Asymmetric Pricing Caused by Collusion

Very Preliminary First Draft

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Abstract

We consider a simple oligopoly model with differentiated firms facing stochastic demand and unexpected, persistent cost shocks. Restricting attention to focal equilibria in which firms use a demand tail test to enforce collusion, we show that the probability of price adjustments after shocks will typically be skewed in favor of positive adjustments, implying asymmetric pricing. However, very large cost decreases will be transmitted instantaneously. Simulations reveal that the model predicts observed patterns of city level average gasoline prices well.

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1 Introduction

This article aims to improve our understanding of asymmetric pricing, according to which output prices tend to adapt differently to input cost increases than to input cost decreases. More specifically, asymmetric pricing is usually associated with negative cost shocks being passed along to consumers in a “slower” fashion than positive ones. Bacon (1991) coined the term rockets and feathers for this phenomenon, since output prices rise like rockets when there is a positive cost shock, but fall like feathers when there is a negative one. This is a remarkable observation, since classical economic theory predicts an instantaneous and symmetric response of prices to changes in costs, no matter what the particular structure or degree of concentration of a given market is.

During the last two decades, numerous empirical studies confirmed the so called pattern of rockets and feathers. Because of the rich amount of data available and some theoretical reasons that make gasoline a good candidate to test for asymmetric pricing, many studies have concentrated on the gasoline market. Examples include Johnson (2002); Bacon (1991); Karrenbrock (1991); Borenstein et al. (1997) and Lewis (2009). However, there is growing support that asymmetric pricing is not restricted to few specialized markets, but is a very broad phenomenon. Yang and Ye (2008, p. 547) state that other markets where adjustment asymmetry can be observed include fruit and vegetables, beef and pork, and banking. For example, examining deposit interest rates in the banking sector, Hannan and Berger (1991) find a significant delayed response of deposit rates to positive exogenous changes in interest rates compared to negative changes, which is similar in spirit to a downward price rigidity in goods markets. Controlling for market structure, the authors also find a strong positive correlation between adjustment asymmetry and concentration, raising suspicion that asymmetric pricing might go along with an abuse of market power.

In a comprehensive study of 77 consumer and 165 producer goods, Peltzman (2000) shows that asymmetric pricing can be found in more than two thirds of the markets the researcher examined. He also points out that positive cost shocks usually have twice the immediate impact on output prices than negative ones, and that this asymmetry tends to last for at least five to eight months.

All of these findings suggest that asymmetric pricing is a crucial element of modern markets that is not explained by standard theory. By providing a theoretical model of asymmetric price transmission, this paper attempts to fill at least some of this gap.

The general public as well as government authorities often attribute asymmetric price transmission to an abuse of market power, i.e., implicit or explicit collusion. One reason for this is that asymmetric pricing implies a significant welfare redistribution from consumers to producers, at
least in the short run. Another reason is that the phenomenon is not well understood from a theoretical perspective and alternative explanations are seldom considered in the empirical literature. However, as some existing theoretical models of asymmetric pricing have shown, collusion is not a necessary prerequisite for rockets and feathers. Moreover, it seems that only few authors have yet attempted to provide a formal model of asymmetric price adjustment caused by collusion. The model we present tries to improve this theoretical deficit. We show that asymmetric pricing can be the result of tacit collusion under fairly general conditions. As long as firms’ demand is a sufficiently accurate indicator of their rivals’ pricing behavior, it is not even necessary to monitor the prices of their competitors to enforce collusion. In addition, firms do not have to adopt complicated, dynamic punishment routines to maintain asymmetric price adjustment. It suffices to determine one unique and cost invariant demand threshold; if demand falls below that threshold, tacit collusion is optimally abandoned until the next negative cost shock happens.

We base our model on an idea discussed in Borenstein et al. (1997). There, the authors suggest that asymmetric pricing might be caused by imperfect coordination on focal points, where coordination happens in the spirit of the model of non-cooperative collusion outlined by Green and Porter (1984). Starting from a competitive regime, where oligopolistic firms price (relatively) close to marginal cost, these firms might use negative cost shocks as coordinating device to increase their profits, especially in markets with low price elasticities of aggregate demand (e.g. retail gasoline). Typically, due to a multiplicity of equilibria, non-cooperative equilibria, where firms can accrue larger profits than in the competitive one-shot equilibrium, will be hard to reach if firms have no means of explicit communication (for example because of effective antitrust regulation). But negative cost shocks can greatly facilitate coordination, because they are faced by every firm in the relevant market and are common knowledge. If firms simply do not react to such negative shocks, which is as easy as a coordination tactic can get, this leads to an immediate increase of margins, and, if demand is inelastic, an increase in profits. On the other hand, firms see no reason to stick to competitive prices when their input costs increase, as even if every other competitor did so, increasing one’s own price will lead to an increase in (expected) profits. Thus, at least starting from competitive prices, firms’ selling prices would react much quicker to positive cost shocks than to negative ones, which directly leads to asymmetric price adjustment as can be observed in the data.

But by the above logic, output prices would actually never react to negative cost shocks. This

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1Support for this can be found in Lewis (2009, p. 18) and Eckert (2002, p. 53). A notable exception is given by Wölfing (2008), who considers collusive asymmetric price adjustment in supply function equilibrium of the German electricity spot market.
is were the mechanism introduced by Green and Porter (1984) comes into play. It is probably realistic to assume that in many oligopolistic markets, firms find it impossible to perfectly observe the prices of their rivals. This can be the case because of frequent price changes, significant spatial separation, or unobservable components of product characteristics\(^2\). But then, if the probability of detecting an undercutting rival is sufficiently low, firms would have to find a different mechanism to enforce collusion on prices that are above the competitive level, because undercutting would pay off otherwise. A simple way to do so is to observe one’s own demand. If it drops significantly in one period, while demand usually tends to be quite stable, it is a reasonable assumption to make that one or more of one’s rivals have undercut. This will lead to punishment and downward adjustment of prices to the new competitive level induced by the downward cost shock.

Now the final ingredient to asymmetric price transmission is the incorporation of fluctuating demand. If there is a low but positive probability that demand falls short of some punishment threshold even though every firm colludes on high prices, all negative cost shocks will eventually be transmitted to output prices. The speed of transmission will only depend on firms’ ability to infer their rivals’ behavior from their own demand. In the model outlined below, the noisier the distribution of demand, the more firms there are in a market and the less patient firms are, the faster negative cost shocks will be transmitted to output prices. Moreover, large negative cost shocks, ceteris paribus, cannot support collusion because firms don’t find it incentive compatible anymore not to undercut (and this is common knowledge).

The structure of the remainder of this article is as follows. The paragraph below gives an overview over some of the existing theoretical models of asymmetric pricing, most of which do not consider collusion as explanation. In the next section, we outline the general setup of the model and derive the equilibrium of the stage game. In Section 3, we consider a repeated version of the stage game and prove that, under certain parameter conditions, a unique demand tail test equilibrium exists where firms can collude on supra-competitive price levels after a negative cost shock has hit the market. In Section 4, we report qualitative results of the model. Section 5 then extends the basic model to a multitude of separated submarkets and shows that this can

\(^2\)For example, a gasoline station may regularly charge a somewhat lower price than a nearby competitor without violating a (tacit) collusive agreement. The market shares of both stations might still be similar because the cheaper station has a worse location or no adjacent shop that consumers value. But would a further decrease of the lower priced station’s prices then imply undercutting? Not necessarily, since the station might have reduced its services even more, which could be unobservable to the higher priced firm. In order to maintain the same market share, it thus needs to decrease its prices even more, although the other firm would not notice the effect through its own demand (and thus, one can hardly speak of undercutting).
produce realistic price patterns that are close to what can be observed in city level retail gasoline
data. A short summary is provided in Section 6.

**Related Theoretical Literature.** Our theoretical knowledge of asymmetric pricing is lim-
ited. Ball and Mankiw (1994) were among the first to pick up the problem. In their model,
positive trend inflation leads to adjustment asymmetry of nominal prices. This is because when
firms’ desired relative output prices fall, they can simply wait for inflation to do the job. This
is possible because other firms, whose desired relative price levels remain unchanged, will adapt
their nominal prices in accordance with the inflation rate to keep real prices on the same level.
On the other hand, when a firm’s desired relative output price rises, it has to pursue a large and
immediate rise of nominal prices in an attempt to both offset inflation and change its relative
price in the market. This is how a perceived asymmetric price adjustment emerges. As, at least
in most Western economies, the inflation rate tends to be small, the model of Ball and Mankiw
seems incapable of explaining rockets and feathers when price changes are frequent and their size
overshadows any possible change of nominal prices due to inflation, especially in the short run.
One example for such a market is retail gasoline.

Eckert (2002) does not model true asymmetric pricing, but shows that Edgeworth price cycles
can closely resemble rockets and feathers in highly competitive markets. This is because even
when costs remain unchanged, the firms in his model will slightly undercut each other in an
attempt to increase their market share. When prices are low enough, firms start to price at cost
and a war of “attrition” starts. Finally, one firm relents and prices suddenly jump back to their
maximum levels. For observers who have no information about production costs whatsoever, the
resulting price pattern can be confused with asymmetric pricing.\(^3\)

In Cabral and Fishman (2008), consumers cannot observe firms’ costs. However, they know that
cost changes are positively correlated across sellers. In the model, a change in prices leads to
increased consumer search and sticky prices. Also, the signal that goes along with a price change
is different for price increases and decreases. Asymmetric price adjustment is the result.

Yang and Ye (2008) consider a mechanism of consumer search with learning. In their model,
positive cost shocks immediately reveal the true cost state to searchers, meaning that prices
adjust to the new equilibrium level within one period. In contrast, negative cost shocks are only
learned by informed consumers and some non informed consumers who happen to observe a low

\(^3\)A very interesting related analysis is also provided by Doyle et al. (2008).
price by chance. This continues for every period where costs remain low. As a result, prices
decline much slower and gradually in response to negative cost shocks.
Tappata (2009) provides a consumer search model that is related to the work of Yang and Ye
(2008), but doesn’t rely on asymmetric learning. In his model, consumers search less when
they expect costs to be high, as firms’ optimal price dispersion, given a fixed search intensity
of consumers, is less under costs that are close to the monopoly price. Because of this, if costs
are persistent, the elasticity of demand faced by firms is different under low and high costs. If
costs unexpectedly rise, many consumers search and firms immediately adapt their selling prices.
On the other hand, if costs unexpectedly drop, only few consumers search and firms can delay
the downward adjustment of their output prices, leading to higher profits. Asymmetric pricing
results.
Finally, a model of Lewis (2009) suggests that asymmetric pricing might emerge because of
consumers’ distorted expectations about the underlying price distribution. In Lewis’ “reference
price” search model, the search decision of consumers is based on the average market price of the
preceding period. When a cost increase leads prices to rise, consumers’ expectations about the
distribution of prices will be too low, leading them to search “too much”. This implies smaller
margins and a smaller optimal price dispersion, as well as a fast adaption of output prices to cost
changes. On the other hand, consumers search “too little” when prices are falling, giving rise to
the opposite behavior and an overall observation of asymmetric pricing.

[insert more models, e.g. Wölfing (2008)]

2 Model Setup and Equilibrium of the Stage Game

Consider a market with \( i = 1, \ldots, N \) profit maximizing and risk neutral firms who compete in
prices \( p_i \) (of some single homogeneous good produced) in a dynamic environment. Firms cannot
directly observe current as well as bygone prices of their rivals. Instead, they observe their own
demand, which is an imperfect signal about the prices set by others. In order to make collusion
on higher than competitive prices possible, firms try to infer the prices of their competitors from
their own, noisy demand.\(^4\)

\(^4\)The model is robust to the introduction of a stochastic, imperfect observability of prices as long as the
probability of observing one’s rivals’ prices is sufficiently low. We argue that in a market with frequent price

Infinite horizon time is modeled as discrete, with \( t = 1, 2, \ldots \). In each period, all \( N \) firms face a
common marginal cost $c_t$. While costs are assumed to be persistent with probability $\rho < 1$ and will thus be changing with positive probability after each period, it is assumed that firms have no knowledge about the distribution of cost shocks whatsoever and base their decisions solely on the expected discounted profit stream they can accrue until the next cost shock happens. Firms discount future profits with a common discount factor $\kappa \in (0, 1)$, and the effective discount factor used for all their decision problems is given by $\delta = \rho \kappa$.

The consumer side is characterized by a continuum of identical consumers with a random total mass $\tilde{D}_t$ (henceforth called “total demand”) that is drawn from a stationary probability distribution $F$ in each period. As with prices, it is assumed that firms are unable to observe $\tilde{D}_t$ directly. Although some of the models’ results can be stated in terms of a general demand distribution function, for the rest of the paper, we consider one particular class of distribution functions borrowed from Porter (1983). These single parameter demand distributions are characterized by a constant mean of one with differing degrees of variance. The advantage of this is that we can get closed form solutions for all relevant expressions that arise in the article, while we can still analyze a variety of different demand conditions. In short, we impose that

$$F(\theta) = \mathbb{P}(\tilde{D} \leq \theta) = \left(\frac{\theta}{\beta + 1}\right)^{\beta} \quad \text{for} \quad 0 < \theta < \frac{\beta + 1}{\beta}, \quad \beta > 1. \quad (1)$$

It can easily be derived that $f(\theta) := F'(\theta) = \frac{\beta F(\theta)}{\theta}$ and $f''(\theta) := F''(\theta) = \frac{(\beta - 1)\beta F'(\theta)}{\theta^2}$, where the last expression is unambiguously positive (implying convexity of $F$) if $\beta > 1$, as assumed. Also, one can see that $\mathbb{E}(\theta) = 1$, while $\text{Var}(\theta) = \frac{1}{\beta(\beta + 2)}$. Because of the latter two properties, a decrease in $\beta$ signifies a mean preserving spread of the considered distribution function. Figure 1 depicts $F(\theta)$ for varying degrees of noise $\beta$.

Consumers have unit demand and their utility from consuming the homogeneous good produced by firms is equal to $\nu$. We assume that $\nu$ is sufficiently large such that firms’ pricing constraint, given by consumers’ valuation, is never binding in equilibrium. Thus, in the model, consumers will always prefer buying over not buying and total demand is perfectly price inelastic at each point in time. Due to spatial differentiation (alternatively, differentiation in tastes) of consumers, the (random) demand of firm $i$ if it prices at $p_i$ and all other firms price at some vector $p_{-i} = (p_1, ..., p_{i-1}, p_{i+1}, ..., p_N)$, can be written as

$$\tilde{D}_i = \tilde{D} * s_i(p_i, p_{-i}), \quad (2)$$

changes and spatial separation, this is a realistic assumption. For simplicity, we set this probability to zero for the remainder of this paper.
Figure 1: Distribution function of total market demand $\theta$ for different values of $\beta$. 
with firm $i$’s market share $s_i$ (and thus, firm $i$’s expected demand) in any period deterministically
given by the linear specification

$$s_i(p_i, p_{-i}) = \frac{1}{N} - \alpha(p_i - \bar{p}).\text{ (3)}$$

The simple market share function $s_i(p_i, p_{-i})$ summarizes the idea that in a market with $N$ other-
wise identical firms, the difference of any firm’s market share to the average market size $\frac{1}{N}$ only
depends (linearly) on the difference of the firm’s price to the average market price $\bar{p} = \frac{1}{N} \sum_{j=1}^{N} p_j$,
where the exogenous parameter $\alpha > 0$ is introduced to account for the intensity of competition
in the market.\text{ (6)} If a firm’s price is greater (smaller) than the average market price, it will receive
a market share that is smaller (greater) than the average market size. It is easy to show that
$\sum_{j=1}^{N} s_j = 1$, which implies that total demand is price inelastic.

We will focus on symmetric equilibria in pure strategies. This implies that only deviations
from a common price level $p$ need to be considered when determining firms’ market shares. Given
that all other firms price symmetrically at $p$, a unilateral deviation of firm $i$ to $p_i$ will result in
the firm getting some market share $s_i$, while all other firms get an equal split of the remainder,
i.e., $s_j = \frac{1-s_i}{N-1} \quad \forall j \neq i$. More specifically, by setting firm $i$’s market share to zero (one) for prices
$p_i$ where it would become negative (greater than one) according to equation 3, the latter directly
implies that

$$s_i(p_i; p, ..., p) = \begin{cases} 
1 & \text{if } p_i < p - \frac{1}{\alpha} \\
\frac{1}{N} - \frac{\alpha(N-1)}{N} (p_i - p) & \text{if } p_i \in [p - \frac{1}{\alpha}, p + \frac{1}{\alpha(N-1)}] \\
0 & \text{if } p_i > p + \frac{1}{\alpha(N-1)}.
\end{cases} \text{ (4)}$$

Given the above structure of firms’ market shares (expected demand), it is easy to determine
the symmetric stage game Nash equilibrium for any level of unit cost $c$. As in a symmetric

\text{5This market share function is well defined if and only if $p_i \leq \frac{1+\alpha \sum_j p_j}{\alpha(N-1)}$ for some $i$, the firms}
\text{satisfying this inequality would get a market share of less than zero when using the original specification. A well defined}
\text{market share function might then, for example, be characterized by $s_i(p_i, p_{-i}) = \left\{ \begin{array}{ll}
\frac{\frac{\alpha}{\alpha(N-1)}(p_i - p)}{\frac{\alpha}{\alpha(N-1)}(p_i - p) + \frac{1}{\alpha}} & \text{if } p_i \leq \frac{1+\alpha \sum_j p_j}{\alpha(N-1)} \\
0 & \text{else.}
\end{array} \right.$}

\text{6It can also be interpreted as the inverse of some travel cost parameter $\tau$ that discourages consumers from}
\text{buying at lower priced firms.}
equilibrium all firms get a positive market share, using standard techniques leads to the unique symmetric stage game Nash equilibrium

\[ p^*(c) = \frac{1}{\alpha(N - 1)} + c, \quad (5) \]

with associated expected profits of

\[ \Pi^* = \frac{1}{\alpha N(N - 1)}. \quad (6) \]

Equilibrium prices in the stage game are costs plus some markup that is negatively affected by competition intensity and the number of firms in the market.\(^7\) Competitive profits decrease linearly in competition intensity and quadratically in the number of firms in the market. Moreover, they are independent of costs.

Note that as \( p^*(c) \) is a strictly increasing function in \( c \), its inverse exists and is trivially given by \( c(p) = (p^*)^{-1}(p) = p - \frac{1}{\alpha(N - 1)}. \) This means that every possible price level is associated with a hypothetical cost level that would exactly imply that price level if prices were competitive. Thus, if we analyze situations were all firms price at some greater than competitive price level \( p > p^*(\bar{c}) \) when current costs are \( \bar{c} \), we can equivalently think about firms colluding on some old competitive price level \( p^*(c(p)) > p^*(\bar{c}) \).\(^8\) In such a situation, define \( \Delta c = c - \bar{c} > 0 \). Doing so, firms’ expected per period profits can be determined to be

\[ \Pi^C = \frac{1}{\alpha N(N - 1)} + \frac{\Delta c}{N}, \quad (7) \]

\(^7\)Setting \( \alpha = \frac{1}{2}, N = 2 \) and comparing the resulting equilibrium price to the equilibrium price obtained in the model of horizontal differentiation introduced by Hotelling (1929), it can be seen that the two prices coincide. The reason for this is that the (relative) demand function given by equation 3 is a generalization of Hotelling type models to \( N \) dimensions, where it is assumed that firms are located at the vertices of an \( N \)-dimensional regular simplex and consumers are uniformly distributed along its edges, with linear travel costs. Then, for strictly positive market shares of all firms in the market, a marginal decrease of the price of one firm will lead to the same loss of demand for each competing firm, which is one of the main properties of equation 3.

\(^8\)As overall demand is perfectly inelastic, it is obvious that firms would never want to stick to a price level that is smaller than the competitive one, as this would violate common rationality. Although equilibria can be constructed where this is exactly the case, it is shown in the following section that sticking to a price lower than the competitive one can never constitute an equilibrium when firms employ a simple trigger sales strategy to enforce collusion.
which is strictly larger than $\Pi^*$. Moreover, applying firm $i$’s maximization problem when firm $i$’s costs are $\tilde{c}$ and $p = p^*(c) > p^*(\tilde{c}) \quad \forall j \neq i$, it is trivial to show that the best response of firm $i$ is be given by

$$p^D = \frac{1}{\alpha(N-1)} + \frac{c + \tilde{c}}{2} > p^*(\tilde{c}),$$

implying expected profits of

$$\Pi^D = \frac{1}{\alpha N(N-1)} + \frac{\Delta c}{N} + \frac{(\Delta c)^2}{4N} > \Pi^C > \Pi^*.$$  

Later, it will also be quite handy to have an explicit expression for the expected profit a firm gets if it prices at $p_i$ and all other firms price collusively on some (hypothetical) old competitive price $p = p^*(c)$ under current costs $\tilde{c}$. Using the market share function given in equation 4 and inserting the expression for the collusive price $p^*(c)$ found in equation 5, it follows that

$$\Pi_i(p_i) = \begin{cases} 
(p_i - \tilde{c}) & \text{if } p_i < c - \frac{N-2}{\alpha(N-1)} \\
(p_i - \tilde{c}) \left[ \frac{2}{N} - (p_i - c) \frac{\alpha(N-1)}{N} \right] & \text{if } c - \frac{N-2}{\alpha(N-1)} \leq p_i \leq c + \frac{2}{\alpha(N-1)} \\
0 & \text{if } p_i > c + \frac{2}{\alpha(N-1)},
\end{cases}$$

where for brevity, $\Pi_i(p_i) = \Pi_i(p_i; p^*(c), ..., p^*(c))$.

Defining $p^L := c - \frac{N-2}{\alpha(N-1)}$, $p^H := c + \frac{2}{\alpha(N-1)}$, one can see that for $p_i \in [p^L, p^H]$,

$$\left. \frac{\partial \Pi_i(p_i)}{\partial p_i} \right|_{p_i = p^*(c)} = -\frac{\alpha(N-1)\Delta c}{N} < 0,$$

while

$$\left. \frac{\partial^2 \Pi_i(p_i)}{\partial p_i^2} \right|_{p_i = p^*(c)} = -\frac{2\alpha(N-1)}{N} < 0.$$  

If every other firm keeps pricing on some (hypothetical) old competitive price that exceeds the current competitive price, a firm could increase its profits by unilaterally deviating to some lower price level. Moreover, the incentive to deviate increases in the size of the cost gap. Since individual profits are strictly concave, they must exhibit a unique maximum.
3 Equilibrium of the Repeated Game

It has often been argued that firms might find it hard to coordinate on a specific collusive equilibrium if there exists a multiplicity of equilibria, as is typically the case in infinitely repeated games under sufficiently high discount factors. For markets with low price elasticities of demand, firms are in principle able to coordinate on a whole range of supra-competitive prices by employing trigger strategies. However, a priori it seems unclear on which of these equilibria firms should collude on, which might result in collusion not to be implemented at all. Of course, firms might try to coordinate on the equilibrium which maximizes their expected discounted profit stream given market conditions. For the case of unobservable stochastic price shifts in a Cournot oligopoly – a market structure closely related to ours – this problem has been dealt with in great sophistication by Porter (1983), Abreu et al. (1986) and Abreu et al. (1990).

The drawback of these models is that they deal exclusively with the case of invariant production costs and are therefore not directly applicable for modeling asymmetric price transmission. Moreover, even when costs are constant, the computational abilities of the involved enterprises would have to be exceptional to allow for calculating the optimal strategy. As a consequence, we believe that firms’ strategies will in practice not be characterized by the strategies that maximize their expected profits. Rather than this, it seems natural to assume that firms would pick strategies that are easy to implement, robust to cost shocks and that still yield supra-competitive profits.

Thus, as has been suggested by Borenstein et al. in their seminal paper of 1997 and as was recently adopted in a related model of Sherman (2011), we restrict our attention to one special type of equilibrium: focal point pricing that is enforced via a simple demand tail test. Due to coordination problems, firms will either price competitively or collude on a common focal point price as long as their demand stays above a certain threshold, i.e., unless their random demand $\tilde{D}_i$ falls too far in the lower tail of its distribution given that all firms collude on $p$. If employed correctly, this effectively discourages unilateral deviation, while the strategy is very easy to implement. As Sherman (2011) puts it, focal point pricing is essentially passive and thus requires a minimum of coordination.

9In our terminology, a focal point is given by a common price level $p$ that at some point was used by every firm in the market. Typically, (old) competitive price levels will constitute such focal points.
We believe that introducing this type of focal point strategies can improve our understanding of asymmetric price adjustment because compared to existing theoretical models, a great deal of complexity is added by allowing for a continuum of different cost states (in the available literature, typically only one high and one low cost state is considered). For example, in our model, costs can fall several times in a row, or there can be non-monotonic behavior of costs\(^\text{10}\), which has the potential to create patterns of price adjustment that are much closer to what is observed in reality than in related models.

Let us begin with a formal characterization of firms’ behavior. We restrict firms’ strategies to a state contingent pricing behavior with punishment of low demand, where the only information that matters is that of the current and that of the previous period. At the beginning of each period \(t\), firms observe their last period’s realized demand \(D_{i,t-1}\), last period’s costs \(c_{t-1}\), and current costs \(c_t\). Now, given any old price \(p\) a firm charged last period, its new price \(p_{i,t}\) is given by

\[
p_{i,t}(D_{i,t-1}, c_t, c_{t-1}, p) = \begin{cases} 
p^*(c_t) & \text{if } p \neq p^*(c_{t-1}) \text{ and } D_{i,t-1} < k(.) \text{ or } \\
p_t \text{ such that } E(\Pi^S_i(p_t; k(.))) > E(\Pi^S_i(p; k(.))) & \\end{cases}
\]

where \(E(\Pi^S_i(p; k(.)))\) is a short notation for the expected discounted profit stream firm \(i\) would get when pricing at \(p\), under the assumption that all other firms price at \(p\) as well, as long as their individual demand stays above \(k(.) = k(p, c_{t-1})\), i.e., under the assumption that the other firms use \(p\) as a focal point. As the notation suggests, we do not restrict \(k\) to some fixed level: it is allowed to vary with the (relevant) state of the economy, which means that it can, in principle, depend on both previous period’s price level and previous period’s costs.\(^\text{11}\) However, as will be seen below, a nice feature of our model is that the unique trigger value \(k^*\) that is compatible with sustaining supra-competitive prices in our proposed focal point equilibrium does not depend on the state of the economy.

\(^{10}\)E.g., after costs have fallen and coordination on a higher than competitive price level is established, costs rise again, but sufficiently small such that collusion can be maintained.

\(^{11}\)Since firms compare their lagged demand \(D_{i,t-1}\) with the punishment threshold \(k\), also \(k\) should depend on the lagged state of the economy.
To sum up the above strategy specification, firms will adapt their prices to the (possibly new) competitive level $p^*(c_t)$ whenever they didn’t price competitively in the previous period and their last period’s demand fell below a certain threshold $k$ (punishment of potential deviation) or whenever they realize that not sticking to the old price $p$ could give them higher expected profits than doing so, under the assumption that all other firms stick to that price (incentive compatibility of pricing supra-competitive).\footnote{In order to avoid confusion, it has to be emphasized that the outlined strategy is not dependent in any way on what the other firms actually do, i.e., what their strategy is. The calculation of a firm’s expected profit stream given that all other firms collude on $p$, which is an arbitrary assumption, can be carried out no matter whether this is how the other firms actually behave. All that is interesting is whether the above strategy, if employed by every firm in the market, constitutes a mutual best response. If so, it is an equilibrium, no matter whether firms’ assumptions are correct or not.}

Clearly, because of the aggregate nature of demand shocks in the model, firms must price the same at all times when every firm adopts the above strategy. In each period, either the necessary conditions to collude on some non-competitive price level are fulfilled for all firms, or each firm will price competitively. Thus, the state of the economy in some period $t$ can always be perfectly described by the current, common price level $p_t$ and the current costs $c_t$. Overall, we can now state

**Proposition 3.1.** Starting from some common price level $p$, the strategy combination in which every firm prices according to equation 13 constitutes an equilibrium of the repeated game, no matter how the parameters of the economy are and no matter how the demand threshold $k$ is chosen.

**Proof.** An equilibrium of the repeated game is characterized by a strategy combination in which each firm’s strategy is a best response to all other firms’ strategies in any possible situation that arises. Since pricing at $p^*(c_t)$ is a best response to all other firms doing so, it is immediately clear that whenever prices were not competitive in the previous period and demand was below the critical threshold $k(.)$, firms’ specified action of adapting their prices to $p^*(c_t)$ (punishment of potential undercutting) must be optimal, i.e., firms’ strategies are a best response to each other in this situation. The same argument holds true whenever there exists some $p_i$ such that any firm’s expected profit stream, assuming that all other firms stick to the old price level $p$, could be increased by unilaterally deviating to $p_i$. As every firm will realize this, every firm will start to price competitively by assumption, which is also a mutual best response. Finally, if there exists no $p_i$ that allows for a profitable unilateral deviation from $p$ (assuming that all other firms stick
to \( p \), every firm will indeed stick to \( p \) by our specification. But since no profitable unilateral deviation to this is possible, this again constitutes a mutual best response.

Note that although firms are rational and thus know that no firm will ever deviate from the above equilibrium, it is still optimal for them to punish by pricing competitively if demand was less than \( k \) in the previous period. There are two reasons for this. First, if all other firms start to price competitively, a firm’s best response is also given by pricing competitively. Second, if firms didn’t follow through with punishing low demand (i.e., if that threat was not credible), no firm would have an incentive to behave collussively in the first place, since deviation would pay.

Now, given the above (equilibrium) setup of firms’ strategies, we will derive one necessary and one sufficient condition for the existence of a focal point equilibrium in which firms can actually manage to collude on supra-competitive prices \( p \) when the economy is in suitable states. Moreover, we will characterize these states. \(^{13}\) What first needs to be determined is which values of the demand threshold \( k \), if any, allow for maintaining a price level of \( p \) when no punishment was induced by low demand in the previous period. First of all, it is trivial to see that \( k \) must be strictly larger than zero. If this was not the case, there would be no punishment of deviation from \( p > p^*(\bar{c}) \) to a lower price that gives a higher expected per period profit (compare with equation 11). Thus, by our specified strategy combination, firms would always have some scope for unilateral deviation and would thus price competitively in each period.

For positive demand thresholds, the probability of collusion to be maintained if a firm sticks to \( p = p^*(c) \) is always higher than if the firm deviates to any price lower than that. This is intuitively straightforward: deviation to a lower price increases one’s own market share at the cost of the other firms and thus the likelihood that their (random) demand exceeds some threshold value \( k > 0 \), which is always strictly less than one by our specification of the demand distribution function, decreases.

In the following, we denote by \( q(k) \) the probability that the demand of the other firms is at least \( k \) under collusion on \( p = p^*(c) \) and denote by \( r(p_i, k) \) the probability that the demand of the other firms is at least \( k \) if one deviates to some \( p_i < p^*(c) \). It is then straightforward to show that

\[
q(k) = 1 - F(Nk) \quad \text{and} \quad (14)
\]

\(^{13}\)As in section 2, let \( \Delta c(p, c_t) = c(p) - c_t \) denote the difference of last period’s hypothetical cost level \( c \) (given prices of \( p \)) to current costs \( \bar{c} = c_t \).
\[ r(p_i, k) = 1 - F \left( \frac{Nk}{\frac{N-2}{N-1} + \alpha(p_i - c)} \right) \] \quad (15)

The expected discounted profit stream of pricing at \( p_i \) is given by the expected immediate profit of pricing at \( p_i \) (which can be accrued for certain), plus the expected profit in the next period, which will be given by \( \Pi_i(p_i) \) with probability \( r(p_i, k) \) and which changes to \( \Pi^* \) with probability \( 1 - r(p_i, k) \), times the effective discount factor (the probability \( \rho \) that a new period with unchanged costs is reached times firms’ discount factor \( \kappa \)), plus the expected profit in the third period, which will be given by \( \Pi_i(p_i) \) if and only if the other firms’ demand reached at least \( k \) in both the first and the second period, i.e., with probability \( r^2(p_i, k) \), and which will otherwise be given by \( \Pi^* \), times the effective discount factor squared, and so on. Formally,

\[
\mathbb{E}(\Pi_i^S(p_i, k)) = \Pi_i(p_i) + \delta [r(p_i, k) \Pi_i(p_i) + (1 - r(p_i, k)) \Pi^*] \\
+ \delta^2 [r^2(p_i, k) \Pi_i(p_i) + (1 - r^2(p_i, k)) \Pi^*] + \ldots
\]

which can be simplified to

\[
\mathbb{E}(\Pi_i^S(p_i, k)) = \frac{\Pi_i(p_i) - \Pi^*}{1 - \delta r(p_i, k)} + \frac{\Pi^*}{1 - \delta}.
\] \quad (16)

In order for collusion on some higher than competitive price level \( p = p^*(c) > p^*(\tilde{c}) \) to be sustainable, \( k \) must be chosen such that the first derivative of equation 16 with respect to \( p_i \) at \( p_i = p^*(c) \) is equal to zero. As \( p^*(c) \) lies in the interior of firms’ pricing range, this is the necessary (but not sufficient) condition for colluding on \( p^*(c) \) to be optimal in terms of firms’ expected profit stream (if potential undercutting is punished by using a demand threshold \( k \)). This leads to

**Proposition 3.2.** (Necessary condition) Firms can only stick to a supra-competitive price \( p^*(c) > p^*(\tilde{c}) \) in the specified focal point equilibrium if the demand threshold \( k \) employed by firms is given by

\[ k^* := \left( \frac{\frac{1}{2} - \frac{1}{\beta N - 1}}{\frac{\beta}{N - 1}} \right)^{\frac{1}{2}} \frac{\beta + 1}{\beta N}. \] \quad (17)

**Proof.** See Appendix. \( \square \)

\(^{14}\)Comparing these expressions, by the monotonicity of the distribution function \( F \) of total market demand, \( r(p_i, k) \) turns out to be smaller than \( q(k) \) if and only if \( p_i < p^*(c) \), which confirms the intuition from above.
As mentioned above, one interesting property of the above necessary condition for the sustainability of supra-competitive prices in equilibrium is that any solution $k^*$ will be independent of the state of the economy (as well as of competition intensity $\alpha$). The former implies that even when costs fall further after an initial decrease (or increase again, for that matter), not deviating from the old competitive price, if it remains incentive compatible, doesn’t require adaption of $k^*$, which is unique and fixed for all time. One can imagine that this fact would greatly facilitate coordination among firms in markets that are similar to ours.

Since it is assumed that $\beta > 1$, $k^*$ is only well defined if $\frac{1}{\beta} - 1 \geq 0$, which is the case if $\beta > N - 1$.

In other words, given some number of firms $N$ in the market, the demand distribution $F$ should not be too noisy. Only in that case, firms have a somewhat reliable chance to correctly identify deviations of their competitors and collusion doesn’t break down too often in face of random demand shocks. Notice, moreover, that since $\delta < 1$ (and thus the nominator of the above fraction is strictly positive), $k^* = 0$ can never be the result of equation 17. As explained above, this is because no punishment of deviation would always destroy the incentive compatibility of sticking to prices that are above the competitive level.

We have seen that $k^*$ exists if $\beta > N - 1$. However, very high values of $k^*$ that cannot even be reached when demand is at its maximum (given the noisiness $\beta$ of the distribution of total demand) can definitely not allow for maintaining supra-competitive prices: if collusion must always break after the first period of its implementation, there can be no effective punishment of deviation in the first period, destroying incentive compatibility of sticking to any $p > p^*(\tilde{c})$.

Given the chosen shape of $F$, individual demand can never exceed $\frac{\beta + 1}{\beta N}$. Comparing $k^*$ with this expression leads to

**Proposition 3.3.** *(Sufficient condition)* There are states of the economy in which firms manage to sustain supra-competitive prices in a focal point equilibrium if $k = k^*$ and firms’ effective discount factor is sufficiently large, i.e.,

$$\delta > \delta := \frac{N - 1}{\beta}.$$  

Proof. To prove this, one needs to check whether for any $\delta > \delta$, there exists at least one pair $(p, c_t)$ such that firms prefer sticking to $p$ over pricing at any other price $p_i$, given that they assume that every other firm also sticks to $p$ (when the demand threshold $k^*$ is used). This will become apparent in the analysis of states that are suitable for collusion on supra-competitive prices that is carried out below. \qed
Let us now analyze under which conditions firms will actually continue to price at \( p \) when this is not the competitive price level. Proposition 3.2 asserts that choosing \( k = k^* \) is a necessary, but not sufficient condition for collusion on some supra-competitive price level \( p \) to be possible in the analyzed equilibrium. For sufficiency, continuing to price at \( p = p^*(c) \) needs to be a global maximum of firms’ expected profit stream \( E(\Pi_i^S(p_i, k^*)) \), assuming that all other firms continue to price at \( p \).

In any case, \( p^*(c) \) must at least constitute a local maximum (rather than a local minimum) of the expected profit stream function. To see this, consider the following

**Lemma 3.4.** \( p^*(c) \) is a local maximum of the expected profit stream function \( E(\Pi_i^S(p_i, k^*)) \) if the distribution function \( F \) of total market demand is convex in its argument.

**Proof.** See Appendix. \( \square \)

**Corollary 3.5.** As \( F \) is strictly convex for \( \beta > 1 \), as assumed, \( p^*(c) \) is in fact a local maximum of the expected profit stream function.

For a better understanding of the sufficient conditions that make pricing at some old price \( p^*(c) \) the global best response to the assumption that all other firms’ stick to the same price \( p^*(c) \) (given that the effective gap in costs is \( \Delta c = c - \tilde{c} \)), it is useful to take a closer look at the shape of a single firm’s profit function when its rivals continue to price at \( p^*(c) \) under low costs \( \tilde{c} \). Given the market share function defined in equation 3, the expected profit function \( E(\Pi_i(p_i)) \) of a single firm will consist of four distinct regions (Figure 2).

In region I, firm \( i \)’s price is so low compared to the price of all other firms, namely \( p_i \leq p^L \), that it captures all of the market, implying an expected per period demand of one. Obviously, as firm \( i \) cannot increase its demand anymore by undercutting to a price that is even lower than \( p^L \), pricing in the interior of region I can never be optimal. Similarly, in region IV, firm \( i \)’s price is so high compared to the price of all other firms, namely \( p_i \geq p^H \), that it loses all of its market share and makes a profit of zero in every period until collusion is abolished. As the firm could get a strictly positive profit until collusion breaks by pricing at \( p^*(c) \), for example, no price in region IV can ever constitute a global maximum of firm \( i \)’s profit function.
Figure 2: Firm $i$'s expected profit stream (solid blue line), firm $i$'s per period surplus over competitive profits (solid orange line) and the overall percentage probability that collusion is maintained (solid green line) for different prices $p_i$. The parameters that were used to obtain this picture are $\alpha = 0.03$, $\beta = 4$, $N = 3$, $c = 100$, $\tilde{c} = 60$ and $\delta = 0.9$. 
In region II, firm \( i \) does not capture the whole market anymore (as in region I), but the probability that firms keep colluding in later periods remains zero. Using equations 15, 17 and 1, it is not hard to show that

\[
r(p_i, k^*) = 1 - \left( \frac{\frac{1}{\delta} - 1}{\frac{N-2}{N-1} + \alpha(p_i - c)} \right)^{\frac{1}{\beta}}.
\]  

(19)

Setting this expression equal to zero and solving for \( p_i \), one can calculate

\[
p_M = c - \frac{N - 2}{\alpha(N - 1)} + \left( \frac{\frac{1}{\delta} - 1}{\frac{N-2}{N-1} - 1} \right)^{\frac{1}{\beta}}/\alpha,
\]  

(20)

which is the threshold above which this probability becomes strictly positive. Inserting \( r(p_i, k^*) = 0 \) into equation 16, one can see that region II is given by a quadratic function in \( p_i \), which is comprised of its immediate, quadratic per period profit function \( \Pi_i(p_i) \) (compare with equation 10) plus an infinite stream of the discounted competitive profit level \( \Pi^* \), starting from the next period.

Whether the profit stream function in region II is monotonically increasing, decreasing or exhibits an interior maximum depends on the location of the optimal undercutting price \( p_D \) relative to region II. If \( p_D \) is smaller than \( p_L \), it is monotonically decreasing; if it is larger than \( p_M \), it must be monotonically increasing; and if it falls inside region II, \( p_D \) constitutes its maximum. Clearly, if \( p_D > p_M \), the global best response can never be in region II because by marginally increasing its price \( p_i \) starting from \( p_M \), firm \( i \) can increase its per period profits (which reach their maximum at \( p_D > p_M \), by assumption) and, at the same time, also increase the probability that collusion is maintained \( r(p_i, k^*) \).

Finally, region III is characterized by the range of prices \( p_i \in (p_M, p_H) \) where firm \( i \) maintains a positive chance that collusion on the high cost level is sustained by the other firms in the market and where firm \( i \) prices low enough that it still gets a positive market share in collusive periods. Thus, an increase (decrease) of prices in region III will always have two effects on the expected profit stream the firm gets. First, it will have an ambiguous effect on the per period surplus over competitive profits \( \Pi_i(p_i) - \Pi^* \), depending on the value of the optimal (in terms of per period profits) price \( p_D \). Second, it will have an unambiguously negative (positive) effect on the probability \( r(p_i, k^*) \) that the other firms keep colluding, in accordance with the firm’s change in market share. Furthermore, given our model specification, one can prove the following
**Lemma 3.6.** The only possible candidate for a global maximum of firm $i$’s profit function in region III is given by $p_i = p^*(c)$.

*Proof.* See Appendix.

Now that the four regions of a firm’s profit function have been described, it is possible to pin down sufficient conditions on $\Delta c$ – given the other model parameters – that guarantee that sticking to $p^*(c)$ is optimal when a firm assumes that all other firms do so as well (and the demand threshold $k^*$ is used). In the above analysis, one could see that there are only three possible candidates for a global optimum of firms’ profit function: $p^*(c)$ (the collusive price), $p^L$ (the highest possible price where the whole market is captured) and $p^D$ (the price that maximizes per period profits for interior market shares). This greatly facilitates the analysis, and after some calculations, one can show

**Proposition 3.7.** For high effective discount factors

$$\delta \geq \hat{\delta} := \frac{\beta N - [\beta - (N - 1)]}{\beta N},$$

(21)

sticking to the collusive price is always incentive compatible for firms, no matter how far prices are above the competitive price level. For intermediate effective discount factors that lie in the interval $[\hat{\delta}, \hat{\hat{\delta}})$, where

$$\hat{\delta} := \frac{(N - 1)(\beta + 2)}{(N + 1)\beta},$$

(22)

collusive prices can be maintained if and only if

$$\Delta c \leq \Delta c := \frac{(1 - \hat{\delta})(N - 1)\beta}{\alpha\{\beta N(1 - \hat{\delta}) - [\beta - (N - 1)]\}}.$$

(23)

Finally, for small effective discount factors $\delta < \hat{\delta}$, collusive prices can be maintained if and only if

$$\Delta c \leq \Delta c := \frac{4[(\beta - (N - 1))]}{\alpha\beta(1 - \delta)(N - 1)}.$$

(24)

*Proof.* See Appendix.
4 Model Results and Comparison to Station Level Gasoline Price Data

In what follows, we will restrict our attention to the case where \( \delta \geq \hat{\delta} \). As can easily be shown, this automatically guarantees that \( \delta > \hat{\delta} \) and hence that a focal point pricing strategy as defined in section 3 can be a best response to firms. The restriction facilitates the remaining analysis because one only has to deal with two distinct cases: either \( \delta \in [\hat{\delta}, \hat{\hat{\delta}}) \) and there exists some unique upper threshold \( \Delta \) above which collusion on old focal points cannot be maintained, or \( \delta > \hat{\hat{\delta}} \) and firms will never optimally deviate from sticking to a collusive price level unless a punishment phase is triggered by low demand.

While it is possible that both \( \delta > \hat{\delta} \) and \( \delta < \hat{\delta} \), the analysis of this scenario leads to the same qualitative results as the analysis of the case where \( \delta \in [\hat{\delta}, \hat{\hat{\delta}}) \), except for the rather trivial result that smaller negative cost shocks suffice to induce a downward adaption of prices because firms value future profits less (and thus have a higher incentive to deviate from collusive prices). We will thus ignore that case, although technically it can be dealt with no more difficulty than the other ones.

We are now able to analyze the degree of asymmetric price adjustment that emerges in the model. For this, we first need to formally define what we consider to be asymmetric pricing. Below, we will give three different characterizations of it and prove that the model is in general able to generate all of them.

**Definition 4.1.** Price adjustment is called *steady state impulse asymmetric with respect to* \( c, dc \) if, starting from any steady state price equilibrium given current costs \( c \), an unexpected and persistent positive cost shock of size \( dc \) induces a different expected length until price adjustment to the new steady state price equilibrium is completed, compared to a negative shock of size \( -dc \). It is said to be *negative* (respectively, *positive*) *steady state impulse asymmetric with respect to* \( dc \) if in expectation, a negative (respectively, positive) cost shock leads to a longer (respectively, shorter) expected length of transition.

As Definition 4.1 is easiest to analyze, let us start with discussing it. In the previous section, it was shown that as long \( \delta > \frac{N-1}{2} \), the proposed focal point strategies form an equilibrium of the repeated game where collusion on higher than competitive prices \( p > p^*(c) \) is possible as long
as firms’ individual demand stays above a threshold $k^*$ (and, for low $\delta$, the effective gap in costs $\Delta c$ is not too large). Using equations 1 and 17, we find that the probability of demand falling short of $k^*$ can be written as

$$P(k < k^*) = \frac{\frac{1}{\delta} - 1}{N - 1} \in (0, 1),$$

which is positive. This implies that any price $p > p^*(c)$ can never be a steady state price equilibrium given $c$, as the probability of adjustment to the competitive level is 1 for $t \to \infty$. That is, Definition 4.1 only applies for the competitive price level $p^*(c)$.

Now, by applying Lemma 3.6, one can see that for $p = p^*(c)$, a positive cost shock $dc$ must lead to immediate adaption of prices to the new competitive level $p^*(c + dc)$ in any case. On the other hand, as firms’ strategies are specified such that their previous period’s demand does not matter for their pricing decision whenever last period’s prices were competitive, there will only be an immediate adaption of prices to a negative cost shock $-dc$ if collusion is not incentive compatible. By Proposition 3.3, this can only be the case if $\delta < \hat{\delta}$ and $dc > \Delta c$ (if $\delta \in [\tilde{\delta}, \hat{\delta})$) or $dc > \Delta c$ (if $\delta < \tilde{\delta}$).

5 Extension to Multiple Submarkets and Comparison to City Level Gasoline Price Data

6 Summary
References


Matthew Lewis. Asymmetric price adjustment and consumer search: An examination of the retail gasoline market. The Ohio State University, Department of Economics, September 2009.


Joshua Sherman. When asymmetric price response is more pronounced under head-to-head competition. Department of Economics, Bar-Ilan University, Ramat Gan, Israel, February 2011.

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7 Appendix A: Proofs

Proof of Proposition 3.2. The necessary condition for \( p^*(c) \) to constitute an equilibrium is given by
\[
\frac{\partial \Pi_i^S(p_i, k)}{\partial p_i} = 0.
\]
This is equivalent to the condition
\[
\Pi_i'(p^*(c))[1 - \delta q(k)] + \delta [\Pi^C - \Pi^*] \left( \frac{\partial \pi_i(p_i, k)}{p_i} \right)_{p_i = p^*(c)} = 0.
\]
Using equation 11 to substitute \( \Pi_i'(p^*(c)) \) and comparing equations 6 and 7 to see that \( \Pi^C - \Pi^* = \frac{\alpha \delta}{Nk^2} \), as well as calculating \( \frac{\partial \pi_i(p_i, k)}{p_i} \) to be \( f(Nk)N\kappa \) (where \( f(Nk) \) is the first derivative of \( F \) evaluated at the value \( N\kappa \)), the above first order condition simplifies to
\[
\frac{N - 1}{N} [1 - \delta + \delta F(Nk)] - \delta kf(Nk) = 0,
\]
which must hold for general demand distribution functions \( F \). Using equation 1, this condition can be explicitly solved for \( k \), which leads to the expression in the proposition. \( \Box \)

Proof of Lemma 3.4. For this, it is convenient to define \( u(p_i) \) as the nominator and \( v(p_i) \) as the denominator of the first, price dependent term of \( \Pi_i^S(p_i, k) \), that is, \( u(p_i) = \Pi_i(p_i) - \Pi^* \) and \( v(p_i) = 1 - \delta r(p_i, k) \). Clearly, \( v(p_i) \) is a strictly positive function. But if some function \( v \) is strictly positive, it can easily be shown that values \( \chi \) satisfying the first order condition for an extremum of \( \frac{\alpha}{N} \) (as \( p^*(c) \) does, given \( k^* \)), will satisfy the second order condition for a local maximum (i.e., \( \frac{\alpha}{N} \) being concave at its extremum \( \chi \)) if and only if \( u''v - uv'' > 0 \). As
\[
u(p^*(c)) = \frac{\alpha}{N} > 0, u''(p^*(c)) = -\frac{2\alpha(N-1)}{k^2} < 0 \] (compare with equation 12) and \( v(p^*(c)) = 1 - \delta + \delta F(Nk) = 1 - \delta q(k) > 0 \), this will typically depend on the functional form of \( v''(p^*(c)) \), which can be determined to be
\[
\delta Nk^2 [NKF'(Nk) + 2\alpha f(Nk)].
\]
Obviously, if \( F \) is convex, the term in brackets will be positive, which implies that \( u''v - uv'' \) will unambiguously be negative, proving that \( p^*(c) \) constitutes a local maximum given \( k = k^* \) and a convex distribution function of total demand. \( \Box \)

Proof of Lemma 3.6. By convexity of \( F \) (and given our choice of \( k = k^* \)), we know that continuing to price at \( p^*(c) \) must constitute a local maximum in region III. However, the functional form of \( \Pi_i^S \) in region III is fairly complicated. While it seems, in general, not possible to analytically determine local extremal points of the above function other than \( p^*(c) \) (if there are any), we show below that the profit stream function in region III must be strictly monotonically increasing for \( p_i \in [p^D, p^*(c)] \) and strictly monotonically decreasing for \( p_i \in (p^*, p^H] \). Because of that, \( p^*(c) \) must constitute a unique maximum in the region \([p^D, p^H]\). Thus, even when \( p^D \) lies in region III as well, \( p^*(c) \) must also be the unique global maximum of the whole region. To see this, note that decreasing \( p_i \) further than \( p^D \) can never be optimal because both the probability of collusion to be sustained and the per period surplus over competitive profits \( \Pi_i(p_i) - \Pi^* \) would fall.

We now show that in region III of a firm’s price range, \( \Pi_i^S(p_i, k^*) \) must be strictly monotonically increasing for \( p_i \in [p^D, p^*(c)] \) and strictly monotonically decreasing for \( p_i \in (p^*, p^H] \), which completes the proof. Begin
by defining $\tilde{\rho} := p^*(c) + \frac{\delta}{\epsilon}$, where $\epsilon$ can be positive or negative. Then, inserting this expression into equations 10, 11 and 19, respectively, one gets
\begin{equation}
\Pi_i(\tilde{\rho}) - \Pi^* = \frac{\Delta c}{N} - \epsilon(N - 1)\Delta c - \frac{\epsilon^2(N - 1)}{\alpha N},
\end{equation}
\begin{equation}
\Pi'_i(\tilde{\rho}) = - \frac{\alpha(N - 1)\Delta c}{N} - \frac{2(N - 1)\epsilon}{N} - \frac{1 - \delta}{\left(\frac{\beta}{N-1} - 1\right)(1 + \epsilon)^\beta},
\end{equation}
\begin{equation}
1 - \delta r(\tilde{\rho}, k^*) = 1 - \delta + \frac{1 - \delta}{\left(\frac{\beta}{N-1} - 1\right)(1 + \epsilon)^\beta}.
\end{equation}

Also, taking the first derivative of $r(p_i, k^*)$ (obtained from equation 19) with respect to $p_i$ and evaluating this expression at $\tilde{\rho}$, it follows that
\begin{equation}
r'(\tilde{\rho}, k^*) = \frac{\alpha \beta \left(\frac{1}{2} - 1\right)}{\left(\frac{\beta}{N-1} - 1\right)(1 + \epsilon)^{\beta+1}}.
\end{equation}

Finally, plugging in the above results in the numerator of the first derivative of firm $i$’s profit stream function for region III (obtained from equation 16), i.e., calculating
\begin{align*}
\frac{\partial \Pi^3_i(p_i, k^*)}{\partial p_i}
&= \left\{ \Pi'_i(p_i)[1 - \delta r(p_i, k^*)] - \delta \Pi'_i(p_i) - \delta r(p_i, k^*) \frac{\partial r(p_i, k^*)}{\partial p_i} \right\}
\bigg|_{p_i = \tilde{\rho}}
\end{align*}

one can see that
\begin{align*}
\text{sgn} \left( \frac{\partial \Pi^3_i(p_i, k^*)}{\partial p_i} \right)
&= \left[ \frac{\alpha(N - 1)\Delta c}{N} - \frac{2(N - 1)\epsilon}{N} \right] \left[ 1 - \delta + \frac{1 - \delta}{\left(\frac{\beta}{N-1} - 1\right)(1 + \epsilon)^\beta} \right] + \delta \left[ \frac{\Delta c}{N} - \epsilon(N - 1)\Delta c - \frac{\epsilon^2(N - 1)}{\alpha N} \right] \left[ \frac{\alpha \beta \left(\frac{1}{2} - 1\right)}{\left(\frac{\beta}{N-1} - 1\right)(1 + \epsilon)^{\beta+1}} \right].
\end{align*}

Rearranging terms, it must hold that
\begin{align*}
\text{sgn} \left( \frac{\partial \Pi^3_i(p_i, k^*)}{\partial p_i} \right)
&= \left[ \frac{\alpha(N - 1)\Delta c}{N} - \frac{2(N - 1)\epsilon}{N} \right] \left[ 1 - \delta + \frac{1 - \delta}{\left(\frac{\beta}{N-1} - 1\right)(1 + \epsilon)^\beta} \right] + \delta \left[ \frac{\Delta c}{N} - \frac{\epsilon(N - 1)\Delta c}{\alpha N} + \frac{\epsilon^2(N - 1)}{\alpha N} \right] \left[ \frac{\alpha \beta \left(\frac{1}{2} - 1\right)}{\left(\frac{\beta}{N-1} - 1\right)(1 + \epsilon)^{\beta+1}} \right].
\end{align*}

Using that $\tilde{\rho} = p^*(c)$ for $\epsilon = 0$, from the definition of $k^*$ (namely that $\frac{\partial \Pi^3_i(p_i, k^*)}{\partial p_i} \bigg|_{p_i = p^*(c)} = 0$), it follows that the above expression must be equal to zero for $\epsilon = 0$. 

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Moreover, for any $\epsilon > 0$, it clearly holds that the sum of the last two terms must be negative. Also, for $-1 < \epsilon < 0$, \footnote{$\epsilon > -1$ is equivalent to $\tilde{\rho} > p^L$, which must always be satisfied in region III.} it is easy to show that the sum of the last two terms must be positive for all (relevant) values of $\epsilon$ that keep the price above $p^D$. To see this, note that for $\epsilon < 0$, \(\epsilon(N-1)\Delta c + \frac{\epsilon^2(N-1)}{\alpha N}\) can only become positive (leading to an ambiguous sign of the sum of the last two terms) if $\epsilon < -\alpha \Delta c$, or $\tilde{\rho} < \frac{1}{\alpha(N-1)} + \epsilon < p^D$, i.e., this can only happen for values of $\tilde{\rho}$ that are not needed for the proof (since pricing below $p^D$ can never be optimal in region III).

Proof of Proposition 3.7. Note that the expected profit stream associated with the boundary non-collusive price $p^L$ will only exceed the expected profit stream associated with $p^D$ if the latter lies in region I, i.e., $p^D < p^L$. The reason for this is that demand cannot be increased anymore for prices smaller than $p^*$, rendering pricing at $p^D < p^L$ pointless. Solving $p^D < p^L$, this is equivalent to $\Delta c > \frac{2}{\alpha}$. That is, for $\Delta c$ larger (smaller) than $\frac{2}{\alpha}$, $p^L$ must yield a higher (lower) profit stream than $p^D$, and it is sufficient to compare $\Pi_L^\beta(p^*(c), k^*)$ to $\Pi_L^\beta(p^L, k^*)$ to determine whether sticking to the collusive price is incentive compatible or not.

Let us begin by examining the case of large cost shocks, i.e., $\Delta c \geq \frac{2}{\alpha}$. Inserting $p^*(c)$ in equation 16 and using equations 6, 7 and 19, it is not hard to verify that

$\Pi_L^\beta(p^*(c), k^*) = \frac{\Delta c[\beta - (N-1)]}{\beta N(1-\delta)} + \frac{\Pi^*}{1-\delta},$ \hspace{1cm} (33)

whereas inserting $p^L$ in equation 16, using equations 6 and 9 as well as noting that $r(p^L, k^*) = 0$, one can see that

$\Pi_L^\beta(p^L, k^*) = \Delta c \left[ \frac{N-1}{\alpha N} + \frac{\Pi^*}{1-\delta} \right].$ \hspace{1cm} (34)

\footnote{Since, for $\epsilon = 0$, the last two terms of equation 30 drop out and also the whole expression is equal to zero (by the definition of $k^*$), this is a trivial consequence.}
Solving $\Pi_i^S(p^*(c), k^*) < \Pi_i^S(p^L, k^*)$ for $\Delta c$, one runs into two possibilities. First, if

$$\delta \geq \frac{\beta N - (\beta - (N - 1))}{\beta N} = \hat{\delta},$$

firms are patient enough and costs are sufficiently persistent such that no matter how large $\Delta c$ gets, firms find it always optimal to stick to the collusive price. If this is not the case, i.e. $\delta < \hat{\delta}$, one can easily show that incentive compatibility of sticking to the collusive price is violated whenever

$$\Delta c > \frac{(1 - \delta)(N - 1)\beta}{\alpha\beta N (1 - \delta) - [\beta - (N - 1)]} = \Delta c.$$

Note that for the above inequality, $\Delta c \geq \frac{2}{\alpha}$ holds if and only if

$$\delta \geq \frac{(N - 1)(\beta + 2)}{(N + 1)\beta} = \tilde{\delta},$$

where it can be shown that $\hat{\delta} < \tilde{\delta}$. Only if inequality 22 is satisfied, i.e., $\delta \in \left[\hat{\delta}, \tilde{\delta}\right]$, it is valid to compare the profit stream of pricing at $p^L$ to the profit stream of pricing at $p^*(c)$ in order to determine whether sticking to the collusive price level is incentive compatible or not.

Next, consider the case of small cost shocks, i.e., $\Delta c < \frac{2}{\alpha}$. Then, one has to compare $\Pi_i^S(p^*(c), k^*)$ (as calculated in equation 33) with $\Pi_i^S(p^D, k^*)$. By using equations 6 and 9 (and again noting that $r(p^D, k^*) = 0$ for $p^D$ in region II), the latter is given by

$$\Pi_i^S(p^D, k^*) = \frac{\Delta c}{N} + \frac{(\Delta c)^2\alpha(N - 1)}{4N} + \frac{\Pi^*_{\text{I}}}{1 - \delta}. \quad (35)$$

Solving $\Pi_i^S(p^*(c), k^*) < \Pi_i^S(p^D, k^*)$ for $\Delta c$, it follows that

$$\Delta c > \frac{4(\beta \delta - (N - 1))}{\alpha\beta (1 - \delta)(N - 1)} = \Delta c$$

is required for collusion to optimally get abandoned by firms. Consistent with the previous result, this inequality can only be fulfilled for $\Delta c < \frac{2}{\alpha}$ if $\delta < \hat{\delta}$. All of this results together prove the proposition.

□