The difference of manipulability indexes in IC and IANC model

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Abstract We consider the problem of manipulability of social choice rules in impartial anonymous and neutral culture model (IANC). We derive some properties of anonymous and neutral equivalence classes in order to evaluate maximal difference of Kelly’s index in impartial culture (IC) and IANC model. The value of Kelly’s index was calculated in IANC and compared with the same index in IC model. Using the results of theoretical study, we analyze how the relative manipulability of social choice rules changes in these probabilistic models.

1 Introduction

Procedures aggregating individual preferences into a collective choice differ in their vulnerability to manipulations. We say that manipulation can occur if any voter can achieve better voting result for himself by misrepresenting his preferences. The detailed research into the problem of manipulability was started by Gibbard (1973) and Satterthwaite (1975), whose publications are the most considerable in this area. They proved that any social choice rule with at least three possible outcomes and without a dictator is manipulable. Satterthwaite gave a definition of strategy-proof procedure. It is a voting scheme in which no manipulation can occur. These studies have given rise to a number of extensions and generalizations of Gibbard-Satterthwaite theorem.

Following Gibbard and Satterthwaite, the possibility of constructing a satisfactory social choice procedure were studied by Barbera (1977). He proved the theorem which says that social choice rule that satisfies unanimity condition and does not leave “too much” to chance must be either uniformly manipulable or dictatorial.

After them, the question of manipulability was widely investigated by J. Kelly. In Kelly (1977) it was proved that without an assumption of singlevaluedness (the quality of social choice rule having only one element as a result) such rules that satisfy both non-dictatorship and strategy-proofness could exist. In Kelly (1988) two approaches for measuring manipulability were introduced: counting the number of profiles where manipulation is possible and counting the number of profiles where manipulation is very unlikely to occur though it is possible. The first approach was developed in Kelly (1993), where social choice rules are compared in their vulnerability to manipulations.

This line of investigation has been followed by Aleskerov and Kurbanov (1999) and Aleskerov et al. (2010). The first paper contains the results of computational experiments that reveal the degree of manipulability of social choice rules. In addition, the authors introduced some new indexes for evaluating manipulability. In the paper Aleskerov et al. (2010), which is basic to this study, the research into manipulability is
developed in two directions. It extends the number of voters in computational experiment and uses different methods of expanding preferences.

All the listed articles focus on individual manipulations under impartial culture assumption. Impartial culture model was introduced in social choice literature by Guilbaud (1952). This model assumes that a set of all preference profiles is used for generating public preferences. Another important probabilistic model is impartial anonymous culture model (IAC), first described in Kuga, Nagatani (1974) and Gehrlein, Fishburn (1976). The question of manipulability of social choice rules in IAC model was thoroughly studied by Pritchard, Wilson (2007), Lepelley, Valognes (2002), Favardin, Lepelley (2006), Slinko (2006). These four publications are devoted to the study of coalitional manipulations.

In this paper we consider impartial anonymous and neutral culture model (IANC), in which both names of voters and names of alternatives are immaterial. In this model, some preference profiles are regarded as equivalent in terms of permutations of individuals and alternatives. Therefore, the set of all preference profiles is split up into equivalence classes. The first investigation of this model was started in Egecioglu (2005) and extended in Egecioglu, Giritligil (2009). They introduced a way of calculating the number of anonymous and neutral equivalence classes and an algorithm for their random generating. However, this model has not been thoroughly analyzed yet. Particularly, a way of analyzing the difference of indexes in IC and IANC without conducting a computational experiment has not been investigated in literature.

We consider the difference of Kelly’s manipulability index in impartial culture model and impartial anonymous and neutral culture model. We derive this difference from properties of anonymous and neutral equivalence classes, such as the minimal and the maximal number of elements in equivalence classes and the number of such classes.

Then we move on to the example of manipulability index for four social choice rules in IC and IANC for the case of three alternatives. We compare their relative manipulability and the difference of indexes for each procedure in both models. After that, we explain it in terms of anonymous and neutral culture model.

Numerical experiments in IANC model have, indeed, rather high computational complexity. The method of evaluating the difference of indexes will enable us to compare manipulability of social choice rules in IANC model without any experiments.

2 Definitions, notions and theoretical basis

In this paper we use the system of notions for impartial anonymous and neutral culture model introduced in Egecioglu (2005). First, there is a set of alternatives $A$, consisting of $m$ elements, and a set of individuals (or voters) $N = \{1, 2, ..., n\}$ with $n$ elements. Preference profile is defined as a matrix consisting of $n$ vectors that represent voters’ preferences by ordering $m$ alternatives. Preference profile is signed by $\tilde{P} = \{P_1, P_2, ..., P_n\}$, and preferences of $i$-th individual are signed by $P_i$.

The total number of different preference profiles is $(m!)^n$. Impartial culture model assumes that each voter selects his preferences out of $m!$ possible linear orders and each of $(m!)^n$ preference profiles is equally likely. The set of all preference profiles with $n$ voters and $m$ alternatives is signed by $\Omega(m, n)$.

As it was mentioned earlier, in Impartial Anonymous and Neutral Culture model there is no difference between voters and between names of alternatives. For example, in this model the following three profiles are considered as the same representation of preferences:
Therefore, we have a partition of $\Omega(m, n)$ into anonymous and neutral equivalence classes (ANECs). Any preference profile from a given ANEC can be taken as the representative profile (or root). In other words, ANEC is a set of preference profiles which can be generated from each other by permuting voters’ preferences and renaming alternatives.

Permutation of voters (or columns) is signed by $\sigma$, permutation on the set of alternatives is $\tau$. The pair of permutations $\sigma$ and $\tau$ is denoted by $(\sigma, \tau)$. $G$ is the group of all permutations $(\sigma, \tau)$, it acts on the set of all preference profiles. There exist $n!$ permutations of voters and $m!$ permutations of alternatives, therefore, the number of permutations $(\sigma, \tau)$,

$$|G| = n!m!.$$  

A partition $\lambda$ of $n$ is defined as a weakly decreasing sequence of positive integers $\lambda = (\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots \geq \lambda_z)$, where $\lambda_i$ is called a part of $\lambda$. For example, $(3, 2, 1, 1)$ is a partition of 7 into 4 parts. The type of a partition is denoted by $1^{a_1}2^{a_2}\ldots n^{a_n}$, which means that a partition $\lambda$ has $a_i$ parts of size $i$ for each $i$ from 1 to $n$.

Each permutation can be represented by a cycle decomposition. Therefore, $\sigma$ defines a partition $\lambda$ of $n$, and $\tau$ defines a partition $\mu$ of $m$ in such a way that parts of partitions $\lambda$ and $\mu$ are the lengths of cycles in $\sigma$ and $\tau$ respectively. The sum $\alpha = \alpha_1 + \alpha_2 + \ldots$ is the total number of cycles in permutation $\sigma$.

The image of a profile $\tilde{P}$ under the permutation $g = (\sigma, \tau)$ is denoted by $\tilde{P}^g$. Anonymous and neutral equivalence class $\theta_{(\tilde{P})}$ can be defined as a subset of $\Omega$: $\{\tilde{P}^g \mid g \in G\}$. Profiles $\tilde{P}_1, \tilde{P}_2$ are regarded as equivalent if there exists such permutation $g \in G$ that $\tilde{P}_1^g = \tilde{P}_2^g$.

If for a given permutation $g$ there exists a profile $\tilde{P}$, such that $\tilde{P}^g = \tilde{P}$ then $\tilde{P}$ is called a fixed-point of $g$. The set of all fixed points for $g$ is $F_g = \{P \in \Omega \mid P^g = P\}$.

For a given profile $\tilde{P}$ the set of all permutations that do not change $\tilde{P}$ is called the stabilizer subgroup of $\tilde{P}$ and defined as

$$G_{\tilde{P}} = \{g \in G \mid \tilde{P}^g = \tilde{P}\}.$$  

$G_{\tilde{P}}$ is a group. The number of elements in the orbit of $\tilde{P}$ (or its equivalence class) can be evaluated as a ratio

$$|\theta_{(\tilde{P})}| = |G|/|G_{\tilde{P}}|.$$  

$k \mid b$ means that $k$ divides $b$ evenly. Indicator function $\chi(S)$ of statement $S$ is defined as follows

$$\chi(S) = \begin{cases} 1, & \text{if } S \text{ is True,} \\ 0, & \text{if } S \text{ is False.} \end{cases}$$  

$GDC(\lambda)$ means a greatest common divisor of the part of $\lambda$. $LCM(\lambda)$ is a least common multiple of the parts $\lambda$. 

\[
\begin{array}{cccccc}
P_1 & P_2 & P_1' & P_2' & P_1'' & P_2'' \\
y & y & x & x & y & y \\
\tilde{P} = x & z & \tilde{P}' = y & z & \tilde{P}'' = z & x \\
z & x & z & y & x & z \\
\end{array}
\]
Binomial coefficient for an integer \( k, \ 0 \leq k \leq x \):
\[
\binom{x}{k} = C^k_x = \begin{cases} 
\frac{x!}{k!(x-k)!}, & x \in \mathbb{Z}, \\
0, & x \not\in \mathbb{Z}. 
\end{cases}
\]

The number of anonymous and neutral equivalence classes for \( n \) voters and \( m \) alternatives was found in Egecioglu (2005). It is given by
\[
R(m, n) = \sum_{\mu} z^{-1} \left( \frac{n + m!}{d} - \frac{m!}{d} - 1 \right),
\]
where \( d = \text{LCM}(\mu) \).

For \( n \) and \( m! \) being relatively prime the number of equivalence classes
\[
R(m, n) = \frac{1}{m!} \left( \frac{n + m! - 1}{m! - 1} \right).
\]

In addition, we give some definitions on manipulability. Preference profile where all individuals express their true preferences except the \( i \)-th individual is denoted by \( \tilde{P}_i = \{P_1, ..., P_i, P'_i, P_{i+1}, ..., P_m\} \). \( P'_i \) is the deviation of \( i \)-th individual from his true preferences \( P_i \).

The social choice (or the outcome of aggregating procedure) with respect to the profile \( \tilde{P} \) is denoted by \( C(\tilde{P}) \). As in Aleskerov et al. (2010), the case of multiple choice is considered. That means that the result of aggregating procedure might consist of several elements. Consider a preference profile \( \tilde{P}'' \) from the example provided above. How to decide, what is better for the first individual: the set \( \{x, y, z\} \) or \( \{z\} \)? To answer this question there are several methods of expanding preferences. In this study there will be two of them considered, lexicographic methods introduced in Pattanaik (1978).

In \textit{Leximin} method the worst alternatives of two sets are compared, the set where the better alternative is contained is considered as the better set. If they are the same then the second-worst alternatives are compared and so on. In \textit{Leximax} method of expanding preferences, the best alternatives are compared, than the second-best alternatives and so on. According to the first method, \( \{x, y, z\} \) is worth than \( \{z\} \), according to \textit{Leximax}, \( \{x, y, z\} \) is better than \( \{z\} \). Thus, \( EP_i \) denote expanded preferences of individual \( i \).

In the case of multiple choice manipulation is defined as follows: if for individual \( i \)
\[
C(\tilde{P}_i) EP_i C(\tilde{P}) \]
, than manipulation takes place.

Kelly’s index of manipulability is the ratio
\[
K = \frac{d_0}{(m!)^n},
\]
where \( d_0 \) is the number of profiles in which manipulation is possible.

In IANC model this index is calculated over the set of roots of equivalence classes
\[
K_{IANC} = \frac{r_0}{R(m, n)},
\]
where \( r_0 \) is the number of roots in which manipulation is possible, and \( R(m, n) \) is the total number of roots.
3 Evaluating the difference of Kelly’s index

In this section, we reveal some properties of anonymous and neutral equivalence classes in order to evaluate maximal difference of indexes in IC and IANC models. First, we consider what properties cause this difference to appear. Let us consider a hypothetical example of a set $\Omega$ consisting of ten preference profiles. Assume that there are four ANECs: two classes of cardinality 2, one class has cardinality 5 and the rest one has only one preference profile.

![Diagram showing equivalence classes and their probabilities]

We can suppose that only profiles from the biggest equivalence class are manipulable. Consequently, manipulability index in IC model is 0.5; in IANC model this index is equal to 0.25, because only one of four equivalence classes is manipulable. So, we can see that this difference results from inequality of equivalence classes. In IANC model all equivalence classes are equally likely.

Therefore, manipulability index in IC exceeds index in IANC if the average cardinality of equivalence classes that are manipulable exceeds the average cardinality of all equivalence classes. As a consequence, manipulability index is less in IC than in IANC, if the average cardinality of equivalence classes that are manipulable is less than the average cardinality of all equivalence classes.

To start with, we consider equivalence classes that have the least and the greatest cardinality.

**Theorem 3.1 (Anonymous and neutral equivalence class with the minimal number of elements)** The minimal number of elements in anonymous and neutral equivalence class is $m!$. This class is unique for the case of $n \geq 3$.

**Proof** Let us consider the case of $n = 2$. Suppose, in some reference profile voters have similar preferences. For this profile stabilizer group is the set of permutations of voters, its cardinality equals $2!$. Consequently, the number of elements in anonymous and neutral equivalence class is

$$\theta_{\mu} = \frac{|G|}{|G_\mu|} = \frac{2! m!}{2!} = m!.$$

Then, let us take an example of a profile with two alternatives and two voters, which have different preferences. For that profile the cardinality of stabilizer group also equals 2 and the number of equivalence classes with minimal number of elements is more than 1. Generally, the number of such classes grows with the growing of $m$. However, the maximal number of stabilizing permutations is 2, one of them is an identity permutation. Suppose, it is not true and there exists one more permutation in a stabilizer of profile $\tilde{P}$ with two different voters’ preferences. This permutation must permute alternatives, not voters. Thus, we have two different permutations of alternatives, both not changing profile $\tilde{P}$, but this is impossible. We can conclude, that the cardinality of minimal equivalence class does not exceed $m!$.

Let us consider the case of 3 voters. We must prove that there can not exist such a profile, that voters’ preferences are not similar and the cardinality of its stabilizer is equal or exceeds $n!$. If $\sigma$ is a permutation of voters, $\tau_1$ and $\tau_2$ are different permutations of
alternatives, then both $g_1 = (\sigma, \tau_1)$ and $g_2 = (\sigma, \tau_2)$ can not be stabilizing permutations for any profile. Therefore, the cardinality of stabilizer can not exceed $n!$.

Suppose, there exists a profile $\tilde{P}$ in which at least two voters have different preferences and the stabilizer for $\tilde{P}$ has exactly $n!$ elements. For each permutation of voters $\sigma$ there must exist a permutation of alternatives $\tau$, such that $g = (\sigma, \tau)$ is a stabilizing permutation for $\tilde{P}$. Let us try to build a permutation $\tau$ for each of permutations of voters

\[(1 \ 2)(3), (1 \ 3)(2), (1)(2 \ 3).\]

Each of the listed permutations of voters fixes one column, for that reason, $\tau$ can not permute alternatives, then $\tau$ for each $\sigma$ is an identity permutation. Thus, the only profile that can be a fixed-point for these permutations is a profile consisting of similar preferences. So, we came to a contradiction. These permutations of 3 voters are the parts of permutations of $n > 3$, therefore, our claim turns out to be true for any number of voters $n \geq 3$.

Since all profiles consisting of similar preferences belong to the same equivalence class, we can conclude, that it is unique. An equivalence class with a minimal number of elements is denoted by $\theta_{\min}$. The cardinality of such class

\[|\theta_{\min}| = \frac{m!n!}{n!} = m!\]
Theorem 3.3 The number of maximal anonymous and neutral equivalence classes in the case \( m! > n \) is equal

\[
K_{\text{max}} = \left( \frac{(m!)!}{(m!-n)!} \right) \cdot \sum_{g \in G} \chi \left( HOK(\lambda) = HOJD(\lambda) = HOK(\mu) \right) \cdot h \cdot \prod_{k=0}^{n-1} (m! - k \cdot HOK(\mu)) \cdot \frac{1}{m!n!},
\]

\[
h = \begin{cases} 
\frac{1}{2}, & \text{if } t = 2, \\
1, & \text{otherwise}.
\end{cases}
\]

We then move our attention to the problem of calculating the number of equivalence classes, such that average cardinality of these classes exceeds the average cardinality of all equivalence classes \( \theta_{av} \). It is enough to know the number and the cardinality of maximal equivalence classes to evaluate the difference if there are no other classes \( \theta \), such that \(|\theta| > |\theta_{av}| \). In this case, the maximal difference of Kelly’s index in IC and IANC models is evaluated as follows

\[
\Delta_{\text{IANC}} = \frac{K_{\text{max}} \cdot n!}{R(m,n) - (m!)^{n-1}},
\]

which is an absolute value of the difference between manipulability index in IANC and in IC model with the assumption that only preference profiles from maximal equivalence classes are manipulable.

If there is a second maximal cardinality of equivalence class \( \theta \), such that \(|\theta| > |\theta_{av}| \), and third maximal cardinality does not exceed the average, then we must add the difference of indexes cause by second maximal classes.

\[
\Delta_{\text{IANC}} = \frac{K_{\text{max}} \cdot n!}{R(m,n) - (m!)^{n-1}} - \frac{K_{\text{max}} \cdot n!}{2(m!)^{n-1}}.
\]

If \( n < m! \) then the cardinality of second maximal equivalence classes is twice less.

Generally, the number of equivalence classes \( \theta \), such that \(|\theta| > |\theta_{av}| \) is growing, but the share of difference they add in the total \( \Delta_{\text{IANC}} \) is getting less.

4 Manipulability of social choice rules in IC and IANC models

Using the results of theoretical study from the previous section, we calculate maximal difference of Kelly’s indexes in IC and IANC and compare it with an actual difference of this index for four social choice rules. First, we give a formal definition of these rules.

1. Plurality Rule. This rule chooses an alternative that is the best for maximal number of voters.

\[
a \in C(\overline{P}) \iff \forall x \in A \ n^+ (a, \overline{P}) \geq n^+ (x, \overline{P})
\]

where \( n^+ (a, \overline{P}) = \text{card} \{ i \in N | \forall y \in A \ aP_i y \} \).

2. Approval Voting. Social choice is an alternative that is placed on \( q \)'s place or higher in preferences of the maximal number of voters.

\[
a \in C(\overline{P}) \iff \forall x \in A \ n^+ (a, \overline{P}, q) \geq n^+ (x, \overline{P}, q)
\]

where \( n^+ (a, \overline{P}) = \text{card} \{ i \in N | \forall y \in A \ aP_i y \} \).

3. Borda’s Rule. For each alternative in \( i \)-th preferences the number \( r_i (x, \overline{P}) \) is counted as follows
\[ r_i(x, \bar{P}) = \text{card}\{b \in A : xPb\}. \]

The sum of \( r_i(x, \bar{P}) \) over all \( i \in N \) is called a Borda’s count.

\[ r(a, \bar{P}) = \sum_{i=1}^{n} r_i(a, \bar{P}). \]

Borda’s rule chooses an alternative with maximal Borda’s count.

\[ a \in C(\bar{P}) \Leftrightarrow [\forall b \in A, \ r(a, \bar{P}) \geq r(b, \bar{P})]. \]

4. Black’s procedure. Chooses a Condorset winner, if it exists, and the winner of Borda’s rule otherwise.

We compute Kelly’s indexes both in impartial culture and impartial anonymous and neutral culture model using Leximin and Leximax method of expanding preferences in 3-alternatives voting. After that, we calculate the difference of these indexes

\[ \Delta K_{\text{IANC}} = \frac{d_0}{(m!)^n} - \frac{r_0}{R(m,n)}, \]

Figure 4.1 represents differences calculated for Leximin method. The maximal difference is represented by the lowest and the highest border on figure 4.1. As can be seen from this graph, the difference is negative only for approval voting rule. This fact can be explained as follows. Preference profiles in which manipulation is possible often belong to equivalence classes with a small number of elements.

Plurality rule has the highest level of difference for all \( 3 \leq n \leq 10 \). These two facts cause the changes in the relative manipulability of social choice rules. Figures 4.2 and 4.3 illustrate behavior of Kelly’s index in IC and IANC. Approval rule turns out to be the most manipulable in IANC model, while under IC assumption it is a second least manipulable. The relative position of plurality rule changed in opposite direction. However, Black’s procedure is the least manipulable in both cultures.

![Figure 4.1 The difference of Kelly’s index in IC and IANC, Leximin](image-url)
5 Concluding remarks

Anonymity and neutrality are the fundamental axioms in social choice theory. IANC model based on these axioms assumes that both names of voters and names of alternatives are immaterial. We introduce combinatorial instruments that allow us to study the properties of social choice rules under IANC assumption. Since computational experiments in this model have rather high complexity, we present an alternative way of analyzing properties of anonymous and neutral social choice rules.

Using combinatorial methods, we evaluate the number and the cardinality of anonymous and neutral equivalence classes with maximal and minimal number of elements. We evaluate maximal difference between Kelly’s indexes in IC and IANC model. As a result, we estimate the relative manipulability of social choice rules under IANC assumption taking into account their specific qualities and calculating the maximal difference of manipulability indexes.

We also analyze actual difference of manipulability indexes of four social choice rules in IC and IANC model with Leximin and Leximax extension methods.
References