Share Auctions and the Division of Value in Corporate Reorganizations*

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Abstract

I propose a new mechanism to divide a reorganized firm's value between its claimants according to a given priority rule. In this mechanism, the reorganized firm is sold in a special kind of share auction, with the proceeds distributed to claimants according to the given priority rule. I show that this mechanism can enforce a large class of priority rules, even when the firm's claimants face severe financial constraints.

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1 Introduction

A key question in the design of bankruptcy law is what priority rule should govern the division of a reorganized firm's value among its claimants. A large body of theoretical research suggests that the division of value in a reorganization can affect incentives before, during, and after the reorganization.[1] An equally important question is how the desired priority rule should be enforced. In general, ex-post bargaining among a firm's claimants need not result in the division of value desired ex ante. Rather, the outcome is determined by the claimants' outside options and the procedural details of the reorganization process, giving bankruptcy law a central role in enforcing the desired priority rule.

As an example, consider the absolute priority rule (APR), widely (although not universally) held to be a proper goal of bankruptcy law.[2] In practice, the outcomes of Chapter 11 reorganizations frequently deviate from the APR in favor of shareholders.[3] Moreover, specific features of Chapter 11, such as claimants' ability to delay agreement on a reorganization plan, and the eventual conversion to a Chapter 7 liquidation, are readily identified as contributing to such deviations.[4] Consequently, much of the research on bankruptcy law design and reform has focused on mechanisms that enforce the APR, such as the auction proposals of Baird (1986), Jackson (1986), and Jensen (1991) or the mechanisms proposed by Roe (1983), Bebchuk (1988), and Adler and Ayres (2001).

The fundamental problem for the enforcement of priority rules is that the bankruptcy judge does

[1] In particular, the division of value can affect the incentives of a bank lender-equity holder coalition to continue or liquidate (White (1980)); creditors' incentives to continue lending after the onset of financial distress (Gertner and Scharfstein (1991)), monitor the borrower (Cornelli and Felli (1997)), or ration credit (Longhofer (1997)); or a manager-shareholder's incentives to entrench herself (Bebchuk and Picker (1993)), invest in firm-specific human capital (Berkovitch, Israel, and Zender (1997, 1998)), reveal her private information about the firm's financial situation (Povel (1999)), delay a bankruptcy filing (White (1989), Baird (1991)), make risky investments (Bebchuk (2002)), or exert effort following a reorganization (Ayotte (2007)).

[2] See, e.g., Jackson (1986) for a discussion.

[3] For empirical evidence see, e.g., Franks and Torous (1989, 1994); Eberhart, Moore, and Roenfeldt (1990); Weiss (1990); LoPucki and Whitford (1990); Betker (1995); and Tashjian, Lease, and McConnell (1996).

[4] For formal models see, e.g., Baird and Picker (1991), Bebchuk and Chang (1992), Bergman and Callen (1995), and Eraslan and Yilmaz (2007).

not know the firm's value—it is, in the language of contract theory, *unverifiable*. In a reorganization, claimants exchange their old claims for new ones, and in order to directly enforce a particular priority rule in such an exchange, the judge would generally need to know the firm's value. Suppose, for example, that the firm's creditors are to receive equity in the reorganized firm in exchange for forgiving some fixed amount of its debt. Then, under a priority rule such as the APR, the fraction of the firm's equity that creditors should get is decreasing in the firm's value. Moreover, claimants may not report their valuations truthfully—in the APR example, creditors benefit from a lower valuation, while shareholders benefit from a higher valuation. Faced with this informational disadvantage, the bankruptcy judge must enforce the desired priority rule indirectly, through a mechanism that elicits the relevant information from the firm's claimants and other informed parties.

1.1 A new share auction mechanism

In this paper I propose a new mechanism that divides a reorganized firm's value among its claimants according to a given priority rule. In this mechanism, all existing claims on the firm (including equity) are cancelled and new shares in the resulting debt-free firm are issued in their place. Instead of distributing the new shares directly to the firm's claimants, however, they are distributed through a special kind of share auction. In this auction, both claimants and non-claimants can participate, and each bidder submits a decreasing schedule of cash bids (one bid for each share), allowing him to bid for as little as one share or as much as the entire firm. The auction differs from a standard share auction in two important ways. First, as payments are collected from the winning bidders, they are distributed to the firm's claimants according to the given priority rule, with each claimant's share of the total offsetting his own payment, if any. Second, if a winning bidder defaults, then the affected shares are not sold to the highest remaining bidders. Rather, they are distributed directly to the firm's claimants according to the given priority rule, using the auction price as an estimate of their value.

In my main result I show that this mechanism enforces the given priority rule, in the sense that the reorganized firm's value is divided according to that priority rule in every equilibrium. In the mechanism, competition between bidders drives the auction price up to the firm's value and, as a result, the entire value is extracted from the winning bidders and distributed to the firm's claimants according to the given priority rule. As explained immediately below, this result holds for a large class of priority rules and remains true even under severe financial constraints.

This mechanism is, unlike the mechanisms proposed by Bebchuk (1988) and Adler and Ayres (2001), not limited to implementing the APR for a standard debt-and-equity capital structure. Rather, it can implement any priority rule as long as, under that rule, each claimant's share is increasing in the reorganized firm's total value. At a practical level, one can therefore view this mechanism as an improvement in the contracting technology: it allows the ex-post enforcement of almost any priority rule mandated by bankruptcy law or agreed on ex ante by the firm and its claimants, perhaps as a result of a particularly complex capital structure.

At a more fundamental level, the large class of priority rules that can be implemented by this mechanism allows us to separate the question of how to implement the optimal priority rule from the question of which priority rule is the optimal one. Specifically, with this mechanism, unlike with the proposals mentioned above, one need not take a stand on which priority rule is the optimal one (the APR or some particular deviation from the APR, for example) before solving the problem of how to implement it. This separation of the two questions is useful because, as the literature on the incentive effects of different priority rules (see footnote 1) attests, it is far from clear which priority rule is the optimal one.

In another contrast with previously proposed mechanisms, this mechanism works even when some or all of the firm's claimants have little cash of their own, no outside financing is available, and no outside bidders participate. In particular, a senior creditor with deep pockets cannot exploit junior creditors with limited resources to get for himself a larger share of the firm's value than that prescribed by the priority rule. This stands in stark contrast to the effects of financial constraints in standard auctions (see, e.g., Che and Gale (1998)), where they may allow the bidder with the deepest pockets to acquire the object for sale at a much lower price than he otherwise could.

Financial constraints pose problems for the reorganization mechanisms proposed by Bebchuk (1988) and Adler and Ayres (2001) (see, e.g., Skeel (1993) for a more detailed critique), as well as for the auction proposals of Baird (1986), Jackson (1986), and Jensen (1991). In Bebchuk's mechanism, for example, junior creditors do not get any value at all unless they either collectively have access to sufficient funds to repay in full all debt senior to theirs or can sell their repayment options to outside investors. In many cases—especially those where multiple firms in the same industry face bankruptcy (see, e.g., Shleifer and Vishny (1992))—these conditions are unlikely to be satisfied. In my mechanism, however, the desired priority rule can always be enforced as long as each claimant has some positive amount of cash, no matter how little.

Two essential features of the mechanism ensure that financial constraints do not matter. First, as described above, the entire auction proceeds are immediately returned to the firm's claimants according to the desired priority rule. Therefore, the claimants cannot, as a group, ever exhaust their financial resources through participation in the auction—they are essentially buying the reorganized firm from themselves. In particular, this implies that, no matter what the outcome of the auction, there is always at least one claimant who has not exhausted his financial resources.

Second, the mechanism exploits the fact that ownership of the firm is perfectly divisible by issuing and selling a large number of shares in the reorganized firm, rather than selling the firm in one piece. Most importantly, this feature increases competition between bidders, thus driving the auction price up to the firm's value. It does so by enabling a financially constrained bidder to bid for just a tiny fraction of the firm (i.e., to bid a positive amount for the first few shares and zero for the remaining shares). This ensures that a bid below the firm's value can never win: as long as each share is small enough, there will be at least one claimant who has sufficient financial resources (see immediately above) to beat that bid and (profitably) acquire one additional share.

The mechanism responds to default in a novel way. Since bidders are financially constrained, they may be forced to default on some of their winning bids. Instead of discarding these bids and accepting the highest remaining bids, however, the mechanism allocates the affected shares according to the desired priority rule, using the auction price as an estimate of their value.[5] Most importantly, this feature ensures that a deep-pocketed bidder cannot gain from submitting artificially low bids and losing to a financially constrained bidder, only to force that bidder into default. At the same time, this feature also ensures that no bidder is penalized for another bidder's default, as would be the case if the affected shares were simply removed from the auction.

1.2 Other division mechanisms

My mechanism builds on the idea, first suggested by Roe (1983), that a quick, inexpensive, and fair reorganization is best attained by automatically converting all claims on the bankrupt firm into equity and using some market-based mechanism to ensure that the shares in the reorganized firm are distributed according to the APR. In his original paper, Roe proposes that ten percent of the shares in the reorganized firm be offered for sale and the resulting market price, along with the APR, be used to determine the distribution of the remaining ninety percent.

In the same spirit, Bebchuk (1988, 2000) proposes a mechanism in which each participant is given an option to purchase shares in the reorganized firm which, when exercised optimally, yields a payoff that is consistent with the APR. More recently, Adler and Ayres (2001) propose a mechanism in which each participant may offer to buy or sell shares in the reorganized firm at a fixed price. New shares are issued to claimants of successively lower priority, thus diluting the value of the shares already outstanding, until supply equals demand. The price at which each participant may buy or sell shares is set in such a way that his equilibrium payoff is consistent with the APR.

A key feature of these mechanisms is that participants are given carefully designed rights to buy claims on the reorganized firm that, when exercised optimally, result in payoffs that are consistent with the desired division. Financially constrained participants, however, may not be able to exercise

[5] This feature is reminiscent of the reorganization mechanism proposed by Roe (1983), in which a fraction of the reorganized firm's equity is sold in an auction, with the auction price determining how the remaining equity is distributed. Here, however, bidders' incentives to manipulate the auction price are not a problem since default does not occur in equilibrium.

their rights and, as a result, they may not receive the value they are entitled to according to the APR. Therefore, these mechanisms rely heavily on a participant's ability to exercise his rights, either with his own funds or with borrowed funds, an ability that is likely to be especially limited in bankruptcy.[6] These mechanisms are also specifically designed to enforce "sequential" priority rules such as the APR that derive from the priority structure of several classes of debt and equity, and cannot easily enforce other priority rules.[7]

1.3 Fully fledged bankruptcy reform proposals

The share auction mechanism presented here is only one part of a complete bankruptcy procedure. Specifically, I have not taken a stand on who should initiate the bankruptcy process (the debtor or the creditor), how the merit and priority of each claim on the firm should be determined (secured claims, for example, pose a particular challenge), who should control the firm while it is in bankruptcy (incumbent management, creditors, or a court-appointed trustee, for example), or whether any decisions about allocation of control or deployment of assets should be a formal part of the bankruptcy process. While court proceedings may be necessary to address these issues, the mechanism effectively takes the issue of division off the bargaining table. As such, it could play the role of division in existing proposals for bankruptcy reform, such as those of Aghion, Hart, and Moore (1992) or Hart et al. (1997). Alternatively, the mechanism could replace Chapter 7 liquidation as a default option when a reorganization plan cannot be confirmed, thereby avoiding a fire sale of the firm's assets.

[6] While the authors make the case that their mechanisms should work under financial constraints, they do, however, rely on someone having the financial resources to either lend to the participants or acquire their options in a well-functioning market.

[7] In principle, previous proposals *can* implement any priority rule in the larger class considered here. This is because any such priority rule can be approximated by a countable set of debt contracts, ordered by priority, each with a face value of $\varepsilon > 0$. Such a capital structure, however, could be extremely unwieldy and the execution of the mechanisms commensurately complicated.

2 A share auction with financially constrained agents

2.1 The economic environment

There is a single object (the bankrupt firm) and a finite set of *agents* N (creditors, shareholders, and outsiders). Ownership of the firm can be divided between the agents as desired by issuing a large enough number of shares and distributing them accordingly. An *allocation* q is an $|N|$-dimensional vector $\left(q_1, q_2, \ldots, q_{|N|}\right)$ where $q_i \in [0,1]$ is the fraction of the object allocated to agent i and

$$\sum_{i \in N} q_i \leq 1. \tag{1}$$

Each agent values ownership in the same way: the entire object is worth v and the fraction q_i is worth $q_i v$, where v, his *valuation*, can be any strictly positive number. Agent i's preferences over allocations and money are formally represented by the utility function $u_i(q, x_i) = q_i v + x_i - C(x_i)$ where x_i denotes his monetary wealth. The function C represents financing costs. If $x_i < 0$, then agent i's cash balance is negative and the financing cost $C(x_i)$ is strictly positive and increasing in $-x_i$. If $x_i \geq 0$, then he does not need to borrow and $C(x_i) = 0$. I also assume that agents can borrow very small amounts at near-zero cost. Formally, as x_i approaches zero from below, then C also approaches zero.[8] Finally, agent i is initially endowed with a non-negative cash budget w_i.

2.2 A share auction mechanism

The agents move simultaneously. Agent i submits a positive, non-increasing, and left-continuous[9] *bid function* $b_i : (0,1] \to \mathbb{R}_+$ and a *default point* $\hat{q}_i \in (0,1]$. The bid function b_i specifies his willingness to pay for an incremental fraction of the object. Therefore, the most he is willing to pay for the fraction $q_i \in [0,1]$ is

$$\int_0^{q_i} b_i(s)\, ds. \tag{2}$$

[8] I invoke this assumption in the last step of the proof of Claim 2 on page 23.
[9] I invoke left-continuity in the proof of Claim 6 on page 26.

The default point $\hat{q}_i$ specifies the last incremental fraction of the object for which agent i will honor his bid if it is accepted. For example, if, for every $s \in (0, 0.7]$, his bid $b_i(s)$ is accepted but $\hat{q}_i = 0.5$, he will, for every $s \in (0.5, 0.7]$, default on his bid $b_i(s)$. Default allows an agent to express his valuation of a fraction of the object that is larger than what his budget allows him to acquire, but without the risk of actually exceeding his budget and incurring financing costs if all his bids were to be accepted. Default points also play a role in breaking ties. As defined above, agent i must submit a strictly positive default point $\hat{q}_i$ and therefore cannot default completely.[10]

Based on the agents' actions, the object is divided among them and cash transfers are made between them.

2.2.1 Primary allocation

An allocation q^* for which the sum of agents' bids is highest (after ties are broken) is chosen. Formally,

$$q^* \in \arg\max_q \left\{ \sum_{i \in N} \int_0^{q_i} b_i(s)\, ds \right\}. \tag{3}$$

The maximization problem in (3) can be solved by pooling all agents' bids and then accepting them in descending order until the entire object has been allocated. The last accepted bid defines the *clearing price* (or *lowest winning bid*) p^*, formally defined as

$$p^* = \sup \left\{ p : \sum_i \sup \{ \{ q_i : b_i(q_i) \geq p \} \cup \{0\} \} \geq 1 \right\}. \tag{4}$$

If multiple allocations exist that satisfy (3), then there is excess demand at p^*. When this is the case, the tie is broken by first accepting, at the same rate for all agents, bids for incremental fractions below each agent's default point and then accepting, at the same rate for all agents, bids for incremental fractions above each agent's default point.[11]

[10] I invoke strict positivity of $\hat{q}_i$ in the proof of Claim 1 on page 22.

[11] A formal definition of q^* is as follows: Once all bids that strictly exceed p^* have been accepted, agent i's allocation q_i^* is equal to $q_i^{\min}$, defined by

$$q_i^{\min} = \sup \{ \{ q_i : b_i(q_i) > p^* \} \cup \{0\} \}.$$

Example 1 *Suppose that there are three agents who all submit the bid function*

$$b_i(s) = \begin{cases} 100 & \textit{if } s \in (0, 0.1] \\ 90 & \textit{if } s \in (0.1, 0.4] \\ 80 & \textit{if } s \in (0.4, 1] \end{cases}$$

for $i = 1, 2, 3$ *and the default points* $\hat{q} = (\hat{q}_1, \hat{q}_2, \hat{q}_3) = (0.5, 0.2, 0.2)$*. First, bids that are equal to 100 are accepted, which allocates 0.3 of the object—0.1 to each agent. Second, bids that are equal to 90 are accepted, up to each agent's default point. This allocates an additional 0.5 of the object—0.3, 0.1, and 0.1 to agents 1, 2, and 3, respectively. At this point, all of agent 1's bids that are equal to 90 have been accepted and agent 2 and 3's default points have been reached. Finally, the remainder of the object is divided equally between agents 2 and 3, giving 0.1 to each, yielding the allocation* $q^* = (0.4, 0.3, 0.3)$*.*

If $\sum_i q_i^{\min} = 1$, then the entire object has been allocated. (Note that, by the definition of p^* and $q_i^{\min}$, $\sum_i q_i^{\min} > 1$ is not possible.) If $\sum_i q_i^{\min} < 1$, then bids that are equal to p^* must be accepted. Let $q_i^{\max}$, defined by

$$q_i^{\max} = \sup\left\{\{q_i : b_i(q_i) \geq p^*\} \cup \{0\}\right\},$$

be the largest fraction that can be allocated to agent i before bids that are strictly smaller than p^* must be accepted. If

$$\sum_i q_i^{\min} < 1 \leq \sum_i \max\left\{q_i^{\min}, \min\left\{\hat{q}_i, q_i^{\max}\right\}\right\},$$

then the entire object can be allocated without exceeding any agent's default point. In this case, agent i's allocation is

$$q_i^* = \max\left\{q_i^{\min}, \min\left\{\hat{q}_i, q_i^{\max}, q_i^{\min} + \inf\left\{x : \sum_i \max\left\{q_i^{\min}, \min\left\{\hat{q}_i, q_i^{\max}, q_i^{\min} + x\right\}\right\} \geq 1\right\}\right\}\right\}.$$

If, however,

$$\sum_i \max\left\{q_i^{\min}, \min\left\{\hat{q}_i, q_i^{\max}\right\}\right\} < 1 \leq \sum_i q_i^{\max},$$

then it is not possible to allocate the entire object without exceeding at least one agent's default point. In this case, agent i's allocation is

$$q_i^* = \max\left\{q_i^{\min}, \min\left\{q_i^{\max}, \hat{q}_i + \inf\left\{x : \sum_i \max\left\{q_i^{\min}, \min\left\{q_i^{\max}, \hat{q}_i + x\right\}\right\} \geq 1\right\}\right\}\right\}.$$

(Note that, by the definition of p^* and $q_i^{\max}$, $\sum_i q_i^{\max} < 1$ is not possible and therefore the entire object is always allocated at this point.)

Once q^* has been determined, the object is divided among the agents. If no agent defaults (if $\hat{q}_i \geq q_i^*$ for all i), then agent i gets the fraction q_i^*. Otherwise, the object is allocated in two steps. In the first step agent i gets his *primary allocation* $\hat{q}_i^* = \min\{\hat{q}_i, q_i^*\}$—his allocation q_i^* or his default point $\hat{q}_i$, whichever is smaller. Before defining the second step of the allocation, I define the cash transfers.

2.2.2 Payments

Cash transfers between the agents are determined by the primary allocation $\hat{q}^* = \left(\hat{q}_1^*, \hat{q}_2^*, \ldots, \hat{q}_{|N|}^*\right)$ and the agents' bids. A *payment rule* $t = \left(t_1, t_2, \ldots, t_{|N|}\right)$ determines agents' *gross payments*, with t_i denoting agent i's gross payment. The sum of all gross payments that are collected from the agents is then redistributed among the agents according to a *sharing rule* $m = \left(m_1, m_2, \ldots, m_{|N|}\right)$ where m_i is a function that represents agent i's *share*. Agent i's *net payment* is therefore

$$t_i - m_i\left(\sum_{j\in N} t_j\right). \tag{5}$$

A share m_i can be *positive*, in which case $m_i(x) > 0$ for some x, or it can be *zero*, in which case $m_i(x) = 0$ for every x. I assume that there are at least two agents whose shares are positive. I also assume that m is *strictly increasing* in the sense that every positive share m_i is a strictly increasing function.[12] The sharing rule ensures that the mechanism has a balanced budget by requiring that

[12] This entails some loss of generality since a simple financial structure comprised of debt and equity implies shares that are constant on some interval. Every such *weakly* decreasing sharing rule m, however, can be approximated by a strictly increasing sharing rule $\hat{m}$, defined by

$$\hat{m}_i(x) = m_i([1-\varepsilon]x) + \frac{\varepsilon x}{|N_+|}$$

where i is an agent whose original share m_i is positive, N_+ is the set of all such agents, and $\varepsilon > 0$. I invoke the assumption of a strictly increasing sharing rule throughout the proof of my main result. Every claim except Claim 3 on page 24, however, remains true even under the assumption of a weakly increasing sharing rule.

all gross payments that are collected from the agents are returned to them, or

$$\sum_{i \in N} m_i (x) = x \tag{6}$$

for all x.

I consider one particular payment rule, based on Clarke (1971), Groves (1973), and Vickrey (1961). Under this payment rule, which generalizes the single-unit second-price auction to a perfectly divisible multi-unit setting, agent i pays the value of the other agents' bids that would have been accepted had he not been awarded the fraction $\hat{q}_i^*$. To define his gross payment explicitly, first define the *composite bid function* b_{-i} as the bid function that would result if all agents $j \neq i$ were to pool their bids and form a single bid function using the highest bids in the pool. Formally, b_{-i} is defined by

$$\int_0^{q_{-i}} b_{-i} (s) \, ds = \max_{\{q: \sum_{j \neq i} q_j = q_{-i}\}} \left\{ \sum_{j \neq i} \int_0^{q_j} b_j (s) \, ds \right\} \tag{7}$$

for all $q_{-i} \in (0, 1]$. Much as in (3), the maximization problem in (7) can be solved by pooling the bids of all agents $j \neq i$ and accepting them in descending order until the fraction q_{-i} has been allocated among them. Agent i's gross payment can now be written as

$$t_i = \int_{1-\hat{q}_i^*}^{1} b_{-i} (s) \, ds, \tag{8}$$

the sum of the other agents' bids that would have been accepted had he not been awarded the fraction $\hat{q}_i^*$.[13] This results in the net payment

$$\int_{1-\hat{q}_i^*}^{1} b_{-i} (s) \, ds - m_i \left(\int_{1-\hat{q}_i^*}^{1} b_{-i} (s) \, ds + \sum_{j \neq i} \int_{1-\hat{q}_j^*}^{1} b_{-j} (s) \, ds \right) \tag{9}$$

[13] By definition, the payment in this setting is

$$t_i = \max_{\{q: \sum_{j \neq i} q_j = 1\}} \left\{ \sum_{j \neq i} \int_0^{q_j} b_j (s) \, ds \right\} - \max_{\{q: \sum_{j \neq i} q_j = 1 - \hat{q}_i^*\}} \left\{ \sum_{j \neq i} \int_0^{q_j} b_j (s) \, ds \right\}$$

where agent i's share of all gross payments collected has been subtracted from his own gross payment.

Example 2 *Consider again the previous example. Since* $q^* = (0.4, 0.3, 0.3)$ *and* $\hat{q} = (0.5, 0.2, 0.2)$*, the resulting primary allocation is* $\hat{q}^* = (0.4, 0.2, 0.2)$*. Since each agent submits the same bid function, the composite bid function for any given pair of agents* $j \neq i$ *is*

$$b_{-i}(s) = \begin{cases} 100 & \text{for } s \in (0, 0.2] \\ 90 & \text{for } s \in (0.2, 0.8] \\ 80 & \text{for } s \in (0.8, 1] \end{cases}$$

which results in the gross payments $t = (34, 16, 16)$ *(an agent must pay 80 for the first 0.2 of the object, 90 for the next 0.6, and 100 for the remaining 0.2).*

2.2.3 Secondary allocation

If at least one agent defaults (if $\hat{q}_i < q_i^*$ for some i), then the fraction $1 - \sum_j \hat{q}_j^*$ of the object remains unallocated after the primary allocation has been made. Agent i's *secondary allocation* consists of the fraction

$$\frac{m_i(p^*) - m_i\left(q_i^* p^* + \sum_{j \neq i} \hat{q}_j^* p^*\right)}{p^*} \tag{10}$$

where the clearing price p^* is still defined by (4).[14] Unlike the primary allocation, the secondary allocation is awarded without any corresponding cash payment.

Example 3 *Consider again the previous example, in which* $q^* = (0.4, 0.3, 0.3)$*,* $\hat{q}^* = (0.4, 0.2, 0.2)$*,*

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which, after applying the definition of b_{-i} in (7), can be written as

$$\begin{aligned} t_i &= \int_0^1 b_{-i}(s)\,ds - \int_0^{1-\hat{q}_i^*} b_{-i}(s)\,ds \\ &= \int_{1-\hat{q}_i^*}^1 b_{-i}(s)\,ds \end{aligned}$$

which coincides with (8).

[14] If $p^* = 0$ (which cannot occur in equilibrium), then let the secondary allocation be zero.

and $p^* = 90$. *Suppose that*

$$m_1(x) = \tfrac{1}{2}\max\{x - 80, 0\}$$
$$m_2(x) = \tfrac{1}{2}x$$
$$m_3(x) = \tfrac{1}{2}\min\{x, 80\}$$

and consider the secondary allocation, according to which the 0.2 of the object on which the agents default is to be distributed.[15] *Agent 1 does not default* ($\hat{q}_1^* = q_1^*$)*. His secondary allocation is therefore*

$$\frac{\frac{1}{2}\max\{90 - 80, 0\} - \frac{1}{2}\max\{0.8 \times 90 - 80, 0\}}{90} = 0.0\overline{5}.$$

Agents 2 and 3 default on 0.1 of the object each. Their secondary allocations are therefore

$$\frac{\frac{1}{2}90 - \frac{1}{2}0.9 \times 90}{90} = 0.05$$

and

$$\frac{\frac{1}{2}\min\{90, 80\} - \frac{1}{2}\min\{0.9 \times 90, 80\}}{90} = 0,$$

respectively.

The idea behind the secondary allocation is to divide the fraction on which the agents default according to the sharing rule m, using the clearing price p^* as an estimate of its value. This idea takes the view that estimated value equal to $\sum_j \hat{q}_j^* p^*$ has already been divided among the agents by returning the auction proceeds to them, with agent i's share being $m_i\left(\sum_j \hat{q}_j^* p^*\right)$. It remains to distribute $p^* - \sum_j \hat{q}_j^* p^*$, and agent i's share of the remainder is $m_i(p^*) - m_i\left(\sum_j \hat{q}_j^* p^*\right)$ which,

[15] For simplicity of exposition, this sharing rule is not strictly increasing. It is, however, approximated by the strictly increasing sharing rule

$$m_1(x) = \tfrac{1}{2}\max\{x[1-\varepsilon] - 80, 0\} + \varepsilon x/3$$
$$m_2(x) = \tfrac{1}{2}x[1-\varepsilon] + \varepsilon x/3$$
$$m_3(x) = \tfrac{1}{2}\min\{x[1-\varepsilon], 80\} + \varepsilon x/3$$

where $\varepsilon > 0$.

when divided by the total estimated value p^* that is available for distribution, gives the fraction

$$\frac{m_i(p^*) - m_i\left(\sum_j \hat{q}_j^* p^*\right)}{p^*}. \tag{11}$$

To penalize agent i for his own default, however, (11) is reduced by the fraction

$$\frac{m_i\left(q_i^* p^* + \sum_{j \neq i} \hat{q}_j^* p^*\right) - m_i\left(\sum_j \hat{q}_j^* p^*\right)}{p^*}, \tag{12}$$

yielding the fraction given by (10). The fraction (12) is seized by the auctioneer or sold to parties who did not participate in the auction and are not affiliated with any of its participants.[16] Finally, an agent who defaults is subject to a penalty c, regardless of the size of his default.

Agent i's payoff, ignoring the financing cost C and the default penalty c, is therefore

$$\begin{aligned} & \int_{1-\hat{q}_i^*}^{1} [v - b_{-i}(s)]\, ds + m_i\left(\int_{1-\hat{q}_i^*}^{1} b_{-i}(s)\, ds + \sum_{j \neq i} \int_{1-\hat{q}_j^*}^{1} b_{-j}(s)\, ds\right) \\ + \quad & \frac{m_i(p^*) - m_i\left(q_i^* p^* + \sum_{j \neq i} \hat{q}_j^* p^*\right)}{p^*} v. \end{aligned} \tag{13}$$

3 An implementation result

A mechanism *implements* a sharing rule m if, for any valuation v, the mechanism has an equilibrium and in every equilibrium, agent i's payoff is $m_i(v)$. In the following proposition, I claim that my mechanism implements m as long as m is strictly increasing and as long as every agent whose share is positive has a strictly positive budget, no matter how small that budget is. This implies that, although it allows participation by outsiders (agents with zero shares), my mechanism only requires

[16] If agent i were not penalized for default in this way, he could deviate from any equilibrium outcome by setting $b_i(s)$ very high for every $s \in (0,1]$ and $\hat{q}_i \simeq 0$, thereby winning the entire object but defaulting on virtually every bid. Since p^* determines the secondary allocation and $p^* = b_i(1)$ in this case, he could set p^* to maximize his own payoff as long as it exceeded the other agents' bids. For example, agent 1 from the previous example would like to set the clearing price as high as possible to maximize his secondary allocation.

participation by the firm's creditors and shareholders (agents with positive shares), even if they have virtually no financial resources. It also follows from my claim that, in equilibrium, agents stay within their budgets and do not default.

Proposition 1 *Fix a strictly increasing sharing rule m, a financing cost function C, and a profile of budgets $w = (w_1, w_2, \ldots, w_{|N|})$ such that $w_i > 0$ if $m_i(x) > 0$ for some x. Then, for any valuation v, a Nash equilibrium of the share auction mechanism defined above exists and, in every such equilibrium and for every $i \in N$, agent i's payoff is $m_i(v)$.*

Corollary 1 *In equilibrium, no default occurs and every agent's payment is within his budget.*

3.1 An informal proof of Proposition 1

Using the previous example, I will give a step-by-step illustration of the proof of Proposition 1 by showing how every outcome that does not result in the desired payoffs cannot be an equilibrium. The steps correspond to the claims that make up the formal proof, which follows and also proves the corollary and equilibrium existence.

3.1.1 Losing bids equal the clearing price

The first step (which corresponds to Claim 1 of the formal proof) is to show that, for every agent j, the losing bids of his opponents' composite bid function b_{-j} must equal the clearing price p^*. To see why this is true, note that the payoff of agent $i \neq j$ (given by (13)) is increasing in agent j's gross payment, which is given by the losing bids of the composite bid function b_{-j}. The composite bid function b_{-j} (defined by (7)) in turn depends on agent i's own bids. Therefore, all agents $i \neq j$ have a joint incentive to set the losing bids of the composite bid function b_{-j} as high as possible.

Example 4 *Consider the previous example. Agent 2's payoff is*

$$\begin{aligned} & \int_{0.8}^{1} [v - b_{-2}(s)]\, ds + \frac{1}{2}\left[\int_{0.6}^{1} b_{-1}(s)\, ds + \int_{0.8}^{1} b_{-2}(s)\, ds + \int_{0.8}^{1} b_{-3}(s)\, ds\right] \\ + \quad & \left[\frac{1}{2}p^* - \frac{1}{2}0.9p^*\right]\frac{v}{p^*} \end{aligned}$$

which is strictly increasing in b_{-1} and b_{-3}. Since both b_{-1} and b_{-3} (defined in Example 2) depend on agent 2's own bids and both are strictly below p^* for every $s \in (0.8, 1]$ ($p^* = 90$ here), he could strictly increase his payoff by simply raising $b_2(s)$ for every $s \in (0.4, 1]$. As a result, it must be that, in equilibrium, $b_{-1}(s) = p^*$ for every $s \in (0.6, 1]$ and $b_{-3}(s) = p^*$ for every $s \in (0.7, 1]$. (Even if agent 2's payoff is increasing in $b_{-3}(s)$ only for $s \in (0.8, 1]$, these bids can only equal p^* if those for $s \in (0.7, 0.8]$ also do.)

Agent 3's payoff is

$$\begin{aligned} & \int_{0.8}^{1} [v - b_{-3}(s)]\, ds + \frac{1}{2}\min\left\{\int_{0.6}^{1} b_{-1}(s)\, ds + \int_{0.8}^{1} b_{-2}(s)\, ds + \int_{0.8}^{1} b_{-3}(s)\, ds, 80\right\} \\ + \quad & \left[\frac{1}{2}\min\{p^*, 80\} - \frac{1}{2}\min\{0.9p^*, 80\}\right]\frac{v}{p^*} \end{aligned}$$

which is strictly increasing in b_{-2}. By the same logic as above, it must therefore be that, in equilibrium, $b_{-2}(s) = p^*$ for every $s \in (0.7, 1]$.

3.1.2 No default in equilibrium

The next step (which corresponds to Claim 2 of the formal proof) is to show that default will not occur in equilibrium, or that $\hat{q}_i \geq q_i^*$ for every i. Consider an agent i who defaults. He could deviate by setting the bids on which he defaults—$b_i(s)$ for every $s \in (\hat{q}_i, q_i^*]$—just below the clearing price p^*. Since, by the previous step, his opponents' highest losing bids are equal to p^*, the bids on which he deviates will therefore lose. Since only bids on which he defaults are changed, his primary allocation remains the same. His secondary allocation, however, can only increase since he will no longer be penalized for default, and he avoids the default penalty c. Finally, since the deviation bids can be arbitrarily close to the original clearing price p^*, there always exists a deviation for which the decrease in his opponents' gross payments is smaller than the default penalty c, yielding a strict increase in his payoff.

Example 5 *Consider the previous example. A bid profile that satisfies the previously established equilibrium condition is*

$$b_1(s) = \begin{cases} 100 & \text{for } s \in (0, 0.1] \\ 90 & \text{for } s \in (0.1, 0.4] \\ 80 & \text{for } s \in (0.4, 1] \end{cases} \quad b_i(s) = \begin{cases} 100 & \text{for } s \in (0, 0.1] \\ 90 & \text{for } s \in (0.1, 1] \end{cases}$$

for $i = 2, 3$ *and* $\hat{q} = (0.5, 0.2, 0.2)$*, again resulting in* $q^* = (0.4, 0.3, 0.3)$ *and* $\hat{q}^* = (0.4, 0.2, 0.2)$*. Agent 2's payoff is still given in Example 4 and, by the previous step, now takes the value*

$$\begin{aligned} & \int_{0.8}^{1} [v - 90]\, ds + \frac{1}{2}\left[\int_{0.6}^{1} 90 ds + \int_{0.8}^{1} 90 ds + \int_{0.8}^{1} 90 ds\right] \\ + \quad & \left[\frac{1}{2} \times 90 - \frac{1}{2} 0.9 \times 90\right] \frac{v}{90} \end{aligned}$$

since the clearing price p^* *equals* 90*. He could deviate to* $b_2(s) = 90 - \varepsilon$ *for every* $s \in (0.2, 1]$*. As a result,* $q^* = (0.4, 0.2, 0.4)$ *and* $\hat{q}^* = (0.4, 0.2, 0.2)$*. His payoff would then be*

$$\begin{aligned} & \int_{0.8}^{1} [v - 90]\, ds + \frac{1}{2}\left[\int_{0.6}^{1} 90 ds + \int_{0.8}^{1} 90 ds + \int_{0.8}^{1} [90 - \varepsilon]\, ds\right] \\ + \quad & \left[\frac{1}{2} \times 90 - \frac{1}{2} 0.8 \times 90\right] \frac{v}{90} \end{aligned}$$

where the only changes are a strict increase in his secondary allocation and a negligible decrease in agent 3's gross payment.

3.1.3 Lowest winning bids are equal across agents

The next step (which corresponds to Claim 3 of the formal proof) is to show that, for every agent i whose share m_i is positive, the lowest winning bid of his opponents' composite bid function, $b_{-i}(1-q_i^*)$, must equal the clearing price p^* (which, by definition, is the lowest of all winning bids). This is equivalent to showing that there cannot be an agent i whose lowest winning bid, $b_i(q_i^*)$, is lower than all other agents' winning bids. If there were in fact such an agent, then his lowest winning bid would define the clearing price p^*. According to the first step, the clearing price in turn defines every agent's gross payment since agent $j \neq i$ cannot raise his losing bids above p^* without winning a larger fraction of the object. Agent i could, however, raise his winning bids, thereby raising the clearing price, and therefore also raise his losing bids without winning an additional fraction of the object. This increases his opponents' gross payments and, since his share is assumed to be strictly increasing, his payoff. Because there is no default and therefore no secondary allocation, the resulting increase in the clearing price has no independent effect on his payoff.

Example 6 *Consider the previous example. A bid profile that satisfies both previously established equilibrium conditions is*

$$b_1(s) = \begin{cases} 100 & \textit{for } s \in (0, 0.1] \\ 90 & \textit{for } s \in (0.1, 0.4] \\ 80 & \textit{for } s \in (0.4, 1] \end{cases} \qquad b_i(s) = \begin{cases} 100 & \textit{for } s \in (0, 0.1] \\ 95 & \textit{for } s \in (0.1, 0.3] \\ 90 & \textit{for } s \in (0.3, 1] \end{cases}$$

for $i = 2, 3$ *and* $\hat{q} = (0.4, 0.3, 0.3)$, *resulting in* $q^* = \hat{q}^* = (0.4, 0.3, 0.3)$ *and* $p^* = 90$. *Given the clearing price* p^*, *agents 2 and 3 cannot increase their opponents' gross payments by raising their losing bids. Agent 1, however, could do so. This is because his lowest winning bid (90) is smaller than that of his opponents' (95). Therefore, he could deviate to* $b_1(s) = 95$ *for every* $s \in (0.1, 1]$, *thereby strictly increasing his payoff. Instead of getting*

$$0.4\,[v-90]+\frac{1}{2}\max\{90-80,0\}$$

he would get

$$0.4\,[v-90]+\frac{1}{2}\max\{0.4\times 90+0.6\times 95-80,0\}\,.$$

3.1.4 Price cannot exceed value

The next step (which corresponds to Claim 4 of the formal proof) is to show that the clearing price p^* cannot be greater than the valuation v. If this were the case, then an agent i for whom $q_i^*>0$ would have a profitable deviation. He could set all his bids just below p^* and, by the first step, thereby lose on all his bids. Because p^*, and therefore agent i's gross payment, is greater than v, this deviation is profitable. This standard auction argument is not complete, however, because the total proceeds, and agent i's share of them, may decrease when he deviates. Specifically, as the example shows, agent i's opponents may default on the bids that win as a result of his deviation. This loss is offset by an increase in the secondary allocation, however.

Example 7 *Consider the previous example. A bid profile that satisfies all previously established equilibrium conditions is*

$$b_i\,(s)=\begin{cases}100 & \textit{for } s\in(0,0.05]\\ 90 & \textit{for } s\in(0.05,1]\end{cases}$$

for $i=1,2,3$ *and* $\hat{q}=(0.06,0.49,0.45)$*, resulting in* $\hat{q}^*=q^*=(0.06,0.49,0.45)$ *and* $p^*=90$*. Suppose that* $v=85$*. Agent 3's payoff is then*

$$\int_{0.55}^{1}[85-90]\,ds+\frac{1}{2}\min\left\{\int_{0.94}^{1}90ds+\int_{0.51}^{1}90ds+\int_{0.55}^{1}90ds,80\right\}.$$

If he were to deviate to $b_3(s) = 90 - \varepsilon$ for every s, then his deviation payoff would be

$$\frac{1}{2}\min\left\{\int_{0.94}^{1} 90ds + \int_{0.51}^{1} 90ds, 80\right\} + \left[\frac{1}{2}\min\{90, 80\} - \frac{1}{2}\min\{0.55 \times 90, 80\}\right]\frac{85}{90}.$$

Note that the total proceeds decrease by the entire amount of agent 3's pre-deviation gross payment. This is because agents 1 and 2 default on every bid of theirs that wins as a result of agent 3's deviation. Their default, however, yields a secondary allocation for agent 3 which, together with avoiding the loss on his primary allocation, more than offsets his loss on his share of the auction proceeds. To see why agent 3's deviation is profitable, consider the change in his payoff, which can be written as

$$\left[0.45 - \frac{\frac{1}{2}\min\{90, 80\} - \frac{1}{2}\min\{0.55 \times 90, 80\}}{90}\right][90 - 85]$$

and is strictly positive.

3.1.5 At least one agent is not financially constrained

The next step (which corresponds to Claim 5 of the formal proof) is to show that there always exists at least one agent who has the financial resources to acquire an additional fraction of the object (and who has not already acquired all of it). This follows directly from the fact that all gross payments are returned to the agents through the sharing rule and the assumption that every agent has a positive budget.

3.1.6 Price cannot be less than value

The next step (which corresponds to Claim 6 of the formal proof) is to show that the clearing price p^* cannot be smaller than the valuation v. By the previous step, there is at least one agent who could deviate from such an equilibrium by raising some of his losing bids to acquire a larger fraction. By the step before that, the lowest winning bid of this agent's opponents must equal the clearing price p^* and therefore, by our assumption, also be smaller than the valuation v. This agent could therefore profitably raise some of his losing bids and thereby beat some of his opponents' lowest winning bids.

Example 8 *Consider the previous example but suppose that* $v = 95$*. Further suppose that* $w = (0.40, 0.01, 0.50)$*, which implies that agents 1 and 3 have completely exhausted their budgets. This, however, in turn implies that agent 2 cannot have exhausted his budget, no matter how small it is. Agent 2 could therefore deviate to* $b_2(s) = 90 + \varepsilon$ *for all* $s \in (0, 0.50]$ *(and* $\hat{q}_2 = 0.50$*) and win an additional 0.01 of the object without exceeding his budget.*

As a result, the clearing price p^* must equal the valuation v in every equilibrium. Moreover, since every agent's gross payment equals the clearing price, the auction proceeds equal v. Therefore, since the proceeds are distributed according to m, agent i's payoff is exactly $m_i(v)$.

3.1.7 Why is a strictly increasing sharing rule needed?

Although, for simplicity, I rely heavily on the assumption that the sharing rule is strictly increasing, it is only necessary when establishing the third step. The following example illustrates why this is the case, even when agents' budgets are not close to zero.

Example 9 *Consider the previous example. Let* $v = 100$ *and* $w_i = 8$ *for* $i = 1, 2, 3$ *and consider the strategy profile*

$$b_1(s) = \begin{cases} 100 & \text{if } s \in (0, 0.1] \\ 80 & \text{if } s \in (0.1, 1] \end{cases} \quad b_2(s) = \begin{cases} 100 & \text{if } s \in (0, 0.6] \\ 80 & \text{if } s \in (0.6, 1] \end{cases} \quad b_3(s) = \begin{cases} 80 & \text{if } s \in (0, 0.3] \\ 80 & \text{if } s \in (0.3, 1] \end{cases}$$

and $\hat{q} = (0.1, 0.6, 0.3)$*. As a result,* $(q_1^*, q_2^*, q_3^*) = (\hat{q}_1^*, \hat{q}_2^*, \hat{q}_3^*) = (0.1, 0.6, 0.3)$*. Since every losing bid equals 80 and there is no default, agent i's payoff is*

$$q_i^*[100 - 80] + m_i(80).$$

This outcome does not divide the object's value according to the sharing rule: the agents' payoffs are 2*,* 52 *and* 46*, respectively but, according to the sharing rule, they should be* 10*,* 50 *and* 40*, respectively. Nonetheless, this is a Nash equilibrium. To see why, first note that agents 1 and 2 have both exhausted their budgets. Therefore, neither one can take advantage of the profitable deviation that otherwise would be available to him: raise some of his losing bids to* $80+\varepsilon$*, beat some of agent 3's winning bids and win an additional fraction of the object at a price of* 80*. Second, note that agent 3, while not financially constrained, has no incentive to deviate in the same way: to do so, he would have to raise some of his losing bids to* $100+\varepsilon$*, beat some of agent 1 and 2's winning bids and win an additional fraction at a price of* 100*, which is not profitable since that price equals the object's true value. The reason such an equilibrium can arise in this example is that agent 3 has no incentive to raise his losing bids (and thereby force agents 1 and 2 to pay* 100 *for the fraction they win) because his share* m_3 *is flat for auction proceeds greater than* 80*.*

3.2 A formal proof of Proposition 1

Fix a profile of equilibrium actions: for every agent i, a bid function b_i and a default point $\hat{q}_i$, in which case his payoff, ignoring the financing cost C and the default penalty c, is

$$\int_{1-\hat{q}_i^*}^{1} [v - b_{-i}(s)]\, ds + m_i \left(\int_{1-\hat{q}_i^*}^{1} b_{-i}(s)\, ds + \sum_{j \neq i} \int_{1-\hat{q}_j^*}^{1} b_{-j}(s)\, ds \right) \tag{14}$$
$$+ \quad \frac{m_i(p^*) - m_i\left(q_i^* p^* + \sum_{j \neq i} \hat{q}_j^* p^*\right)}{p^*} v.$$

I prove the proposition and its corollary through a series of claims. In what follows, let b_i' and $\hat{q}_i'$ denote a deviation bid function and default point, respectively, and let $q_i'^*$, p'^* and $\hat{q}_i'^*$ denote the resulting deviation outcome, if different from the equilibrium outcome. Let ε denote a small, strictly positive number.

Claim 1 *For every agent i and every $s \in (1 - q_i^*, 1]$, $b_{-i}(s) = p^*$.*

Proof. Suppose, to the contrary, that, for some agent j and some $s \in \left(1-q_j^*, 1\right]$, $b_{-j}(s) < p^*$ and note that, since b_{-j} is non-increasing, this implies that, for every $s' \in (s, 1]$, $b_{-j}(s') < p^*$ and $q_j^* > 0$ (and therefore also $\hat{q}_j^* > 0$).[17] Then, instead of getting the equilibrium payoff (14), agent $i \neq j$ could, by deviating to $b_i'(s) = p^* - \varepsilon$ for every $s \in (q_i^*, 1]$, get the deviation payoff

$$\int_{1-\hat{q}_i^*}^{1} [v - b_{-i}(s)]\, ds + m_i \left(\int_{1-\hat{q}_i^*}^{1} b_{-i}(s)\, ds + \sum_{j \neq i} \int_{1-\hat{q}_j^*}^{1} b'_{-j}(s)\, ds \right) \tag{15}$$
$$+ \quad \frac{m_i(p^*) - m_i\left(q_i^* p^* + \sum_{j \neq i} \hat{q}_j^* p^*\right)}{p^*} v.$$

where, for every agent $j \neq i$ and every $s \in \left(1-\hat{q}_j^*, 1\right]$, $b'_{-j}(s) \geq p^* - \varepsilon$. Since $\hat{q}_j^* > 0$ and m_i is strictly increasing, the deviation payoff (15) is strictly greater than the equilibrium payoff (14) for ε small enough. Furthermore, since this deviation strictly decreases agent i's net payment, his financing cost C can only decrease. ■

According to the previous claim, agent i's equilibrium payoff, ignoring the financing cost C and the default penalty c, is

$$\hat{q}_i^* [v - p^*] + m_i \left(\hat{q}_i^* p^* + \sum_{j \neq i} \hat{q}_j^* p^* \right) \tag{16}$$
$$+ \quad \frac{m_i(p^*) - m_i\left(q_i^* p^* + \sum_{j \neq i} \hat{q}_j^* p^*\right)}{p^*} v.$$

Claim 2 *For every agent i, $\hat{q}_i \geq q_i^*$.*

Proof. Suppose, to the contrary, that, for some agent i, $\hat{q}_i < q_i^*$. Then, instead of getting the equilibrium payoff (16), agent i could deviate to $b_i'(s) = p^* - \varepsilon$ for every $s \in (\hat{q}_i, 1]$ and $b_i'(s) = b_i(s)$

[17] The case $b_{-j}(s) > p^*$ is ruled out by the fact that $b_{-j}(s)$ is a losing bid and p^* is the lowest winning bid.

otherwise and get the deviation payoff

$$
\begin{aligned}
& \hat{q}_i^* \left[v - p^*\right] + m_i \left(\hat{q}_i p^* + \sum_{j \neq i} \int_{1-\hat{q}_j'^*}^{1} b'_{-j}(s)\, ds \right) \\
+ \quad & \frac{m_i(p^*) - m_i\left(\hat{q}_i p^* + \sum_{j \neq i} \hat{q}_j'^* p^*\right)}{p^*} v
\end{aligned}
\tag{17}
$$

where, for every agent $j \neq i$, $\hat{q}_j'^* \geq \hat{q}_j^*$, for every $s \in \left(1 - \hat{q}_j'^*, 1\right]$, $b'_{-j}(s) \geq p^* - \varepsilon$ and $\sum_{j \neq i} \left[\hat{q}_j'^* - \hat{q}_j^*\right] \leq q_i^* - \hat{q}_i^*$.[18] The change in his payoff is

$$
\begin{aligned}
& m_i \left(\hat{q}_i^* p^* + \sum_{j \neq i} \hat{q}_j'^* p^* \right) - m_i \left(\hat{q}_i^* p^* + \sum_{j \neq i} \hat{q}_j^* p^* \right) \\
+ \quad & \left[m_i \left(q_i^* p^* + \sum_{j \neq i} \hat{q}_j^* p^* \right) - m_i \left(\hat{q}_i^* p^* + \sum_{j \neq i} \hat{q}_j'^* p^* \right) \right] \frac{v}{p^*} \\
- \quad & \left[m_i \left(\hat{q}_i^* p^* + \sum_{j \neq i} \hat{q}_j'^* p^* \right) - m_i \left(\hat{q}_i^* p^* + \sum_{j \neq i} \int_{1-\hat{q}_j'^*}^{1} b'_{-j}(s)\, ds \right) \right].
\end{aligned}
\tag{18}
$$

Since m_i is non-decreasing (i's share may be zero), $q_i^* > \hat{q}_i^*$, $\sum_{j \neq i} \left[\hat{q}_j'^* - \hat{q}_j^*\right] \leq q_i^* - \hat{q}_i^*$ and, for every agent $j \neq i$, $\hat{q}_j'^* \geq \hat{q}_j^*$, the terms in the first two lines are both non-negative. His payoff increases by the default penalty c and, as ε approaches zero, the term in the third line approaches zero, rendering the deviation strictly profitable. Furthermore, as ε approaches zero, the upper bound on the possible increase in his net payment, and therefore on that in his financing cost C, approaches zero. ■

According to the previous claims, agent i's equilibrium payoff, ignoring the financing cost C and the default penalty c, is

$$
q_i^* \left[v - p^*\right] + m_i(p^*) . \tag{19}
$$

[18] By Claim 1, all bids that are losing bids before the deviation are equal to p^*. Therefore, p^* will not change when agent i deviates, since some of these previously losing bids will then become winning bids, one of which will be the lowest winning bid p*

Claim 3 *If agent i's share m_i is positive, then,* $b_{-i}(1-q_i^*) = p^*$.

Proof. Suppose, to the contrary, that, for some such agent i, $b_{-i}(1-q_i^*) > p^*$[19]. Then, instead of getting the equilibrium payoff (19), agent i could deviate to $b_i'(s) = b_{-i}(1-q_i^*)$ for every $s \in (0, q_i^*]$ and $b_i'(s) = b_{-i}(1-q_i^*) - \varepsilon$ otherwise and get the deviation payoff

$$q_i^* [v - p^*] + m_i \left(q_i^* p^* + \sum_{j \neq i} q_j^* [b_{-i}(1-q_i^*) - \varepsilon] \right) \tag{20}$$

which, since m_i is strictly increasing, is strictly greater than the equilibrium payoff (19) for ε small enough. Furthermore, since this deviation strictly decreases agent i's net payment, his financing cost C can only decrease. ■

Claim 4 $p^* \leq v$.

Proof. Suppose, to the contrary, that $p^* > v$ and note that, by Claim (1), $b_{-i}(s) = p^*$ for every agent i and every $s \in (1-q_i^*, 1]$. Then, instead of getting the equilibrium payoff (19), some agent i for whom $q_i^* > 0$ could deviate to $b_i'(s) = p^* - \varepsilon$ for every s and get the deviation payoff

$$m_i \left(\sum_{j \neq i} \int_{1-\hat{q}_j'^*}^{1} b_{-j}'(s)\, ds \right) + \frac{m_i(p^*) - m_i\left(\sum_{j \neq i} \hat{q}_j'^* p^*\right)}{p^*} v \tag{21}$$

where, for every agent $j \neq i$, $\hat{q}_j'^* \geq q_j^*$, for every $s \in \left(1 - \hat{q}_j'^*, 1\right]$, $b_{-j}'(s) \geq p^* - \varepsilon$ and $\sum_{j \neq i} \hat{q}_j'^* \leq 1$. The change in his payoff is

$$\begin{aligned} & \left[q_i^* - \frac{m_i(p^*) - m_i\left(\sum_{j \neq i} \hat{q}_j'^* p^*\right)}{p^*} \right] [p^* - v] \\ - & \left[m_i \left(\sum_{j \neq i} \hat{q}_j'^* p^* \right) - m_i \left(\sum_{j \neq i} \int_{1-\hat{q}_j'^*}^{1} b_{-j}'(s)\, ds \right) \right]. \end{aligned} \tag{22}$$

[19] The case $b_{-i}(1-q_i^*) < p^*$ is ruled out by the fact that $b_{-i}(1-q_i^*)$ is a winning bid and p^* is the lowest winning bid.

Since $p^* > v$, $\sum_{j \neq i} \hat{q}_j'^* \geq \sum_{j \neq i} q_j^* = 1 - q_i^*$ and $m_i(p^*) - m_i([1 - q_i^*] p^*) < q_i^* p^*$, the term in the first line is strictly positive. As ε approaches zero, the term in the second line approaches zero, rendering the deviation strictly profitable. Furthermore, since this deviation strictly decreases agent i's net payment, his financing cost C can only decrease. ■

Claim 5 *There exists an agent i such that m_i is positive, $q_i^* < 1$, and $t_i - m_i\left(\sum_{j \in N} t_j\right) < w_i$.*

Proof. Suppose, to the contrary, that, for every agent i whose share m_i is positive, $q_i^* = 1$ or $t_i - m_i\left(\sum_{j \in N} t_j\right) \geq w_i$.

Case 1 For some such agent i, $q_i^* = 1$. Then, for every such agent $j \neq i$, $q_j^* = 0$ and therefore $t_j = 0$ and $t_j - m_i\left(\sum_{k \in N} t_k\right) < w_j$, a contradiction.

Case 2 For every such agent i, $q_i^* < 1$. Since, by the definition of the sharing rule m, $\sum_i \left[t_i - m_i\left(\sum_{j \in N} t_j\right)\right] = 0 < \sum_i w_i$, there exists some such agent i such that $t_i - m_i\left(\sum_{j \in N} t_j\right) < w_i$, a contradiction. ■

Claim 6 *$p^* \geq v$.*

Proof. Suppose, to the contrary, that $p^* < v$. By Claim 5, there exists an agent i such that m_i is positive, $q_i^* < 1$, and $t_i - m_i\left(\sum_{j \in N} t_j\right) < w_i$. Then, instead of getting the equilibrium payoff (19), agent i could, for some $\delta > 0$, deviate to $b_i'(s) = p^* + \varepsilon$ for every $s \in (0, q_i^* + \delta]$, $b_i'(s) = b_i(s)$ otherwise and $\hat{b}_i' \geq q_i^* + \delta$ and get the deviation payoff

$$q_i^*[v - p^*] + \int_{1-q_i'^*}^{1-q_i^*} [v - b_{-i}(s)]\, ds + m_i\left(q_i^* p^* + \int_{1-q_i'^*}^{1-q_i^*} b_{-i}(s)\, ds + \left[1 - q_i'^*\right] p^*\right) \tag{23}$$

where $q_i'^* > q_i^*$ since, by Claim 3, $b_{-i}(1 - q_i^*) = p^*$ and, by assumption, b_{-i} is left-continuous. For ε small enough, $b_{-i}(s) \in [p^*, v)$ for every $s \in (1 - q_i'^*, 1 - q_i^*]$ and therefore the deviation payoff (23) is strictly greater than the equilibrium payoff (19). Furthermore, this deviation does not violate agent i's budget constraint for δ small enough and therefore the financing cost C remains zero. ■

According to the previous claims, agent i's equilibrium payoff, ignoring the financing cost C and the default penalty c, is $m_i(v)$.

Claim 7 *For every agent i, $t_i - m_i\left(\sum_{j\in N} t_j\right) \leq w_i$.*

Proof. Suppose, to the contrary, that, for some agent i, $t_i - m_i\left(\sum_{j\in N} t_j\right) > w_i$ and note that this implies that $q_i^* > 0$ and that he incurs a strictly positive financing cost C. Instead of getting the equilibrium payoff $m_i(v) - C$, he could, by deviating to $b_i'(s) = p^* - \varepsilon$ for every s, get the deviation payoff

$$m_i\left(\sum_{j\neq i}\int_{1-\hat{q}_j'^*}^{1} b_{-j}'(s)\,ds\right) + \frac{m_i(p^*) - m_i\left(\sum_{j\neq i}\hat{q}_j'^* p^*\right)}{p^*}v \tag{24}$$

where, for every agent $j \neq i$, $\hat{q}_j'^* \geq q_j^*$, for every $s \in \left(1-\hat{q}_j'^*, 1\right]$, $b_{-j}'(s) \geq p^* - \varepsilon$, $\sum_{j\neq i}\hat{q}_j'^* \leq 1$ and $p^* = v$. As ε approaches zero, the deviation payoff (24) approaches $m_i(v)$, which is strictly greater than the equilibrium payoff $m_i(v) - C$. ■

Claim 8 *The action profile $b_i(s) = v$ for every s and $\hat{q}_i = m_i(v)/v$ for every agent i is a Nash equilibrium.*

Proof. Since $\sum_j \hat{q}_j = 1$, $q_i^* = \hat{q}_i^* = \hat{q}_i$. Furthermore, since $\hat{q}_i = m_i(v)/v$ and $p^* = v$, agent i's net payment is zero. Agent i's payoff is therefore

$$\hat{q}_i[v - p^*] + m_i(p^*) = m_i(v). \tag{25}$$

Instead of getting the equilibrium payoff $m_i(v)$, agent i could deviate to b_i' and $\hat{q}_i'$ and get the deviation payoff

$$\hat{q}_i'^*[v - p^*] + m_i\left(\hat{q}_i'^* p^* + \sum_{j\neq i}\int_{1-\hat{q}_j'^*}^{1} b_{-j}'(s)\right) \tag{26}$$
$$+ \quad \frac{m_i(p'^*) - m_i\left(q_i'^* p'^* + \sum_{j\neq i}\hat{q}_j'^* p'^*\right)}{p'^*}v$$

where, for every agent $j \neq i$ and every $s \in \left(1 - \hat{q}_j'^{*}, 1\right]$, $b'_{-j}(s) \leq p^*$. If $q_i'^{*} < 1$, then $p'^{*} = p^*$. If $q_i'^{*} = 1$, then $p'^{*} \geq p^*$ but note that then the secondary allocation is zero. Since $p^* = v$, this deviation payoff cannot exceed the equilibrium payoff $m_i(v)$ in either case, even when the deviation does not violate agent i's budget constraint. ■

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