Term Structure of Discount Rates under Multivariate \(\vec{s}\)-Ordered Consumption Growth

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Abstract

The statistical relationship between future changes in consumption can be exploited to derive, under certain assumptions on investor preferences, an unambiguous effect on the yield-curve shape of zero-coupon bonds, \textit{viz.}, the term structure of discount rates. Thus, an increase in concordance in uncertain consumption growth has a negative impact on the yield-curve slope if, and only if, the representative investor is correlation averse (Gollier, \textit{Pricing the future}, to appear). Using multivariate \(\vec{s}\)-concave stochastic orderings, this note generalizes this relationship to multivariate higher-order risk preferences. The result under concordance is included for bivariate (1,1)-concave orders. The effect on the yield curve decreases absolutely with initial consumption for a given stochastic deterioration in the random addends to initial consumption. In an approximate representation of the interest rate for the univariate case, the effects on the yield curve are controlled by the Ross coefficients of risk aversion.

Keywords: term structure of discount rates, multivariate stochastic orderings, higher-order risk aversion

JEL classification: H43, E43, D81

1 Introduction

The statistical relationship between future changes in consumption can be exploited to derive unambiguous effects on the shape of the yield curve of zero-coupon bonds, and thus the term

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structure of discount rates, if the representative investor’s preferences exhibit certain characteristics. One way to capture this statistical relationship uses the concept of concordance. Concordance describes the intensity of association between the variables in the domain of a function at comparable levels of their identical discrete univariate domains (e.g., Epstein and Tanny 1980, Tchen 1980). Increases in concordance between two future (random) addends to present consumption can be shown to have a negative effect on the yield curve if and only if the representative investor is correlation averse.¹

This note derives a result that extends the bivariate case under concordance to an arbitrary number \( n \) of random addends to present consumption \((n \geq 2)\) and higher-order univariate and multivariate risk preferences. For this, the theory of general stochastic orderings of the \( \vec{s} \)-concave type is exploited.² General stochastic orderings – or, simply, (stochastic) orders – state the equivalence between the stochastic dominance relationship between two (vectors of) random variables with equal support and the ordering of the expectations of certain classes of functions of the two (vectors of) random variables. The vector \( \vec{s} \equiv (s_1, ..., s_n) \) of positive integers \( s_i, i = 1, \ldots, n \), indicates the orders of stochastic dominance between the corresponding components of the two random vectors. In economic or actuarial contexts, the functions referred to are typically interpreted as utility functions. As a result, the sign conditions on the derivatives of the functions, included in the definition of general stochastic orderings, have an interpretation in terms of (higher-order) risk attitudes. These orders allow thus a partial ordering of probability distributions on outcomes based on only partial information of the decision maker’s utility.

The main result is that, given two alternative vectors of random addends to initial consumption that are ordered in a \( \vec{s} \)-concave stochastic ordering with \( \sum_{i=1}^{n} s_i \) even (odd), the effect on the yield-curve slope of a shift to the dominating vector is negative (positive) if, and only if, the representative agent is multivariate-\( \vec{s} \) risk averse. An agent is called multivariate-\( \vec{s} \) risk averse if the \( (\sum_{i=1}^{n} s_i)^{th} \) derivative of her utility function is non-negative (non-positive) for \( \sum_{i=1}^{n} s_i \) odd (even). The same alternating effect on the yield-curve slope arises under univariate preferences.

¹A correlation averse individual prefers a lottery with equally probable mixed good and bad outcomes to a lottery that yields either good or bad outcomes with equal probabilities. The announced statement is a variant for additive consumption growth of Gollier’s (2010a) proposition that states the effect on the yield curve for log consumption and requiring that the representative investor’s relative prudence is larger than unity.

²The theory of univariate \( s \)-convex stochastic orderings for real-valued measurable functions has been developed by Denuit et al. (1998), and generalized to the bivariate case by Denuit et al. (1999a,b). This note relies on the multivariate cases of Denuit and Mesfioui (2010) and Denuit, Eeckhoudt, Tsetlin, and Winkler (2010), where the latter treat the concave counterpart. The concepts referred to are introduced in detail in section 2.
if and only if the representative investor is \((\sum_{i=1}^{n} s_i)^{th}\)-degree risk averse, in the sense of Ekern (1980). The connection between the case under concordance and the treatment that refers to general stochastic orderings can be made for the bivariate case, using a result of Denuit, Eeckhoudt, and Rey (2010). Thus, the concordance order of Tchen (1980) between the original random couple and a random couple that arises from the former through a mean-preserving increase in concordance, has an equivalent representation based on a \((1,1)\)-increasing concave order. The strength of the impact on the yield curve of a deterioration of the random addends to initial consumption in the \(\bar{s}\)-concave order decreases absolutely with rising initial consumption. The latter result extends similar findings by Eeckhoudt et al. (2009) and Denuit and Rey (2010), that referred to an expected utility premium, to the case of an expected marginal utility premium. I further show that, in the case of univariate risk preferences, the effects on the economy’s equilibrium interest rate can be separated in the summands of a Taylor approximation of it, each of which is controlled by the corresponding Ross coefficient of \(k^{th}\)-degree risk aversion.

Section 2 sets out the conceptual basis for the subsequent analysis. Section 3 provides the main result. Section 4 relates the main result to the literature and discusses some aspects of it. Section 5 concludes.

2 Conceptual Basis

A stochastic ordering \(\preceq_s\) between two \(n\)-dimensional random vectors, \(X, Y\), is typically defined as the ordering of the expectations of measurable functions \(u\) of some class \(U_s^{S^n}\) defined over \(S^n\), with \(S \subseteq \mathbb{R}\), having either of the two random vectors as their argument,

\[
X \preceq_s Y \iff \text{Eu}(X) \leq \text{Eu}(Y) \quad \forall \ u \in U_s ,
\]

provided that the respective expectations exist.\(^3\) If \(X \preceq_s Y\) holds, then \(X\) is said to be smaller than \(Y\) in the stochastic ordering \(\preceq_s\) generated by \(U_s\). Alternatively, \(Y\) is said to dominate \(X\) with respect to \(U_s\).

The (multivariate) \(\bar{s}\)-concave stochastic ordering \(\preceq_{\bar{s}_{-cv}}\) refers to the class of all differentiable

\(^3\)As set out in footnote 2, the definition of multivariate stochastic orders in this section follows Denuit, Eeckhoudt, Tsetlin, and Winkler, and Denuit and Mesfioui.
$s$-concave functions $u : \mathcal{S}^n \rightarrow \mathbb{R}$, defined as

$$\mathcal{U}_{s-cv}^n := \left\{ u \mid (-1)^\sum_{i=1}^n s_i \frac{\partial \sum_{i=1}^n s_i}{\prod_{i=1}^n \partial x_i} u(\vec{x}) \leq 0 \right\}.$$  

The $\vec{s}$-increasing concave order $\succeq_{\vec{s}-icv}$ refers to the class of all differentiable $\vec{s}$-increasing concave functions $u : \mathcal{S}^n \rightarrow \mathbb{R}$,

$$\mathcal{U}_{\vec{s}-icv}^n := \left\{ u \mid (-1)^\sum_{i=1}^n k_i \frac{\partial \sum_{i=1}^n k_i}{\prod_{i=1}^n \partial x_i} u(\vec{x}) \leq 0 \text{ for } k_1 = 0, 1, \ldots, s_i, i = 1, \ldots, n, \sum_{i=1}^n k_i \geq 1 \right\}.$$  

The term concave relates to the non-positive second derivatives of the functions in $\mathcal{U}_{s-cv}^n$ and $\mathcal{U}_{\vec{s}-icv}^n$, the additional term increasing for the latter relates to the non-negative signs of the first derivatives. Close relatives are the corresponding classes of convex functions where, respectively, all highest derivatives and all derivatives are non-negative. Below, also the notation $f^{(k)}$ will be used for the mixed partial derivative of function $f$ up to $k_i$ in component $i$ for $i = 1, \ldots, n$.

For the relationship between multivariate orders and univariate stochastic dominance between positive linear combinations of their respective components, the following relation holds.\(^4\)

**Remark 1**

Let $X, Y$ be two $n$-dimensional vectors with components $X_i, Y_i$ for $i = 1, \ldots, n$. It holds that

$$X \succeq_{\vec{s}-cv} Y \Rightarrow \sum_{i=1}^n \alpha_i X_i \succeq_{(\sum_{i=1}^n s_i)-cv} \sum_{i=1}^n \alpha_i Y_i \text{ for all } \alpha_i \geq 0, i = 1, \ldots, n.$$  

**Proof.** For the multivariate function $v(\vec{z}) = u(\sum_{i=1}^n \alpha_i z_i)$ with $u \in \mathcal{U}_{s-cv}^n$, it is $v^{(\vec{s})}(\vec{z}) = u^{(\sum_{i=1}^n s_i)}(\sum_{i=1}^n \alpha_i z_i) \prod_{i=1}^n \alpha_i s_i$, so that $v \in \mathcal{U}_{\vec{s}-cv}^n$. \(\blacksquare\)

In the following, I will interpret all functions $u$ as utility functions. For components $s_i$ of $\vec{s}$, $i = 1, \ldots, n$, sufficiently high, the (mixed partial) derivatives contained in the definitions of the classes of functions have an interpretation in terms of higher-order uni- or multivariate risk aversion.\(^5\)

\(^4\) Remark 1 holds analogously for increasing-concave functions. Denuit and Mesfionii prove the increasing-convex case also for general non-negative functions $\psi \in \mathcal{U}_{\vec{s}-icx}^n$, where $\psi(\vec{x}) = \sum_{i=1}^n \alpha_i x_i$, $\alpha_i \geq 0$ for $i = 1, \ldots, n$, is a special case. Eckhoudt et al. (2009: Theorem 3) prove a version of the bivariate increasing concave case.

\(^5\) The first part of Definition 1 refers to Ekern (1980), except for the formulation with weak inequality here. The second part formally extends Richard’s (1975) multivariate risk aversion, contained for $\vec{s} = (2, \ldots, 2)$, to higher orders.
Definition 1

An agent is called $s$th-degree risk averse if and only if $(-1)^su(s) \leq 0$ with $s > 1$. An agent is called multivariate-$\mathbf{s}$ risk averse if and only if $(-1)^{\sum_{i=1}^{n}s_i}u(\mathbf{s}) \leq 0$ with $s_i \geq 1$ for $i = 1, \ldots, n$ and $\sum_{i=1}^{n}s_i > n$.

Definition 1 covers, component-wise for every $i \in \{1, \ldots, n\}$, $n \in \mathbb{N}$, for $s_i = 2$ the usual notion of risk aversion (in the sense of a concave utility function), for $s_i = 3$ prudence, for $s_i = 4$ temperance, for $s_i = 5$ edginess, and for $s_i \geq 6$ the corresponding higher orders of risk apportionment (e.g., Eeckhoudt and Schlesinger 2006). Moreover, $u \in \mathcal{U}_{n-icv}$ with $s \to \infty$ corresponds to mixed risk aversion in the sense of Caballé and Pomansky (1996). For $n = 2$, bivariate-$(s_1, s_2)$ risk aversion coincides for $(s_1, s_2) = (1, 1)$ with correlation aversion (as in Epstein and Tanny 1980), and for $(s_1, s_1) = (1, 2)$ ($(s_1, s_1) = (2, 1)$) with cross-prudence in the first (second) outcome and for $(s_1, s_2) = (2, 2)$ with cross-temperance (as in Eeckhoudt et al. 2007).

For the univariate case with $z \in \mathcal{S}$, it will be referred to $-u^{(n+1)}(z)/u^{(n)}(z)$ as the (Arrow-Pratt) coefficient of absolute $n$th-degree risk aversion, to $-zu^{(n+1)}(z)/u^{(n)}(z)$ as the (Arrow-Pratt) coefficient of relative $n$th-degree risk aversion, and to $-u^{(n+1)}(z)/u'(z)$ as the Ross coefficient of (absolute) $n$th-degree risk aversion.\(^6\)

Without loss of generality, I focus in the following on random variables $X_i, Y_i$ with compact supports $[a_i, b_i]$ on the real line, $a_i < b_i$, and further restrict the attention to $a_i = 0$, for $i = 1, \ldots, n$. For the proof of Lemma 1, I assume that the random vectors $\mathbf{X}$ and $\mathbf{Y}$, with probability density functions $f_{\mathbf{X}}(\mathbf{x})$ and $f_{\mathbf{X}}(\mathbf{y})$, respectively, have the same $h$-variate marginals for $h = 1, \ldots, n-1$,

$$
\int_{n-h}^{n} \cdots \int_{n-h}^{n} f_{\mathbf{X}}(\mathbf{t}) \prod_{\ell=1}^{n-h} dt_{i(\ell)} = \int_{n-h}^{n} \cdots \int_{n-h}^{n} f_{\mathbf{Y}}(\mathbf{t}) \prod_{\ell=1}^{n-h} dt_{i(\ell)}, \tag{2}
$$

where $i(\ell) \neq i(\ell')$ for $i \in \{1, \ldots, n\}$, $\ell \neq \ell'$ and all $\ell \in \{1, \ldots, n-1\}$.

\(^6\)The absolute and relative coefficients of $n$th-degree risk aversion that refer to successive derivatives of the univariate utility function have been introduced, respectively, by Caballé and Pomansky (1996) and Eeckhoudt and Schlesinger (2008). Ross (1981) proposes stronger criteria that also allow for comparisons of risk aversion with regard to reductions in risk, instead of its elimination. (Pratt (1990) notes some caveats to Ross’ approach that apply to its later extensions, too.) Modica and Scarso (2005) extend Ross’ approach to downside risk aversion. Jindapon and Neilson (2007) develop an original approach to higher-order Arrow-Pratt and Ross risk aversion based on comparative statics. Demiit and Eeckhoudt (2010b) generalize the Ross/Modica-Scarso approach to the $n$th order, and also provide the link to $s$-increasing concave orders and Ekern increases in risk.
3 Main Result

In order to determine the necessary and sufficient conditions on investor preferences for an unambiguous effect on the yield-curve slope of a shift in changes in future consumption that are ordered according to a $\bar{s}$-concave ordering, I consider the following simple setting (cf. Gollier 2008, 2010a). A representative agent has to decide about the allocation of consumption between the initial period 0 and the period $T$ in which a zero-coupon bond matures in the face of uncertain exogenous consumption growth. The only possibility to smooth consumption is, hence, between the initial period and the terminal period of the bond. The representative agent’s (univariate) utility function $u$ is $\sum_{i=1}^{n} s_i$ times differentiable, increasing and decreasingly concave. The Euler condition for this problem is

$$u'(c_0) = e^{-\delta T} Eu'(C_T)e^{r_t T},$$

(3)

where $C_t$ is consumption at date $t$, which is only certain – and thus denoted $c_0$ – at date 0, $\delta$ is the individual rate of pure preference for the present, and $r_T$ is the per-period rate of return (or yield) of the bond at date 0. Thus, equation (3) equates the welfare cost of reducing consumption by one monetary unit, which is invested in the bond, and the welfare benefit that such an investment yields. An investment of one unit at date 0 will increase consumption at date $T$ by $e^{r_T T}$. Expected marginal utility at date $T$ will amount to $Eu'(C_T)e^{r_T T}$, which is discounted to the present at rate $\delta$. The equilibrium interest rate in this economy derives from equivalence transformations of equation (3) as

$$r_T = \delta - \frac{1}{T} \ln \left( \frac{Eu'(C_T)}{u'(c_0)} \right).$$

(4)

For frictionless and efficient financial markets, $r_T$ coincides with the socially efficient discount rate for maturity $T$. Expecting positive consumption growth, so that $EC_T > c_0$, $\ln(.)$ is negative. Hence, a rise (decrease) in the expected marginal utility of future consumption $Eu'(C_T)$ will reduce (increase) the long-term risk-free rate $r_T$.

For the interpretation of the results below, the three effects determining the interest rate in the absence of predictability shall be briefly restated. In addition to the positive impatience

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$^7$Equation (4) represents the standard asset-pricing formula for risk-free bonds (e.g., Cochrane 2005).
effect (as expressed by $\delta$), two competing effects occur. Higher expected consumption in the future makes individuals save less, and thus consume more, in the present. This wealth effect raises the equilibrium interest rate. However, the accumulating uncertainty in the future makes individuals save more. This precautionary effect reduces the equilibrium interest rate. The total effect on the interest rate depends on the relative strength of the two latter effects.

In order to investigate the effect of a shift in consumption growth on the economy’s equilibrium interest rate $r_T$ when the future addends to present consumption are related in an $\vec{s}$-concave order, the bond’s time to maturity is divided in $n$ subperiods $(t_{i-1}, t_i]$, $i = 1, ..., n$. Let $Z_i \in \{X_i, Y_i\}$ denote the random addend to initial consumption in subperiod $i$, with characteristics as defined in section 2. Consumption at maturity is then

$$C_T = c_0 + Z_1 + \ldots + Z_n.$$  \hfill (5)

The effects on the yield curve can be seen by comparing the long-term interest rates $r_T$ for the two alternative vectors of random addends to aggregate consumption, $X$ and $Y$, that are related in a certain stochastic ordering. The following proposition states the main result.

**Proposition 1**

Any shift in the vector of random addends to initial consumption from $X$ to $Y$, where $Y$ is larger than $X$ in the $\vec{s}$-concave order, with $\sum_{i=1}^n s_i$ even [odd], reduces [raises] the long-term risk-free rate if and only if the representative agent is multivariate-$\vec{s}$ risk averse.

For the proof, the following lemma is helpful.

**Lemma 1**

For any function $u \in U_{s-cv}^{S_n}$, and any pair of random vectors, $X$ and $Y$, where the first precedes the second in the $\vec{s}$-concave ordering, it holds that

$$Eu^{(1,...,1)}(X) - Eu^{(1,...,1)}(Y) \begin{cases} \geq 0 & \text{for } \sum_{i=1}^n s_i \text{ odd} \\ \leq 0 & \text{for } \sum_{i=1}^n s_i \text{ even} \end{cases}.$$ \hfill (6)

The proof of Lemma 1 relies on a statement of $\vec{s}$-increasing concave orderings that includes explicit sign conditions on the integrated left tails of the distributions of the two random vectors $X$ and $Y$. The following two characterizations provide the link between the definition of $\preceq_{s-cv}$ as in (1) with $U_s = U_{s-cv}^{S_n}$ and the sign conditions. Characterization 1 provides conditions
characterizing the multivariate $\mathcal{S}$-increasing concave order via lower partial moments, without reference to utilities.\textsuperscript{8}

**Characterization 1**

Let $\mathbf{X}$ and $\mathbf{Y}$ be random vectors with support contained in $[0, \bar{b}]$, with $\bar{b} \in \mathcal{S}^n \subset \mathbb{R}_+^n$, and denote $x_+ = \max\{x, 0\}$. Then, $\mathbf{X} \succeq_{\mathcal{S}-icv} \mathbf{Y}$ if and only if

$$(-1)^{\sum_{i=1}^n k_i - n} \left[ E \left( \prod_{i=1}^n (X_i - t_i)^{k_i - 1}_+ \right) - E \left( \prod_{i=1}^n (Y_i - t_i)^{k_i - 1}_+ \right) \right] \geq 0 \quad (7)$$

for all $t_i \in [0, b_i]$ if $k_i = s_i$, and $t_i = b_i$ if $k_i = 1, \ldots, s_i - 1, i = 1, \ldots, n$.

The proof is given in Appendix A for completeness.\textsuperscript{9}

An alternative characterization can be stated involving integral conditions. For a random vector $Z$ with distribution function $F_Z(\mathbf{Z})$, the $n$-variate integrated left tails of $Z$ derive, starting from $F_Z^{[1,\ldots,1]} \equiv F_Z$, for $k_1, \ldots, k_n \geq 1$ and all $i = 1, \ldots, n$ as

$$F_Z^{[k_1,\ldots,k_i+1,\ldots,k_n]}(\mathbf{z}) = \int_0^{z_i} F_Z^{[k_1,\ldots,k_i,k_{i+1},\ldots,k_n]}(z_1,\ldots,t_i,\ldots,z_n)dt_i, \quad (8)$$

where the superscript vector in square brackets indicates the number of integrations with respect to the elements of $\mathbf{z}$. Note that for $n = 1$, equation (8) states the standard integral condition for univariate stochastic dominance. A problem of technical nature is that $F_Z^{[k]}$ may be infinite for $n > 1$. However, each side of equation (8) is finite if the other is. In particular, $F_Z^{[k]}$ is finite if $E \left( \prod_{i=1}^n Z_i^{k_i} \right)$ exists. This can be seen from the following representation, which derives using induction and Fubini’s Theorem,\textsuperscript{10}

$$F_Z^{[k_1,\ldots,k_n]}(\mathbf{t}) = \frac{E \left( \prod_{i=1}^n (Z_i - t_i)^{k_i - 1}_+ \right)}{\prod_{i=1}^n (k_i - 1)!}. \quad (9)$$

Inserting representation (9) into the conditions in Characterization 1 yields

\textsuperscript{8}The functions of the form $\prod_{i=1}^n (z_i - t_i)^{k_i - 1}_+$ under the conditions of Characterization 1 are in $\mathcal{U}^{\mathcal{S}^n}_{\mathcal{S}-icv}$ and constitute the minimal generators of this class of functions (Müller 1997, Denuit and Mesfioui).

\textsuperscript{9}Denuit and Mesfioui, and Denuit, Eeckhoudt, Tsetlin, and Winkler contain statements of Characterization 1 for the convex and concave cases, respectively, without explicit sign conditions. The authors only prove necessity.

\textsuperscript{10}Scarsini (1985) and O’Brien and Scarsini (1991), for example, provide explicit proofs of particular cases.
Characterization 2  
Let $X$ and $Y$ be random vectors with support contained in $[0, \vec{b}]$, with $\vec{b} \in S^n \subseteq \mathbb{R}_+^n$. Then,

$$X \preceq_{\vec{s} \text{-icv}} Y \iff (-1)^{\sum_{i=1}^n k_i} [F_X^{[k_1, \ldots, k_n]}(\vec{t}) - F_Y^{[k_1, \ldots, k_n]}(\vec{t})] \geq 0$$

for all $t_i \in [0, b_i]$ if $k_i = s_i$, and $t_i = b_i$ if $k_i = 1, \ldots, s_i - 1$, $i = 1, \ldots, n$.

Using Characterization 2, Lemma 1 can now be proven.

Proof of Lemma 1. The sign conditions in Lemma 1 derive by induction. Let $D^n_{(s)}$ denote the difference of the expectations in conditions (6) for the $n$-dimensional random vectors $X, Y$.

Consider the $(\sum_{i=1}^n s_i - n)$-fold integration by parts of $D^n_{(s)}$, first for $n = 1$ and $s_1 \in \{1, 2\}$.

$$D^1_{(1)} = \int u^{(1)}(t) \ d[F_X(t) - F_Y(t)] \geq 0 \quad (10a)$$
$$D^1_{(2)} = -\int u^{(2)}(t) \ d[F_X^{[2]}(t) - F_Y^{[2]}(t)] \leq 0 \quad (10b)$$

The sign of $D^1_{(1)}$ arises because $u^{(1)} \geq 0$ and because the difference of the integrated left tails is non-negative, the sign of $D^1_{(2)}$ arises because $u^{(2)} \leq 0$ and because the difference of the integrated left tails is non-positive (cf. Characterization 2). Moreover, in line (10b) the definition of distribution functions implies that $\int d(F_X - F_Y) = 0$.

For a generic odd $n$, $s_1 \in \{1, 2\}$, and $s_j = 1$ for $j = 2, \ldots, n$, where the role of $s_1$ among the $s_i$, $i = 1, \ldots, n$, is chosen without loss of generality, it holds with analogous arguments that

$$D^n_{(1, \ldots, 1)} = \int \cdots \int u^{(1, \ldots, 1)}(\vec{t}) \ d[F_X(\vec{t}) - F_Y(\vec{t})] \geq 0 \quad (10c)$$
$$D^n_{(2,1,\ldots,1)} = -\int \cdots \int u^{(2,1,\ldots,1)}(\vec{t}) \ d[F_X^{[2,1,\ldots,1]}(\vec{t}) - F_Y^{[2,1,\ldots,1]}(\vec{t})] \leq 0, \quad (10d)$$

and for general $\vec{s}$ with $\sum_{i=1}^n s_i$ even

$$D^n_{(\vec{s})} = -\int \cdots \int u^{(\vec{s})}(\vec{t}) \ d[F_X^{[\vec{s}]}(\vec{t}) - F_Y^{[\vec{s}]}(\vec{t})] \leq 0. \quad (10e)$$

Lines (10d) and (10e), moreover, use the assumption of equal $h$-dimensional marginals for $h = 1, \ldots, n - 1$. Obviously, for $D^{n+1}_{(1,\ldots,1)}$ and $D^{n+1}_{(2,1,\ldots,1)}$ the signs are reversed. For $D^{n+1}_{(\vec{s})}$, with
$\sum_{i=1}^{n} s_i$ even, the following condition arises:

$$D_{(s)}^{n+1} = \int \cdots \int u^{(s)}(\mathbf{t}) \, d \left[ F_X^{[s]}(\mathbf{t}) - F_Y^{[s]}(\mathbf{t}) \right] \leq 0 . \quad (10f)$$

For $n$ and $\sum_{i=1}^{n} s_i$ odd, it holds that

$$D_{(s)}^{n} = \int \cdots \int u^{(s)}(\mathbf{t}) \, d \left[ F_X^{[s]}(\mathbf{t}) - F_Y^{[s]}(\mathbf{t}) \right] \geq 0 \quad (10g)$$

$$D_{(s)}^{n+1} = -\int \cdots \int u^{(s)}(\mathbf{t}) \, d \left[ F_X^{[s]}(\mathbf{t}) - F_Y^{[s]}(\mathbf{t}) \right] \geq 0 . \quad (10h)$$

Finally, note that the conditions (10) do not make use of sign conditions for $k_i < s_i$, for $i = 1, \ldots, n$, and only exploit the conditions with $\mathbf{K} = \mathbf{s}$ from Characterization 2. As a consequence, conditions (6) hold for any $u \in U_{-cv}^{s_n}$.

By Remark 1, Lemma 1 also holds if the argument of $u^{(1, \ldots, 1)}$ is stated in a univariate way (as, for example, specified in equation (5) for future consumption). With this substitution, Proposition 1 follows by applying Lemma 1 to the equation (4) of the equilibrium interest rate.

The version of Lemma 1 with univariate argument of $u^{(1, \ldots, 1)}$ yields the following corollary.

**Corollary 1**

*Any shift in the random addends to initial consumption from $X$ to $Y$, where $Y$ is larger than $X$ in the $\mathbf{s}$-concave order, with $\sum_{i=1}^{n} s_i$ even [odd], reduces [raises] the long-term risk-free rate if and only if the representative agent is $(\sum_{i=1}^{n} s_i)^{th}$-degree risk averse.*

Hence, according to Proposition 1, the effect on the yield-curve slope of a shift in the vector of random addends to initial consumption alternates in sign depending on whether the sum of the orders of the stochastic orderings between the corresponding elements of the two vectors of random addends are even or odd and the representative investor exhibits multivariate-$\mathbf{s}$ risk aversion. Due to the additive composition of future consumption (equation (5)), the analogous conditions hold, according to Corollary 1, if the representative agent is, instead, $(\sum_{i=1}^{n} s_i)^{th}$-degree risk averse. In both cases, if $\sum_{i=1}^{n} s_i$ is even, the effect tends to be negative, and if $\sum_{i=1}^{n} s_i$ is odd, the effect tends to be positive.
4 Discussion

In this section, I first relate the above findings to results in the literature that refer to the concept of concordance. Then, I derive a comparative-statics result on the variation of the strength of the effect of a given stochastic deterioration among the random addends to initial consumption on the equilibrium discount rate under varying initial consumption. Finally, I provide an illustration of the effects for the univariate case at an approximate representation of the equilibrium discount rate, and an interpretation of the alternating character of the effects referring to Ekern risk increases of order one to three.

4.1 Relationship to Result Based on Concordance

Gollier (2010a) proves a version of Corollary 1 for the bivariate case \((n = 2)\) with \(s_1 = s_2 = 1\), referring to the concept of concordance.\(^{11}\) A marginals-preserving increase in concordance (MPIC) between two discrete random variables \(Z_i, i = 1, 2\), with \(Z_i \in \{X_i, Y_i\}\) and \((Y_1, Y_2) \succeq (X_1, X_2)\) without loss of generality, is defined as any transformation of their joint distribution such that \(F_{(Y_1,Y_2)}\) is obtained from \(F_{(X_1,X_2)}\) by shifting probability mass \(\varepsilon\) from a small neighborhood of the points \((X_1,Y_2)\) and \((Y_1,X_2)\) to a small neighborhood of \((X_1,X_2)\) and \((Y_1,Y_2)\). Such an increase in concordance raises the correlation among the two random variables but does not affect their marginal distributions.

Denuit, Eeckhoudt, and Rey (2010) provide the link between such an increase in concordance and a shift between two pairs of random couples that are related in the \(\preceq_{(1,1)}\) order. They formalize (an increase in) concordance with reference to a couple of binary random variables \((I_1, I_2)\), with \(Pr[I_i = 0] = 1 - Pr[I_i = 1] \equiv p_i\) for \(i = 1, 2\) and the joint distribution

\[
Pr[I_1 = 0, I_2 = 0] = p_1 p_2 + \rho \\
Pr[I_1 = 1, I_2 = 0] = (1 - p_1) p_2 - \rho \\
Pr[I_1 = 0, I_2 = 1] = p_1 (1 - p_2) - \rho \\
Pr[I_1 = 1, I_2 = 1] = (1 - p_1) (1 - p_2) + \rho ,
\]

\(^{11}\)See Gollier (2010b) for an application of the concordance concept in the determination of the effect of varied economic convergence on the socially efficient discount rate.
where, for \( p_1 \leq p_2 \) without loss of generality, \( \rho \in [-p_1p_2, p_1(1-p_2)] \) is a correlation parameter.

In the joint distribution, compared to the case of mutual independence of \( I_1 \) and \( I_2 \), \( \rho \) is added to the probability mass at \((0,0)\) and \((1,1)\), and subtracts the same quantity from the probability mass at \((0,1)\) and \((1,0)\). The constraints on \( \rho \) guarantee that the probability distribution generated by this transformation continues to be well defined. The resulting distributions are ordered in the concordance order of Tchen (1980). An increase in \( \rho \) implies a MPIC in the above sense.

To establish the link between an increase in \( \rho \) and general stochastic orderings, Denuit et al. consider a bivariate random vector \( B_\rho \equiv ((1-I_1) S_1 + I_1 T_1, (1-I_2) S_2 + I_2 T_2) \), where the random variables \( S_1, S_2, T_1 \) and \( T_2 \) are mutually independent, independent from \((I_1,I_2)\), and such that \( S_i \preceq_{s_i-icv} T_i \) for \( i = 1,2 \). The authors compare this vector to an analogous random vector \( B_{\rho'} \) in which \((I'_1,I'_2)\), with distribution depending on \( \rho' \geq \rho \), replaces \((I_1,I_2)\). For the proof, they exploit the equivalent representation of the bivariate \( \preceq_{(s_1,s_2)-icv} \)-order as

\[
E[u_1(X_1)u_2(X_2)] \geq E[u_1(Y_1)u_2(Y_2)]
\]  

(11)

for all the non-positive \( u_i \in U_{s_i-icv} \), \( i = 1,2 \), and random pairs \((X_1, X_2)\) and \((Y_1, Y_2)\) with equal support (cf. Denuit and Eeckhoudt 2010a: Proposition 4.1). \( E[u_1(.)u_2(.)] \) depends positively on \( \rho \), as can be seen when substituting in condition (11) the two components of \( B_\rho \) for, respectively, \( X_1 \) and \( X_2 \) (or \( Y_1 \) and \( Y_2 \)), and taking into account the mutual independence of \( S_1, S_2, T_1 \) and \( T_2 \), as well as the orderings \( S_i \preceq_{s_i-icv} T_i \) for \( i = 1,2 \). Hence, an increase from \( \rho \) to \( \rho' \) makes the random couple \( B_\rho \) decrease in the \( \preceq_{(s_1,s_2)-icv} \)-sense, so that \( B_{\rho'} \preceq_{(s_1,s_2)-icv} B_\rho \). As a consequence, decision makers with utilities in \( U_{(s_1,s_2)-icv} \) dislike any increase in correlation in this sense.\(^{12}\) The case that has been treated before is contained for \((X_1, X_2) = (a_1 I_1, a_2 I_2)\), where \( a_1, a_2 \) are arbitrary positive constants, and \( s_1 = s_2 = 1 \).

Given this connection between increases in concordance and decreases in the \( \preceq_{(s_1,s_2)-icv} \) order, Corollary 1 covers as special cases both the above mentioned version of Gollier’s (2010a) result and its possible generalization based on Denuit et al.’s random vector \( B_\rho \).

\(^{12}\)Denuit et al. also note that \( B_\rho \) can be seen as a four-state lottery \(((S_1, S_2), (S_1, T_2), (T_1, S_2), (T_1, T_2))\) with probabilities as indicated for \((I_1, I_2)\) above, which shows that an increase in \( \rho \) makes the extreme outcomes \((S_1, S_2)\) and \((T_1, T_2)\) more likely, at the expense of the mixed outcomes \((S_1, T_2)\) and \((T_1, S_2)\).
4.2 Effects under Varying Initial Consumption

How does the strength of the effect on the equilibrium discount rate alter under a given stochastic deterioration between the vectors of random addends when the initial consumption varies? The following remark states the respective comparative-statics result.

**Remark 2**

Consider Lemma 1 for the case with univariate argument, i.e., \( u^{(1,...,1)}(\overline{z}) = u^{(1,...,1)}(\omega + \sum_{i=1}^{n} z_i) \), where \( \omega > 0 \) is an arbitrary constant. Then,

\[
\frac{d}{d\omega} \left| E u^{(1,...,1)} \left( \omega + \sum_{i=1}^{n} X_i \right) - E u^{(1,...,1)} \left( \omega + \sum_{i=1}^{n} Y_i \right) \right| \leq 0 \quad \forall \omega \text{ and } \sum_{i=1}^{n} s_i \geq 1.
\]

**Proof.** Observe that in the case with univariate argument in the analogs of equations (10) in the proof of Lemma 1 only the integrand \( u^{(1,...,1)}(\omega + \sum_{i=1}^{n} t_i) \) depends on \( \omega \). The equivalence follows. ■

With the substitution \( \omega = c_0 \), Remark 2 means that the (depending on whether the sum \( \sum_{i=1}^{n} s_i \) is odd or even, positive or negative) effect of a deterioration of the random addends to initial consumption in the \( \overline{s} \)-concave order on the expected marginal utility of final consumption tends to decrease absolutely in strength as the level of initial consumption increases if, and only if, the representative agent is \((\sum_{i=1}^{n} s_i)^{th}\) and \((\sum_{i=1}^{n} s_i + 1)^{th}\)-degree risk averse. Obviously, the conditions on the preferences in Remark 2 are slightly more restrictive than in Corollary 1.13

Remark 2 parallels some findings in the recent literature. Eeckhoudt et al. (2009: Theorem 3) show that bivariate stochastic dominance \((X_1, X_2) \preceq_{(s_1,s_2)-icv} (Y_1, Y_2)\) implies a preference for “disaggregating harms” in the form of \((Y_1 + Y_2, X_1 + X_2) \preceq_{(s_1,s_2)-icv} (X_1 + Y_2, Y_1 + X_2)\).14 Their proof refers to the expected-utility premium \(g(w) \equiv E[u(w+X)] - E[u(w+Y)]\), showing that, for \(X \preceq_{s-icv} Y\) and \(u \in U^{S}_{(s+t)-icv}\), \(g(w) \in U^{S}_{(s+t)-icv}\). Denuit and Rey (2010) show that the pain, measured by \(-g(w)\), from a deterioration of the random addends to initial wealth \(w\) in the \(s\)-increasing concave order decreases with rising initial wealth if the agent’s utility \(u \in U^{S}_{(s+t)-icv}\). Moreover, they establish, referring to the concepts defined in subsection 4.1, that the expected-utility pain from a mean-preserving increase in correlation decreases with

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13Remark 2 could also be proven for general \( s_0 \geq 1 \) and the multivariate case. For the multivariate case, however, the interpretation of \( \omega \) is less clear in the present context.

14The authors state their result referring to 50–50 lotteries of the indicated outcomes, and using the concept of risk apportionment (Eeckhoudt and Schlesinger 2006) that can express risk preferences of any order without reference to any particular model of preferences.
rising initial wealth if the agent’s utility \( u \in \mathcal{U}_{(s_1 + s_2 + 1) - \text{icv}} \) \( \mathcal{S}^2 \times [-p_1 p_2, p_1(1-p_2)] \) (so that the bivariate utility of initial wealth and the correlation parameter \( \rho, U(w, \rho) \), is supermodular). The result in Remark 2, obtained for somewhat more general preferences, relates to these findings in that it refers to an expected marginal utility premium, which explains the alternating sign of the effects here.

4.3 Illustrations of Effects for Univariate Case

A direct illustration of the relationship between the signs of the derivatives of the investor’s utility function and the equilibrium interest rate derives, for the univariate case, from the following approximation of \( r_T \) in equation (4).\(^{15}\) Exploiting a \((\sum_{i=1}^{n} s_i)\)\(^{th}\)-order Taylor approximation of \( E u'(C_T) \) and first-order Taylor approximations of \( \ln E u'(C_T) \) and \( E u'(C_T) \), the equilibrium interest rate can be represented as

\[
r_T \approx \delta - \frac{1}{T} \sum_{k=1}^{\sum_{i=1}^{n} s_i} \frac{E(C_T - c_0)^k}{k!} \cdot \frac{u^{(k+1)}(c_0)}{u'(c_0)}. \tag{12}
\]

If the random addends to the present consumption \( c_0 \) in the \( n \) subperiods are independent and identically distributed, only the first three terms on the right-hand side of approximation (12) are relevant. Then, the strength of the (positive) wealth effect (second term) is controlled by the coefficient of absolute risk aversion, while the strength of the (negative) precautionary effect (third term) depends on the Ross coefficient of prudence. If the \( n \) random addends to present consumption are ordered in a \( \mathcal{S}\)-concave order, the effect of a change in random growth on \( r_T \) is controlled by the Ross coefficients of \( k\)\(^{th}\)-degree risk aversion up to \( k = \sum_{i=1}^{n} s_i \). For Ekern increases in risk, where only the \((\sum_{i=1}^{n} s_i)\)\(^{th}\) moment of the distribution of \( C_T \) changes, the difference depends only on the Ross coefficient of \((\sum_{i=1}^{n} s_i)\)\(^{th}\)-degree risk aversion (cf. also Denuit and Eeckhoudt 2010b).

Viewed differently, the terms beyond the third term in equation (12) can also be seen as shaping the original precautionary effect. If these further terms together with the third term are denoted extended precautionary effect, an increasing (decreasing) yield curve results if the wealth effect dominates (is dominated by) the extended precautionary effect. Obviously, when

\(^{15}\)Gollier (2010a) provides an alternative representation where the effects are controlled by the product of all (Arrow-Pratt) coefficients of relative risk aversion up to the respective degree.
the random addends to present consumption are ordered in a \( \mathbf{s} \)-concave order and the agent is multivariate-\( \mathbf{s} \) risk averse, the equilibrium interest rate would rather be expected to decline if \( \sum_{i=1}^{n} s_i \) is even than if \( \sum_{i=1}^{n} s_i \) is odd.

At least until fourth-order (univariate) Ekern increases in risk, Remark 1 on the relationship of multivariate and univariate stochastic orders can be exploited to explain the alternating sign of the effect on the equilibrium interest rate with increasing \( \sum_{i=1}^{n} s_i \).\(^{16}\) Thus, an increase in second-order risk merely tends to increase the riskiness of final consumption \( C_T \). Using the words of Milgrom (1981), this constitutes ‘bad news’ for the investor. Similarly, a third-order increase in risk is ‘good’ news because of the associated reduction in upside risk. A fourth-order increase in risk is ‘bad’ news because of the implied increase in outer risk.

5 Conclusion

The shape of the yield curve of zero-coupon bonds is determined by the stochastic properties of the growth of aggregate consumption and the investors’ preferences. If two alternative vectors of random addends to initial consumption are related in a \( \mathbf{s} \)-concave order, with \( \sum_{i=1}^{n} s_i \) even (odd), a shift to the dominating vector has a negative (positive) effect on the yield-curve slope if, and only if, the representative agent is multivariate-\( \mathbf{s} \) risk averse. This result covers, as a special case, the negative effect arising under marginals-preserving increases in concordance among two random addends and correlation averse investor preferences, a version of a previous finding by Gollier (2010a). Because total consumption arises as the sum of present and future consumption, the alternating effect of the general case also holds if the representative agent is (univariate) \( \sum_{i=1}^{n} s_i \)-degree risk averse. I further show that the strength of the impact on the yield curve of a deterioration of the random addends to initial consumption in the \( \mathbf{s} \)-concave order decreases absolutely with rising initial consumption. In the univariate case, the different risk-related effects shaping the yield-curve slope can be separated in the summands of a Taylor approximation of the equilibrium discount rate, each of which is controlled by the corresponding Ross coefficient of \( k^{th} \)-degree risk aversion.

The relevance of higher-order multivariate stochastic orderings in aggregate consumption

\(^{16}\)These three kinds of risk increases have been introduced, respectively, by Rothschild and Stiglitz (1970), Menezes et al. (1980), and Menezes and Wang (2005). To the best of my knowledge, increases in risk of degree higher than four have not been studied, and thus labeled, explicitly in the literature.
growth (beyond the bivariate case with $s_1 = s_2 = 1$) is an open empirical question. Comparing internationally the statistical properties of consumption growth should provide ample evidence over at least a number of decades. Interestingly, also higher-degree risk attitudes, beyond (second-degree) risk aversion, are empirically few understood. Deck and Schlesinger (2011) and Noussair et al. (2011) conduct experiments that find (univariate) prudence and temperance as prevailing attitudes in student samples and a representative sample of the Dutch population. Bostian and Heinzel (2010) provide the first parametric structural estimates of (Arrow-Pratt) prudence coefficients based on a dynamic stochastic general equilibrium (DSGE) model. Using US macroeconomic data, they find evidence for decreasing relative risk aversion and decreasing relative prudence, although the magnitudes of the declines are small. Even fewer research has thus far concerned correlation aversion and higher-degree concepts related to it. A first incentivized-choice experiment on intertemporal correlation aversion, by Andersen et al. (2010), confirms its prevalence for a representative sample of the Danish population.

Appendix

A Proof of Characterization 1

**Necessity.** $Eu(Z)$, with $u \in U^{S^n}_{s_{-icv}}, Z_i \in [0, b_i], b_i > 0$, for $i = 1, \ldots, n$, has, when viewed as a function of $Z_1$ around 0 for fixed $z_2, \ldots, z_n$, an exact representation based on a Taylor series expansion and remainder as

$$Eu(Z) = \sum_{k_1=0}^{s_1-1} \frac{E(Z_1^{k_1})}{k_1!} u^{(k_1, 0, \ldots, 0)}(0, z_2, \ldots, z_n) + \int_0^{b_1} \frac{E((Z_1 - t_1)^{s_1-1})}{(s_1-1)!} u^{(s_1, 0, \ldots, 0)}(t_1, z_2, \ldots, z_n)dt_1.$$
Applying, similarly, Taylor series expansions to \( u^{(k_1,0,\ldots,0)}(0, z_2, \ldots, z_n) \) and \( u^{(s_1,0,\ldots,0)}(t_1, z_2, \ldots, z_n) \), when viewed as functions of \( Z_2 \) around 0 for fixed \( z_3, \ldots, z_n \), yields

\[
Eu(Z) = \sum_{s_1=0}^{s_1-1} \sum_{s_2=0}^{s_2-1} \frac{1}{s_1!s_2!} E \left( \frac{Z_{k_1}^{s_1} Z_{k_2}^{s_2}}{k_1!k_2!} \right) u^{(k_1,k_2,0,\ldots,0)}(0,0, z_3, \ldots, z_n)
\]

\[
+ \sum_{s_1=0}^{s_1-1} \int_0^{b_2} E \left( \frac{Z_{k_1}^{s_1} (Z_2 - t_2)_{s_2}^{s_2-1}}{s_1!} \right) u^{(k_1,s_2,0,\ldots,0)}(0, t_2, z_3, \ldots, z_n) dt_2
\]

\[
+ \sum_{s_2=0}^{s_2-1} \int_0^{b_1} E \left( \frac{(Z_1 - t_1)_{s_1}^{s_1-1} Z_{k_2}^{s_2}}{(s_1-1)!k_2!} \right) u^{(s_1,k_2,0,\ldots,0)}(t_1, 0, z_3, \ldots, z_n) dt_1
\]

\[
+ \int_0^{b_1} \int_0^{b_2} E \left( \frac{(Z_1 - t_1)_{s_1}^{s_1-1} (Z_2 - t_2)_{s_2}^{s_2-1}}{(s_1-1)!(s_2-1)!} \right) u^{(s_1,s_2,0,\ldots,0)}(t_1, t_2, z_3, \ldots, z_n) dt_1 dt_2
\]

Repeating this preceding component by component gives the general expansion formula:

\[
Eu(Z) = \sum_{S} \sum_{i \in S} \sum_{k_i=0}^{s_i-1} \int_0^{b_i} E \left( \frac{\prod_{i \in S} Z_i^{k_i} \prod_{j \in \overline{S}} (Z_j - t_j)_{s_j}^{s_j-1}}{k_i! \prod_{j \in \overline{S}} (s_j - 1)!} \right) \frac{\partial \sum_{i \in S} k_i + \sum_{j \in \overline{S}} s_j}{\partial x_i^{k_i} \prod_{j \in \overline{S}} \partial x_j^{s_j}} u(t_S) \prod_{j \in \overline{S}} dt_j
\]

where \( S \) and \( \overline{S} \) form a partition of \( \{1,2,\ldots,n\} \) (i.e., \( S \cup \overline{S} = \{1,2,\ldots,n\} \) and \( S \cap \overline{S} = \emptyset \)), \( n_S = \#S, t_S = \sum_{i \in S} t_e, \) and \( e_i = (0,0,\ldots,0,1,0,\ldots,0) \). Specifying the expansion of \( Eu(Z) \) for \( Z \in \{X,Y\} \), and comparing \( Eu(X) \leq Eu(Y) \) as in definition (1) taking into account the sign conditions on the partial derivatives of \( u \in U^{S_n}_{S-ice} \) shows the necessity of inequalities (7).

**Sufficiency** follows by analogy from the uni- and bivariate cases treated, respectively, in Denuit et al. (1998) and Denuit et al. (1999b).  

**References**


