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Certainty and Uncertainty in the Taxation of Risky Returns

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Abstract

I extend the general equilibrium techniques that have been applied to proportionate taxes to gain better understanding of non-proportionate taxes, such as those having loss disallowances or progressivity features. I frame the question of how to go about this extension by analogizing proportionate taxes to financial forwards and more general taxes to structured financial options, and I find that option pricing theory and methods carry over naturally. In general, I find that the burden of an income tax has a "certainty equivalent value" that can be calculated as the price of a corresponding option, and I find that non-proportionate income taxes generally burden risky returns just as option prices generally reflect the volatility inherent in risky assets. I develop in detail the case of a tax that is proportionate for gains but allows no deduction for losses, and I find that riskier investments are disfavored and that such a tax exerts pressure for homogeneity of investment portfolios across taxpayers. In addition, I find that the effect is amplified for options, with puts and calls, for example, generally burdened by non-proportionate taxes to a much greater degree than the underlying risky asset itself. As a result, synthetic divisions of ownership using options result in greater tax burdens than undivided ownership. In particular, debt financing is disfavored relative to 100% equity financing under such a system. Beyond application to this specific type of non-proportionate tax, the new theoretical approach I lay out and the concept of the certainty equivalent value I develop provide the tools necessary for enhanced understanding and analysis of the precise way in which a wide range of taxes burden risky returns.

Keywords: Income Taxation; Proportionate Income Taxation; Risky Return to Assets; Domar-Musgrave; General Equilibrium

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1 Overview

It is well understood in the law and public finance literature that a constant-rate tax levied proportionately on investment income is equivalent to a tax on the risk-free return to initial investment wealth, at least under idealized assumptions about the supply of assets, the interest rate for borrowed money, and the de minimis nature of transaction costs.[1] What is less well understood is how precisely this result changes when the tax is not proportionate. This paper addresses the gap in understanding by extending the general equilibrium approach developed by Kaplow (1991, 1994) to encompass continuous-time trading and then using option pricing techniques to analyze systematically non-proportionate methods of taxation.

The principle result of this paper is that the tax levied on an asset's return under a non-proportionate system of taxation can be translated into an equivalent up-front lump sum tax and that this "certainty equivalent" value can be calculated in a precise and systematic way. Analysis of the nature of the certainty equivalent value yields the further result that it grows with the risk of the return but does not depend on the expected size of the return. Thus, non-proportionate taxes burden risk taking, with greater risks corresponding to greater tax burdens, regardless of the expected level of returns. This effect is amplified for options, with puts and calls, for example, generally burdened by non-proportionate taxes to a much greater degree than the underlying risky asset itself. In addition, if taxpayers divide rights in a risky asset among themselves synthetically using options, the sum of their individual tax burdens will generally be larger than it would be if each had an undivided right to a fraction of the asset. An example of such a division is debt financing of an investment,[2] and a non-proportionate tax thus generally places a heavier tax burden on separate debt and equity investors than those who do not finance ownership with debt.

In general, the burden of a non-proportionate tax falls on risk taking and tends to fall most heavily on those who do not diversify their holdings. Financial portfolios consisting of many diversified assets are preferred, from a tax perspective, to investments in smaller numbers of assets that bear significant idiosyncratic risk. Moreover, outright ownership of a risky asset is preferred over a partial interest with option-like characteristics. This paper makes these findings precise and quantifies the burden a non-proportionate tax places on investment. It also describes a general methodology that can be used to assess the burden

[1]Domar and Musgrave (1944, 1945) lay out the result in its original form, and Kaplow (1991, 1994) extends it to a general equilibrium setting. Additional discussion and development in the legal literature can be found, for example, in Warren (1996) and Weisbach (2004, 2005). See Avi-Yonah (2004) for a discussion of some of the limitations resulting from borrowing costs and transaction costs.

[2]In terms of financial products, equity can be thought of as a call option on the underlying risky asset and debt can be thought of as a short put option on the underlying asset coupled with a long position in a riskless bond.

imposed by a non-proportionate tax system on a wide variety of investment strategies.

1.1 Related Literature

The fact that a non-proportionate income tax disfavors risky investment is not new to the literature. It is well-known that a tax disallowing losses, for example, preferentially favors gains over losses and generally distorts taxpayer behavior from what it would be if there were a proportionate tax, or no tax at all. Moreover, it is understood that any non-linearity in a tax system, such as a progressive rate structure, can impose a significant burden on risk taking. The work of Stiglitz (1969, 1972) and Sandmo (1985) are important theoretical studies of the effect of taxation on risk taking. The subsequent work of Kaplow (1991, 1994) introduces a general equilibrium methodology and establishes principles for determining equivalence between tax regimes. In addition, the work of Schenk (2000) and Zelenak (2006) recognizes that the usual arguments about the effect of proportionate taxes do not apply to non-proportionate taxes. Beyond the theoretical results and discussion, there is also an empirical literature, and the work of Gentry and Hubbard (2005) shows striking empirical evidence that entry into entrepreneurial activity by taxpayers is sensitive to convexity in the progressive rate schedule, with such convexity serving to discourage risk taking.

This paper moves beyond the existing literature by applying the general equilibrium techniques developed by Kaplow (1991, 1994) in the setting of non-proportionate taxes. This innovation is accomplished by extending the existing model to allow for trading in investment assets between the time of initial investment and final consumption, and by introducing techniques from financial economics to evaluate non-proportionate taxes as options on underlying risky assets. The key result of this approach is the identification of a certainty equivalent value for a non-proportionate tax on an investment portfolio. Such certainty equivalent values can be compared across investment strategies and quantify the degree to which a tax burdens risk taking of different types.

The results of this paper are also related to work in the corporate finance literature, where it has been long recognized that understanding of taxes on corporate income can be informed by the option pricing theory of financial economics. Green and Talmor (1985) and Majd and Myers (1986) present important research in this regard. The current paper is distinct from the corporate finance literature in that it employs the general equilibrium approach developed by Kaplow to analyze taxes in a context that takes into account fully the behavior of all parties, including the government. In addition, the current paper focusses on the question of the burden of a tax on individual investors instead of corporate investors. Nonetheless, there is significant connection between the corporate finance literature and the methodology set forth in this paper, and it is hoped that the two sets of literature may be able to benefit from and inform each other in the future.

1.2 Motivation in Terms of Financial Instruments

The approach taken by this paper and the methodology developed can be explained by interpreting tax burdens on investments in terms of financial instruments. Before turning to the details of the economic model involved, it is helpful to provide background and explain the motivation behind what follows.

As has been noted, the analysis of proportionate taxes dates back to the classic results of Domar and Musgrave (1944, 1945), and it has been broadly extended and placed in a general equilibrium setting.[3] The fundamental underlying idea of all this analysis can be expressed in the language of finance in terms of a forward contract on a risky asset and a riskless bond. For every unit of risky asset a taxpayer holds, his future obligation to pay taxes is the same as a short position in τ units of a forward contract on the asset, where τ is the constant tax rate, coupled with a short riskless bond position under which he must pay in the future τ times the difference between the forward price and the current price of the asset.[4] The government, on the other hand, has a right to collect taxes paid that is the same as the long forward position and long riskless bond position complementary to the short positions of the taxpayer. The short forward position can be perfectly hedged by borrowing at the risk-free rate to invest in an offsetting position in the risky asset, and the long forward position can be perfectly hedged by the shorting an offsetting amount of the risky asset and investing at the risk-free rate. Both of these hedges are costless, and so the value of the forward contract is actually zero. Thus, the total cost of a proportionate tax is simply the same as that of a tax depending only on the risk-free return to capital.

If a system of taxation is not proportionate, however, the usual analysis no longer applies since the obligation to pay tax and the right to collect tax are no longer the same as simple forward positions in the risky asset coupled with a position in a riskless bond. For example, if tax is levied on gains but no offsetting deduction is allowed for losses, then the taxpayer is short a call option on the risky asset while the government has the complementary long position in the same option. It is still possible to take positions in the risky asset that hedge

[3]See Kaplow (1991, 1994), who innovates and broadens the analysis by incorporating government action into the equilibrium and thereby obtains results that are truly of a general equilibrium character.

[4]A forward contract, or more simply forward, means a contract under which one party sells and the other party buys an asset at a specified price at a specified exercise time. The specified price is called the "forward price" and is required to be the future value that would be obtained if the current cost of the asset were invested at the risk-free rate of return until the specified exercise time.

the option position represented by the tax,[5] but the cost of hedging the option depends not only on the risk-free rate but also on the risk inherent in the underlying risky asset.[6]

A tax system can fail to be proportionate for a variety of reasons. As discussed in the preceding paragraph, it may not allow for loss-offsetting deductions. In fact, any progressive tax system also fails to be proportionate, since different levels of return are taxed at different rates. Moreover, a system that allows taxpayers discretion about the timing or rate of tax applicable to returns also fails to be proportionate. A common example of this is the choice taxpayers have about when to realize gains and losses on capital asset – they may often choose, perhaps, to realize losses sooner and gains later. Another example is the ability of U.S. firms to decide when to keep earnings of a foreign subsidiary permanently reinvested abroad – they may often choose, perhaps, to permanently reinvest earnings in foreign jurisdictions with low tax rates so as to avoid additional tax upon repatriation of those earnings.

Each of the foregoing examples of non-proportionate taxes can be expressed as a deterministic function of the stochastic return on the risky assets in question. That is to say, even though risky asset returns are uncertain at the start, once the return is known, there is a definite and known way in which the tax on the return can be computed. This computation may involve choice on the part of the taxpayer, but if the taxpayer's preferences are known from the start, then the choice he will make given any particular levels of return is also known from the start.

If the tax on a risky asset is deterministic, conditioned upon knowing the asset return level, then the tax payable can, in general, be hedged with a (possibly dynamic) position in the underlying asset. The tax can be thought of as a complex option on the risky asset, and the cost of this option can be determined by option pricing techniques that compute the cost of the hedging strategy in present value terms. This cost is referred to as the certainty equivalent value of the tax, and it quantifies, in present value terms, the burden imposed by

[5]The hedging required will generally evolve dynamically. At each point in time, a particular position in the risky asset will hedge the option position, but the size of the required hedge will change over time depending upon how the asset value changes.

[6]The present value of the cost of hedging an option is the same as the price of the option. The techniques that can be used to determine this price rely on replicating the option dynamically using hedging positions in the asset and then evaluating the present value cost of such a dynamic hedging strategy. This cost generally depends on the amount of risk inherent in the underlying asset, as well as other factors. For example, if the returns of the risky asset are log-normally distributed, then the Black-Scholes formula can be applied to determine the option price, and if the risk-free rate is zero and the asset pays no dividends, an at-the-money call option costs $\frac{1}{\sqrt{2\pi}} S_0 \sigma \sqrt{t}$, where S_0 is the current price of the asset in dollars, σ is the volatility (risk) of the stock, and t is the time to expiration of the option. Since $1/\sqrt{2\pi} \approx 0.40$, this is approximately equal to 40% of the volatility of the stock in dollars over the term of the option. See Bodie et al. (2005) and Hull (2000) for further details and discussion of option hedging and pricing.

the tax on the individual investor, and the value of the right of the government to collect the tax from the investor. This value will generally reflect the volatility, or risk, of the underlying asset in a specific way, meaning that the manner in which a non-proportionate income tax burdens risky returns is understandable in a precise and quantifiable way.

In order to implement the ideas of the foregoing discussion in a rigorous way, it is necessary to have a conceptual framework that allows for the modeling of various systems of taxation. The starting place is the general equilibrium model introduced by Kaplow (1991, 1994), and this paper builds on that foundation. The existing model is extended to allow for continuous trading of investment assets between the time of initial investment and the time of final consumption. This extension allows option pricing techniques to be applied to determine precisely the above-described certainty equivalent value for a tax. In Section 2, the extension of the existing model is described in detail, certainty equivalent values are shown to capture the cost of a tax regime in a precise way, and the extended model is applied to recover well-known results in the case of a proportionate tax. In Section 3, the extended model is applied to investigate the burden imposed by a non-proportionate tax that disallows loss offsets, and it is demonstrated through numerical examples that such a tax places a heavier burden on riskier assets, that this effect is amplified in the case of options on risky assets, and that portfolio diversification is favored by such a tax. Section 4 concludes.

2 Extending the General Equilibrium Approach

In this section, the general equilibrium approach pioneered by Kaplow (1991, 1994) is extended in two ways. First, the possibility of trading and rebalancing of investment portfolios is allowed between the starting time, at which initial investments of capital are determined, and the ending time, at which tax is levied by the government and consumption of investment value occurs. Second, the possibility of a non-proportionate tax on returns to investment is introduced. The calculation of the certainty equivalent value of such a tax is described using option valuation techniques and to construct a hedge for the tax liability that is updated dynamically over the course of time from $t = 0$ to $t = 1$. Finally, the techniques developed are applied to recover the well-known equivalence results of Domar and Musgrave (1944, 1945) and Kaplow (1991, 1994) in the case of a proportionate tax regime.

2.1 The Basic Model and Multiple Time Steps

Following Kaplow (1991, 1994), the model is based on an economy beginning at time $t_0 = 0$ and ending at time $t_1 = 1$. At the first time point, individuals receive an amount to be invested, and not consumed, until time t_1. At time t_1, the government levies a tax on investment returns and individuals consume the entirety of their after-tax investments. The government is also permitted to invest during the period from t_0 to t_1, and individuals and the government allocate their investment between a riskless asset and a risky asset. Long and short positions can be taken by any investor.[7]

This paper extends the usual model by allowing for multiple times between t_0 and t_1 at which costless trading in investment assets by individual investors and the government can occur. The requirement still holds, however, that no consumption of investment assets is allowed until time t_1. Thus, the trading that is allowed only rebalances investment holdings and does not permit investors to consume investment wealth early. The addition of the additional time steps allows investors to manage their portfolios actively, giving them a richer and more realistic set of opportunities than is possible if only a single initial portfolio allocation choice is made. In addition, it allows for the use of option valuation techniques necessary to quantify the effects of non-proportionate taxes on investment earnings.

Throughout most of this paper, certain assumptions are made about the times at which trading can occur and the way that asset prices evolve from one such time to the next. It is assumed that there are $n - 1$ equally spaced times between t_0 and t_1, namely $t_{i/n} = i/n$, for $i = 1, \ldots, n-1$. At each time $t_{i/n}$ for $0 \leq i \leq n$, the value of the risky asset either increases

[7] Note that a short position in the riskless asset amounts to borrowing, and individuals and the government pay the same interest rate on borrowing in this model.

by a factor u_n or decreases by a factor d_n between time $t_{i/n}$ and $t_{(i+1)/n}$, where

$$u_n = \exp\left(\mu/n + \sigma\sqrt{1/n}\right) \quad \text{and} \quad d_n = \exp\left(\mu/n - \sigma\sqrt{1/n}\right). \tag{1}$$

These values are chosen so that, in the limit as $n \to \infty$, if there is an equal likelihood of u_n and d_n at each time increment, the distribution of asset values at time $t = 1$ is lognormally distributed such that underlying normal distribution has mean μ and standard deviation σ. If a_0 is the initial value of the risky asset, this specification for asset value evolution implies that there are $i + 1$ possible values for the asset value $a_{i/n}$ at time $t_{i/n}$, namely $a_0 u_n^j d_n^{i-j}$ for $0 \leq j \leq i$.

In contrast to the risky asset, the change in size of the riskless asset from time $t_{i/n}$ and $t_{(i+1)/n}$ is constant. It is assumed that the rate of growth in each interval is $\exp(r/n)$ so that the value of the riskless asset at time i is $b_{i/n} = \exp(ir/n)$.

The particular choice for restricting asset values to the discrete approximation of a normal distribution specified above allows for the valuation of options on the risky asset to be made with the binomial tree method discussed in Hull (2000, Chapter 9).[8] As will become clear, however, little depends on this particular choice of approximation as $n \to \infty$ and continuous portfolio rebalancing is permitted. The binomial approach facilitates exposition and illustrates the principles involved, but option pricing and hedging in continuous time work as well, assuming that costless trading in continuous time is possible.

2.2 Certainty Equivalent of a Tax Regime

Consider a tax regime in which the government imposes a tax on investment returns at time t_1. The tax regime is specified by a deterministic rule that reckons an amount of tax for each investor based upon the final value of his investments and the initial value of his investments. An example of such a regime is a proportionate tax that levies a constant-rate tax on gains and allows an offset for losses at the same constant rate. Another example is a non-proportionate tax that levies a constant-rate tax on gains but disallows any offset for losses. The goal of this section is to define a certainty equivalent value of the tax regime for each investor. This amount quantifies the burden of the tax borne by an investor in a single present value figure, and the sum of these amounts over all investors quantifies the aggregate value of taxes to be paid to the government in present value terms.

The notion of the certainty equivalent value is based upon the concept of tax regime

[8] For this method to be applied, "risk neutral" probabilities for u_n and d_n are used rather than equal probabilities. See Hull (2000) for details.

equivalence introduced in Kaplow (1991, 1994). The idea is to determine an up-front lump sum amount for each investor and a set of offsetting portfolio adjustments for each individual and the government with certain properties.[9] To wit, if the government foregoes a tax at time t_1 and instead levies a tax at time t_0 in the lump sum amounts and if the offsetting portfolio adjustments are made by all parties, then: (1) the after-tax investment portfolio is the same at time t_1 for each investor as it would have been under the original tax regime; (2) the final investment portfolio value for the government is equal to the amount of tax the government would have collected under the original tax regime; and (3) the portfolio adjustments are all offsetting so that the government and investors in the aggregate do not alter the net holdings of any asset at any time. If it is possible to determine up-front lump sum amounts and portfolio adjustments with these properties, then a state of general equilibrium under the original tax regime corresponds to a state of general equilibrium under the alternative tax system with portfolio adjustments. Indeed, the change from the original tax regime to the new system allows markets to clear and does not alter after-tax outcomes for investors or the government at time t_1. Because the objective functions maximized by investors and the government depend only on these final outcomes, a state of general equilibrium will be preserved by a change from the original regime to the new system. In this sense, the new system is equivalent to the original tax regime, and the up-front lump sum amounts represent the certainty equivalent value of tax for each investor.

It is now necessary to determine whether it is in fact possible to find up-front lump sum amounts and portfolio adjustments of the sort described in the preceding paragraph. Proceed first under the assumption that the binomial model described in Section 2.1 holds so that there are $i+1$ possible values for the risky asset at time $t_{i/n}$.[10] In this model, a final pre-tax target portfolio specification is a function f that has a non-negative value for a portfolio assigned to each of the $n+1$ final possible states of the world, which correspond to the $n+1$ final possible values for the risky asset. It is a standard result that for any such function f, there is an up-front amount of money and a dynamic plan of investment that will lead deterministically to the final values specified in f.[11] The function f can be thought of as representing a complex option on the risky asset with payoff function f, and the up-front

[9] The question of whether such amounts and such adjustments exists, and how to calculate them if they do exist, is discussed below.

[10] The more general case of continuous time is handled below.

[11] The idea is to work backwards with a methodology called "dynamic programming". The $n+1$ final values for the portfolio are specified at time t_1. From this it is possible to determine the n values and portfolio choices at time $t_{1-1/n}$ that necessarily give rise to the final portfolios at time $n+1$. Continuing to work backward, at time t_0 there is a single price that is the up-front amount of money required. The investment steps taken at each point constitute the required dynamic plan of investment. See Hull (2000, Chapter 9) for further details.

amount can be thought of as the price of the option. In a state of general equilibrium under the original tax regime, each investor has chosen a final pre-tax payoff function f that has an option price equal to the investor's initial investment amount. The investor purchases this option and follows the dynamic investment strategy necessary to obtain deterministically the desired final pre-tax payoff specified by f.

Just as each investor specifies a target final pre-tax payoff function f, so the government has a final payoff function associated with the investor, denoted f_τ, which results from the levying of tax at time t_1 on the values given by f. The government could obtain this same final payoff, however, by foregoing taxation at time t_1 and instead investing the option price of f_τ at time t_0 and following the dynamic investment strategy necessary to obtain deterministically the final payoff specified by f_τ. Moreover, the government could obtain the option price of f_τ at time t_0 by levying an up-front lump sum tax on the investor. The investor would then be left with the option price of f minus the option price of f_τ to invest at time t_0, but if he were to take this amount and invest it according to his original dynamic strategy, adjusted by positions to counter exactly the government's new investments, the final after-tax payoff to the investor would be the after-tax amount under the original tax regime, namely the after-tax payoff function $f - f_\tau$. The option value of f_τ is thus the certainty equivalent value of the original tax regime for the investor. If the government levies an up-front lump sum tax in this amount, and if appropriate portfolio adjustments are made, then a general equilibrium under the original tax regime will be transformed into a general equilibrium under this new system.

The foregoing derivation of certainty equivalent values may seem restrictive in that it relies on the particular choice of asset price evolution represented by the binomial model. Much more general results hold, however. What is necessary is that markets be complete in the sense that it is possible to construct a dynamic replicating portfolio of any desired final payoff function f. If asset prices follow a suitably well-behaved distribution, such as a lognormal distribution, then such dynamic replicating portfolios always exist in continuous-time trading. More complex cases, such as those involving stochastic volatility, for example, may require the addition of more assets beyond the basic risky asset in order to make the market complete. As long as the model is made expansive enough to have complete markets, however, the general approach described here works. Even if there are multiple risky assets, the process of pricing a final payoff option through the procedure of valuing the cost of replicating portfolios works, and this pricing procedure can be used to determine the certainty equivalent values of a tax regime.

2.3 Revisiting the Proportionate Tax

It is useful to see how the concept of certainty equivalent value developed above applies to a proportionate tax regime. Under such a regime, tax is levied at a constant rate τ, and so the tax payment from an individual to the government at time t_1 is

$$\tau\,(P_1 - P_0), \tag{2}$$

where P_i is the value of the individual's portfolio at time t_i. Note that it is possible for this payment to be negative, if $P_1 < P_0$, and in this case the government allows an offset of losses sustained.

For each investor, the certainty equivalent value of this tax regime is the option price of the government's payoff function f_τ, which is based upon the investor's payoff function f. Because of the structure of this tax, it follows that $f_\tau = \tau(f - P_0)$, and so the option price of f_τ is simply τ times the difference between the option price of f and the option price of P_0. The option price of f is P_0, since this is the initial amount invested to obtain the final payoff f, and the option price of the constant payoff P_0 is $(b_0/b_1)P_0$, which is just the present value of P_0.[12] Combining these findings, it follows that the option price of f_τ is $\tau P_0\,(1 - (b_0/b_1))$. If one writes $b_0/b_1 = \exp(-r)$, the certainty equivalent value of the proportionate tax is thus seen to be $\tau P_0(1 - \exp(-r))$, which is also equivalent to a value of $\tau P_0(\exp(r) - 1)$ payable at time t_1. This is the well-known result of Domar and Musgrave (1944, 1945) and Kaplow (1991, 1994) that a proportionate tax is equivalent to a tax on the risk-free rate of return to initial investment value.

It is useful to express the foregoing result in the language of financial instruments as well. From this perspective, the government's tax payoff function is a combination of forward contract for a payoff in the amount f at time t_1 and a riskless bond, namely

$$f_\tau = \tau\,(f - P_0) = \tau\,\overbrace{(f - (b_1/b_0)P_0)}^{\text{Forward Payoff}} + \tau\,\overbrace{((b_1/b_0)P_0 - P_0)}^{\text{Bond Payoff}}. \tag{3}$$

The forward contract has present value zero,[13] and the present value of the bond payoff is $\tau P_0\,(1 - (b_0/b_1))$, and this is the present option value for f_τ calculated above.

[12] Since the riskless asset has value b_0 at time t_0 and value b_1 at time t_1, the present value of a certain payoff at time t_1 is equal to b_0/b_1 times the amount of that certain payoff.

[13] The price of the forward contract is zero since the present option value of f is P_0, the amount the investor initially has available to invest, and the present value of a certain payoff of $(b_1/b_0)P_0$ is also P_0. Thus the option value of the difference of these two sets of payoffs is zero.

3 Applications to a Non-Proportionate Tax

In this section, some applications and results of the model in Section 2 are described in the case of a tax that is proportionate for gains but that disallows loss offsets. The approach is to consider specific numerical examples to gain insight into the effects of a non-proportionate tax. The findings are robust to a variety of choices for the parameters underlying the numerical example, and they thus appear typical of the behavior of this non-proportionate tax in general.

The first set of results below highlights the fact that such a non-proportionate tax burdens the risk but not the expected return of investments. The second set of results illustrates the high tax burden on options under such a non-proportionate tax. Finally, the third set of results considers the situation of two risky assets and demonstrates that diversification of portfolio holdings across risky assets is generally favored.

3.1 Burden on the Risky Asset

Consider a tax regime in which gains are taxed at a constant rate τ but in which no loss offsets are allowed. Thus, the tax payment from an individual to the government at time t_1 is

$$\tau \max(P_1 - P_0, 0), \tag{4}$$

where P_i is the value of the individual's investment portfolio at time t_i. The purpose of this section is to gain an understanding of the way in which this non-proportionate tax burdens risk by analyzing a range of numerical examples.

Consider two simple strategies that can be employed by an investor. The first strategy, denoted "Bond" is one in which the investor puts all of his initial investment in the riskless asset and leaves it there until time t_1. The second strategy, denoted "Stock" is one in which the investor puts all of his initial investment in the risky asset and leaves it there until time t_1. Much more complex strategies are of course possible, but it is informative to start with these basic choices. Given parameters for the asset returns and the rate of tax on gains, it is possible to use the methods of Section 2 to compute the certainty equivalent value of the tax on each of these strategies. Assume that

$$\mu = 8\%, \quad \sigma = 20\%, \quad r = 5\%, \quad \text{and} \quad \tau = 35\%, \tag{5}$$

where μ and σ are the quantities appearing in (1) that describe the lognormal distribution followed by the risky asset's value, and where r is the continuously compounded rate of return on the riskless asset. Under these assumptions, the certainty equivalent values of the two simple strategies are

$$\text{CEV(Bond)} = 1.70\% \quad \text{and} \quad \text{CEV(Asset)} = 3.66\%,$$

where both amounts are expressed as a percentage of the initial investment value, P_0.[14] It is notable that the certainty equivalent value for the Stock strategy is more than twice the value for the Bond strategy. In fact it is typical that strategies with more volatility have higher tax cost, and since the Bond strategy has no volatility, it is the lowest cost strategy available.

Moving beyond the two simplest strategies, consider a mixed strategy in which an investor places a fraction w of his initial investment value in the Stock strategy and the remaining fraction $1 - w$ in the Bond strategy. The investor does not trade any investments over the course of time from t_0 through t_1. Figure 1 illustrates how the certainty equivalent value of such a strategy varies with the choice of weight w. When $w = 0\%$ and $w = 100\%$, the certainty equivalent values calculated above for the Bond strategy and the Stock strategy are recovered, respectively. In between, however, it is notable that the certainty equivalent value remains roughly constant for small values of w. This corresponds to the fact that, for small values of w, potential losses in the risky asset are generally outweighed by deterministic gains in the riskless asset. As long as the net change in value of the portfolio over time is positive, the non-proportionate tax is the same as a proportionate tax, and the proportionate tax yields the same certainty equivalent value for any choice of w. Thus, the non-proportionate tax does not discourage all risk taking equally, but rather it burdens most heavily those portfolios that are most likely to lead to net portfolio losses. The non-proportionate tax therefore treats preferentially portfolios that are hedged in such a way as to make losses less likely.

In addition to considering the trade-offs between the Bond and Stock strategies, it is interesting to examine the extent to which the tax burden depends on the volatility of the risky asset. Figure 2 illustrates how the certainty equivalent value of the tax on a portfolio invested 100% in the risky asset varies with the size of τ and also with the risk inherent in the asset, as measured by σ. Not surprisingly, the certainty equivalent value grows linearly

[14]The calculation of these amounts was accomplished using the binomial option pricing model described in Section 2 with $n = 500$ time increments. In general, this is the methodology used for calculating all certainty equivalent values in this Section 3.1 and in Section 3.2.

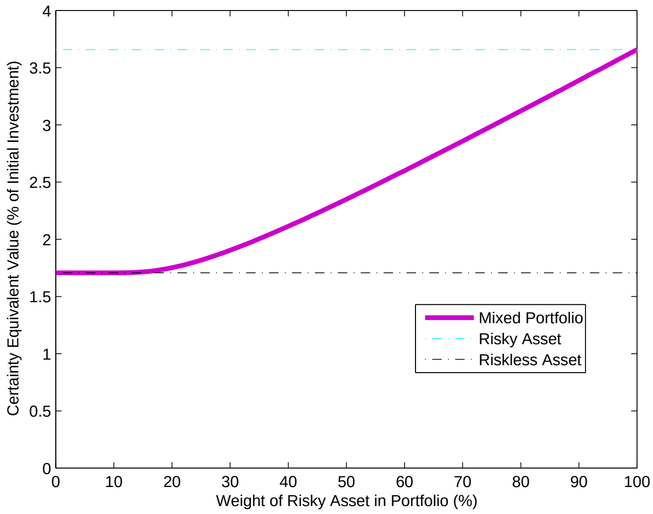
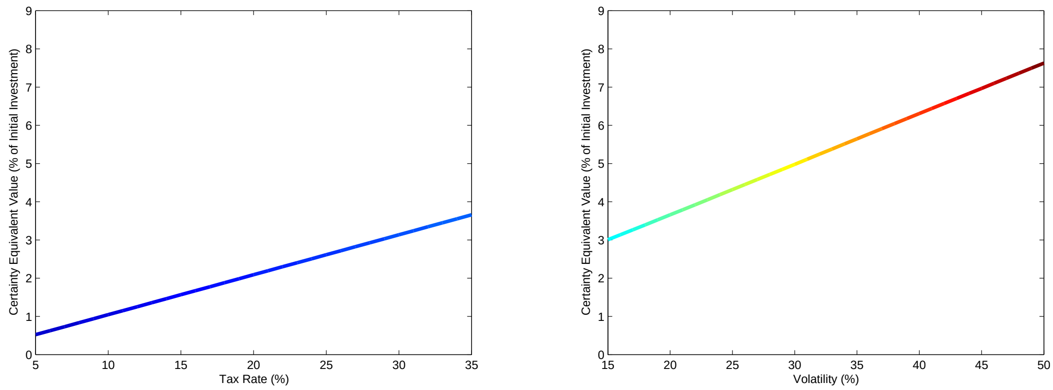


Figure 1: The figure illustrates the certainty equivalent value for the non-proportionate tax in the case of a portfolio constructed to have an initial weight, w, invested in the risky asset and a constant complementary weight, $1 - w$, invested in the riskless asset at time t_0, with no further trading in assets occurring from time t_0 through time t_1. The weight varies from 0% to 100% along the horizontal axis, and the certainty equivalent value is expressed as a percentage of the initial investment value, P_0.

with the tax rate τ. Less obviously, the certainty equivalent value also grows roughly linearly with σ. This is a reflection of the fact that the non-proportionate tax acts like a call option, with a payment due that is proportionate to the positive change in portfolio value from time t_0 to t_1 but zero if this change is negative. The price of a call option generally increases with volatility, and so the more volatile the underlying asset, the higher the certainty equivalent value of the tax imposed.

Figure 3 illustrates in three dimensions how the certainty equivalent value of the Stock strategy varies with the tax rate, τ, and with the volatility, σ, of the risky asset. In general, the same pattern observed in Figure 2 holds – for any fixed rate of tax, the certainty equivalent value increases with volatility. It is interesting to note that none of these findings depend on the value $\mu = 8\%$ specified in (5). The tax cost depends only on the risk, σ, and not on the expected return level, μ.



(a) Tax Cost vs. Tax Rate (b) Tax Cost vs. Volatility

Figure 2: The figure on the left shows how how the tax cost of a position invested 100% in the risky asset varies with the nominal tax rate imposed on gains. The tax cost is equal to the certainty equivalent value is expressed as a percentage of the initial investment value, P_0. The figure on the right shows how the tax cost of a position invested 100% in the risky asset varies with the volatility, σ, of the asset.

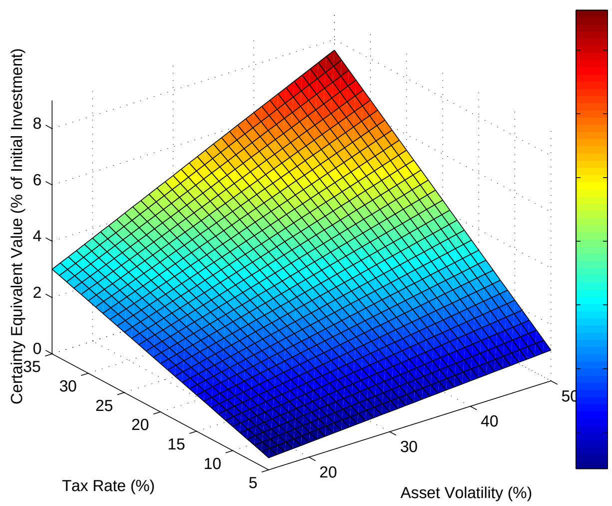


Figure 3: The figure provides a three dimensional illustration of how the tax cost, measured as the certainty equivalent value, of a portfolio invested 100% in the risky asset changes with the nominal tax rate, τ, and the volatility of the risky asset, σ.

3.2 Analysis of Options and Put-Call Parity

Moving beyond combinations of the simple Stock and Bond strategies, it is interesting to investigate how the non-proportionate tax described in (4) burdens the portfolios of an investor who targets a final payoff function f that depends on the risky asset in more complex ways. To this end, the burden on final payoffs functions in the form of the payoffs for put and call options are now considered.

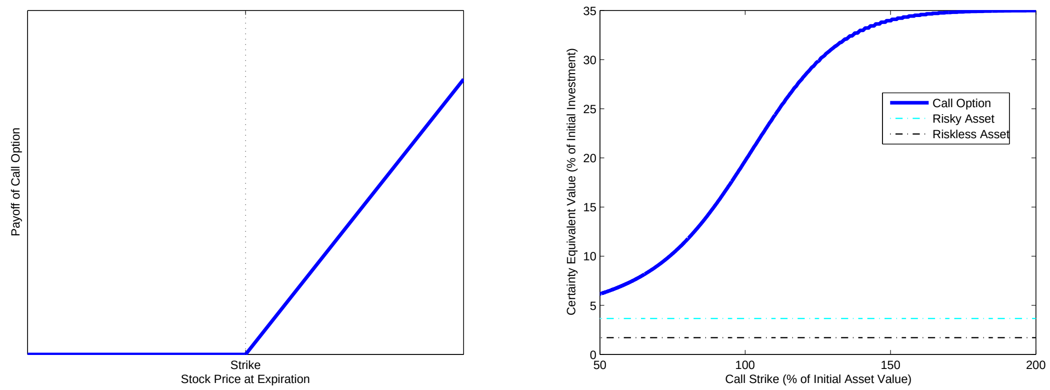
Figure 4 illustrates a typical payoff function for a call option (on the left) and the certainty equivalent value for a portfolio with final payoff function equal to a call option on the risky asset (on the right). The horizontal axis in the second graph indicates the strike, as a percentage of the initial value of the risky asset, of the option. As the strike price decreases, the payoff on the call option is closer to the payoff on the underlying risky asset, and ultimately a call option with a strike price of zero is the same as an interest in the underlying risky asset. Accordingly, the certainty equivalent value of a portfolio of call options tends to that of the Stock portfolio described in Section 3.1 as the strike price tends toward zero. As the strike price increases, however, the certainty equivalent value of the call option becomes substantially larger.

The maximum certainty equivalent value for a call option in Figure 4 tends to $\tau = 35\%$ as the strike price grows large. To see why this is so, note that the price on a call option with a high strike price is small.[15] If the option expires out of the money, the realized final payoff is zero, but if the option expires in the money, the payoff is generally substantially in excess of the cost of the option. In the first alternative, the portfolio has a net loss, and so the tax payable is zero, but in the second alternative, the tax payable is τ times the difference between the payoff amount and the cost of the option. This last value is close to τ times the payoff amount (since the cost of the option is small), and so it follows that the tax payoff function f_τ is much like τ times the payoff function of a call option with the same strike, so that $f_\tau \to \tau f$, as the strike price becomes large. Thus the certainty equivalent value, which is just the option value of f_τ, is approximately equal to τ times the option value of f, which in turn is equal to τ times the initial investment value, P_0. As a result, the tax burden tends to $\tau = 35\%$ as the strike becomes large.

In general, the tax burden on a call option under a non-proportionate tax is greater than the burden on the underlying risky asset. In the specific example considered, this burden on the call option is a significant multiple of the burden on the risky asset, with the burden, as a percentage of P_0, increasing to the full nominal tax rate of $\tau = 35\%$ as the strike of the

[15] Because the cost of an individual option is small, a portfolio invested exclusively in this type of option consists of investment in a large number of such options.

option grows larger.



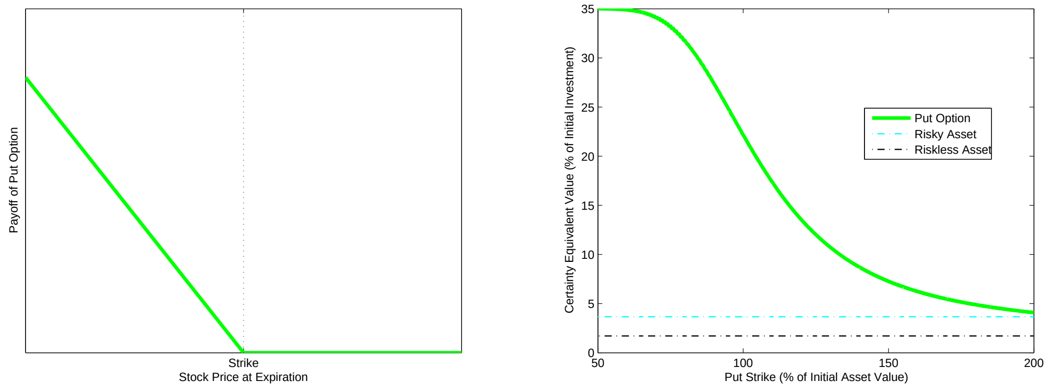
(a) Payoff Function for a Typical Call

(b) Tax Cost of a Call

Figure 4: The figure on the left shows a typical payoff function for a call option, with nothing paid if the risky asset value is not greater than the strike price, and with the difference between the risky asset value and the strike price received otherwise. The figure on the right shows the tax cost, measured as the certainty equivalent value, for an investment portfolio with final payoff function f equal to a call option payoff on the risky asset. The strike of the call option varies along the horizontal axis and is expressed as a percentage of the current value of the risky asset. The certainty equivalent value is expressed as a percentage of initial investment value.

The burden on put options is similar to that described for call options, except that the burden is greater for lower strike values than higher strike values. Figure 5 illustrates a typical payoff diagram for a put option (on the left) and the certainty equivalent value for a portfolio with final payoff function equal to a put option on the risky asset (on the right). An argument similar to the one above for call options shows that put options with a low strike price lead to a tax payoff function f_τ that is approximately like τf, and this shows that the certainty equivalent value for a put option tends to $\tau = 35\%$ as the strike price tends to 0. For large strike prices, the certainty equivalent value for a put option portfolio tends to that for a portfolio invested exclusively in the risky asset. In this way, the burden on the put option is much like a mirror image of the burden on a call option.

It is a natural question to ask whether a short option position might produce a result different from a long position, and so the case of a short put option is now investigated. Suppose that an investor selects a final payoff function f with the property that f has the same payoff as a certain number of short put positions and a long position in the riskless asset in an amount that has a final payment equal to the strike price of the put times the number



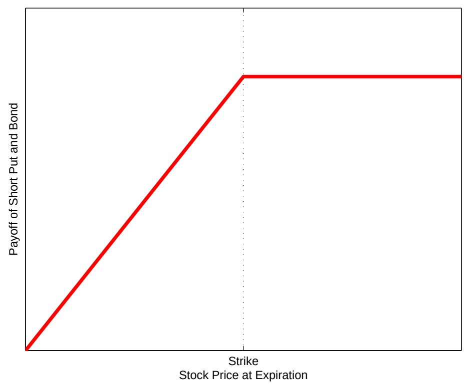
(a) Payoff Function for a Typical Put (b) Tax Cost of a Put

Figure 5: The figure on the left shows a typical payoff function for a put option, with nothing paid if the risky asset value is not less than the strike price, and with the difference between the strike price and the risky asset value received otherwise. The figure on the right shows the tax cost, measured as the certainty equivalent value, for an investment portfolio with final payoff function f equal to a put option payoff on the risky asset. The strike of the put option varies along the horizontal axis and is expressed as a percentage of the current value of the risky asset. The certainty equivalent value is expressed as a percentage of initial investment value.

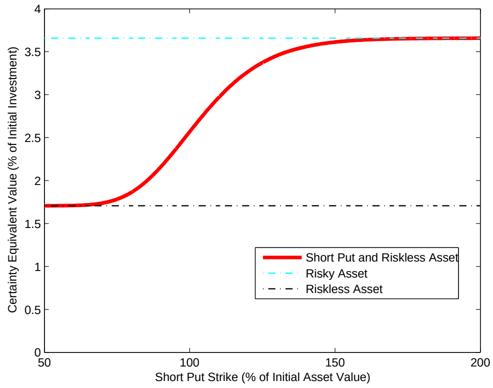
of short put positions.[16] This strategy is designed so that the long bond positions prevent the investment portfolio from ending with a negative total amount for any final value of the risky asset. The typical payoff diagram for such a combination of short puts and riskless assets is shown on the left in Figure 6. On the right, the figure shows the certainty equivalent value for this portfolio, with the strike varying along the horizontal axis. The figure indicates that the burden on this portfolio increases with increasing strike, the opposite of what happens for the long put portfolio. Moreover, the burden grows from a level equal to that of the Bond strategy to a level equal to that of the Stock strategy. Thus the short put position, which is effectively a "short volatility" strategy, allows an investor to achieve a tax burden below that of a position invested 100% in the risky asset, but it is never lower than the burden on a position invested 100% in the riskless asset.

The foregoing results can be combined to determine what the tax burden is on a synthetic division of ownership of the risky asset among taxpayers who use the put-call parity

[16]The specific number of positions taken is the number that can be financed by the initial investment amount, P_0, available to the investor. Thus, the number is chosen so that the entire portfolio consists of such positions.



(a) Payoff Function for a Typical Short Put and Bond



(b) Tax Cost of a Short Put and Bond

Figure 6: The figure on the left shows a typical payoff function for a short put option combined with a long position in the riskless asset in an amount equal to the strike of the put. The payoff is the same as that of the risky asset up to the level of the strike, and thereafter the payoff is limited to the amount of the strike. The figure on the right shows the tax cost, measured as the certainty equivalent value, for an investment portfolio with final payoff function f of the type illustrated in the figure on the left. The strike of the put option varies along the horizontal axis and is expressed as a percentage of the current value of the risky asset. The certainty equivalent value is expressed as a percentage of initial investment value.

relationship, namely,

$$\text{(Risky Asset)} \;=\; \text{Call} \;+\; \text{(Riskless Asset)} \;-\; \text{Put.}$$

In this equation, the put and call have the same strike price, and amount of the riskless asset is chosen so that the final payoff amount of this asset is equal to the common strike price of the put and the call. The payoff of the risky asset is exactly equal to the combined payoff of the positions on the right hand side of this equation, and so an investment in the right hand portfolio is the same as an investment in the risky asset.

Consider two alternative scenarios. In the first, there is a group of investors each with an equal initial investment value of P_0 and each investing his entire portfolio in the riskless asset. Assuming the parameter values specified in (5), the certainty equivalent value of the tax burden on each investor is 3.66% of the initial investment value. In the second scenario, this same group of investors reorganizes itself so that a certain fraction own portfolios of call options on the risky asset with a strike price of 100,[17] and the remaining fraction have portfolios consisting of short put options on the risky asset with a strike price of 100 and long riskless asset positions that have a final payoff amount equal to 100. The fractions in the second scenario are chosen so that the aggregate pre-tax position of the second group is the same as that of the first group. Specifically, this means that 10.45% of the group members have the call portfolios and the remaining 89.55% have the short put and long riskless asset positions.[18] Because the aggregate pre-tax position of the group has not changed, it might seem that the aggregate certainty equivalent value of the tax for the group should not change. However, those who hold call positions now have a certainty equivalent value of 19.70% of the initial investment value and those who hold short put positions have a certainty equivalent value of 2.57% of the initial investment value. In weighted average terms across the entire group, the aggregate certainty equivalent value is now

$$19.70\% \left(\frac{10.45}{100} \right) + 2.57\% \left(\frac{89.55}{100} \right) = 4.36\%.$$

Thus the overall tax burden on the group has increased from 3.66% to 4.36%, an increase of nearly 20%.

The result of the last paragraph is typical of the situation when put-call parity is used

[17] For purposes of this example, assume that the initial value of one unit of the risky asset is 100.

[18] This follows under the assumed parameters of (5) since the price of a call option struck at 100 is 10.45, and the price of a short put option plus a long riskless asset position of the type described is 89.55.

to reorganize the division of ownership across a group with options. There is generally an aggregate increased tax burden in terms of the aggregate certainty equivalent value. The increased burden is not spread evenly, however. As the foregoing example illustrates, some individuals may have a much higher burden (as the call holders did), and some a significantly lower burden (as the put holders did).

This result is of particular interest because a division of ownership using debt financing is essentially the same as the type of division based on put-call parity described the above example. The equity holders have a call option on the underlying risky asset, and the debt holders have the remaining short put and long riskless asset positions. The above analysis thus shows that, in this non-proportionate tax regime, debt financing results in a larger aggregate tax burden, as measured by the certainty equivalent value, than does 100% equity ownership.

Figure 7 illustrates the results of the example above for various alternative values of the strike value for the options involved. As can be seen from the figure, the largest increase in tax burden coming from the synthetic strategy occurs for strikes close to 100% of the initial value of the risky asset.

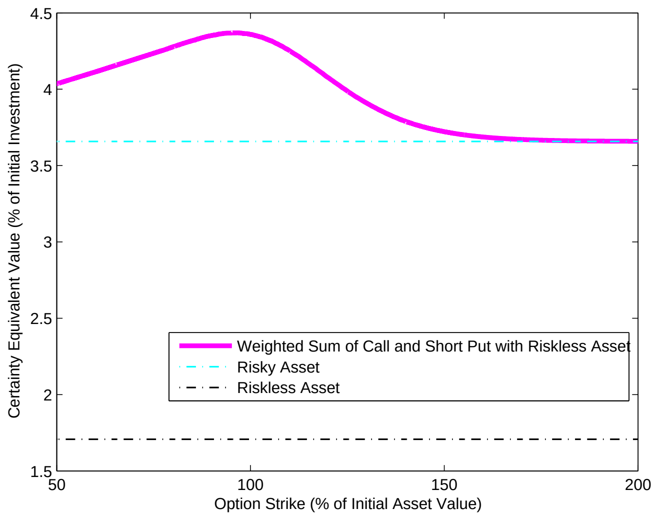


Figure 7: The figure shows the tax burden, measured as the average certainty equivalent value, for the group of investors described under the second scenario of the put-call parity example of Section 3.2. Some of these investors hold call options on the risky asset, and some have short put options with the same strike as the calls and long riskless asset positions with final payoffs equal to the strike of the calls. The relative number of each type of investor is chosen so that, in the aggregate, the pretax position of the group is equivalent to ownership of the risky asset. The strike value used for the calculation varies along the horizontal axis. The certainty equivalent value is expressed as a percentage of initial investment value.

3.3 The Case of Two Risky Assets

As a final numerical example, consider the case of two risky assets. Until now, the focus of the analysis has been on only one risky asset and one riskless asset. However, it is possible to add a second risky asset to the set of available investments and to calculate certainty equivalent values for tax burdens on the possible investment portfolios. Option values can no longer be calculated using the simple binary tree employed in the previous sections. Other techniques are available, however, and the computations performed below use the Monte Carlo simulation techniques described in Hull (2000, Chapter 16).

It is assumed for purposes of the numerical examples in this section that each of the two risky assets has a lognormal distribution and that the mean and standard deviation of the underlying normal distribution for both assets are $\mu = 8\%$ and $\sigma = 20\%$, respectively. Also, the continuously compounded rate of return on the riskless asset is $r = 5\%$, and the constant tax rate on gains is $\tau = 35\%$. The degree of correlation between the returns of the two risky assets is varied in the examples that follow.

Suppose first that the returns on the two risky assets are uncorrelated and that an investor chooses a positive fraction of his initial investment value to invest in each of the three available assets. Suppose further that the investor does not change investment holdings from time t_0 through time t_1. Each possible portfolio of this type is completely described by the initial investment choice, namely the aggregate fraction of the initial value invested in the two risky assets (with the balance being invested in the riskless asset) and the fraction of the risky amount allocated to the second risky asset. Figure 8 illustrates the certainty equivalent values for such portfolios. It is evident from the figure that riskless portfolios tend to have the lowest tax burden, while the highest burden is obtained for portfolios that hold exclusively just one of the risky assets. For any given level of aggregate investment in risky assets, the minimum tax burden is obtained at an equal division of holdings between the two risky assets.

If the risky assets have a non-zero correlation, the same computations can be performed. The figure on the left in Figure 9 illustrates the case in which the correlation is 50%, and the figure on the right illustrates the case in which the correlation is -50%. In both cases, it remains true that the highest tax burden is borne by portfolios concentrated in a single risky asset. Also, the lowest certainty equivalent value for any particular level of aggregate risky investment is achieved when a portfolio is evenly divided between the two risky assets.[19] In addition, examination of the same portfolio weightings at different levels of correlation

[19] The fact that an even division minimizes the certainty equivalent value in all cases considered here is a result of the fact that the assets chosen for this numerical example have the same underlying standard deviation for their returns.

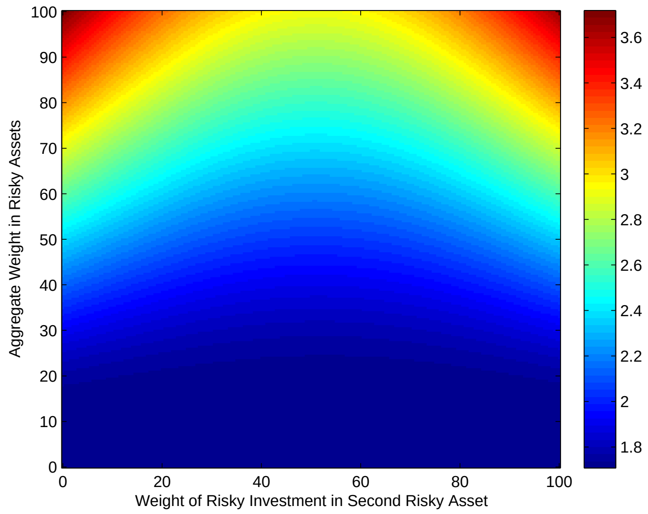
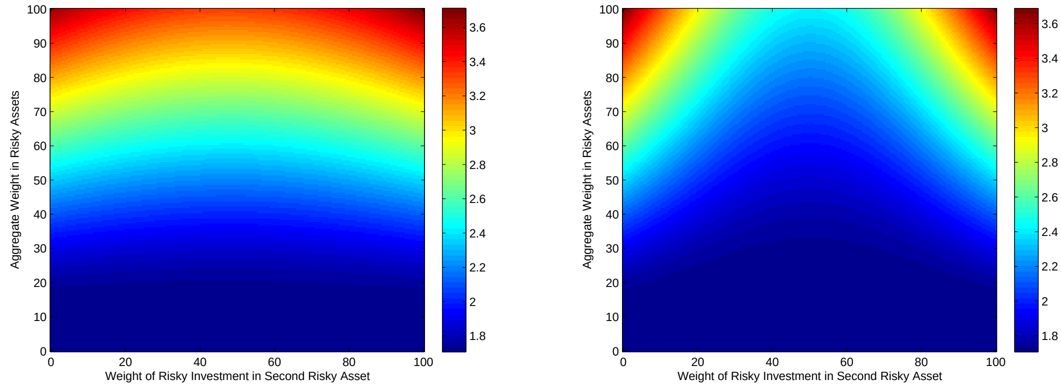


Figure 8: The figure shows the tax burden, as measured by the certainty equivalent value, on portfolios that contain a mix of two uncorrelated risky assets and a riskless asset. The aggregate weight of the portfolio invested in the two risky assets is indicated along the vertical axis. The weight of the fraction of the risky portion of the portfolio that is allocated to the second risky asset is indicated along the horizontal axis. The colors represent the certainty equivalent value of the tax on the portfolio as a percentage of initial asset value.

shows that more negative correlations lead to lower certainty equivalent values, and more positive correlations lead to higher certainty equivalent values. In general, the burden of the non-proportionate tax falls more heavily on less diversified portfolios.



(a) Correlation of 50% (b) Correlation of −50%

Figure 9: The figure on the left shows the tax burden, as measured by the certainty equivalent value, on portfolios that contain a mix of two risky assets, with a correlation of 50%, and a riskless asset. The aggregate weight of the portfolio invested in the two risky assets is indicated along the vertical axis. The weight of the fraction of the risky portion of the portfolio that is allocated to the second risky asset is indicated along the horizontal axis. The figure on the right is the same except that the correlation between the two risky assets is −50%. In both figures, the colors represent the certainty equivalent value of the tax on the portfolio as a percentage of initial asset value.

4 Conclusion

The methodology developed in this paper extends the general equilibrium techniques of Kaplow (1991, 1994) by incorporating trading in investor asset portfolios and applying option pricing techniques to evaluate the burden of non-proportionate taxes on investors in terms of certainty equivalent values. This technique allows the effects of a tax to be calculated in a precise and systematic way and allows for detailed examination of the relative burdens of the tax across investors and investment strategies.

As the numerical examples of Section 3 show, a tax that is proportionate for gains but does not allow an offset for losses burdens risk in a variety of ways. Assets of lower risk are favored over those of higher risk, independent of what the respective expected return levels of the assets may be. Moreover, this effect is amplified for options, and investment portfolios designed to have payoffs equal to those of put or call options are generally subject to substantial tax burdens. The high burden on options also translates into a burden on a synthetic division of risky asset ownership through put-call parity. In general, an arrangement in which some taxpayers have call positions on an underlying risky asset and others have the remaining interest results in a higher aggregate certainty equivalent value than would occur if all such taxpayers instead had interests directly in the risky asset itself. Thus, for example, a debt financing arrangement under such a tax would generally lead to a larger aggregate certainty equivalent tax burden for debt holders and equity holders than would be the case if debt financing were not involved, since the equity is a call option on an underlying risky asset and the debt is the remaining interest in the risky asset. Finally, it is observed that when multiple risky assets are considered, the burden of a non-proportionate tax falls most heavily on undiversified risky portfolios. Thus the tax favors diversification and the elimination of idiosyncratic risk.

The examples developed in Section 3 are indicative of the manner in which the methodology developed in this paper can be fruitfully used. It is possible, in general, to analyze tax burdens and determine certainty equivalent values using option pricing techniques for a wide range of potential tax regimes. Application of the ideas laid out herein should allow for better understanding of the burdens imposed by departures from proportionality in the current tax system and help inform which proposed reforms would be most beneficial.

Overall, it appears that the technique of calculating certainty equivalent values using option pricing techniques in a general equilibrium setting provides a powerful framework for evaluating tax regimes and assessing the impact they have on investment opportunities and risk taking. It is hoped that the approach taken here will serve as the basis for further analysis of taxes using similar methods.

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