta$

We start by discussing how $T$ and $D$ move in the space $(T, D)$ as $\varepsilon_\gamma$ is made to continuously change. From (8) and (9), the locus of points $(T(\varepsilon_\gamma), D(\varepsilon_\gamma))$ as $\varepsilon_\gamma$ varies in $[1, +\infty)$ describes a half-line $\Delta$ in the plane $(T, D)$, starting at $(T(1), D(1)) = (T_1, D_1)$, when $\varepsilon_\gamma = 1$, with a slope $S > 0$ under Assumption

5Indeterminacy occurs when the number of eigenvalues strictly lower than one in absolute value is larger than the number of predetermined variables.

6When $T$ and $D$ cross the line $AC$ other types of bifurcations may occur (pitchfork or saddle-node). However, since we have assumed the existence of at least one steady state (Proposition 1) and at most of two (Proposition 3) these other bifurcations are ruled out.
1, equal to:

\[ S = \frac{\sigma (1 - \theta a_K) - \theta(1 - s)(1 - a_K)}{\sigma} \]  

(10)

Note that the half-line \( \Delta \) shifts when \( \sigma \) changes, because its slope \( S \) and its initial point \((T_1, D_1)\) depend on \( \sigma \).

5.2.2 The half-line \( \Delta_1 \)

We now discuss the behavior of \((T_1, D_1)\) as \( \sigma \) varies. From (8) and (9), the locus of points \((T_1(\sigma), D_1(\sigma))\) obtained as \( \sigma \) decreases from \( +\infty \) to \( s \) describes a half-line \( \Delta_1 \), starting at \((T_1(+\infty), D_1(+\infty))\) where

\[ T_1(+\infty) = 1 + \frac{[1 - \theta a_K (1 + \chi) (1 - a_L)]}{(1 + \chi) (1 - a_L)} \]  

(11)

\[ D_1(+\infty) = \frac{1 - \theta a_K}{(1 + \chi) (1 - a_L)} \]  

(12)

Note that we have \( D_1(+\infty) = T_1(+\infty) - 1 \) in the following cases: (i) without capital income taxation or when capital tax rates are constant, so that \( a_K = 0 \), or (ii) without labor income taxation and consumption externalities, i.e. when \( a_L = 0 \) and \( \chi = 0 \).

The slope \( S_1 \) of the half-line \( \Delta_1 \), is given by:

\[ S_1 = \frac{s(1 - \theta a_K) - \theta(1 - s)(1 - a_K)}{s - \theta(1 - s)(1 + \chi)(1 - a_L)(1 - a_K)} \]  

(13)

The next Lemma, that can be easily proved using (10), Assumption ?? and simple analytical computations, will help us understanding why results are different according to whether \( a_K \geq 0 \) or \( a_K < 0 \).

**Lemma 1** Under Assumption 1 and defining \( \sigma_T \equiv -\frac{(1-s)(1-a_K)}{a_K} \), we have the following:

1. If \( 0 < a_K < \frac{1 - s}{1 - 2s} \), then \( 0 < S < 1 \) for all \( \sigma \in (s, +\infty) \) and \( \lim_{\sigma \rightarrow +\infty} S < 1 \)

2. If \( a_K = 0 \), then \( 0 < S < 1 \) for all \( \sigma \in (s, +\infty) \) and \( \lim_{\sigma \rightarrow +\infty} S = 1 \)

3. If \( a_K < 0 \), then:

   (a) \( 1 < S < 1 - \theta a_K \) if and only if \( \sigma \in (\sigma_T, +\infty) \), and \( \lim_{\sigma \rightarrow +\infty} S > 1 \)

   (b) \( S = 1 \) if and only if \( \sigma = \sigma_T \)
Below, we present the proofs of Propositions 2, 3 and 4 using geometrical arguments whenever possible, by referring to Figures 1-3. As explained below Proposition 2 is a particular case of Propositions 3. Therefore we present first the proof of Proposition 3.

5.2.3 Proof of Proposition 3

When only labor income is taxed, $\alpha_k = \phi_K = 0$, or when capital income tax rates are constant, $\phi_K = 0$, i.e. when $\alpha_K = 0$, from 10 and 13 we have that

\[
S = \frac{\sigma - \theta(1 - s)}{\sigma} \tag{14}
\]

and

\[
S_1 = \frac{s - \theta(1 - s)}{s - \theta(1 - s) (1 + \chi)(1 - a_L)} \tag{15}
\]

Remember that the half-line $\Delta$ points upwards to the right as $\varepsilon_\gamma$ increases from 1 to $+\infty$. In this case, from (14), or from Lemma 1, it is easy to see that $0 < S < 1$ for all $
abla \sigma \in (s, +\infty)$ and $\lim_{\sigma \to +\infty} S = 1$. Also, as $\sigma$ decreases from $+\infty$ to $s$, the half line $\Delta$ becomes less steep (see (14)) and its initial point $(T_1, D_1)$ moves upwards along the half line $\Delta_1$. Note that in this case $D_1(+\infty) = T_1(+\infty) - 1$, i.e. the half-line $\Delta_1$ starts on the line $(AC)$. See (11) and (12). Below we present our analysis considering different intervals for the parameter $\chi$, that we call configurations, and that correspond to different intervals for $S_1$, the slope of the half-line $\Delta_1$.

For $0 < \chi < \chi^a$, the half line $\Delta_1$ that starts (for $\sigma = +\infty$) on the line $(AC)$ between $A$ and $C$, has a slope lower than 1, i.e., lower than the slope of $(AC)$, and points upwards, lying therefore on the right of $(AC)$. See Figure 2. As $\alpha_k = 0$, then $0 < S < 1$ (see Lemma 1) and the half-line $\Delta$ lies below line $(AC)$ and above line $(AB)$. Hence, the steady state is a saddle.

For $\chi^a < \chi < \chi^b$ we have $|S_1| > 1$. Therefore, the half-line $\Delta_1$ that starts on the line $(AC)$, between $A$ and $C$, points upwards with a slope $S_1$ strictly greater than 1 or strictly smaller than $-1$, crossing neither $(AB)$, nor $(AC)$ (see Figure 1). However, since $\Delta_1$ crosses the segment $[BC]$, there is a critical value for $\sigma$, $\sigma_{H_1} > s$ such that $D_1(\sigma_{H_1}) = 1$ (see (??)). If $\sigma \leq \sigma_{H_1}$, the half-line $\Delta$ starts on $\Delta_1$ above $[BC]$ and since it points upwards to the right, it crosses the line $(AC)$, above point $C$, for $\varepsilon_\gamma = \varepsilon_{\gamma_T}$. Accordingly, the steady state is a source for $1 \leq \varepsilon_\gamma < \varepsilon_{\gamma_T}$, undergoes a transcritical bifurcation for $\varepsilon_\gamma = \varepsilon_{\gamma_T}$ and becomes a saddle for $\varepsilon_\gamma > \varepsilon_{\gamma_T}$. If $\sigma > \sigma_{H_1}$, $(T_1(\sigma), D_1(\sigma))$ is inside the triangle $(ABC)$ and the half-line $\Delta$ must also cross the line $(BC)$,
but Hopf bifurcations only occur if the crossing point is on the left of point $C$, so that the half-line $\Delta$ crosses $[BC]$ in its interior. We can see geometrically that, by continuity, for $\sigma$ higher but close to $\sigma_{H_1}$ the half-line $\Delta$ crosses the segment $[BC]$ in its interior. However, for higher values of $\sigma > \sigma_{H_1}$ this may not happen. As $\phi_k = 0$ we have have $S < 1$, and $\lim_{\sigma \to +\infty} S = 1$ (see Lemma 1). Therefore in this case the half-line $\Delta$ does not cross point $C$, i.e. for all $\sigma \in (\sigma_{H_1}, +\infty)$ the half-line $\Delta$ crosses the segment $[BC]$ in its interior and then crosses $(AC)$ above point $C$.

For $\chi^b < \chi < \chi^c$ we have $S_1 \in (-1, S_B)$. Therefore, in this configuration, the half-line $\Delta_1$ points upwards to the left, crossing line $(AB)$ above point $B$ (see Figure 2). Let us now define $\sigma_F$ as the critical value such that $1 + D_1(\sigma_F) + T_1(\sigma_F) = 0$. See (19). Since $a_k = 0$ we have $0 < S < 1$ (see Lemma 1). When $\sigma < \sigma_F$, $(T_1(\sigma), D_1(\sigma))$ is below line $(AB)$ and above $B$. Since the half-line $\Delta$ points upwards it does not cross $[BC]$, but crosses $(AB)$ before crossing $(AC)$. When $\sigma_F \leq \sigma \leq \sigma_{H_1}$, $(T_1(\sigma), D_1(\sigma))$ is above (or over) $(AB)$ and $[BC]$. Then, $\Delta$ only crosses $(AC)$. As in the previous configuration, when $\sigma > \sigma_{H_1}$, the half-line $\Delta$ crosses first $[BC]$ and then $(AC)$ above $C$.

For $\chi^c < \chi < \infty$ we have that $S_1 \in (S_B, 0)$, with $S_B \in (-1, 0)$. Therefore, the slope $S_1$ is negative and greater than $-1$, and the half line $\Delta_1$, that points upwards to the left, crosses line $(AB)$ below point $B$ (see Figure 3). In this configuration, a new critical value, $\sigma_{H_3}$, the value of $\sigma$ such that the half line $\Delta$ goes through point $B$, becomes relevant. In Appendix 6.5 we prove that in this configuration, there exists a unique critical value $\sigma_{H_3} \in (\sigma_{H_1}, \sigma_F)$ such that the half-line $\Delta$ goes through point $B$ and crosses $[BC]$ on the right of $B$ for $\sigma > \sigma_{H_3}$. Since $a_k = 0$ we have $0 < S < 1$ (see Lemma 1). Hence, for $\sigma < \sigma_{H_3}$, $\Delta$ starts on the left-side of $(AB)$, crosses $(AB)$ above $B$ and then crosses $(AC)$. For $\sigma_{H_3} < \sigma < \sigma_F$, $\Delta$ also starts on the left-side of $(AB)$, but crosses $(AB)$ below $B$, the segment $[BC]$, and $(AC)$ above $C$. Then, for $\sigma > \sigma_F$, $(T_1(\sigma), D_1(\sigma))$ is inside $(ABC)$, and $\Delta$ crosses $[BC]$ and $(AC)$ above $C$.

All these results are summarized in Table 2.

5.2.4 Proof of Proposition 2

Without taxes, $\alpha_i = \phi_i = 0$, or when tax rates are constant, $\phi_i = 0$, so that $a_i = 0$, for $i = l, k$. Note that this case is a particular case of the one considered above where $a_L = 0$. Therefore Proposition 2 follows immediately from Proposition 3 by setting $a_L = 0$. Note also that in this case $S$ is given
by (14) and from (13) \(S_1\) becomes:

\[
S_1 = \frac{s - \theta(1 - s)}{s - \theta(1 - s)(1 + \chi)}
\]

(16)

5.2.5 Proof of Proposition 4

When only capital income is taxed, \(\alpha_L = \phi_L = 0\), or when labor income tax rates are constant, \(\phi_L = 0\), i.e. when \(a_L = 0\), \(S\) is given by (10) and the slope \(S_1\) of the half-line \(\Delta_1\), is given by:

\[
S_1 = \frac{s(1 - \theta a_K) - \theta(1 - s)(1 - a_K)}{s - \theta(1 - s)(1 + \chi)(1 - a_K)}
\]

(17)

In this case results are different for different values of \(a_K\).

**The case where** 1 < \(a_K\) \(< \min \left\{ \frac{1}{\theta}, \frac{1}{1 - 2\chi} \right\} \).

In this case from (17) it is easy to see that 0 < \(S_1\) < 1 for any 0 < \(\chi\) < +\(\infty\), i.e. the half line \(\Delta_1\) has a slope lower than 1. Also, using (11) and (12) we have that it starts (for \(\sigma = +\infty\)) inside the triangle \(ABC\). It is also easy to prove that the half line \(\Delta_1\) always crosses the line \(BC\), i.e., as \(\sigma\) decreases from +\(\infty\) to \(s\), it crosses first the line \(AC\) for \(\sigma = \sigma_a\) and then the line \(BC\) to right of point \(C\) when \(\sigma = \sigma_{H_1}\). From Lema (1) we have that 0 < \(S < 1\) for all \(\sigma \in (s, +\infty)\) and \(\lim_{\sigma \to +\infty} S < 1\). Moreover \(\partial S/\partial \sigma < 0\) when \(a_K > 1\). Also it is easy to verify that in this case, when \(\chi > \chi_1\) we always have \(S \geq S_1\), whereas for \(\chi < \chi_1\) we have \(S \geq S_1\) if \(\sigma \leq \sigma_{SS_1}\) and \(S < S_1\) if \(\sigma > \sigma_{SS_1}\).

Therefore for \(\chi > \chi_1\) the half line \(\Delta\) that starts on the half line \(\Delta_1\) points upwards with a slope \(S_1 < S < 1\). When \(s < \sigma \leq \sigma_a\) it never crosses \(AC\) so that the steady state is always a saddle. For \(\sigma_a < \sigma < +\infty\), as in this case \(\varepsilon_{\gamma_T} < \varepsilon_{\gamma_H}\), the half line \(\Delta\) crosses first the line \(AC\) and then \(BC\), i.e. the steady state is first a sink and then a saddle.

For \(\chi < \chi_1\) we have to consider two cases. If \(\sigma_{H_1} < \sigma_{SS_1} < \sigma_a\), for \(s < \sigma \leq \sigma_a\) the half line \(\Delta\) that starts on the half line \(\Delta_1\) points to the right with a slope lower than 1; for \(s < \sigma \leq \sigma_{SS_1}\) it goes above the half line \(\Delta_1\) and for \(\sigma_{SS_1} < \sigma \leq \sigma_a\) it goes below it, never crossing the line \(AC\). Therefore the steady state is a saddle. For \(\sigma_a < \sigma \leq +\infty\), the half line \(\Delta\) points to the right with a slope \(S < S_1\), crossing the line \(AC\) below the half line \(\Delta\). Therefore the steady state is first a sink and then a saddle. If \(\sigma_{SS_1} > \sigma_a\), for \(s < \sigma \leq \sigma_a\) the half line \(\Delta\) that starts on the half line \(\Delta_1\) points to the right with a slope \(S > S_1\), but lower than 1, never crossing the line \(AC\) so that the steady state is a saddle. For \(\sigma_a < \sigma \leq \sigma_{SS_1}\), the half line \(\Delta\) still points to right with a slope \(S > S_1\), but lower than 1, crossing first \(AC\)
and then $BC$. Therefore the steady state is first a sink and then a saddle. For $\sigma_{SS_1} < \sigma \leq \infty$, the half line $\Delta$ points to the right with a slope $S < S_1$, crossing the line $AC$ below the half line $\Delta$ so that the steady state is first a sink and then a saddle.

We can therefore conclude that when $1 < a_K < 1/\theta$ the steady state is a saddle for $s < \sigma \leq \sigma_a$, whereas for $\sigma_a < \sigma < \infty$ the steady state is first a sink and then a saddle.

**The case where** $0 < a_K < 1$.

From Lemma (1) we have that $0 < S < 1$ for all $\sigma \in (s, +\infty)$ and $\lim_{\sigma \to +\infty} S < 1$. Moreover, in this case, $\partial S/\partial \sigma > 0$, i.e., as $\sigma$ decreases from $+\infty$ to $s$, the half line $\Delta$ becomes less steep (see (10)) and its initial point $(T_1, D_1)$ moves upwards along the half line $\Delta_1$. Also, as $a_K > 0$ using (11) and (12) it is easy to show that the half line $\Delta_1$ starts (for $\sigma = +\infty$) inside the triangle $ABC$. Below we present our results considering different intervals for the parameter $\chi$, that we call configurations, and that correspond to different intervals for $S_1$, the slope of the half-line $\Delta_1$.

For $0 < \chi < \chi^a$, the half line $\Delta_1$ has a slope lower than 1, i.e., lower than the slope of $(AC)$. See (17). In this case for $\chi > \chi_1$ we always have $S < S_1$, whereas for $\chi < \chi_1$ we have $S \geq S_1$ if $\sigma \geq \sigma_{SS_1}$ and $S < S_1$ if $\sigma < \sigma_{SS_1}$. Also, for $0 < \chi < \chi_2$ the half line $\Delta_1$ crosses line $AC$ below line $BC$, i.e., as $\sigma$ decreases from $+\infty$ to $s$, it crosses first the line $AC$ for $\sigma = \sigma_a$ and then the line $BC$ to right of point $C$ when $\sigma = \sigma_{H_1}$. For $\chi > \chi_2$ the half line $\Delta_1$ crosses line $AC$ above $BC$, i.e. it crosses first the segment $BC$ for $\sigma = \sigma_{H_1}$, a Hopf occurring, and then line $AC$ for $\sigma = \sigma_a$. Since $0 < \chi_2 < \chi_1 < \chi^a$, for $0 < \chi < \chi_2$, as $\sigma_{H_1} < \sigma_{SS_1} < \sigma_a$, for $s < \sigma \leq \sigma_a$ the half line $\Delta$ that starts on the half line $\Delta_1$ and points to right, goes below the half line $\Delta_1$ for $s < \sigma \leq \sigma_{SS_1}$ and for $\sigma_{SS_1} < \sigma \leq \sigma_a$ goes above it, never crossing the line $AC$. Therefore the steady state is a saddle. For $\sigma_a < \sigma \leq \infty$ the half line $\Delta$ points to right with a slope $S > S_1$, but lower than 1. As in this case $\varepsilon_{\gamma_H} < \varepsilon_{\gamma_T}$, it crosses first $AC$ and then $BC$. Therefore the steady state is first a sink and then a saddle. For $\chi_2 < \chi < \chi_1$, as $\sigma_a < \sigma_{H_1} < \sigma_{H_2} < \sigma_{SS_1}$, for $s < \sigma \leq \sigma_a$ the half line $\Delta$ that starts on the half line $\Delta_1$ points to right with a slope $S < S_1$ so that the steady state is a saddle. For $\sigma_a < \sigma < \sigma_{H_1}$ the half line $\Delta$ still points to right with a slope $S < S_1$ crossing the line $AC$ above point $C$. Therefore the steady state is first a source and then a saddle. When $\sigma_{H_1} < \sigma < \sigma_{H_2}$ the half line $\Delta$ that starts on the half line $\Delta_1$ inside the triangle $ABC$, points to right with a slope $S < S_1$ crossing line $AC$ below point $C$. Therefore the steady state is first a sink and then a saddle. When $\sigma_{H_2} < \sigma < \infty$ the half line $\Delta$ that starts on the half line $\Delta_1$ inside the triangle $ABC$, points to right, first with a slope $S < S_1$ for $\sigma_{H_2} < \sigma < \sigma_{SS_1}$ crossing
first the segment $BC$ and then line $AC$ above point $C$. For $\sigma_{SS_1} < \sigma < \infty$, we have $S < S_1$ but the half line $\Delta$ still crosses first the segment $BC$ and then line $AC$ above point $C$. Therefore for $\sigma_{H_2} < \sigma < \infty$ the steady state is first a sink, then a source and then a saddle. For $\chi_1 < \chi < \chi^a$ the half line $\Delta_1$ has a slope $S < S_1$. As $\sigma_a < \sigma_{H_1}$ for $s < \sigma \leq \sigma_a$ the half line $\Delta$ points to the right below $S_1$ so that the steady state is a saddle. For $\sigma_a < \sigma < \sigma_{H_1}$ the half line $\Delta$ points to the right with a slope $S < S_1$ crossing the line $AC$ above point $C$. Therefore the steady state is first a source and then a saddle. For $\sigma_{H_1} < \sigma < \sigma_{H_2}$ the half line $\Delta$ points to the right with a slope $S < S_1$ crossing the line $AC$ below point $C$. Therefore the steady state is first a sink and then a saddle. When $\sigma_{H_2} < \sigma < \infty$ the half line $\Delta$ points to the right with a slope $S < S_1$ crossing first the segment $BC$ and then the line $AC$ above point $C$. Therefore the steady state is first a sink, then a source and then a saddle.

For $\chi^a < \chi < \chi^b$ we have $|S_1| > 1$. Therefore, the half-line $\Delta_1$ that starts (for $\sigma = +\infty$) inside the triangle $ABC$, points upwards with a slope $S_1$ strictly greater than 1 or strictly smaller than −1, crossing neither $(AB)$ nor $(AC)$, but crossing the segment $[BC]$ for $\sigma = \sigma_{H_1} > s$. If $s < \sigma \leq \sigma_{H_1}$, the half-line $\Delta$ starts on $\Delta_1$ above $[BC]$ and since it points upwards to the right, it crosses the line $(AC)$, above point $C$, for $\varepsilon_\gamma = \varepsilon_{\gamma_T}$. Accordingly, the steady state is a source for $1 \leq \varepsilon_\gamma < \varepsilon_{\gamma_T}$, undergoes a transcritical bifurcation for $\varepsilon_\gamma = \varepsilon_{\gamma_T}$ and becomes a saddle for $\varepsilon_\gamma > \varepsilon_{\gamma_T}$. If $\sigma > \sigma_{H_1}$, $(T_1(\sigma), D_1(\sigma))$ is inside the triangle $(ABC)$ and the half-line $\Delta$ must also cross the line $(BC)$. For $\sigma_{H_1} < \sigma < \sigma_{H_2}$ the half-line $\Delta$ crosses first the segment $[BC]$ in its interior an then line $AC$ above point $C$, so that the the steady state is first a sink, then a source and then a saddle. When $\sigma_{H_2} < \sigma < \infty$ the half line $\Delta$ crosses first line $AC$ and then $BC$ to the right of $C$. Therefore the steady state is first a sink and then a saddle.

For $\chi^b < \chi < \chi^c$ we have $S_1 \in (-1, S_B)$. Therefore, in this configuration, the half-line $\Delta_1$ that starts (for $\sigma = +\infty$) inside the triangle $ABC$, points upwards to the left, crossing line $(AB)$ above point $B$ for $\sigma = \sigma_F$. See (19). When $\sigma < \sigma_F$, $(T_1(\sigma), D_1(\sigma))$ is below line $(AB)$ and above $B$. Since the half-line $\Delta$ points upwards it does not cross $[BC]$, but crosses $(AB)$ before crossing $(AC)$. When $\sigma_F \leq \sigma \leq \sigma_{H_1}$, $(T_1(\sigma), D_1(\sigma))$ is above $(AB)$ and $[BC]$. Then $\Delta$ only crosses $(AC)$. As in the previous configuration when $\sigma > \sigma_{H_1}$ the half-line $\Delta$ also crosses $[BC]$. For $\sigma_{H_1} < \sigma < \sigma_{H_2}$ the half-line $\Delta$ crosses first the segment $[BC]$ in its interior and then line $AC$ above point $C$. When $\sigma_{H_2} < \sigma < \infty$ the half line $\Delta$ crosses first line $AC$ and then $BC$ to the right of $C$.

For $\chi^c < \chi < \infty$ we have $S_1 \in (S_B, 0)$, with $S_B \in (-1, 0)$. Therefore, the slope $S_1$ is negative and greater than −1, and the half line $\Delta_1$, that points
upwards to the left, crosses line \((AB)\) below point \(B\) (see Figure 3). In this configuration, a new critical value, \(\sigma_{H_3}\), the value of \(\sigma\) such that the half line \(\Delta\) goes through point \(B\), becomes relevant. In Appendix 6.5 we prove that in this configuration, there exists a unique critical value \(\sigma_{H_3} \in (\sigma_{H_1}, \sigma_F)\) such that the half-line \(\Delta\) goes through point \(B\) and crosses \([BC]\) on the right of \(B\) for \(\sigma > \sigma_{H_3}\). Since \(0 < S < 1\) (see Lemma 1) for \(\sigma < \sigma_{H_3}\), the half line \(\Delta\) starts on the left-side of \((AB)\), crosses \((AB)\) above \(B\) and then crosses \((AC)\) above \(C\). For \(\sigma_{H_3} < \sigma < \sigma_F\), \(\Delta\) also starts on the left-side of \((AB)\), but crosses \((AB)\) below \(B\), the segment \([BC]\), and \((AC)\) above \(C\). For \(\sigma_F < \sigma < \sigma_{H_2}\), \((T_1(\sigma), D_1(\sigma))\) is inside \((ABC)\), and \(\Delta\) crosses \([BC]\) and \((AC)\) above \(C\). For \(\sigma > \sigma_{H_2}\) again \((T_1(\sigma), D_1(\sigma))\) is inside \((ABC)\), and \(\Delta\) crosses first line \(AC\) and then \(BC\) to the right of \(C\).

All these results are summarized in Table 3.

**The case where** \(a_K < 0\)

From Lemma (1) we have \(0 < S < 1\) iff \(\sigma \in (s, \sigma_T)\), \(S = 1\) iff \(\sigma = \sigma_T\) and \(1 < S < 1 - \theta a_K\) iff \(\sigma \in (\sigma_T, +\infty)\), with \(\lim_{\sigma \to +\infty} S > 1\). Since \(a_K < 1\), \(\partial S/\partial \sigma > 0\), i.e., as \(\sigma\) decreases from \(+\infty\) to \(s\), the half line \(\Delta\) becomes less steep (see (10)) and its initial point \((T_1, D_1)\) moves upwards along the half line \(\Delta_1\). Also, as \(a_K < 0\) using (11) and (12) it is easy to show that the half line \(\Delta_1\) starts (for \(\sigma = +\infty\) below \(BC\) to the right of \(AC\) and above \(AB\). Below we present our results considering different intervals for the parameter \(\chi\), that we call configurations, and that correspond to different intervals for \(S_1\), the slope of the half-line \(\Delta_1\).

For \(-\theta a_K < \chi < \chi^d\) we have \(S_1 > 1\). Therefore, the half-line \(\Delta_1\) points upwards with a slope \(S_1\) strictly greater than 1. Also, for \(-\theta a_K < \chi < \chi_2(< \chi^d)\) the half line \(\Delta_1\) crosses line \(AC\) above \(BC\), i.e. it crosses first the segment \(BC\) on the right of \(C\) for \(\sigma = \sigma_{H_1}\), and then line \(AC\) for \(\sigma = \sigma_a\). So for \(-\theta a_K < \chi < \chi_2\), since \(s < \sigma_T < \sigma_a < \sigma_{H_1} < +\infty\), \(^7\) for \(s < \sigma < \sigma_T\) the half-line \(\Delta\) that starts on the half line \(\Delta_1\) above \(BC\) points to the right with a slope \(0 < S < 1\) (see (1)) crossing line \(AC\). Therefore the steady state is first a source and then a saddle. For \(\sigma_T < \sigma < \sigma_a\) the half-line \(\Delta\) that still starts on the half line \(\Delta_1\) above \(BC\), points to the right with a slope higher than 1. Therefore it does not cross line \(AC\) and the steady state is always a source. For \(\sigma > \sigma_a\) the half-line \(\Delta\) starts on the half line \(\Delta_1\) above \(BC\) points to the right with a slope higher than 1, crossing \(AC\). Therefore the steady state is first a saddle and then a source.

For \(\chi_2 < \chi < \chi^c\), since for \(\chi > \chi_2\) the half line \(\Delta_1\) crosses line \(AC\) below line \(BC\) (i.e., as \(\sigma\) decreases from \(+\infty\) to \(s\), it crosses first the line \(AC\) for \(\sigma = \sigma_a\) and then the line \(BC\) when \(\sigma = \sigma_{H_1}\)), and since \(\sigma_T < \sigma_{H_1}\)

\(^7\)Note that for \(a_K < 0\) we have \(\sigma_a > \sigma_T\).
for $\chi < \chi^e$, we have $s < \sigma_T < \sigma_{H_1} < \sigma_a < +\infty$. Then, for $s < \sigma < \sigma_T$ the half-line $\Delta$ that starts on the half line $\Delta_1$ above $BC$ points to the right with a slope $0 < S < 1$ (see (1)) crossing line $AC$. Therefore the steady state is first a source and then a saddle. For $\sigma_T < \sigma < \sigma_{H_1}$ the half-line $\Delta$, that still starts on the half line $\Delta_1$ above $BC$, points to the right with a slope higher than 1 so that the steady state is always a source. For $\sigma_{H_1} < \sigma < \sigma_a$, $(T_1(\sigma), D_1(\sigma))$ is inside $(ABC)$ and $\Delta$ points to the right with a slope higher than 1, crossing $[BC]$ but not $AC$. Therefore the steady state is first a sink and then a source. For $\sigma > \sigma_a$ $(T_1(\sigma), D_1(\sigma))$ is in the saddle region below $AC$ and $BC$ and above $AB$, and $\Delta$ still points to the right with a slope higher than 1. For $\sigma_a < \sigma < \sigma_{H_2}$, the half line $\Delta$ crosses first $AC$ and then segment $BC$ in its interior. Therefore the steady state is first a saddle, then a sink and then a source. For $\sigma > \sigma_{H_2}$, $\Delta$ crosses first $BC$ to the right of $B$ and then $AC$ above point $C$. Therefore the steady state is first a saddle and then a source.

For $\chi^e < \chi < \chi^d$, since $\chi > \chi_2$, $\sigma_{H_1} < \sigma_a$ (the half line $\Delta_1$ crosses $AC$ below $BC$) and $\sigma_T > \sigma_{H_1}$ since $\chi > \chi^e$. Therefore we have $s < \sigma_{H_1} < \sigma_T < \sigma_a < +\infty$. Then, for $s < \sigma < \sigma_{H_1}$ the half-line $\Delta$ that starts on the half line $\Delta_1$ above $BC$ points to the right with a slope $0 < S < 1$, crossing line $AC$. Therefore the steady state is first a source and then a saddle. For $\sigma_{H_1} < \sigma < \sigma_T$, $(T_1(\sigma), D_1(\sigma))$ is inside $(ABC)$ and $\Delta$ points to the right with a slope higher than 1, crossing only $[BC]$. Therefore the steady state is first a sink and then a source, and then a saddle. For $\sigma_T < \sigma < \sigma_a$, $(T_1(\sigma), D_1(\sigma))$ is still inside $(ABC)$, but $\Delta$ now points to the right with a slope higher than 1, crossing only $[BC]$. Therefore the steady state is first a sink and then a source. For $\sigma > \sigma_a$ $(T_1(\sigma), D_1(\sigma))$ is in the saddle region below $AC$ and $BC$ and above $AB$, and $\Delta$ still points to the right with a slope higher than 1. For $\sigma_a < \sigma < \sigma_{H_2}$, the half line $\Delta$ crosses first $AC$ and then segment $BC$ in its interior. Therefore the steady state is first a saddle, then a sink and then a source. For $\sigma > \sigma_{H_2}$, $\Delta$ crosses first $BC$ to the right of $B$ and then $AC$ above point $C$. Therefore the steady state is first a saddle and then a source.

For $\chi^d < \chi < \chi^b$ we have $-\infty < S_1 < -1$. Since $\chi > \chi_2$ as $\sigma$ decreases from $+\infty$ to $s$, the half line $\Delta_1$ crosses first the line $AC$ for $\sigma = \sigma_a$ and then the segment $BC$, in its interior, for $\sigma = \sigma_{H_1}$. Also $\sigma_T > \sigma_{H_1}$ since $\chi > \chi^e$ so that, as in the previous case, $s < \sigma_{H_1} < \sigma_T < \sigma_a < +\infty$. Then, for $s < \sigma < \sigma_{H_1}$ the half-line $\Delta$ that starts on the half line $\Delta_1$ above $BC$ points to the right with a slope $0 < S < 1$, crossing line $AC$. So the steady state is first a source and then a saddle. For $\sigma_{H_1} < \sigma < \sigma_T$, $(T_1(\sigma), D_1(\sigma))$

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We can prove that in this case $\epsilon_{\gamma_T} > \epsilon_{\gamma_H}$.
is inside \((ABC)\) and \(\Delta\) points to the right with a slope \(0 < S < 1\), crossing first \([BC]\) and then \(AC\) above point \(C\).\footnote{We can prove that in this case \(\epsilon_T > \epsilon_M\).} Therefore the steady state is first a sink, then a source, and then a saddle. For \(\sigma_T < \sigma < \sigma_a\), \((T_1(\sigma), D_1(\sigma))\) is still inside \((ABC)\), but \(\Delta\) now points to the right with a slope higher than 1, crossing only \([BC]\). Therefore the steady state is first a sink and then a source. For \(\sigma > \sigma_a\), \((T_1(\sigma), D_1(\sigma))\) is in the saddle region below \(AC\) and \(BC\) and above \(AB\) and \(\Delta\) still points to the right with a slope higher than 1. For \(\sigma_a < \sigma < \sigma_{H_2}\), the half line \(\Delta\) still crosses first \(AC\) and then segment \(BC\) in its interior. Therefore the steady state is first a saddle, then a sink and then a source. For \(\sigma > \sigma_{H_2}\), \(\Delta\) crosses first \(BC\) to the right of \(B\) and then \(AC\) above point \(C\). Therefore the steady state is first a saddle and then a source.

For \(\chi^b < \chi < \chi^c\) we have \(S_1 \in (-1, S_B)\). Therefore, in this configuration, the half-line \(\Delta_1\) that starts (for \(\sigma = +\infty\)) at the right of \(AC\), below \(BC\) and above \(AB\), points upwards to the left, crossing first line \(AC\) for \(\sigma = \sigma_a\), then the segment \(BC\), in its interior, for \(\sigma = \sigma_{H_1}\) and finally line \((AB)\) above point \(B\) for \(\sigma = \sigma_F\). See (19). When \(\sigma < \sigma_F\), \((T_1(\sigma), D_1(\sigma))\) is below line \((AB)\) and above \(B\). Since the half-line \(\Delta\) points upwards with a slope \(0 < S < 1\), it does not cross \([BC]\), but crosses \((AB)\) before crossing \((AC)\). Therefore the steady state is first a saddle, then a source, and then again a saddle. When \(\sigma_F \leq \sigma \leq \sigma_{H_1}\), \((T_1(\sigma), D_1(\sigma))\) is above \((AB)\) and \([BC]\). Then \(\Delta\) only crosses \((AC)\). When \(\sigma > \sigma_{H_1}\) the half-line \(\Delta\) also crosses \(BC\). For \(\sigma_{H_1} < \sigma < \sigma_T\), \((T_1(\sigma), D_1(\sigma))\) is inside \((ABC)\), and \(\Delta\) still points to the right with a slope \(0 < S < 1\), crossing first \([BC]\) and then \(AC\) above point \(C\). Therefore the steady state is first a sink, then a source, and then again a saddle. For \(\sigma_T < \sigma < \sigma_a\) again \((T_1(\sigma), D_1(\sigma))\) is inside \((ABC)\), but \(\Delta\) now points to the right with a slope higher than 1, crossing only \([BC]\). Therefore the steady state is first a saddle, then a sink and then a source. For \(\sigma_a < \sigma < \sigma_{H_2}\) again \(T_1(\sigma), D_1(\sigma)\) is in the saddle region below \(AC\) and \(BC\) and above \(AB\) and \(\Delta\) still points to the right with a slope higher than 1, crossing \(AC\) and then segment \(BC\) in its interior. Therefore the steady state is first a saddle, then a sink and then a source. For \(\sigma > \sigma_{H_2}\) again \(T_1(\sigma), D_1(\sigma)\) is in the saddle region below \(AC\) and \(BC\) and above \(AB\) and \(\Delta\) still points to the right with a slope higher than 1. However now \(\Delta\) crosses first \(BC\) to the right of \(B\) and then \(AC\) above point \(C\). Therefore the steady state is first a saddle and then a source.

For \(\chi^e < \chi < \infty\) we have \(S_1 \in (S_B, 0)\), with \(S_B \in (-1, 0)\). Therefore, the slope \(S_1\) is negative and greater than \(-1\), and the half line \(\Delta_1\) that starts (for \(\sigma = +\infty\)) in the saddle region below \(AC\) and \(BC\) and above \(AB\), points upwards to the left, crossing first line \(AC\) for \(\sigma = \sigma_a\), then line \((AB)\) below point \(B\) for \(\sigma = \sigma_F\), and finally \(BC\) at the left of point \(B\) for \(\sigma = \sigma_{H_1}\). Indeed
in this configuration we have $s < \sigma_{H_1} < \sigma_F < \sigma_T < \sigma_a < +\infty$. For $\sigma < \sigma_{H_1}$, 

$(T_1(\sigma), D_1(\sigma))$ is below line $(AB)$ and above line $BC$. Since the half-line $\Delta$ points upwards with a slope $0 < S < 1$, it crosses $(AB)$ and then $(AC)$. When $\sigma > \sigma_{H_1}$ the half-line $\Delta$ also crosses $BC$. For $\sigma_{H_1} \leq \sigma < \sigma_F$, $(T_1(\sigma), D_1(\sigma))$ is below $(AB)$ and $BC$. $\Delta$ still points upwards with a slope $0 < S < 1$, and we can prove that there exists a unique critical value $\sigma_{H_3} \in (\sigma_{H_1}, \sigma_F)$ such that the half-line $\Delta$ goes through point $B$ and crosses $[BC]$ on the right of $B$ for $\sigma > \sigma_{H_3}$. Then, for $\sigma_{H_1} \leq \sigma \leq \sigma_{H_3}$, $\Delta$ crosses first $BC$, then $AB$ and finally $AC$. For $\sigma_{H_3} \leq \sigma \leq \sigma_F$, the half-line $\Delta$ crosses first $AB$, then the segment $[BC]$ in its interior and finally $AC$. For $\sigma_F \leq \sigma \leq \sigma_T$, $(T_1(\sigma), D_1(\sigma))$ is inside $(ABC)$. The half-line $\Delta$ still points upwards with a slope $0 < S < 1$, crossing first the segment $[BC]$ in its interior and then $AC$. For $\sigma_T \leq \sigma \leq \sigma_a$, $(T_1(\sigma), D_1(\sigma))$ is in the saddle region below $AC$ and $BC$ and above $AB$ and $\Delta$ still points to the right with a slope higher than $1$. For $\sigma_a < \sigma < \sigma_{H_2}$ the half-line $\Delta$ crosses first $AC$ and then the segment $BC$ in its interior whereas for $\sigma > \sigma_{H_2}$ the half-line $\Delta$ crosses first $BC$ on the right of $C$ and then $AB$ above point $C$.

All these results are summarized in Table 4.

### 5.3 Existence of $\sigma_{H_2}$

First recall that $\sigma_{H_2}$ is a value of $\sigma$ such that the half-line $\Delta$ line goes through point $C$, i.e., such that $\epsilon_{\gamma_H} = \epsilon_{\gamma_T}$ (see (22) and (24)).

Consider first that $a_K = 0$, i.e., the case without capital income taxation that corresponds to Proposition 3. In this case, we have $S \in (0, 1)$ for $\sigma \in (s, +\infty)$ and $\lim_{\sigma \to +\infty} S = 1$ (see Lemma 1). Using (22) and (24) it is easy to prove that in this case we obtain $\epsilon_{\gamma_H} < \epsilon_{\gamma_T}$ for all $\sigma \in (s, +\infty)$ so that the half line $\Delta$ always cross the line $(AC)$ above point $C$, and $\sigma_{H_2}$ does not exist.

Consider now the case where $a_K > 0$. Using (22) and (24), we have that $\epsilon_{\gamma_T} \leq \epsilon_{\gamma_H} \Leftrightarrow g(\sigma) \leq 0$, where:

$$
g(\sigma) \equiv \chi a_K (\sigma - \sigma_a)[\sigma(1 - \theta a_K) - \theta(1 - \sigma)(1 - a_K)] - (\chi + \theta a_K)(\sigma - \sigma_{H_1})[a_K \sigma + (1 - \sigma)(1 - a_K)]
$$

(18)

Note that $g'(\sigma) = -\theta a_K (1 + \chi)[a_K (2\sigma - s) + 2(1 - s)(1 - a_K)] < 0$ for $\sigma > \sigma^* \equiv [a_K s - 2(1 - s)(1 - a_K)]/2a_K < \sigma_a$.

When $a_k > 1$ we have $s < \sigma_{H_1} < \sigma_a < +\infty$ and, under Assumption 1, $g(\sigma_{H_1}) < 0$, $g(\sigma_a) < 0$ and $g(+\infty) = -\infty$. As $g'(\sigma) < 0$ for $\sigma > \sigma^* < \sigma_a$ we
conclude that for $a_k > 1$ and $\sigma > \sigma_a$ we always have $\epsilon_{\gamma T} < \epsilon_{\gamma H}$, so that $\sigma_{H_2}$ does not exist.

When $0 < a_K < 1$, for $0 < \chi < \chi_2$, as before $s < \sigma_{H_1} < \sigma_a < +\infty$ and, under Assumption 1, $g(\sigma_{H_1}) < 0$, $g(\sigma_a) < 0$ and $g(\infty) = -\infty$. As $g'(\sigma) < 0$ for any $\sigma > s$, we conclude that for $0 < a_K < 1$ and $0 < \chi < \chi_2$ we always have $\epsilon_{\gamma T} < \epsilon_{\gamma H}$, so that $\sigma_{H_2}$ does not exist. For $\chi_2 < \chi < \chi^a$ we have $\sigma^* < s < \sigma_a < \sigma_{H_1} < +\infty$. As $g'(\sigma) < 0$ for $\sigma > \sigma^*$, $g(\sigma)$ is a decreasing function for $\sigma > s$. As $g(\sigma_a) > g(\sigma_{H_1}) > 0$ and $g(\infty) = -\infty$, we deduce that there is a unique $\sigma_{H_2} \in (\sigma_{H_1}, +\infty)$ such that $g(\sigma_{H_2}) = 0$. By continuity, we have that $\epsilon_{\gamma T} > \epsilon_{\gamma H}$ for $\sigma_{H_1} < \sigma < \sigma_{H_2}$, and $\epsilon_{\gamma T} < \epsilon_{\gamma H}$ for $\sigma_{H_2} < \sigma < +\infty$. For $\chi > \chi^a$ we have $\sigma^* < \sigma_a < s < \sigma_{H_1} < +\infty$. As $g(\sigma_{H_1}) > 0$ and $g(\infty) = -\infty$ again there is a unique $\sigma_{H_2} \in (\sigma_{H_1}, +\infty)$ such that $g(\sigma_{H_2}) = 0$. By continuity, $\epsilon_{\gamma T} > \epsilon_{\gamma H}$ for $\sigma_{H_1} < \sigma < \sigma_{H_2}$, and $\epsilon_{\gamma T} < \epsilon_{\gamma H}$ for $\sigma_{H_2} < \sigma < +\infty$.

Finally, consider that $a_K < 0$. In this case as $\sigma_T$ exists, using 20, we can rewrite $g(\sigma)$ in the following way:

$$g(\sigma) \equiv a_K \{\chi(\sigma - \sigma_a)[\sigma(1 - \theta a_K) - \theta(1 - s)(1 - a_K)] - (\chi + \theta a_K)(\sigma - \sigma_{H_1})(\sigma - \sigma_T)\}$$

As before $g'(\sigma) < 0$ for $\sigma > \sigma^*$ where $\sigma_T < \sigma^* < \sigma_a$.

For $\sigma < \sigma_T$, $\epsilon_{\gamma H} < \epsilon_{\gamma T} \iff g(\sigma) \geq 0$. So for $\chi^e < \chi < \chi^b$ where $s < \sigma_{H_1} < \sigma_T < \sigma_a$, under Assumption 1, $g(s) > 0$, $g(\sigma_{H_1}) > 0$, and $g(\sigma_T) > 0$. As $g'(\sigma) > 0$ for $s < \sigma < \sigma^* (\sigma_T)$ we conclude that in this case we always have $\epsilon_{\gamma T} > \epsilon_{\gamma H}$, so that $\sigma_{H_2}$ does not exist.

For $\sigma > \sigma_a^b$ and $\chi > \chi_2$, $\epsilon_{\gamma H} < \epsilon_{\gamma T} \iff g(\sigma) \leq 0$. As $g'(\sigma) < 0$ for $\sigma > \sigma^*$, $g(\sigma)$ is a decreasing function for $\sigma > \sigma_a$. Also as, under Assumption 1, $g(\sigma_a) > 0$ and $g(\infty) = -\infty$ we deduce that there is a unique $\sigma_{H_2} \in (\sigma_a, +\infty)$ such that $g(\sigma_{H_2}) = 0$. By continuity, we have that $\epsilon_{\gamma H} > \epsilon_{\gamma T}$ for $\sigma_a < \sigma < \sigma_{H_2}$, and $\epsilon_{\gamma H} < \epsilon_{\gamma T}$ for $\sigma_{H_2} < \sigma < +\infty$.

### 5.4 Existence of $\sigma_{H_3}$

Using (22) and (23), we have that $\epsilon_{\gamma H} \geq \epsilon_{\gamma F} \iff h(\sigma) \geq 0$, where:

$$h(\sigma) = [(1 + \chi)(1 - a_L) - (1 - \theta a_K)](\sigma - \sigma_{H_1})[\sigma(2 - \theta a_K) - \theta(1 - s)(1 - a_K)](h) + (2 - \theta a_K)[1 + (1 + \chi)(1 - a_L)](\sigma - \sigma_F)[\sigma(1 - \theta a_K) - \theta(1 - s)(1 - a_K)]$$

By definition, $\sigma_{H_3}$ is a value of $\sigma$ such that $\epsilon_{\gamma H} = \epsilon_{\gamma F}$; therefore it must be a solution of $h(\sigma) = 0$. Since $h(\sigma)$ is a polynomial of degree 2, the equation $h(\sigma) = 0$ has at most two solutions. We limit our analysis to the case

\[\text{Note that, for } a_K < 0, \epsilon_{\gamma T} > 1 \text{ only exists for } \sigma < \sigma_T \text{ and } \sigma > \sigma_a.\]
where \( c < \chi < \infty \) since \( \sigma_{H_3} \) is only relevant under this configuration. The polynomial \( h(\sigma) \) is a convex function of \( \sigma \) since the coefficient of the quadratic term \( \sigma^2 \) is positive.\(^{11}\) We can see geometrically that if there is a \( \sigma_{H_3} > s \) then it must satisfy \( s < \sigma_{H_1} < \sigma_{H_3} < \sigma_F \). Using (??), and Assumptions ??, we see that in this configuration \( h(\sigma_F) > 0 \) and \( h(\sigma_{H_1}) < 0 \). Therefore there is a unique \( \sigma_{H_3} \in (\sigma_{H_1}, \sigma_F) \) such that \( h(\sigma_{H_3}) = 0 \). By continuity, we have that \( \varepsilon_{\gamma_H} > \varepsilon_{\gamma_F} \) for \( \sigma_F > \sigma > \sigma_{H_3} \), and \( \varepsilon_{\gamma_H} < \varepsilon_{\gamma_F} \) for \( \sigma_{H_1} < \sigma < \sigma_{H_3} \).

5.5 Critical values of the parameters

5.5.1 Critical values of \( \chi \)

\( \chi^a \) is the critical value of \( \chi \) such that \( S_1 = 1 \) and its expression is given in Proposition 2 for the no taxation case, in Proposition 3 for the labor income taxation case and in Proposition 3 for the capital income taxation case.

\( \chi^b \) is the critical value of \( \chi \) such that \( S_1 = -1 \) and its expression is given in Proposition 2 for the no taxation case, in Proposition 3 for the labor income taxation case and in Proposition 3 for the capital income taxation case.

\( \chi^c \) is the critical value of \( \chi \) such that \( S_1 = S_B \) and its expression is given in Proposition 2 for the no taxation case, in Proposition 3 for the labor income taxation case and in Proposition 3 for the capital income taxation case.

\( \chi^d \) is the critical value of \( \chi \) such that \( S_1 = \infty \) and is given by:

\[
\chi^d = \frac{s - \theta(1 - s)(1 - a_K)}{\theta(1 - s)(1 - a_K)}
\]

\( \chi^e \) is the critical value of \( \chi \) such that \( \sigma_T = \sigma_{H_1} \) and its expression is given in Proposition 3 for the capital income taxation case.

5.5.2 Definitions and expressions for critical values of \( \sigma \)

\( \sigma_{H_1} \) is the critical value of \( \sigma \) such that \( D_1(\sigma_{H_1}) = 1 \) and is given by:

\[
\sigma_{H_1} \equiv \frac{s (1 + \chi) (1 - a_L) - \theta(1 - s)(1 - a_K)}{(1 + \chi) (1 - a_L) - (1 - \theta a_K)}
\]

\( \sigma_{H_2} \) is a critical value of \( \sigma \) such that the half-line \( \Delta \) goes through the point \( (T, D) = (2, 1) \), i.e., goes through point \( C \). Note that \( \varepsilon_{\gamma_T} = \varepsilon_{\gamma_H} \) for \( \sigma = \sigma_{H_2} \). When \( \phi_k = 0 \) or when \( \alpha_k = 0 \), \( \sigma_{H_2} \) does not exist, i.e. the half-line \( \Delta \) does not cross point \( C \) and always crosses the line \( AC \) above point \( C \).

\(^{11}\)Indeed, this coefficient is given by \( c \equiv (2 - \theta a_K)^2 (1 + \chi) (1 - a_L) \) which is positive under Assumptions 1.
\( \sigma_{H_3} \) is the critical value of \( \sigma \) such that the half line \( \Delta \) goes through the point \( (T, D) = (-2, 1) \), i.e., goes through point \( B \). Note that \( \varepsilon_{\gamma_F} = \varepsilon_{\gamma_H} \) for \( \sigma = \sigma_{H_3} \).

Without capital taxation, i.e., when \( k = 0 \) or when \( \alpha_k = 0 \), we have

\[
\sigma_{H_3} = \frac{s + \theta(1 - s) + \sqrt{s [s - \theta(1 - s)]}}{2}
\]

The critical value \( \sigma_F \) is defined by

\[
1 + D_1(\sigma_F) + T_1(\sigma_F) = 0.
\]

\[
\sigma_F \equiv \frac{s(2 - \theta a_K)(1 + \chi)(1 - a_L) + \theta(1 - s)(1 - a_K)[1 + (1 + \chi)(1 - a_L)]}{(2 - \theta a_K)[1 + (1 + \chi)(1 - a_L)]}
\]

(19)

\( \sigma_T \) is the value of \( \sigma \) for which \( S = 1 \).

\[
\sigma_T \equiv -\frac{(1 - s)(1 - a_K)}{a_K}
\]

(20)

\( \sigma_a \) is the value of \( \sigma \) for which the \( \Delta_1 \) line crosses the line \( AC \).

\[
\sigma_a \equiv \frac{sa_K - (1 - s - a_K)\chi}{a_K\chi}
\]

(21)

\( \sigma_{SS_1} \) is the value of \( \sigma \) for which \( S = S_1 \).

\[
\sigma_{SS_1} \equiv \frac{s - \theta(1 - s)(1 + \chi)(1 - a_K)}{1 - (1 + \chi)(1 - \theta a_K)}
\]

**5.5.3 Expressions for critical values of \( \varepsilon_\gamma \)**

\( \varepsilon_{\gamma_H} \) is such that \( D = 1 \), which is equivalent to:

\[
\varepsilon_{\gamma_H} = \frac{(1 + \chi)(1 - a_L)(\sigma - s)}{\sigma[1 - \theta a_K] - \theta (1 - s)(1 - a_K)}
\]

(22)

\( \varepsilon_{\gamma_F} \) is such that \( 1 + T + D = 0 \). After some computations, we obtain:

\[
\varepsilon_{\gamma_F} = \frac{(1 + \chi)(1 - a_L)[2s + \theta(1 - s - a_K) - \sigma(2 - \theta a_K)]}{\sigma(2 - \theta a_K) - \theta(1 - s)(1 - a_K)}
\]

(23)

\( \varepsilon_{\gamma_T} \) is such that \( 1 - T + D = 0 \). After some computations, we obtain:

\[
\varepsilon_{\gamma_T} = \frac{(1 + \chi)(1 - a_L)[a_K\sigma + (1 - s - a_K)]}{a_K\sigma + (1 - s)(1 - a_K)}
\]

(24)

\textsuperscript{12}In Appendix 6.5 we show conditions for its existence and uniqueness.
References


