Contingent Bonds in Public Debt Management* 

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Preliminary and Incomplete Version 

Abstract 

This paper extends the theory of optimal debt management with state-contingent bonds, which was first developed by Lucas and Stokey (1983), to the case where the government and private agents have different expectations about the probabilities of future realization of government consumption. Deviated from the majority of previous research efforts which take for granted that agents make no systematic mistakes in forming their expectations, we assume agents do not fully trust their government and thus form different expectations from those of the government with respect to the probabilities of either high or low government consumption in the future. We are therefore able to explain the absence of state-contingent bonds in the practice of public debt management even though such bonds 

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have desirable welfare improving features. Our results also shed light on
the famous Equity Premium Puzzle.
1 Introduction [Incomplete]

First of all, this line of research is within neoclassical framework, so the existence of distortionary taxation creates a role for public debt management in the sense that government tries to implement debt plans in order to support the least distortionary tax scheme. “In a dynamic stochastic environment, the theory of optimal taxation is also a theory of optimal debt management.” [Missale, 1999] In the presence of uncertainty about future states of nature, welfare-maximizing policies are such that a joint state-contingent path of tax rates on labour income and public debt returns are determined. In another word, the issuance of state-contingent bonds is designed so as to enable the government to carry out less distortionary tax policies in the face of different economic situations.

The contingency of bonds comes in either explicit or implicit ways. In the case of explicit contingency, debt returns are positively indexed to output, and negatively indexed to government spending. Particularly, when the national output is low or when the government spending is high, such bonds allow the issuer to pay a lower return to the debtors; whereas when the national output is high or when the government spending is low, issuers pay a higher return to the debtors. This risk-return character of contingent bonds is quite favourable to the government if it issues such public bonds, since it reduces the pressure on government spending when the economic situation is bad by partly shifting the debt liabilities to a better economic environment. As a result, contingent public bonds enable the government to realize a smoother path of taxation
across different states of the economy and thus to ensure a smoother path of consumption for the private agents.

Although attractive in their nature, bonds with explicit contingency are seldom issued in practice. Implicitly contingent bonds are more common in reality. Bonds indexed to price level, exchange rates or interest rates of different maturities provide substitutes for explicitly contingent bonds. However, the reason for the absence of bonds with explicit contingency is still not definite. Scholars have attributed such an absence to market imperfections or a moral hazard problem [Missale, 1999].

The research on bonds with explicit contingency was initiated by Lucas and Stokey (1983) where they set up a neoclassical model with government spending shocks as the only exogenous variable and capital excluded from their consideration. They were the first to show the contributive role of negative indexation of debt returns to government spending shocks in implementing tax-smoothing policies. King (1990) and Chari et al. (1994) furthered the research by including capital and productivity shocks in the model and demonstrated that debt à la Lucas and Stokey (1983) is still optimal.

Again, if explicit contingency of bonds is so desirable in optimal policy designs, one would keep asking why such bonds are rarely used by the government, which is also the major motivation of our paper. Scholars have tried to address the issue from different points of view and we were inspired by an argument forwarded by Alessandro Missale in his book [Missale, 1999]. Namely, we will
exploit the trade-off between minimization of debt serving cost and minimization of budgetary cost in our explanation of the problem.

As many other scholars, we realise that the intuitions so far rest on unrealistic assumptions and this might also account for the limitation of the model in explaining the reality. In particular, Lucas and Stokey (1983) assumed that: (1) the government and private agents share the same information of the markets; (2) private agents make no systematic mistakes in forming their expectations; (3) the government can credibly commit to future policy actions; (4) markets are efficient. [Missale, 1999]

In our preliminary attempt, we will relax the second assumption by arguing that private agents might not fully trust their government and hence form different expectations from the ones held by the government. Our argument differs from that of Caprioli (2008), who also believed that private agents possess different expectations from those of the government, in the sense that Caprioli imposed different expectations in future tax rates while we impose different expectations in the realization probabilities of different future states of the economy. Other attempts include the relaxation of the fourth assumption by the scholars promoting the incomplete market argument [Aiyagari et al. 2002] and the relaxation of the first assumption by Sleet (2004) arguing that government might hold private information about its spending needs.

The initial model by Lucas and Stokey (1983) only gives qualitative results on optimal debt management. As an extension of their work, we are able to
demonstrate their results quantitatively. Similar to what Chari et al. (1994) have found, our model à la Lucas and Stokey (1983) confirms the welfare improving features of state-contingent bonds even in a simple two-period model, in the sense that by trading contingent bonds government manages to keep tax rates on labour income constant across time and states of nature in a general equilibrium model. But such desirable features of contingent bonds disappear as soon as we introduce some friction to the general equilibrium model.

A more important finding of this paper is that with the introduction of some friction into the general equilibrium model, it becomes more and more costly for the government to exploit the welfare improving features of contingent bonds as the discrepancy between the different expectations by agents and government about the probabilities of future realization of government consumption increases. This is a very good explanation for the absence of contingent bonds in the practice of public debt management.

Our work also sheds light on the Equity Premium Puzzle. Mehra and Prescott (1985) argued that equity premium derived from general equilibrium model is far smaller than what one would observe from empirical evidence and they suggested that “...some equilibrium model with a friction will be the one that successfully accounts for the large equity premium.” The results of our model confirms their argument specifically when the agents trust their government less and become more pessimistic about future states of the economy.

This paper proceeds in the following manner. Section 2 gives the formuliza-
tion of the general equilibrium model à la Lucas and Stokey (1983) which is a base model where private agents and the government form the same expectations about the realization probabilities of different future states of the economy. Section 3 introduces our equilibrium model with a friction where private agents do not trust the government and possess different expectations about realization probabilities of good or bad future economic situations. Section 4 displays the numerical results from calibration and our major findings. Section 5 concludes.

2 Model à la Lucas and Stokey\(^1\) (1983)

2.1 The Economy

Consider a production economy with no capital and one produced good. The labour input of private agents turns, with a constant returns to scale, into this production output, which is the source of consumption by both the government and private agents. In addition, the government consumption of this good follows a stochastic process, which is the only exogenous policy variable in our model. Government imposes a distortionary income tax on agents' labour supply and balances its budgets by collecting this tax revenue and issuing public bonds.

In our scenario, agents trade in the financial market for bonds which closes before the government makes its decisions. And in the fashion of Arrow-Debreu

\(^1\)The model by Lucas and Stokey (1983) assumes an infinite horizon for the economy and gives only qualitative results, whereas our quantitative demonstration rests on a simple but intuitional two-period version of the original model.
model with complete market under the general equilibrium framework, agents have perfect forecast [rational expectations] about government’s plans and government takes into account agents’ decisions when making its own plans. For this simple two-period model, the intertemporal planning involves only one uncertainty, which is government consumption that takes a either high or low realization value as the two different future states of nature.

Following the intuition of optimal debt management given by Lucas and Stokey (1983), we assume a complete market where government has two debt instruments, \( b^H_t \) and \( b^L_t \), in its debt portfolio to keep distortionary taxes on labour income relatively constant for welfare improving purposes while still satisfy its budget constraints. The two debt instruments are contingent on the two future states of nature, and both with one-period maturity. In a two-period model, in particular, government borrows, at current period \( t \), from agents by taking a short position in government bond \( b^L_t \), which pays one unit of consumption good only if the realization of government consumption for the next period \( t+1 \) is low. At the same time, government also takes a long position in private debt \( b^H_t \), which receives one unit of consumption good only if the realization of government consumption for the next period \( t+1 \) is high.

2.2 Private Agents

The objective function for agents’ optimization problem is:

\[
U(c_t, l_t) + \beta \sum_{i=H,L} \mu_{t+1} U(c^i_{t+1}, l^i_{t+1}),
\]

(1)
which is an intertemporal planning with uncertainty about the realization of the future states of nature, where \( t \) and \( t + 1 \) represent the two periods in our current consideration and \( i \) stands for the different states of nature in the future, with \( \mu^i_{t+1} \) denoting the realization probabilities of each state of nature where \( \mu^H_{t+1} + \mu^L_{t+1} = 1 \), and \( H \) and \( L \) representing the high and low level of government consumption in the second period respectively. As usual, \( c \) and \( l \) are agents’ consumption and labour supply, and \( \beta \) is our discount factor in this intertemporal planning.

Agents maximize their intertemporal utilities on account of the following budget constraints:

\[
\begin{align*}
c_t + p_t^H b_t^H - p_t^L b_t^L &= (1 - \tau_t) l_t + b_t, \\
c_{t+1}^H + b_{t+1}^H &= (1 - \tau_{t+1}^H) l_{t+1}^H,
\end{align*}
\]

(2)

(3)

and

\[
\begin{align*}
c_{t+1}^L - b_{t+1}^L &= (1 - \tau_{t+1}^L) l_{t+1}^L,
\end{align*}
\]

(4)

where \( b_t \) denotes the debt outstanding in the first period while \( p_t^H \) and \( p_t^L \) represent the par value of the two debt instruments \( b_t^H \) and \( b_t^L \) respectively. As usual, \( \tau \) is the tax rate on personal income.

We derive the equilibrium price system as the following:\(^2\):

\[
\tau_t = 1 + \frac{U_{b_t}}{U_{c_t}}
\]

(5)

\(^2\)For the Lagrangean and the first order conditions of agents’ optimization problem, please refer to the appendix 2.2.
$\tau_{t+1}^H = 1 + \frac{U_{t+1}^H}{U_{ct+1}^H}$ \hspace{2cm} (6)

$\tau_{t+1}^L = 1 + \frac{U_{t+1}^L}{U_{ct+1}^L}$ \hspace{2cm} (7)

$p_t^H = \frac{\beta \mu_{t+1}^H U_{ct+1}^H}{U_{ct}}$ \hspace{2cm} (8)

$p_t^L = \frac{\beta \mu_{t+1}^L U_{ct+1}^L}{U_{ct}}$ \hspace{2cm} (9)

2.3 Government

On account of agents’ decisions of their consumption and labour supply, as well as the equilibrium price system, the benevolent government maximizes agents’ intertemporal utilities, subject to its own budget constraints:

$c_t + g_t \leq l_t$ \hspace{2cm} (10)

$c_{t+1}^H + g_{t+1}^H \leq \tau_{t+1}^H$ \hspace{2cm} (11)

$c_{t+1}^L + g_{t+1}^L \leq \tau_{t+1}^L$ \hspace{2cm} (12)

where $g$ denotes government consumption, and the implementability constraint$^3$:

$c_t U_{ct} + l_t U_{lt} + \beta \sum_{i=H,L} \mu_{t+1}^i \left( c_{t+1}^i U_{ct+1}^i + l_{t+1}^i U_{lt+1}^i \right) = b_t U_{ct}.$ \hspace{2cm} (13)

As one can anticipate, government’s optimization problem reduces to the following system of equations$^4$:

$U_{ct} + U_{lt} - \lambda (U_{ct} + c_t U_{ct} - b_t U_{ct} + l_t U_{lt}) = 0$ \hspace{2cm} (14)

$^3$The derivation of the implementability constrain is included in the appendix 2.3.

$^4$Please refer to the appendix 2.3 for the Lagrangean and the first order conditions of government’s optimization problem.
\[ U_{c_{t+1}} + U_{l_{t+1}} - \lambda \left( U_{c_{t+1}^H} + c_{t+1}^H U_{c_{t+1}^H} + U_{l_{t+1}^H} + l_{t+1}^H U_{l_{t+1}^H} \right) = 0 \]  \hspace{1cm} (15) \\
\[ U_{c_{t+1}} + U_{l_{t+1}} - \lambda \left( U_{c_{t+1}^L} + c_{t+1}^L U_{c_{t+1}^L} + U_{l_{t+1}^L} + l_{t+1}^L U_{l_{t+1}^L} \right) = 0. \]  \hspace{1cm} (16)

that, together with the implementability constraint and the government’s budget constraints when they are binding, provides the general equilibrium solutions to our model.

### 2.4 The Equilibrium

To simplify calculation, we assume that there is no outstanding debt in the first period:

\[ b_t = 0. \]  \hspace{1cm} (17)

In the case of the utility function that we assumed:

\[ U(c, l) = \log c - \frac{1}{2} (l)^2, \]  \hspace{1cm} (18)

the system of equations for the solutions of our model further reduces to\(^5\):

\[ \frac{1}{c_t} - l_t + 2\lambda l_t = 0 \]  \hspace{1cm} (19)
\[ \frac{1}{c_{t+1}^H} - l_{t+1}^H + 2\lambda l_{t+1}^H = 0 \]  \hspace{1cm} (20)
\[ \frac{1}{c_{t+1}^L} - l_{t+1}^L + 2\lambda l_{t+1}^L = 0 \]  \hspace{1cm} (21)
\[ 1 - (l_t)^2 + \beta \mu_{t+1}^H \left[ 1 - (l_{t+1}^H)^2 \right] + \beta \mu_{t+1}^L \left[ 1 - (l_{t+1}^L)^2 \right] = 0 \]  \hspace{1cm} (22)
\[ c_t + g_t = l_t \]  \hspace{1cm} (23)

\(^5\)For more details, please refer to the appendix 2.4.
We assume that the realization probabilities of future states of nature and government consumption are exogenous variables, so the solution of the model would be in the form of:

\[
\begin{align*}
    c_t^{H} + g_t^{H} &= i_t^{H} \\
    c_t^{L} + g_t^{L} &= i_t^{L}
\end{align*}
\]

(24)

(25)

3 Model where Agents and Government have Different Expectations

3.1 Beyond the Ideal Economy

A deviation from the standard general equilibrium model with perfect forecast [rational expectations] in our consideration is to assume that agents have incorrect anticipation about the realization probabilities of future states of nature, while government consumption in the different future states of nature are given as known. Such an assumption enables us to incorporate the standard general equilibrium model and our model with a friction into one system of equations which facilitates further interpretations.

When agents have incorrect forecast of the realization probabilities, we assume they believe the probabilities of high and low government consumption
in the second period are different from the true realization probabilities that government holds. The reason why private agents form different expectations from those of the government might be that agents believe that the government is not perfectly honest in its announcement and choose not to fully trust their government. More precisely, agents think the government has a strong tendency to hold back the truth about high government consumption in the future, which is considered a bad state of economy, from the public. Hence, the agents decide to be more pessimistic than what the government claims about the future state of the economy and to form a higher expectation for the bad state of economy in the future and a lower expectation for the good state of economy in the future than what the government would like them to believe, which in the case of a benevolent government is the truth.

3.2 Agents with Different Expectations

The objective function for agents’ optimization problem now becomes:

\[ U(c_t, l_t) + \beta \sum_{i=H,L} \theta_{t+1}^i U(c_{t+1}^i, l_{t+1}^i), \tag{26} \]

where we assume they believe the probabilities of high and low government consumption in the second period are \( \theta_{t+1}^H \) and \( \theta_{t+1}^L \), respectively, which are different from the true realization probabilities \( \mu_{t+1}^H \) and \( \mu_{t+1}^L \) that government holds. Further, \( \theta_{t+1}^H > \mu_{t+1}^H \) and \( \theta_{t+1}^L < \mu_{t+1}^L \) in the case where we have pessimistic agents.

Agents maximize their intertemporal utilities on account of the same budget constraints as in the previous general equilibrium model, but now we have a
different set of equilibrium price system$^6$:

\[ \tau_t = 1 + \frac{U_t}{U_{c_t}} \]  
\[ \tau_{t+1}^H = 1 + \frac{U_{t+1}^{t+1}}{U_{c_{t+1}^H}} \]  
\[ \tau_{t+1}^L = 1 + \frac{U_{t+1}^{t+1}}{U_{c_{t+1}^L}} \]  
\[ p_t^H = \frac{\beta \theta_{t+1}^H U_{c_{t+1}^H}}{U_{c_t}} \]  
\[ p_t^L = \frac{\beta \theta_{t+1}^L U_{c_{t+1}^L}}{U_{c_t}}. \]  

### 3.3 Benevolent Government

The benevolent government maximizes agents’ intertemporal utilities using the correct forecast of realization probabilities for future states of nature $\mu_{t+1}^H$ and $\mu_{t+1}^L$. The objective function for government’s optimization problem thus remains as:

\[ U(c_t, l_t) + \beta \sum_{i=H,L} \mu_{t+1}^i U(c_{t+1}^i, l_{t+1}^i), \]  

while government also takes into account agents’ decisions of their consumption and labour supply, as well as the equilibrium price system when they have incorrect anticipation, which are implied in the new implementability constraint$^7$:

\[ c_t U_{c_t} + l_t U_{l_t} + \beta \sum_{i=H,L} \theta_{t+1}^i \left( c_{t+1}^i U_{c_{t+1}^i} + l_{t+1}^i U_{l_{t+1}^i} \right) = b_t U_{c_t}. \]  

$^6$For the Lagrangean and the first order conditions of agents’ optimization problem, please refer to the appendix 3.2.

$^7$Please refer to the appendix 3.3 for the derivation of the new implementability constraint.
Government’s optimization problem now reduces to the following system of equations:\(^8\):

\[ U_c + U_l - \lambda (U_{c_t} + c_t U_{c_t} - b_t U_{c_t} + U_{l_t} + l_t U_{l_t}) = 0 \]  

(34)

\[ \mu^H_{t+1} (U_{c_{t+1}} + U_{l_{t+1}}) - \lambda \theta^H_{t+1} (U_{c_{t+1}} + c_t U_{c_t} + U_{l_{t+1}} + l_t U_{l_t}) = 0 \]  

(35)

\[ \mu^L_{t+1} (U_{c_{t+1}} + U_{l_{t+1}}) - \lambda \theta^L_{t+1} (U_{c_{t+1}} + c_t U_{c_t} + U_{l_{t+1}} + l_t U_{l_t}) = 0, \]  

(36)

that, together with the new implementability constraint and the government’s budget constraints when they are binding, provides the equilibrium solutions to our model.

### 3.4 The New Equilibrium

To simplify calculation, again, we assume that there is no outstanding debt in the first period:

\[ b_t = 0. \]  

(37)

Still, in the case of the utility function that we assumed:

\[ U(c, l) = \log c - \frac{1}{2} (l)^2, \]  

(38)

the system of equations further reduces to\(^9\):

\[ \frac{1}{c_t} - l_t + 2 \lambda l_t = 0 \]  

(39)

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\(^8\)For the Lagrangean and the first order conditions, please refer to the appendix 3.3.

\(^9\)For more details, please refer to the appendix 3.4.
Again, we assume that different forecasts about the realization probabilities of future states of nature and government consumption are exogenous variables, so the solution of the model would be in the form of:

\[
\begin{align*}
\mu^H_{t+1} \left( \frac{1}{c_{t+1}^H} - l_{t+1}^H \right) + 2 \lambda \theta^H_{t+1} l_{t+1}^H &= 0 \\
\mu^L_{t+1} \left( \frac{1}{c_{t+1}^L} - l_{t+1}^L \right) + 2 \lambda \theta^L_{t+1} l_{t+1}^L &= 0 \\
1 - (l_t)^2 + \beta \theta^H_{t+1} \left[ 1 - (l_{t+1}^H)^2 \right] + \beta \theta^L_{t+1} \left[ 1 - (l_{t+1}^L)^2 \right] &= 0
\end{align*}
\]

\( c_t + g_t = l_t \)  

\( c^H_{t+1} + g^H_{t+1} = l^H_{t+1} \)  

\( c^L_{t+1} + g^L_{t+1} = l^L_{t+1} \)

4 Simulation Results [Incomplete]

The above systems do not offer analytical solutions, so we turn to calibration with Matlab\(^\text{10}\). In order to make a two-dimensional projection possible, we reduce the number of exogenous variables by assigning values to the discount rate, the government consumption levels in both periods and the true realization probabilities of different states of economy in the second period which are also the correct forecasts by the government. In particular, firstly we fix the discount

\(^\text{10}\) Calibration codes with Matlab are included in the appendix 4.1.
factor at $\beta = 0.95^{11}$; secondly we assume that the government consumption in
the first and second periods are $g_t = 0.2$, $g_{t+1}^H = 0.3$, and $g_{t+1}^L = 0.1$, respectively; thirdly we assume the true probabilities of high and low government
consumption in the second period to be equal, resulting in $\mu_{t+1}^H = \mu_{t+1}^L = 0.5$.

Further, if we want to show how the solutions evolve as the discrepancy
between the different perceptions by the government and the agents about the
realization probabilities of the future states of the economy increases, we fix the
government’s belief which are the true probabilities as mentioned earlier and
assume a growing difference between the agents’ belief and the real probabilities
by some simple relation equations:

$$\theta_{t+1}^H = \mu_{t+1}^H + a$$  \hspace{1cm} (46)

and

$$\theta_{t+1}^L = \mu_{t+1}^L - a$$  \hspace{1cm} (47)

where $0 \leq a \leq 0.5$ is a random number compatible with the range of realization
probabilities that assures the randomness of our exogenous probabilities.

Now our system for calibration becomes:

$$\frac{1}{c_t} - l_t + 2\lambda l_t = 0$$  \hspace{1cm} (48)

$^{11}$For the discount factor, $\beta = \frac{1}{1+r}$, we use the arithmetic mean long-term
interest rate of 16 countries in Euro area, as published in January 2011 at
http://www.ecb.int/stats/money/long/html/index.en.html, which leads to an approximate
beta: $\beta = 0.95$ where $r = 4.94$. 

15
\[
\mu_{t+1}^H \left( \frac{1}{c_{t+1}^H} - i_{t+1}^H \right) + 2\lambda \left( \mu_{t+1}^H + a \right) i_{t+1}^H = 0 
\]
(49)

\[
\mu_{t+1}^L \left( \frac{1}{c_{t+1}^L} - i_{t+1}^L \right) + 2\lambda \left( \mu_{t+1}^L - a \right) i_{t+1}^L = 0 
\]
(50)

\[
1 - (l_t)^2 + \beta \left( \mu_{t+1}^H + a \right) \left[ 1 - (i_{t+1}^H)^2 \right] + \beta \left( \mu_{t+1}^L - a \right) \left[ 1 - (i_{t+1}^L)^2 \right] = 0 
\]
(51)

\[
c_t + g_t = l_t 
\]
(52)

\[
c_{t+1}^H + g_{t+1}^H = i_{t+1}^H 
\]
(53)

\[
c_{t+1}^L + g_{t+1}^L = i_{t+1}^L. 
\]
(54)

The figures we have derived from calibration with Matlab are included in the appendix 4.2. The results illustrate the welfare improving features of state-contingent bonds even in a simple two-period model, in the sense that by trading contingent bonds government manages to keep tax rates on labour income constant across time and states of nature in a general equilibrium model. But such desirable features of contingent bonds disappear as soon as we introduce some friction to the general equilibrium model.

With the introduction of some friction into the general equilibrium model, it becomes more and more costly for the government to exploit the welfare improving features of contingent bonds as the discrepancy between the different expectations by agents and government about the probabilities of future realization of government consumption increases. This might explain the absence of contingent bonds in the practice of public debt management. Also, when the agents trust their government less and become more pessimistic about future
states of the economy, risk premium of contingent bonds becomes significantly large.

5 Conclusions [Incomplete]

Our paper starts with an extension of the general equilibrium model with state-contingent bonds by Lucas and Stokey (1983) on the theory of optimal debt management. Even with a simple but intuitive two-period model, we are able to demonstrate quantitatively the nice welfare improving features of state-contingent bonds. In particular, when private agents and government share the same expectations about the probabilities of future realization of government consumption, government manages to keep tax rates on labour income constant across time and states of nature simply by trading contingent bonds.

Next we introduce a friction to the general equilibrium model in the sense that agents do not fully trust their government and form different expectations from those by the government about the probabilities of future realization of government consumption. With this friction, the desirable features of contingent bonds disappear gradually and it becomes more and more costly for the government to exploit the welfare improving features of contingent bonds as the discrepancy between the different expectations by agents and government about the probabilities of future realization of government consumption increases.

The results show the quantitative implications of the qualitative theory by Lucas and Stokey (1983), and more importantly explain why state-contingent
bonds are absent from the practice of public debt management and also shed light on the famous Equity Premium Puzzle by Mehra and Prescott (1985).

[ I feel terribly sorry about this, but I had a major crack down with my laptop last week which took several days to recover and I could not complete the final draft of my paper as planned. So you will find my paper without appendix and references for the moment.]