INVESTOR SENTIMENT, BEHAVIORAL HETEROGENEITY AND STOCK MARKET DYNAMICS

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Abstract
Recent empirical works corroborate importance of sentiment in asset pricing. We further propose that sentiment may not affect everyone in a homogeneous way. In this paper, we construct a sentiment indicator taking into consideration behavioral heterogeneity of interacting investors. From our model simulation, we find sentiment contributes to several financial anomalies such as fat tails and volatility clustering of returns. More importantly, investor sentiment could be a significant source of financial market volatility. Our model with sentiment is also able to replicate different types of crises. Our finding shows that severity of crisis intensifies with investors’ sentiment sensitivity.

Keywords: Investor sentiment, Social interaction, Heterogeneous beliefs, Regime switching, Financial crises

JEL Classification: C61, D84, G12

1. INTRODUCTION

For a long time, Efficient Market Hypothesis (EMH) has been the cornerstone and mainstream belief of modern asset pricing theory. In recent years, however, skepticism against validity of EMH

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has grown in light of its failure to explain several ubiquitous financial regularities, such as excess volatility and systemic under- or over-valuation of stock prices relative to their intrinsic values. This gives rise to the alternative behavioral finance theory which aims to provide rationale behind the unexplained market anomalies.

Behavioral finance challenges the fundamental assumption of the EMH, in that investors are assumed to be boundedly rational and human psychology plays a crucial role in investment decisions. In fact, even before behavioral paradigm came into the limelight in finance and economics, investor sentiment has been perceived as a common phenomenon by financial analysts and market participants. As Shiller (2003) mentioned, perhaps one of the oldest financial theories expressed long ago in nonacademic papers is the price-to-price feedback theory. The feedback theory suggests that an increase in speculative prices is further propagated into bubble when financial successes of some investors are envied by the others and lead to public enthusiasm towards the speculative asset despite such upward spiral in prices is likely to be unsustainable. While conventional wisdom largely supports the idea that investor sentiment may overcome rational thoughts in trading behavior, the sentiment analysis has only started gaining recognition in financial academic research in the past two decades.

One of the pioneering works that formalizes the role of investor sentiment in financial market is the noise trader model proposed by De Long et al. (1990). In their model, uninformed noise traders are prone to the influence of sentiment that is in part unpredictable, while rational investors are wary of the noise trader risk and refrain from aggressive arbitrage, thus contributing to prolonged mispricing in financial market. The subsequent related studies conduct more in-depth analyses on specific channels of investor sentiment. Lux (1995, 1998) explicitly model market mood contagion through social interaction among agents to provide a behavioral explanation for bubbles and crashes. Daniel et al. (1998) and Barberis et al. (1998) construct models of investor sentiment based on psychological evidences to reconcile the empirical findings of overreaction and underreaction of stock prices to news. Daniel el al. (1998) attribute sentiment to overconfidence and self-attribution, whereas Barberis et al. (1998) concentrate on conservatism
and representativeness heuristic. Recently, researchers are trying to quantify the effect of sentiment on financial markets by using empirical data. Baker and Wurgler (2006, 2007) develop a “top down” approach to behavioral finance, by first forming a composite sentiment index based on first principal component of six proxies\(^2\) and then empirically testing effects of the sentiment index on different types of stocks. They find that low sentiment can predict higher returns for a subset of stocks such as small stocks, young stocks, high volatility stocks, distressed stocks and so on. By investigating interactions between the daily media context from *Wall Street Journal* column and stock market from 1984-1994, Tetlock (2007) finds that media content is linked to the behavior of individual investors rather than serving as a proxy for new information about fundamental asset values or proxy for market volatility. Furthermore, he finds that high media pessimism predicts downward pressure on market price and increase of market volatility. To date, there have been many studies, both theoretical and empirical, that prove evidence of investor sentiment effects in financial market (See for examples, Brown and Cliff, 2004; Da et al, 2014; Ho and Hung, 2009; Lee et al., 2002; Neal and Wheatley, 1998; Stambaugh et al., 2012).

Besides these theoretical and empirical works, some studies try to find the relationship between investor’s psychology and market bubbles and crashes in laboratory experiments. Hüsler et al. (2013) discover that laboratory bubbles have a tendency to grow faster than exponential due to positive feedback and over-optimistic expectation of future return. Makarewics (2017) conducts an experiment considering the (dis)trust of investors towards their friends’ mood when they make trading decision, and he finds the friendship network could amplify market turnovers and price oscillations. By conducting a psychological experiment, Lahav and Meer (2012) find induced positive mood could affect price pattern in experimental asset market, and the bubbles generated with induced mood treatment appear bigger than the bubbles in neutral mood treatment.

While the consensus is that investor sentiment can affect asset prices, the question remains as to how prominent the effect can be. More specifically, can sentiment explain financial crisis which

\(^2\) The six proxies are trading volume as measured by NYSE turnover; the dividend premium; the closed-end fund discount; the number and first-day returns on IPOs; and the equity share in new issues
traditional financial models fail to rationalize? How does the sentiment work on the formation of financial crisis? They are particularly pertinent questions to ask given that we are living in an era with more frequent financial crises. Comparing to the sheer amount of published works on investor sentiment, few studies have directly linked sentiment to market crises. Among these are Siegel (1992) and Baur et al. (1996) that focus on U.S. stock market crash of 1987; Zouaoui et al. (2011) who use a panel data of international stock markets and find investor sentiment contributes to higher probability of crises within one-year horizon.

Against such backdrop, we aim to investigate the role of investor sentiment on asset price dynamics as well as crisis formation in the setup of heterogeneous agent model (HAM). HAM is a burgeoning framework under behavioral finance which incorporates interacting agents with heterogeneous trading beliefs. Standard HAM which based on a dichotomy of fundamentalist and chartist beliefs has been proven successful in explaining a number of statistical properties of empirical financial data (see pioneering works by Day and Huang, 1990; Lux, 1995; Brock and Hommes, 1998; Chiarella and He, 2003; He and Westerhoff, 2005). While HAM has gained increased popularity in recent years, only a handful of HAM studies have taken into account investor sentiment (Lux, 2012; Chiarella et al., 2017), let alone a rigorous research on the role of sentiment in financial market. In this paper, we propose a HAM model with sentiment indicator that captures memory of sentiment, social interaction and sentiment shock. Our idea of social interaction is inspired by Lux’s (1995, 1998) notion that agents are not isolated units and speculators (also known as chartists) will rely on both the actual price movements as well as the behavior of their competitors when they form their expectations. As such, we postulate that sentiment may not affect everyone in a homogeneous way. We thus discriminate between speculators and non-speculators, in which the former is susceptible to market sentiment. An endogenous mechanism between sentiment and agent’s belief switching is developed with investors switch between fundamentalist, trend chasing and contrarian traders’ beliefs according

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3 According to Bordo et al. (2001), crisis frequency since 1973 has been double that of the Bretton Woods and classical gold standard periods.
to past performance while the sentiment index is contingent on the fraction of adopted beliefs in the market.

Our contributions are mainly threefold. First, under a multi-agent framework, it is intriguing to allow for heterogeneous responses to sentiment as suggested by anecdotal evidence and study how the sentiment could impact the financial system through interaction among agents. By analyzing the deterministic models, we find the existence of sentiment-related non-fundamental steady states in the systems. Second, through explicitly modelling the sentiment index, we find that investor sentiment is a significant source of financial market volatility. Third, complementing findings of Huang and Zheng (2012), the sentiment channel provides an explanation to the mechanism of regime switching as well as different types of financial crises.

Some highlights of our simulation findings include: (1) sentiment contributes to stylized facts such as fat tails, volatility clustering and long memory dependence of daily returns that are commonly observed in actual stock market; (2) we measure market volatility following Zhu et al. (2009) and find that volatility of market increases with presence of sentiment; (3) our model with sentiment is able to replicate different types of crises, including sudden crisis, disturbing crisis and smooth crisis. Meanwhile, sentiment sensitivity of investors is positively correlated with frequency and magnitude of crisis.

This paper is organized as follows. Section 2 presents the model. Section 3 explores the dynamic of different deterministic skeletons and the stability of fundamental steady states. Section 4 analyses and discusses the results of the model simulation, with focuses on stylized facts, market volatility and crises. Section 5 concludes.

2. THE MODEL

In this section, we set up asset pricing model with a single risky asset to characterize time series momentum and investor sentiment in the financial market. The modelling approach follows

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4 The authors manage to reproduce sudden, smooth and disturbing crises from the simple market-maker, regime-dependent HAM framework with fundamentalist and chartist beliefs.
closely to the current HAM framework by incorporating bounded rationality, belief heterogeneity and adaptive learning process into our model.

In the HAM literature, the market fractions of different types of traders play a pivotal role in determining the market price behavior. According to Lux (1998), the time-varying market fraction of investors is the source of market mood or market sentiment, which may introduce complicated dynamics in the financial market. Based on both theoretical and empirical evidence, our model extends early models by introducing investor sentiment into decision making process of agents. In each trading period, population of agents is assumed to be distributed among three groups, each relying upon different behavioral rules. These include fundamentalists who trade according to fundamental analysis as well as momentum traders and contrarian traders who trade differently based on historical price trend. In particular, we postulate that sentiment affects different types of agents in a heterogeneous way. We assume that fundamentalist group is more rational and is immune from market sentiment, while the chartist group (momentum traders and contrarian traders) is susceptible to market sentiment. Moreover, momentum and contrarian traders also react differently with respect to positive and negative sentiment. An endogenous mechanism between sentiment and agents’ belief switching is developed, in that investors are allowed to switch their beliefs according to past performance while the sentiment index is contingent on the fraction of adopted beliefs in the market. As in Day and Huang (1990), the market price in each period is determined by a market maker who adjusts the price as a function of the excess demand.

2.1. Fundamentalists. Fundamentalists make decision based on fundamental price $\mu_t$. They believe that market price $p_t$ is mean-reverting to the fundamental price and hence they will buy (sell) the stock when the current price is below (above) the fundamental price of the stock. They estimate the fundamental price based on various types of fundamental information, such as dividends, earnings, P/E ratios, economic growth and so forth. In each period, the latest fundamental price is updated with new information arrival which is accessible to the public. The prior of $\mu_t$ is governed by
\[
\mu_t = \mu_{t-1} + e_t
\] (2.1)

where \(e_t\) denotes new information, which follows a normal distribution with mean zero and constant standard deviation \(\sigma\). We assume simple demand functions for all types of agents which are consistent with those in the HAM literature that are derived from heterogeneous expectation and utility maximization. The excess demand of fundamentalists \(D_t^f\) is based on the spread between the latest market price \(p_t\) and the fundamental price \(\mu_t\), which can be written as

\[
D_t^f = A_t (\mu_t - p_t)
\] (2.2)

The time-varying reaction coefficient \(A_t\) captures the behavior of the fundamentalists when price is around the fundamental price. It is assumed to be a nonlinear smooth function of price deviation from fundamental value. Let \(x_t = p_t - \mu_t\) denotes this price deviation, then

\[
A(x_t) = \frac{ax_t^2}{1 + bx_t^4}
\] (2.3)

\(A(x_t)\) function could mimic the change in confidence of fundamentalists. We assume their confidence continuously increases with absolute price deviation \(|x_t|\) in a reasonable zone \(x_t \in (-Z_t, Z_t)\), and the range is determined by parameter \(b\). Within this reasonable zone, fundamentalists firmly hold the fundamental strategy, and the rise of price misalignment makes them feel more confident that price will revert to fundamental value soon, so they should grasp the opportunity to buy or sell stocks to maximize their gain. As suggested by Day and Huang (1990), such behavior is justified by increasing profit opportunities within a reasonable zone. However, if the misalignment further increases and exceeds the reasonable zone, with high uncertainty in the market, fundamentalists may wrongly predict the trend of asset price and gradually lose confidence in financial market. The properties of \(A(x_t)\) function are discussed in Appendix A.

To understand the role of fundamentalists in financial market, we first focus on the case where fundamentalists are the representative agents in the financial market. This assumption is consistent with EMH. In fact, just the nonlinear demand function of fundamentalists alone could generate rich price dynamics in financial market. With the market maker mechanism and adaptive expectation, price function can be written as
\[ p_{t+1} = p_t + \gamma(A * (\mu_t - p_t)) \]  

(2.4)

Assuming \( \mu_{t+1} = \mu_t \), it can be further converted to functions of \( x_t \)

\[ x_{t+1} = x_t - \gamma \frac{a x_t^3}{1 + bx_t^4} \]  

(2.5)

\[ f(x) = \dot{x} = -\gamma \frac{a x^3}{1 + bx^4} \]  

(2.6)

where \( \gamma \) is the speed of price adjustment to excess demand of market maker. In deterministic model, it is easy to see that \( p_t = \mu \) or \( x_t = 0 \) is the unique steady state of equation (2.4). For the stability of this fundamental steady state, non-hyperbolic fixed point with high power increases the difficulty of analysis. According to Dannan et al. (2003) and Murakami (2005), this kind of fixed point is asymptotically stable if condition \( f(x) = 0, f'(x) = 0, f^{(2)}(x) = 0, f^{(3)}(x) < 0 \) holds for equation (2.6). As we can see from Fig. 2.1, when the initial value is close to zero, price eventually converges to the fundamental steady state \( p_t = \mu \).

![Figure 2.1](image)

**FIGURE 2.1.** Convergent price trajectory when initial value close to zero with parameter values

\( a = 0.56, b = 0.01, \gamma = 1, x_0 = 2 \)

However, local stability is not necessarily global stability. If the initial value is far away from zero, with some parameter settings, there are also unstable regions. Within these regions, the system exhibits period doubling followed by a chaotic regime (see Fig. 2.2).
FIGURE 2.2. Chaotic price dynamic in unstable region: chaotic time series of $x_t$ (a); graphical analysis of $x_t$ trajectory (b); bifurcation diagram (c); Lyapunov exponent plot (d). Parameter values $a = 0.56$, $b = 0.01$, $\gamma = 1$, $x_0 = 2.7$

In general, fundamentalist group is a stabilizing force in financial market. Although there are some unstable regions with initial price far away from fundamental value, the chaos originates from nonlinear $A(x_t)$ function and time-varying confidence behavior of fundamentalists. As shown in Fig. 2.3, the phase diagram of $x_t$ indicates the unstable region of price deviation.
2.2. Chartists. There are two types of chartists in financial market, namely momentum traders and contrarian traders. Unlike the fundamentalists, both momentum traders and contrarian traders focus only on their short-term estimated market value, $v_t$, albeit different strategies are adopted by both parties. Similar to fundamentalists, chartists also have time-varying extrapolation rate, but their trading behaviors or confidence levels are sensitive to market sentiment.

2.2.1. Momentum traders. When the current market price is above the short-term value, momentum traders expect future market price to rise and choose to take a long position; conversely, they take a short position. Excess demand of momentum traders is assumed to evolve over time based on the current short-term value by

$$D_t^{mo} = \beta_1 m_t (p_t - v_t)$$

(2.7)

where $\beta_1 m_t$ is the time-varying extrapolation rate of price trend. $\beta_1 > 0$, represents the base extrapolation rate without sentiment effect. $m_t$ is the time-varying sentiment factor, which is updated each period and will affect the trading decision of investors for next period. The sentiment factor is constructed as

$$m_t = 1 + tanh(\kappa(p_t - v_t)) \ast h_1 \ast S_t$$

(2.8)
where the sentiment index $S_t$ is derived from social interaction of different types of agents and random sentiment-related information such as news and policies. $S_t \in [-1,1]$ and construction of sentiment index will be introduced in Section 2.4. $h_1 \in [0,1]$ measures sensitivity of momentum traders to market sentiment. If $h_1 = 0$, it means investor is totally immune to sentiment, and the extrapolation rate is only determined by $\beta_1$. $h_1 = 1$ implies that traders are very sensitive to the sentiment that they perceive in financial market.

Both positive and negative sentiments are expected to elicit opposite impacts on chartists’ long position and short position. More specifically, positive sentiment can enhance the confidence level of momentum traders in long position and raise their cautiousness in short position, while negative sentiment can bolster confidence of momentum traders on short position and weaken it on long position. Put differently, price deviation may work with market sentiment $S_t$ on the same or opposite way. If the price is above $v_t$ while current market sentiment is positive, momentum traders will feel more confident to follow the price trend to buy in. Hence, their confidence level increases with further price deviation given positive market sentiment. On the other hand, if price trend is going up but market sentiment is negative, momentum traders will still follow the trend but with less confidence. This contradiction of momentum traders’ prediction against market sentiment makes them more cautious and reduce their demand for the speculative asset. Besides the market sentiment, the price deviation from the short-term value $p_t - v_t$ also exerts an impact on the investor’s sentiment. To standardize price deviation $p_t - v_t$, we introduce a \textit{tanh} function with range $(-1, 1)$ as well as a scaling factor $\kappa$. Thus, the range of $m_t$ is $[0, 2]$.

We now look more closely at the short-term asset value, $v_t$. We assume that all chartists, both momentum traders and contrarian traders, hold on to an identical short-term asset value. As in Huang et al. (2010) and Huang and Zheng (2012), we assume that chartists adopt the adaptive belief mechanism where they update their expectations on short-term asset value according to different price regimes. They believe in support and resistance levels which are derived from
common rules of technical analysis. Accordingly, we assume that chartists divide price domain $P = [P_{\text{min}}, P_{\text{max}}]$ into $n$ regimes such that:

$$P = \bigcup_{j=1}^{n} P_j = [\bar{p}_0, \bar{p}_1) \cup [\bar{p}_1, \bar{p}_2) \cup \cdots \cup [\bar{p}_{n-1}, \bar{p}_n]$$

(2.9)

where $\bar{p}_j$ for $j = 1, 2, \ldots, n$ represents the different support and resistance levels set by the chartists.

The short-term asset value can be simply extrapolated as the average of the top and the bottom threshold prices:

$$v_t = (\bar{p}_{j-1} + \bar{p}_j)/2 \quad \text{if} \quad p_t \in [\bar{p}_{j-1}, \bar{p}_j)$$

(2.10)

When price fluctuates within the current regime, there are enough reasons for chartists to believe that the short-term asset value will remain unchanged. However, once the price breaks through either the support line or the resistance line, chartists will adjust their expectation on the short-term asset value according to Equation (2.10). This regime dependent phenomenon is commonly found in stock market with chartist's beliefs evolve with regime switching. According to Huang et al. (2010), the short-term asset value for each period is estimated as:

$$v_t = \left(\left\lfloor \frac{p_t}{a} \right\rfloor + \left\lceil \frac{p_t}{a} \right\rceil \right) \cdot \frac{\lambda}{2} \quad \text{if} \quad p_t \in [\bar{p}_{j-1}, \bar{p}_j) \quad \text{and} \quad j = 1, 2 \ldots n$$

(2.11)

To understand the role of momentum traders in financial market, we use the same method as before and assume momentum traders are the representative agents in the market. With the multiple price regimes, price dynamic will be very different from the single regime case. Under the market maker mechanism, the market price with only momentum traders can be expressed as

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5 Donaldson and Kim (1993) have provided empirical evidence of the existence of support and resistance levels in Dow Jones Industrial Average index.
\[ p_{t+1} = p_t + \gamma (\beta_1 m_t \ast (p_t - v_t)) \]  

(2.12)

where we assume the price adjustment rate \( \gamma = 1 \). For one regime case, short-term asset value \( v_t \) is a constant, and support line and resistance line will approach negative and positive infinity, respectively. In this case, the stability of the equation (2.12) is determined by the extrapolation rate \( \beta_1 \). As \( \beta_1 > 0 \), the system is always unstable for \( m_t = 1 \). If \( v_t \) is updated in each period, fluctuation of price generates multiple regimes with different support and resistant lines. The regime switching dynamic is governed by \( \beta_1 \) and \( m_t \). Without sentiment, market price goes up and down within two adjacent regimes for \( \beta_1 < 1 \), and fluctuates in several different regimes for \( \beta_1 > 1 \) (Fig. 2.4). As shown in Fig 2.4, this regime switching behavior is the main reason of the occurrence of crisis between period 10 to period 60. With time-varying sentiment, \( \beta_1 \) must be less than 0.5 to make sure price staying in one regime. Therefore, the sentiment factor \( m_t \) actually increases the probability that price escapes from the initial regime. This implies that in our model, sentiment serves as an important factor to facilitate the regime-switching process and may partly explain mechanism of crisis formation as shown in Fig 2.4.
2.2.2. Contrarian traders. Unlike momentum traders, contrarian investors trade stock based on the hypothesis of market overreaction. Specifically, when current price is higher than short-term value $v_t$, they believe that future market price will drop and therefore take a short position; conversely, they take the long position. We assume contrarian traders use the same method as momentum traders to calculate short-term value $v_t$, hence the demand function of contrarian traders can be expressed as

$$D_t^{co} = \beta_2 c_t (p_t - v_t)$$  \hspace{1cm} (2.13)

Similarly, $\beta_2 c_t$ is the time-varying extrapolation rate of price trend for contrarian traders. $\beta_2 < 0$, represents the base extrapolation rate without sentiment effect. $c_t$ is the time-varying sentiment factor for contrarian traders, which can be written as
\[ c_t = 1 - \tanh(\kappa(p_t - v_t)) * h_2 * S \]  

(2.14)

Similar to \( h_1 \), \( h_2 \) is sensitivity of contrarian traders to market sentiment with a range [0, 1]. Although contrarian traders adopt trading strategy opposite to momentum traders, they are affected by market sentiment in the same way. When market price is above \( v_t \), contrarian traders expect price to decline. If the market sentiment is negative, contrarian traders will be more confident to take the short position. However, positive market sentiment will decrease the confidence level of contrarian investors. Besides the market sentiment, price deviation is another element that contributes to contrarian traders’ sentiment factor \( c_t \).

Usually, we regard contrarian traders as the stabilizing force in financial market. We conduct the same experiment by considering contrarian traders as the only type of investor in the financial market. The price function is expressed as

\[ p_{t+1} = p_t + \gamma (\beta_2 c_t * (p_t - v_t)) \]  

(2.15)

We still assume the price adjustment rate \( \gamma = 1 \). For situation of one regime, short-term value \( v_t \) is unchanged over time, and the stability of the equation (2.15) is determined by the extrapolation rate \( \beta_2 \) and \( c_t \). As shown in Fig. 2.5, for case without sentiment effect, i.e. when \( c_t = 1 \), equation (2.15) is always stable if \(-2 < \beta_2 < 0\). For case with sentiment effect, i.e. when \( c_t = 2 \), the domain of \( \beta_2 \) shrinks to \((-1,0)\) to ensure convergence of market price. These conditions also hold for multiple regime case when \( v_t \) is not constant. In other words, the sentiment factor \( c_t \) may change the stability of the system. For example, if the initial base extrapolation rate \( \beta_2 = -1.6 \), the price will converge to initial short-term value \( v_1 \) after several periods. However, considering a constant sentiment factor \( c_t = 2 \), the whole extrapolation rate will be \(-3.2\) and equation (2.15) becomes unstable. Thus, sentiment may convert contrarian traders from stabilizing force to disturbing force. As illustrated in Fig. 2.5, sentiment may enhance the extrapolation power of contrarian traders, and regime-switching behavior of price happens with emergence of rich price patterns including crisis.
2.3. Belief switching regime. We assume that, at the end of each trading period, agents may switch their belief type or prediction strategy conditional on the performance of three rules. Specifically, the performance measure depends on profit function, which is defined as

$$\pi_{n,t} = (p_t - p_{t-1})D_{t-1}^n$$  \hspace{1cm} (2.16)

For simplicity, we assume the interest rate between period $t - 1$ and period $t$ is one and there is no cost for fundamentalists to acquire additional information. We further introduce additional memory into the performance measure that can be taken as the weighted average of the realized profits, given by

(a) $\gamma = 1, \beta_2 = -1.6$  

(b) $\gamma = 1, \beta_2 = -3.2$

FIGURE 2.5. Price trajectory and phase diagram with only contrarian traders.
\[ U_{n,t} = \varphi U_{n,t-1} + \pi_{n,t} \]  

(2.17)

where \(0 \leq \varphi \leq 1\) represents the strength of memory put into the last-period performance, \(n\) denotes different types of agents.

We let \(\omega_{i,t}\) denotes market fraction of three different types of investors. The fractions of three groups vary endogenously over time according to the choice model with multinomial logit probabilities as introduced by Manski and McFadden (1981) and by Brock and Hommes (1997, 1998):

\[ \omega_{n,t} = \frac{\exp(\rho U_{n,t})}{\sum_{i=1}^{3} \exp(\rho U_{i,t})} \]  

(2.18)

Note that, the new fractions of traders are determined on the basis of the most recent performance measure \(U_{n,t}\). The parameter \(\rho > 0\) is the intensity of choice measuring the sensitivity of agents with respect to the difference of past performance. The higher is \(\rho\), the quicker agents will respond to difference in performance by switching to the most profitable strategy. For finite \(\rho\), \(\omega_{i,t}\) is always positive which implies that not all agents are going for strategy with highest profit.

2.4. Sentiment index. As pointed out by Baker and Wurgler (2007), many factors could be used to construct sentiment index of a market, such as investor mood, news from media, mutual fund flows, trading volume, dividend premium and government policies. We construct the sentiment index by focusing on three main sources, including last-period sentiment index (also known as memory of sentiment), investor mood from social interaction, and sentiment shock such as news, polices, firm innovations and so forth. The index function can be written as

\[ S_t = \eta_1 S_{t-1} + \eta_2 SI_t + \eta_3 \theta_t \]  

(2.19)

where \(\eta_1, \eta_2, \eta_3\) are weights assigned to different factors, such that \(\eta_1 + \eta_2 + \eta_3 = 1\). \(\theta_t\) is the sentiment shock, which we assume to follow a uniform distribution with range [-1, 1]. \(SI\) is social interaction of different types of investors, which influences the current market mood. The idea is
inspired by majority opinion formation in Kirman (1993), Lux (1995), and Lux and Marchesi (2000), in which the majority opinion index is computed as the difference between optimistic and pessimistic individuals. Different from Lux and Marchesi (2000), we assume the opinion index not only contains the opinion of optimistic and pessimistic chartists, but also includes the opinion of fundamentalists. The social interaction should exist among all types of investors, including both fundamentalist and chartist groups. Fundamentalists are optimistic (pessimistic) investor when market price is below (above) their fundamental values. Two different types of chartist may change their opinion under different market states. For instance, when current market price is above the short-term value $v_t$, momentum traders believe price will continue to go up and thus they are optimistic group. In contrast to momentum traders, contrarian traders believe price trend will reverse and hence they are pessimistic group. However, in the next period, if the market price falls below the short-term value $v_{t+1}$, momentum traders will become the pessimistic group while contrarian traders will switch to optimistic opinion. Besides the fractions of different agents, the price deviations from fundamental value $\mu_t$ and short-term value $v_t$ could also affect the opinion index. Large price deviation could make investor opinion more convincing in social interaction. By considering both fractions of agents and magnitude of price deviation, we construct the social interaction index as:

$$S_I_t = \tanh(\kappa(\mu_t - p_t) * \omega_f^I + \tanh(\kappa(p_t - v_t) * (\omega_{m_o}^I - \omega_{c_0}^I))$$

(2.20)

where $\omega_f, \omega_{m_o}, \omega_{c_o}$ are fractions of fundamentalists, momentum traders and contrarian traders in financial market, respectively. $\tanh$ function and $\kappa$ are used to scale the price deviation, so both the range of $S_I_t$ and the range of sentiment index $S_t$ can be constrained to $[-1, 1]$.

2.5. Market maker. Following Day and Huang (1990) and Chiarella and He (2003), we assume net zero supply of the risky asset, and the market price in each trading period is determined by a market maker who adjusts the price as a function of the excess demand. The aggregate market’s excess demand is weighted by population fraction. Hence, for a three-type model (including
fundamentalists, momentum traders and contrarian traders), the price $p_{t+1}$ is set by market maker according to the aggregate excess demand, that is

$$P_{t+1} = P_t + \gamma (\omega^f_t D^f_t + \omega^{mo}_t D^{mo}_t + \omega^{co}_t D^{co}_t)$$

(2.21)

where $\gamma$ represents the speed of price adjustment by the market maker.

3. ANALYSIS OF THE MODEL’S DETERMINISTIC SKELETON

3.1. Market stability analysis. The core of the HAM framework is the existence of heterogeneous agents with different trading strategies. Previous single agent analysis gives us a brief idea of the different roles of fundamentalists, momentum traders and contrarian traders in financial market. To have a better understanding of market with heterogeneous agents, we follow the standard approach in HAM literature and conduct the stability analysis of the underlying model with different market structures. By ignoring the noises, the stability analysis could provide an insight into the effect of trading activities as well as interaction among different types of investors on market stability. Moreover, we will focus on market stability with consideration of investor sentiment effect.

In this section, we give an extensive exploration of two-type (2 types of agent) and three-type (3 types of agent) deterministic models by focusing on the local stability of the fundamental steady states (denoted as $\mu^-$ and $\bar{\nu}$). We also discuss the conditions for existence of fundamental steady state and non-fundamental steady state for both the single regime and multiple-regime cases.

3.1.1. Fundamentalists versus momentum traders. We first consider a market populated only by fundamentalists and momentum traders. This scenario has been investigated by many HAM studies with the standard fundamentalist-chartist framework. Fundamentalists make the trading

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6 Regime here refers to the price window set by chartists. One regime case means short-run asset price $v_t$ is constant, and support and resistance levels approach infinity. Multiple-regime case means $v_t$ is updated in each period.
decision on the basis of fundamental value $\mu_t$. Momentum traders adopt the strategies based on the hypothesis of market underreaction to price trend, and they aim to exploit the opportunities of market continuity. Sentiment index is formed from social interaction of fundamentalists and momentum traders, but it only affects the behavior of momentum traders in the process of making trading decisions. In this case, the system can be modelled as a three-dimensional nonlinear system:

$$
\begin{align*}
    p_{t+1} &= p_t + \gamma \omega_t^f A_t (\mu_t - p_t) + \beta_1 \omega_{t+1} m_t (p_t - v_t) \\
    U_{t+1} &= \varphi U_t + (p_{t+1} - p_t) [(A_t (\mu_t - p_t) - \beta_1 m_t (p_t - v_t)] \\
    S_{t+1} &= \eta_1 S_t + \eta_2 [\tanh(\kappa (\mu_t - p_{t+1})) \omega_{t+1}^f + \tanh(\kappa (p_{t+1} - v_t)) \omega_{t+1}^{mo}] 
\end{align*}
$$

where

$$
\begin{align*}
    A_t &= \frac{a (\mu_t - p_t)^2}{1 + b (\mu_t - p_t)^4} \\
    \omega_t^f &= \frac{\exp(\rho U_t)}{\exp(\rho U_t) + 1}, \omega_t^{mo} = \frac{1}{\exp(\rho U_t) + 1} \\
    m_t &= 1 + \tanh(\kappa (p_t - v_t)) h_1 S_t, \\
    U_t &= U_{t+1}^f - U_{t}^{mo}
\end{align*}
$$

The dynamics in (3.1) are stochastic and there are two sources of noise. The first source, $e_t$ is the noise term of the fundamental value. The second source, $\theta_t$ is stochastic component of market sentiment derived from random news, innovations and policies. When both noise terms are zero, we are able to study the stability of the system, and we adopt the same way to generate deterministic skeletons for the following scenarios. We will first begin by investigating the occurrence of the steady state of system (3.1) in one regime case, in which short-term value $v_t$ is assumed as constant $\bar{v}$ over time.

**Proposition 1.** The system has

a) an unstable fundamental steady state (FSS) with $(p^*, U^*, S^*) = (\mu, 0, 0)$ if $\mu = \bar{v}$; two types of non-fundamental steady states (NFSS) with the form $(p, U, S) = (p_1^*, 0, 0)$, $(p, U, S) = (p_2^*, 0, 0)$ and $p_1^* < \mu$, $p_2^* > \mu$ if $\mu = \bar{v}$. 

20
b) two types of non-fundamental steady states (NFSS) with the form \((p^*, U^*, S^*) = (p_1^*, 0, S_1^*), (p, U, S) = (p_2^*, 0, S_2^*)\) and \(p_1^* < \mu, p_2^* > \mu\) if \(\mu \neq \bar{v}\).

**Proof.** See Appendix B.

In line with previous literature, we find the fundamental steady state \(p^* = \mu\) with a sufficient condition \(\mu = \bar{v}\), and it is proven to be an unstable steady state. With introduction of sentiment, we also find that when market sentiment is allowed to influence decision-making process of momentum traders, the system can be driven toward several steady states with either smaller \((p_1^*)\) or greater \((p_2^*)\) price than fundamental value. These complex attractors are characterized by persistently polarized levels of positive sentiment, negative sentiment and zero sentiment. **Lemma 1** shows the conditions for these sentiment-persistent steady states.

**Lemma 1.** For non-fundamental steady states, the system could achieve positive sentiment equilibria \((S^* > 0)\) if \(\mu - \bar{v} > 0\), negative sentiment equilibria \((S^* < 0)\) if \(\mu - \bar{v} < 0\), zero sentiment equilibria \((S^* = 0)\) if \(\mu - \bar{v} = 0\) and \(\beta_1 \leq \frac{1}{2} ab^{-\frac{1}{2}}\).

**Proof.** See Appendix B.

The high degree of nonlinearity of the equations makes it difficult to explore the stability of the non-fundamental steady states. But from the structure of Jacobean matrix corresponding to this 3D system, we can infer that investor’s feature, encompassed in parameters \(a, b, \beta_1\) and sentiment-related parameters \(h_1, \eta_1, \eta_2\) do not only foster the emergence of non-fundamental steady states, but also significantly affect their number and positions. In this paper, we focus on feature of these sentiment-related steady states rather than the stability conditions.

When condition \(\mu - \bar{v} > 0\) is satisfied, only positive sentiment equilibria are achievable. There are two possible positive sentiment cases in terms of position of equilibrium price. For low-price case, equilibrium price is below the fundamental price \(\mu\) and short-term value \(\bar{v}\) \((p_1^* < \bar{v} \leq \mu)\), so fundamentalists hold the optimistic opinion while momentum traders are pessimist. As the equilibrium configuration can solely exist in an even distribution of fundamentalists and
momentum traders, we have $\omega_t^f = \omega_t^{mo} = \frac{1}{2}$ in steady state. As mentioned in previous section, the sentiment index is determined by both fraction of different types of investors and price deviations from fundamental value $\mu$ and short-term value $\bar{v}$. With $p_1^* < \bar{v} < \mu$, price and short-term value are lower than fundamental value, which indicates bear market state and price deviates further from $\mu$ than $\bar{v}$. The persistent optimism sentiment is sourced from the fundamentalist group. To achieve this positive sentiment steady state, confidence level of fundamentalists must be lower than the extrapolation rate of momentum traders, that is, $A(p_1^*) < \beta_1 m^*$. Conversely, for bull market case with $\bar{v} < \mu < p_2^*$, momentum traders’ optimistic opinion will dominate to form the positive market sentiment. The relation between the confidence level of fundamentalists and momentum traders’ extrapolation rate must satisfy $A(p_2^*) > \beta_1 m^*$.

If we have $\mu - \bar{v} < 0$, only negative sentiment equilibria are achievable. For negative sentiment equilibria, momentum traders’ pessimistic opinion dominates the market with $A(p_1^*) > \beta_1 m^*$ for bear market case ($p_1^* < \mu$), and pessimism from fundamentalists sweeps the market with $A(p_2^*) < \beta_1 m^*$ for bull market case ($p_2^* > \bar{v}$). When $\mu = \bar{v}$ holds in one regime case, we can have both fundamental steady state and non-fundamental steady states. NFSS is featured by neutral sentiment, and must satisfy the condition $A(p^*) = \beta_1$, which means fundamentalists have the same confidence level as the extrapolation rate of momentum traders. In this case, the number of NFSS is determined by parameter $a$, $b$ and $\beta_1$. Specifically, if $\beta_1 < \frac{1}{2} ab^{-\frac{1}{2}}$, four NFSSs exists; If $\beta_1 = \frac{1}{2} ab^{-\frac{1}{2}}$, number of NFSS reduces to two; If $\beta_1 > \frac{1}{2} ab^{-\frac{1}{2}}$, no NFSS exists.

For multiple-regime case, we allow short-term value to update in each period. Price could fluctuate within one regime or among different regimes. The regime switching behavior has been analyzed in previous section through different types of chartists, and one can further explore this feature in Huang et al. (2010) and Huang and Zheng (2012). As discussed in Huang and Zheng (2010), there are many equilibria with multiple regimes, which make it rather complicated to investigate the stability of the system if regime switching of price is considered. Hence, we only
focus on the possible price trajectory to form the steady states for multiple-regime case. There are three possible scenarios:

**Scenario 1.** If $\mu > v_1$ and they are located at different price regimes, one NFSS is possible to emerge in the first regime ($v_1$ regime) with $p^*_1 < v_1$ and $A(p^*_1) < \beta_1 m(p^*_1)$. For price higher than $v_1$ in the first regime, price would increase within this regime, and finally escape from the old regime to a new one. Other NFSSs are possible to exist in a higher price regime with $\mu > v_t > p^*_t$ and $A(p^*_t) < \beta_1 m(p^*_t)$. Or the case is converted to **Scenario 2** or **3** in next period.

**Scenario 2.** If $\mu < v_1$ and they belong to different price regimes, one NFSS is in the first regime with $p^*_1 < v_1$ and $A(p^*_1) < \beta_1 m(p^*_1)$. For the case when price is lower than $v_1$ in the first regime, price would decrease in this regime until escaping from the old regime to a new one. Other NFSSs are possible in a lower price regime with $p^*_t > v_t > \mu$ and $A(p^*_t) < \beta_1 m(p^*_t)$. Or it may go to **Scenario 1** or **3** in next period.

**Scenario 3.** If $\mu = v_1$, occurrence and positions of NFSSs are determined by parameter $a$, $b$ and $\beta_1$ as well as $\lambda$. NFSS can only emerge in the first regime given $A(p^*) = \beta_1$ and $|p^* - \mu| \leq \lambda$. Otherwise, price will move downward to lower regimes or upward to higher regimes. Then **Scenarios 1** and **2** are all possible to happen in next period.

### 3.1.2. Fundamentalists versus contrarian traders.

This design has been explored in Chiarella and He (2003), He and Li (2015), and they confirm the stabilizing role of contrarian traders. We also adopt this two-type design in this section. The main difference is that we incorporate investor sentiment into the system. We assume only contrarian traders are sensitive to market sentiment, and the three-dimensional nonlinear system can be written as

\[
\begin{align*}
  p_{t+1} &= p_t + \gamma [\omega^f_t A_t(\mu_t - p_t) + \beta_2 \omega^c_t c_t(p_t - v_t)] \\
  U_{t+1} &= \varphi U_t + (p_{t+1} - p_t) [(A_t(\mu_t - p_t) - \beta_2 c_t(p_t - v_t)] \\
  S_{t+1} &= \eta_1 S_t + \eta_2 [\tanh(\kappa(\mu_t - p_{t+1})) \omega^f_{t+1} - \tanh(\kappa(p_{t+1} - v_{t+1})) \omega^c_{t+1}]
\end{align*}
\]

where
\[ A_t = \frac{a(\mu_t - p_t)^2}{1 + b(\mu_t - p_t)^4} \]
\[ \omega_t^f = \frac{\exp(\rho U_t)}{\exp(\rho U_t) + 1}, \omega_t^{f0} = \frac{1}{\exp(\rho U_t) + 1} \]
\[ c_t = 1 - \tanh(p_t - \nu_t) \times h_2 \times S_t, \]
\[ U_t = U_t^f - U_t^{f0} \]

We also start by investigating the stability of the system from one regime case.

**Proposition 2.** The system has

a) a unique fundamental steady state with \((p^*, U^*, S^*) = (\mu, 0, 0)\) if \(\mu = \bar{\nu}\). Fundamental steady state is asymptotically stable for \(-4 < \gamma \beta_2 < 0\), and it undergoes a flip bifurcation for \(\gamma \beta_2 = -4\).

b) three types of non-fundamental steady states with the form \((p^*, U^*, S^*) = (p_1^*, 0, S_1^*)\), \((p^*, U^*, S^*) = (p_2^*, 0, S_2^*)\), \((p^*, U^*, S^*) = (p_3^*, 0, S_3^*)\) and \(p_1^* < \frac{1}{2}(\mu + \bar{\nu}), p_2^* > \frac{1}{2}(\mu + \bar{\nu})\).

**Proof.** See Appendix C.

We find a unique fundamental steady state if \(\mu = \bar{\nu}\), and it is asymptotically stable for \(-4 < \gamma \beta_2 < 0\). For \(\mu \neq \bar{\nu}\), we find three different types of NFSS, and these complex attractors are characterized by persistently polarized levels of positive, negative and neutral sentiment. **Lemma 2** shows these sentiment-persistent steady states in detail.

**Lemma 2** For non-fundamental steady states, the system achieves positive sentiment equilibria \((S^* > 0)\) if \(p^* < \frac{1}{2}(\mu + \bar{\nu})\), negative sentiment equilibria \((S^* < 0)\) if \(p^* > \frac{1}{2}(\mu + \bar{\nu})\) neutral sentiment equilibria \((S^* = 0)\) if \(p^* = \frac{1}{2}(\mu + \bar{\nu})\) and \(\beta_1 \leq \frac{1}{2}ab - \frac{1}{2}\).

**Proof.** See Appendix C.
Different from previous case, equilibrium price of NFSS can only emerge between $\mu$ and $\bar{v}$, and type of NFSS depends on the position of $p^*$. When $p^*$ is below the average of $\mu$ and $\bar{v}$, that is $p^* < \frac{1}{2}(\mu + \bar{v})$, positive sentiment equilibria could be achieved. If $\mu > \bar{v}$, this persistent positive sentiment state is sourced from fundamentalists’ optimistic opinion with $A(p^*) < -\beta_2 c^*$. If $\mu < \bar{v}$, the positive sentiment state is derived from the optimistic belief of contrarian traders with $A(p^*) > -\beta_2 c^*$. When $p^*$ is above the average of $\mu$ and $\bar{v}$ negative sentiment equilibria appear. Opposite to positive sentiment steady state, pessimistic opinion of contrarian traders dominates with $A(p^*) > -\beta_2 c^*$ to generate negative sentiment state if $\mu > \bar{v}$, while pessimistic opinion of fundamentalists will take in charge with $A(p^*) < -\beta_2 c^*$ to generate negative sentiment if $\mu < \bar{v}$.

A special case is NFSS at $p^* = \frac{1}{2}(\mu + \bar{v})$ with neutral sentiment state as in FSS, and the sufficient condition for the occurrence of this NFSS is $A(p^*) = -\beta_2 c^*$. The price zone above the $\max(\mu, \bar{v})$ is declining zone and below the $\min(\mu, \bar{v})$ is rising zone, so price drops in these zones would be attracted to $[\min(\mu, \bar{v}), \max(\mu, \bar{v})]$. Similarly, parameter $a, b, \beta_2$ and sentiment related parameter $h_2, \eta_1, \eta_2$ affect the number and position of NFSS.

For multiple-regime case, price will rise in the first regime under bear market ($\mu > v_1$) and $p_1 < v_1$. The steady state can be achieved in the first regime or price could escape to higher price regimes, and final equilibrium will be reached between fundamental value $\mu$ and the latest short-term value $v_t$. Otherwise, if the initial market is bullish ($\mu < v_1$) with $p_1 > v_1$, price will decrease in the first regime and equilibrium could be achieved in the same regime with lower price or price continuously goes down until equilibrium is achieved in a lower price regime. This kind of price dynamics illustrates the stabilizing role of fundamentalists and contrarian traders. However, the stabilizing forces of these two types of investor do not always go with the same direction. For instance, price initially drops into a window between $\mu$ and $v_t$. If fundamentalists and contrarian traders work on the different directions of price movement, equilibrium may be achieved between latest short-term value and fundamental value. This could provide a possible explanation for why price could fluctuate above or below fundamental price for a long time in real
financial markets. In the regime switching process, investor sentiment plays an important role. It speeds up the regime switching process and expands the magnitude of price change, which could partially explain the existence of excess volatility and even bubbles and crashes in financial markets.

3.13. Momentum traders versus contrarian traders. This kind of two-type design has been rarely explored within HAM framework. In this case, market only consists of two types of chartist, momentum traders and contrarian traders. This rare case may occur during extreme market state, for instance, during long-lasting bubbles or crises, fundamentalists gradually lose their confidence in financial market, so they decide not to enter the market or switch to chartist strategy. Another possible situation is that the market is highly speculative or it is difficult to access the fundamental value of the asset (such as Bitcoin market), then most of the investors would choose chartist strategies. By only including momentum traders and contrarian traders, we can model the system as

\[
\begin{aligned}
 p_{t+1} &= p_t + \gamma (\beta_1 \omega_t^{mo} m_t + \beta_2 \omega_t^{co} c_t)(p_t - v_t) \\
 U_{t+1} &= \varphi U_t + (\beta_1 m_t - \beta_2 c_t)(p_{t+1} - p_t)(p_t - v_t) \\
 S_{t+1} &= \eta_1 S_t + \eta_2 \tanh(\kappa(p_{t+1} - v_{t+1})) G_{t+1}
\end{aligned}
\]

(3.3)

where

\[
\begin{align*}
\omega_t^{mo} &= \frac{\exp(\rho U_t)}{\exp(\rho U_t) + 1}, \quad \omega_t^{co} = \frac{1}{\exp(\rho U_t) + 1} \\
m_t &= 1 + \tanh(\kappa(p_t - v_t)) \ast h_1 \ast S_t \\
c_t &= 1 - \tanh(\kappa(p_t - v_t)) \ast h_2 \ast S_t, \\
U_t &= U_t^{mo} - U_t^{co}, \\
G_t &= \omega_t^{mo} - \omega_t^{co} = \tanh\left(\frac{\rho \kappa}{2}(\varphi U_t + (\beta_1 m_t - \beta_2 c_t)(p_{t+1} - p_t)(p_t - v_t))\right)
\end{align*}
\]
We begin by analyzing the one regime case. As momentum traders have belief opposite to that of contrarian traders, the heterogeneous extrapolation rate of these two chartist groups may affect the price dynamic and stability of steady state.

**Proposition 3.** The system (3.3) has a unique steady state \((p^*, U^*, S^*) = (\bar{v}, 0, 0)\). The steady state is locally and asymptotically stable provided that \(-4 < \gamma(\beta_1 + \beta_2) < 0\), and it undergoes a flip bifurcation for \(\gamma(\beta_1 + \beta_2) = -4\).

**Proof.** See Appendix D.

The unique steady state in this case is very similar to FSS in the fundamentalists versus contrarian traders design. The stability of the equilibrium depends on \(\gamma, \beta_1\) and \(\beta_2\). To satisfy the stability condition, we must have \((\beta_1 + \beta_2) < 0\). It suggests contrarian traders should have stronger extrapolating power to pull price back to short-term value \(\bar{v}\). For a special case \(\beta_1 = -\beta_2\), price would have no fluctuation given same initial utility and fraction for two types of chartist. As their demands will offset each other, the initial price will be the steady state price for this case.

We further explore the multiple-regime case in this design. If momentum traders have stronger extrapolating force \((\beta_1 > -\beta_2)\), \(v_1\) can divide the first regime into rising zone (above \(v_1\)) and declining zone (below \(v_1\)). When price drops in the rising zone, it will increase in next period. Otherwise, price will further decrease in declining zone. If \(\beta_1 < -\beta_2\), contrarian traders have stronger force to pull price toward the short-term value \(\bar{v}\). If the difference of their extrapolating force is small, price will converge to initial regime short-term value \(v_1\); If the difference is large, price may fluctuate among different price regimes. Hence, the regime switching behavior is determined by the extrapolation rate. For a model without sentiment factor, low extrapolation rates or small strength difference between them may lead price to fluctuate within two adjacent regimes. However, if investor sentiment is considered, time-varying extrapolation rate could make both within regime fluctuation and regime switching behavior available as shown in Fig.3.1. Hence, sentiment is an important factor to affect price trajectory in multiple regime design.
FIGURE 3.1. Price trajectory without (left) and with (right) sentiment effect, parameter values $\beta_1 = 1.3, \beta_2 = -1.2$

3.1.4. Fundamentalist, momentum traders and contrarian traders. For the three-type design, we include all kinds of investors, i.e., fundamentalists, momentum traders and contrarian traders into the system. This design has been widely investigated in literature, such as Brock and Hommes (1998), Chiarella and He (2003), and He and Li (2015). This scenario can be treated as a combination of previous two models, so it is expected to generate richer dynamic features. As this model is closer to real financial market, we will use this model as a benchmark for the following numerical analysis. The system can be modelled to a five-dimensional dynamic map as following

\[
\begin{align*}
    p_{t+1} &= p_t + \gamma [\omega_t^f A_t (\mu_t - p_t) + (\beta_1 \omega_t^{mo} m_t + \beta_2 \omega_t^{co} c_t) (p_t - v_t)] \\
    u_{t+1}^f &= \varphi u_t^f + (p_{t+1} - p_t) A_t (\mu_t - p_t) \\
    u_{t+1}^{mo} &= \varphi u_t^{mo} + (p_{t+1} - p_t) \beta_1 m_t (p_t - v_t) \\
    u_{t+1}^{co} &= \varphi u_t^{co} + (p_{t+1} - p_t) \beta_2 c_t (p_t - v_t) \\
    S_{t+1} &= \eta_1 S_t + \eta_2 [\tanh(\kappa (\mu_{t+1} - p_{t+1})) \omega_{t+1}^f + \tanh(\kappa (p_{t+1} - v_{t+1})) (\omega_{t+1}^{mo} - \omega_{t+1}^{co})]
\end{align*}
\]

where

\[
A_t = \frac{a (\mu_t - p_t)^2}{1 + b (\mu_t - p_t)^4},
\]
\[
\omega_{h,t} = \frac{\exp(\rho U_{h,t})}{\sum_{h=1}^{2} \exp(\rho U_{h,t})},
\]
\[
m_t = 1 + \tanh(\kappa(p_t - v_t)) \ast h_1 \ast S_t,
\]
\[
c_t = 1 - tanh(\kappa(p_t - v_t)) \ast h_2 \ast S_t
\]

We start by investigating the stability of the system for one regime case.

**Proposition 4.** The system has

a) a unique fundamental steady state with \((p^*, u^*_f, u^*_m, u^*_c, S^*) = (\mu, \frac{1}{3}, \frac{1}{3}, 0)\) if \(\beta_1 = -\beta_2\).

The Jacobean matrix of this system has five eigenvalues with \(\lambda_1 = 1, \lambda_2 = \varphi\). Fundamental steady state is asymptotically stable for \(|\lambda_3|, |\lambda_4|, |\lambda_5| < 1\).

b) a fundamental steady state with \((p^*, u^*_f, u^*_m, u^*_c, S^*) = (\mu, \frac{1}{3}, \frac{1}{3}, 0)\) if \(\beta_1 \neq -\beta_2\) and \(\mu = \bar{\nu}\). FSS is asymptotically stable for \(-6 < \gamma(\beta_1 + \beta_2) < 0\); Two types of non-fundamental steady states with the form \((p^*, u^*_f, u^*_m, u^*_c, S^*) = (p^*_1, \frac{1}{3}, \frac{1}{3}, S^*_1), (p^*, u^*_f, u^*_m, u^*_c, S^*) = (p^*_2, \frac{1}{3}, \frac{1}{3}, S^*_2)\), and \(p^*_1 < \mu, S^*_1 > 0; p^*_2 > \mu, S^*_2 < 0\) if \(\beta_1 \neq -\beta_2\) and \(\mu \neq \bar{\nu}\).

**Proof.** See Appendix E.

As three types of agents can be regarded as equal forces in financial market, their behavior, especially the extrapolating power of them, may influence the existence and stability of steady state. When \(\beta_1 = -\beta_2\), only fundamental steady state exists. In this case, demands of momentum traders and contrarian traders offset each other if we do not consider sentiment effect, and demand of fundamentalists solely determines the market price. With sentiment effect, the price dynamic generated by this model would become richer, but there is a unique steady state with \(p = \mu\). To analyze the stability of this system, we use the Jacobean matrix. We find that \(\lambda_1 = 1, \lambda_2 = \varphi\) and \(\lambda_3, \lambda_4, \lambda_5\) are functions of standard parameters such as \(\gamma, \rho, \varphi\) and sentiment related parameters.
\( \eta_1, \eta_2, h_1, h_2 \). Fundamental steady state is asymptotically stable if \( |\lambda_3|, |\lambda_4|, |\lambda_5| < 1 \), so sentiment is a factor to influence the stability of the steady state. When \( \beta_1 \neq -\beta_2 \), both fundamental steady state and non-fundamental steady states exist for specific conditions. The sufficient condition for fundamental steady state is \( \mu = \bar{v} \), and this FSS is asymptotically stable for \(-6 < \gamma(\beta_1 + \beta_2) < 0\), which means contrarian traders need to have a stronger extrapolation power to stabilize the market. We have two types of non-fundamental steady state for \( \beta_1 \neq -\beta_2 \) and \( \mu \neq \bar{v} \), the features of them are shown in Lemma 3.

**Lemma 3.** For non-fundamental steady states, the system can achieve both positive and negative sentiment equilibria. If \( \beta_1 > -\beta_2 \), positive (negative) sentiment equilibrium exists at \( p^* < (> \mu \) and \( p^* < (> \bar{v} \). If \( \beta_1 < -\beta_2 \), positive (negative) sentiment equilibrium exists at \( p^* < (> \mu \) and \( p^* > (<) \bar{v} \).

**Proof.** See Appendix E.

For non-fundamental steady states, the position of \( p^* \) with respect to \( \mu \) decides the types of equilibrium. The sentiment generated by two types of chartist will be offset by each other and their effects are cancelled out, so the sentiment of fundamentalists is the main source of equilibrium sentiment. If momentum traders extrapolate stronger (weaker), a positive (negative) sentiment steady state would emerge at \( p^* < \bar{v} \). If contrarian traders extrapolate stronger (weaker), a positive (negative) sentiment steady state would occur at \( p^* > \bar{v} \). Concerning multiple-regime case, the regime switching behavior is similar to two-type case, and sentiment is still an important factor that affects this behavior. Here, we skip the analysis of regime switching for three-type case.

4. NUMERICAL SIMULATION WITH STOCHASTIC MODEL

The analysis performed in the previous section confirms that heterogeneity of investor belief and perception of sentiment can drive the market toward regimes characterized by optimistic or pessimistic market sentiment. To study more features in the financial market such as fat tails,
negative skewness, and long memory of the distributions of returns, we run simulation based on stochastic models with three-type design in system (3.4). We begin this section by briefly reviewing some of the stylized facts in real financial market. Next, we conduct simulation practice on our stochastic models with and without sentiment separately, thereby enable us to compare the fitness between both models to real financial market in terms of its capability to generate stylized facts. More importantly, we are able to explore the role of sentiment effect on these stylized facts. In addition, we also investigate whether sentiment is a source of excess market volatility and different types of crises, and how sentiment could influence the emergence of crises in this section.

4.1. Stylized facts. We calibrate our sentiment model such that it mimics some well-documented stylized facts of financial markets following the practice of existing literature (See for examples, Schmitt and Westerhoff, 2017; Zhu et al, 2009). As summarized by Westerhoff and Dieci (2006), the five salient characteristics of real-world speculative prices include (1) price distortions in the forms of bubbles and crashes; (2) excess price volatility; (3) leptokurtic distribution of returns (characterized by kurtosis exceeding 3); (4) negligible autocorrelation of daily returns; and (5) strong autocorrelation of absolute daily returns. These patterns can be seen in Fig.4.1, which depicts the dynamics of US stock prices and returns\(^7\) based on daily S&P500 index from Jan 3, 2000 to Feb 12, 2019. As seen from the top panel, the evolution of daily stock prices shows both strong price appreciations and crashes in some periods. The second panel displays daily log returns which show evidence of volatility clustering. Distribution of returns in the third panel indicates the presence of fat tails. The last panel plots the autocorrelation functions (ACF) using both raw returns and absolute returns. We can observe the absence of autocorrelations in raw returns and the slow-dampening autocorrelations in absolute returns, implying long memory of daily returns.

We conduct simulation analysis on the basis of the standard parameter setting as shown in Table 4.1. To examine the effect of sentiment on price movements, we compare model with sentiment to model without sentiment by setting the sentiment sensitivity parameter, \(h\), from 1 to

\[ r_t = \log(p_t) - \log(p_{t-1}). \]

\(^7\) Return at time \(t\) as denoted by \(r_t\) is computed by \(r_t = \log(p_t) - \log(p_{t-1}).\)
0, while holding all other parameter values constant. Simulation results of 17,000 periods (daily) price trajectories are shown in Fig. 4.2. Table 4.2 summarizes the descriptive statistics of both actual and artificial market returns.

Table 4.1: Standard Parameter setting

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
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<tbody>
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<td>$\eta_1$</td>
<td>0.4</td>
</tr>
<tr>
<td>$\eta_2$</td>
<td>0.5</td>
</tr>
<tr>
<td>$\eta_3$</td>
<td>0.1</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>12</td>
</tr>
<tr>
<td>$\eta_1$</td>
<td>$a = 1.11 \times 10^{-5}$</td>
</tr>
<tr>
<td>$\eta_2$</td>
<td>$b = 1 \times 10^{-8}$</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>1000</td>
</tr>
</tbody>
</table>

Overall, we find the simulated series with presence of investor sentiment to exhibit much more realistic statistical properties.\(^8\) As shown by the top left diagram of Fig. 4.2, the model with sentiment is capable to produce more volatile prices (indicated by dark line) relative to rather stable fundamental values (indicated by blue line). Prolonged bubbles and crashes can be generated unlike the case without sentiment where prices mostly fluctuate closely around the fundamentals. Meanwhile, by allowing for sentiment, the return trajectories show volatility clustering similar to the actual S&P500 returns, while the distribution of returns from model with sentiment illustrates leptokurtic behavior given the presence of fat tails. From Table 4.2, the kurtosis and skewness for model with sentiment are 6.68 and -0.099, respectively; whereas model without sentiment have underestimated kurtosis and skewness of 2.14 and -0.017, respectively. To check whether there exists long memory dependence in daily returns, we plot ACFs for both raw daily returns (blue line) and absolute daily returns (red line). Without sentiment, the absolute returns have fast-decaying ACF, suggesting there is no long-range dependence for daily returns. However, with sentiment, ACF of absolute returns is strong and persistent even after 50 lags. This finding matches the stylized facts of real stock return, that cross-correlation should be weak for raw returns but strong for absolute returns.

---

\(^8\) This qualitative finding holds for simulation of deterministic model as well. We also conduct robustness check by evaluating several statistical properties of simulated prices and returns for 1000 simulation runs across a range of sentiment sensitivity parameter values. More details can be found in Appendix F.
Table 4.2: Summary statistics of returns

<table>
<thead>
<tr>
<th>Variable</th>
<th>Kurtosis</th>
<th>Skewness</th>
<th>Mean</th>
<th>SD</th>
<th>Min</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>Actual Market</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>S&amp;P500</td>
<td>11.481</td>
<td>-0.215</td>
<td>0.000</td>
<td>0.012</td>
<td>-0.095</td>
<td>0.110</td>
</tr>
<tr>
<td>Artificial Market</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>With sentiment</td>
<td>6.677</td>
<td>-0.099</td>
<td>0.000</td>
<td>0.010</td>
<td>-0.095</td>
<td>0.064</td>
</tr>
<tr>
<td>Without sentiment</td>
<td>2.140</td>
<td>-0.017</td>
<td>0.000</td>
<td>0.005</td>
<td>-0.010</td>
<td>0.010</td>
</tr>
</tbody>
</table>

FIGURE 4.1. The dynamics of daily S&P500 index between Jan 3, 2000 and Feb 12, 2019. The panel shows, the evolution of the stock price index (a), the returns (b), the histogram of returns overlaid by normal curve (c) and the autocorrelation function of
raw returns (red line) together with the autocorrelation function of absolute returns (blue line) (d).
FIGURE 4.2. The dynamics of the models with sentiment (left) and without sentiment (right). The panels show, from top to bottom, the evolution of the stock prices, the returns, the histogram of returns overlaid by normal curve and the autocorrelation function of raw returns (red line) together with the autocorrelation function of absolute returns (blue line), respectively. The simulation run is based on 17000 observations.

4.2. Sentiment and excess volatility. In order to study the effect of sentiment on market volatility, we follow Zhu et al. (2009) and use the standard deviation of the market prices from the fundamental values $SD_{p-\mu}$ as a quantitative measure of market volatility. Standard deviation $SD_{p-\mu}$ is computed as

$$SD_{p-\mu} = \sqrt{\frac{1}{T} \sum_{t=1}^{T} (p_t - \mu_t)^2}$$

Keeping the same standard parameter set, the deviation of the market prices from the fundamental values $SD_{p-\mu}$ is 358.95 for the model with sentiment and 58.46 for the model without sentiment. This implies that the market is more volatile when sentiment effect is included. In general, we find that $SD_{p-\mu}$ becomes larger as sentiment sensitivity parameter $h$ increases from 0 to 1.

To check the robustness of our result, we run the three-type model 1000 times by using Monte Carlo simulation using the standard parameter values with different levels of sentiment sensitivity from 0 to 1 with an interval of 0.2. Fig.4.3 shows the average standard deviations $SD_{p-\mu}$ for 1000 simulation runs at different $h$ values. As we can see, the average value of $SD_{p-\mu}$ increases with
the sentiment sensitivity, suggesting that the volatility of market is positively related to the sensitivity of investors to market sentiment.

![Figure 4.3](image)

**FIGURE 4.3** The effect of sentiment sensitivity on market volatility with 1000 simulations

### 4.3 Sentiment and crises

Financial crisis is another topic we are interested to investigate under the HAM framework. Many researchers have proven and indicated that standard economic theory failed to envisage the occurrence of financial crises, and heterogeneous beliefs and interaction of heterogeneous agents should be taken into account to understand financial crises. Some literature has documented that models with heterogeneous beliefs have extraordinary power in explaining financial crises, and these studies include but not limit to Lux and Westerhoff (2009), De Jong et al. (2009), Huang at el. (2010) and Xiong (2013). In this section, we investigate different financial crises in HAM framework as in Huang et al. (2010), but we focus on the role of sentiment on formation of crises.

In Huang et al. (2010), they successfully replicate three typical types of crises, namely sudden crisis, smooth crisis and disturbing crisis, by using HAM. They find that endogenous price dynamic from agents' interaction might be the reason of these crises, and both fundamentalists and chartists could potentially contribute to the financial crises. Following their logic, we also generate the same three typical types of financial crises using our three-type model with fundamentalists,
momentum traders and contrarian traders\textsuperscript{9}. The difference between our model and theirs is that we have a stochastic model while their model is deterministic.

In a sudden crisis, price abruptly drops from the peak (or near peak) straight down to bottom within a short timeframe. According to Huang et al. (2010), when the price is at the peak, it is highly overvalued. Observing opportunities for profit, investors switch to fundamental strategy excessively and execute great selling forces that cause the steep fall in price. Contrary to their results, we find no contribution of fundamentalists before and during the sudden crisis. As illustrated in Fig.4.4, there is a dramatic price fall between \( t=50 \) and \( t=60 \), during which, the market is dominated by momentum traders without any switching to fundamental strategy. It is the strong negative sentiment that drives the market crash. Due to either exogenous cause or regime switching, price starts to go down at around \( t=45 \), which makes the momentum traders change from buy to sell and create a bearish sentiment. The strong negative sentiment further accelerates selling forces of the momentum traders and subsequently, leads to market panic and causes a sharp decline in price.

\textsuperscript{9} For more accurate replication of crisis patterns, we simulate our model with \( \mu = 200, \rho = 0.7, \gamma = 1.8, \lambda = 10, b = \left( \frac{1}{156} \right)^4 \) while keeping other parameters in line with the standard values.
FIGURE 4.4. Sudden crisis modelling. The panels show, from top to bottom, comparison between simulated price with S&P 500 index from 1987/8/3 to 1987/12/22 (a), simulated sentiment (b), fractions of 3 types of investor (c)
In a smooth crisis, price declines moderately but persistently over a period of time without a visible crash. As shown from Fig. 4.5, the downward trend starts somewhere between $t=30$ and $t=40$ with an increase in the fraction of fundamentalists. After the first selling from fundamentalists,
there are some countermovements in price caused by contrarian traders. When the downward trend becomes more observable, investors cluster to momentum trading strategy and execute more selling forces to push the price further down. During the decline period, sentiment switches between bullish and bearish with more negative values overall. From $t=30$ to $t=70$, the average value of sentiment is -0.3751.
A disturbing crisis is characterized by volatile fluctuations with a downward trend and possible moderate crashes in price. The period of disturbing crisis is somewhere between sudden crisis and smooth crisis. Fig. 4.6 shows a simulation of disturbing crisis in our model. Price fluctuates disturbingly before starting to drop sharply at around $t=37$ due to significant negative sentiment. From $t=45$ to $t=55$, price becomes volatile again as the sentiment fluctuates between positive and negative. After that, the downward trend continues with another strong bearish sentiment. Similar to the other types of crisis, momentum traders are responsible for the downward trend. However, during the crisis, some investors shift to contrarian strategy because of high market volatility.

To further investigate the effect of sentiment on crises, we compare the frequency and magnitude\(^\text{10}\) of crises in simulated series from model with and without sentiment effect in 1000 simulations. To identify the crisis in financial market, we adopt a crisis indicator called CMAX used in Patel and Sarkar (1998) and Zouaoui (2011) with some adjustments. In this method, CMAX is a ratio calculated by dividing current value by the maximum price over the previous $T$ periods, usually $T$ is one to two years.

\[
CMAX_t = \frac{P_t}{\max(P_{t-T} \cdots P_t)}
\]

where $P_t$ is the stock market index at time $t$. CMAX equals one if price rise over the period considered, indicating a bullish market. If price declines over a period, CMAX goes less than one, and crisis is detected each time CMAX drops below a threshold set at the mean of CMAX minus two standard deviations. Both mean and standard deviation are calculated on the whole sample.

---

\(^{10}\) Following Patel and Sarkar (1998) and Zouaoui (2011), we define the magnitude of a crisis as the percentage drop from the peak to the trough. The date of the peak is the month when price reaches its maximum value over $T$-period window prior to the crisis identification, and the date of the trough is month when price reaches its minimum during the crisis.
However, this method may mistakenly identify the bubble correction as a crisis. To fix this problem and make it more suitable for our simulated data, we add one compulsory condition to detect the crisis, which is the current price must be lower than fundamental prices for a certain threshold value. Therefore, the crisis indicator $C_t$ is defined as following:

$$C_t = 1 \text{ if } CMAX_t < \overline{CMAx} - 2\sigma \text{ and } P_t < \tau \times \mu_t$$

$$C_t = 0, \ldots \text{otherwise}$$

where $\sigma$ is the standard deviation of whole sample CMAX, and $0 < \tau < 1$ is the threshold to determine the minimum magnitude to be defined as a crisis. To check the efficiency of this method, we use the S&P 500 monthly data from 1950m1 and 2018m12 and corresponding fundamental value constructed from monthly dividend to detect the occurrence crisis in US stock market\textsuperscript{11}. We set $T = 12 \text{ month}$ and $\tau = 0.9$. Fig.4.7 illustrates the result of crisis detection.

---

\textsuperscript{11} In accordance to Hommes and Veld (2017), fundamental value is estimated using standard Gordon model with S&P500 data from 1950 onwards. Both the real S&P500 prices and dividends on monthly frequency are provided by Shiller (2005).
FIGURE 4.7. Crises detected from 1950 using real S&P 500 data

As shown in Fig.4.7, five periods of crises are identified during the period 1950-2018. The first crash occurs in 1962 known as Kennedy Slide, followed by the second tech-stock crash in 1970 and third in 1973-1974 after the end of Bretton Woods monetary system and oil crisis. Most recent two market crashes are detected in 1987 known as Black Monday and during 2008-2009 global financial crisis.
FIGURE 4.8. Identify crisis in simulated data with sentiment effect $h_1 = h_2 = 1$ (a), without sentiment effect $h_1 = h_2 = 0$ (b)

As CMAX indicator has been proven reliable in identifying crisis in real financial market, we are confident to use it in our simulated data. To be consistent with case of real financial market, we convert our simulated daily data to monthly data and choose $T = 12$ months and $\tau = 0.9$ to detect crisis. To obtain a visual impression on effect of sentiment on crisis, we try to detect the crisis in the previous two simulated time series with and without sentiment, respectively. From Fig.4.8, we find more crises in with-sentiment case than without-sentiment case. To check the robustness of our result and investigate the magnitude of crisis in these two cases, we again conduct Monte Carlo simulation with the three-type model using the standard parameter values across different levels of sentiment sensitivity with interval of 0.2. As shown in Fig.4.9, number of crisis increases with sentiment sensitivity $h^{12}$, which illustrates the significant positive relationship between sentiment sensitivity and frequency of crisis occurrence. We further investigate whether sentiment affects the magnitude of crisis. As we expected, the average magnitude of crisis rises

---

12 There is a slight decrease from $h=0.8$ to $h=1$. One explanation is that the extrapolating power of momentum traders becomes stronger because of high sentiment sensitivity, which could draw price to deviate from fundamental value for a long time. It may increase the magnitude and duration of crisis but decrease the frequency of crisis.
with higher $h$ which corroborates the effect of sentiment on depth of crisis. Hence, we can conclude that sentiment has a significant effect on crisis in terms of frequency and magnitude in our simulations. Our result is consistent with findings in Zouaoui (2011), who find investor sentiment positively influences the probability of the occurrence of stock market crises by using panel data of 16 countries.

![FIGURE 4.9. The effect of sentiment sensitivity on frequency of crisis (left) and average magnitude of crises (right) with 1000 simulations](image1)

In summary, we have demonstrated that investor sentiment contributes to more realistic stylized facts, and our simulation results also show that the market will be more volatile if investors are very sensitive to market sentiment. More importantly, we successfully replicate different crises, and find that sentiment could be an important source of crisis formation. Through 1000 Monte Carlo simulations, we find the evidence that sentiment could amplify both the frequency and magnitude of financial crises.

5. CONCLUSION

In this paper, we set up a sentiment model constituted with memory strength of sentiment, social interaction and sentiment shock components with three different types of agents investing
in a financial market. We conjecture that investor sentiment has heterogeneous influence on different types of investors and the sentiment effect is transmittable through interaction among agents. We then carry out an in-depth analysis on the role of sentiment under HAM framework.

The key findings of this paper are in two respects. First, for the stability analysis of deterministic skeleton, we find both fundamental steady state featured with neutral sentiment and non-fundamental steady states with polarized sentiment (either positive or negative) in two-type and three-type models. In three-type model, sentiment related parameters are able to affect the stability of fundamental steady state when two types of chartists have equal extrapolating power ($\beta_1 = -\beta_2$). Second, reconciling our model with actual US stock market, we discover investor sentiment contributes to more realistic stylized facts of our simulated series. In particular, numerous key indicators, such as negative skewness, leptokurtosis and long memory of returns, market volatility, as well as number of crisis detected increase with sentiment sensitivity. We also try to deduce theoretical underpinning of different types of crises. The key implication that differentiates our finding from the extant HAM literature is that financial crisis can be triggered even without mean-reverting action of fundamentalist. With presence of investor sentiment, just the momentum traders alone can initiate sudden crisis when there is abrupt downward pressure in market sentiment. In other words, the sentiment channel provides an additional explanation underlying different types of financial crises.

While our sentiment HAM model is capable of replicating and even explaining several important features of financial market, it is not without limitation. One constraint is our model is based on single market framework. Nonetheless, given that international financial markets are becoming more and more integrated nowadays, it is imperative to account for sentiment spillover between countries. Our future research may involve extending the sentiment model to a multi-market setup, such that we are able to draw deeper understanding on investor sentiment as well as financial crisis on an international scale.
Appendix A

Confidence function for fundamentalists

\[ A(x_t) = \frac{ax_t^2}{1 + bx_t^4} \]

is a nonlinear smooth and symmetric function of price deviation \( x_t \). The fundamentalists react to the positive and negative trading signals equally and they are less confident when price deviation \( x_t \) is extremely large. The shape of \( A(x_t) \) function is shown in Figure A1, and it follows some general properties:

**Property 1**

\[ A(0) = 0 \text{ and } A(x_t) > 0 \quad \forall x_t \neq 0 \]

When there are price deviations, the fundamentalists will have positive confidence in their mean-reverting strategy.

**Property 2**

If \( x_t > 0 \), \( A'(x) > 0 \) for \( x_t < b^{-\frac{1}{4}} \) and \( A'(x) < 0 \) for \( x_t > b^{-\frac{1}{4}} \)

If \( x_t < 0 \), \( A'(x) < 0 \) for \( x_t > -b^{-\frac{1}{4}} \) and \( A'(x) > 0 \) for \( x_t < -b^{-\frac{1}{4}} \)

Range of increasing confidence level is \([0, A_{max}]\), \( A_{max} = A\left(b^{-\frac{1}{4}}\right) = A\left(-b^{-\frac{1}{4}}\right) = \frac{1}{2} ab^{-\frac{1}{2}} \).

The reasonable zone \((-Z_t, Z_t) = (-b^{-\frac{1}{4}}, b^{-\frac{1}{4}})\).

The confidence level of fundamentalists increases with \(|x_t|\) within reasonable zone \((-b^{-\frac{1}{4}}, b^{-\frac{1}{4}})\), and decreases with \(|x_t|\) outside this zone. The maximum confidence of fundamentalists is achieved when \( x_t = \pm b^{-\frac{1}{4}} \), which is \( \frac{1}{2} ab^{-\frac{1}{2}} \).

**Property 3**
\[
\lim_{x_t \to \pm \infty} A(x_t) = 0
\]

If the price deviation from fundamental value is too large, the fundamentalists will lose confidence and exit the market.

Figure A1: Confidence function for fundamentalist

Appendix B

Proof of Proposition 1. A steady state \((p^*, U^*, S^*)\) of the deterministic system (3.1) in one regime case must satisfy the following equations:

\[
A_t(\mu - p^*) + \beta_1 m^*(p^* - \bar{v}) = 0
\]

\[
S^* = \frac{\eta_2}{2(1 - \eta_1)}[\tanh(\kappa(\mu - p^*)) + \tanh(\kappa(p^* - \bar{v}))]
\]

\[
m^* = 1 + \tanh(\kappa(p^* - \bar{v})) \ast h_1 \ast S^*
\]
If $\mu = \bar{v}$, we have one fundamental steady state $p^* = \mu = \bar{v}$. In this case, $S^* = 0$ and $m^* = 1$. The stability of this fundamental steady state can be inferred from Jacobian matrix $J$ of the system (3.1) as

$$J = \begin{pmatrix} 1 + \frac{1}{2} \gamma \beta_1 & 0 & 0 \\ 0 & \varphi & 0 \\ 0 & 0 & \eta_1 \end{pmatrix}$$

The eigenvalues of diagonal matrix lie on the main diagonal of matrix, so

$$\lambda_1 = 1 + \frac{1}{2} \gamma \beta_1, \ \lambda_2 = \varphi, \ \lambda_3 = \eta_1$$

The fundamental steady state is asymptotically stable when all the eigenvalues of Jacobian matrix at the steady state are less than one. Hence, fundamental steady state is unstable due to $\lambda_1 > 0$ (both $\gamma$ and $\beta_1$ are positive values).

Non-fundamental steady states may also coexist with fundamental steady state when $\mu = \bar{v}$. At the NFSS, $S^* = 0$ and $m^* = 1$, the first equation of system (3.1) can be simplified as $(A_t - \beta_1)(\mu - p^*) = 0$. In this case, $\mu \neq p^*$ and the solutions are from $A_t - \beta_1 = 0$. According to shape of $A_t$, the possible number of solutions is 0, 2, 4. If $\beta_1$ is larger than $A_{max}$ which is $\frac{1}{2}ab^{-\frac{1}{2}}$, there is no solution; If $\beta_1 = A_{max}$, there are two symmetric solutions around fundamental value $\mu$ (positive $x_t$ and negative $x_t$). If $\beta_1 < A_{max}$, four steady states occur with two above $\mu$ and the other two below $\mu$.

**Proof of Lemma 1.** If $\mu \neq \bar{v}$, non-fundamental steady states could also emerge with persistent positive or negative sentiment. $p^*$ must be lower or higher than both $\mu$ and $\bar{v}$ to satisfy the first equation of system (3.1). According to the positions of $\mu$ and $\bar{v}$, we can also derive the market sentiment equilibria as well as the relation between the confidence level of fundamentalists and momentum traders’ extrapolation rate at steady state. When $\mu > \bar{v}$, $S^*$ must be positive due to $\text{tanh}(\kappa(\mu - p^*)) + \text{tanh}(\kappa(p^* - \bar{v})) > 0$ for both lower and higher $p^*$, so we have a positive
sentiment equilibrium. At the same time, \( A(p_1^*) < \beta_1 m^* \) \( (A(p_2^*) > \beta_1 m^*) \) must be satisfied to get lower (higher) equilibrium \( p_1^* (p_2^*) \). Conversely, when \( \mu > \bar{v} \), \( S^* \) is always negative and we have negative sentiment equilibria. Lower (higher) equilibrium \( p_1^* (p_2^*) \) is associated with \( A(p_1^*) > \beta_1 m^* \) \( (A(p_2^*) < \beta_1 m^*) \).

Appendix C

Proof of Proposition 2. Besides the heterogeneous extrapolation rates for momentum traders and contrarian traders, they also react differently to market sentiment. The steady state \( (p^*, U^*, S^*) \) of the deterministic system (3.2) in one regime case must satisfy the following equations:

\[
A_t(\mu - p^*) + \beta_2 c^*(p^* - \bar{v}) = 0
\]

\[
S^* = \frac{\eta_2}{2(1 - \eta_1)} [\tanh(\kappa(\mu - p^*)) - \tanh()]
\]

\[
c^* = 1 - \tanh(\kappa(p^* - \bar{v})) \times h_1 \times S^*
\]

If \( \mu = \bar{v} \), we have unique fundamental steady state \( p^* = \mu = \bar{v} \). In this case, \( S^* = 0 \) and \( c^* = 1 \). The stability of this fundamental steady state can be inferred from Jacobian matrix \( J \) of the system (3.2) as

\[
J = \begin{pmatrix}
1 + \frac{1}{2} \gamma \beta_2, & 0 & 0 \\
0, & \varphi & 0 \\
-\eta_2(1 + \frac{1}{2} \gamma \beta_2), & 0 & \eta_1
\end{pmatrix}
\]

The eigenvalues of Jacobian matrix are

\[
\lambda_1 = 1 + \frac{1}{2} \gamma \beta_2, \quad \lambda_2 = \varphi, \quad \lambda_3 = \eta_1
\]
As $0 < \varphi < 1$ and $0 < \eta_1 < 1$, the stability of the system is determined by $\lambda_1$. It is possible to have $\lambda_1$ inside the unit circle due to $\beta_2 < 0$. When $|\lambda_1| < 1$ or $-4 < \gamma \beta_2 < 0$, the fundamental steady state is locally stable. Note that setting $\gamma \beta_2 = -4$ gives us the flip bifurcation as $\lambda_1 = -1$.

**Proof of Lemma 2.** If $\mu \neq \bar{v}$, three types of non-fundamental steady states may emerge with persistent positive, negative sentiment or zero sentiment. The equilibrium $p^*$ must between the fundamental value $\mu$ and short-term value $\bar{v}$. If price is lower or higher than both $\mu$ and $\bar{v}$, it will be attracted to the zone constructed by $\min(\mu, \bar{v})$ and $\max(\mu, \bar{v})$. According to the position of $p^*$, we can derive three different market steady states with positive, negative or neutral sentiment. As the equilibrium price is within $\left(\min(\mu, \bar{v}), \max(\mu, \bar{v})\right)$, the distances of $p^*$ to $\mu$ and $\bar{v}$ decide the sign of equilibrium sentiment $S^*$. If $p^* < \frac{1}{2}(\mu + \bar{v})$, $S^* > 0$ and a positive sentiment equilibrium is achieved. To know the source of this positive market sentiment, we need to analyze the relative position of $\mu$ and $\bar{v}$. If $\mu > \bar{v}$ and $p_1^* < \frac{1}{2}(\mu + \bar{v})$, we need to have $A(p_1^*) < -\beta_2 c^*$ to satisfy the first equation of system (3.2), and market sentiment is mainly from fundamentalists’ optimistic beliefs. Otherwise, if $\mu < \bar{v}$ and $p_1^* < \frac{1}{2}(\mu + \bar{v})$, $A(p_1^*) > -\beta_2 c^*$ must be satisfied to get the positive sentiment equilibrium, and optimism is derived from relatively aggressive contrarian traders. Another negative sentiment steady state is featured with $p_2^* > \frac{1}{2}(\mu + \bar{v})$. If $\mu > \bar{v}$, equilibrium could be achieved with $A(p_2^*) > -\beta_2 c^*$. If $\mu > \bar{v}$, equilibrium could emerge with $A(p_2^*) < -\beta_2 c^*$. A special case is when $p_3^* = \frac{1}{2}(\mu + \bar{v})$, then $S^* = 0$, and equilibrium is achievable if $A(p_3^*) = -\beta_2$. It means we must have $\beta_2 = -\frac{4a(\mu + \bar{v})^2}{16 + b(\mu + \bar{v})^4}$ to get this steady state.
Appendix D

**Proof of Proposition 3.** The steady state \((p^*, U^*, S^*)\) of the deterministic system (3.3) in one regime case must satisfy the following equations:

\[(\beta_1 m^* + \beta_2 c^*)(p^* - \bar{v}) = 0\]

\[S^* = 0\]

\[m^* = c^* = 1\]

If \(\beta_1 \neq -\beta_2\), we have unique steady state \(p^* = \bar{v}\). The corresponding Jacobian matrix can be derived as

\[
J = \begin{pmatrix}
1 + \frac{1}{2} \gamma (\beta_1 + \beta_2), & 0 & 0 \\
0 & \varphi & 0 \\
0 & 0 & \eta_1
\end{pmatrix}
\]

The eigenvalues of Jacobian matrix are

\[
\lambda_1 = 1 + \frac{1}{2} \gamma (\beta_1 + \beta_2), \quad \lambda_2 = \varphi, \quad \lambda_3 = \eta_1
\]

As \(0 < \varphi < 1\) and \(0 < \eta_1 < 1\), the stability of the system is determined by \(\lambda_1\). For absolute value of \(\lambda_1\) to be less than 1, we must have \(-4 < \gamma (\beta_1 + \beta_2) < 0\) to make the system locally stable. The setting \(\gamma (\beta_1 + \beta_2) = -4\) gives us the flip bifurcation as \(\lambda_1 = -1\).

Appendix E

**Proof of Proposition 4.** The relationship between \(\beta_1\) and \(\beta_2\) could determine the number and type of steady states. If \(\beta_1 = -\beta_2 = \beta\), only fundamental steady state is possible with the following equations:

\[
\omega_f A^*(\mu - p^*) + \beta (\omega_{m0}^* m^* - \omega_{c0}^* c^*)(p^* - \bar{v}) = 0
\]
\[
\omega^*_f = \omega^*_m = \omega^*_c = \frac{1}{3}
\]
\[
u^*_f = \nu^*_m = \nu^*_c = 0
\]
\[
S^* = 0
\]
\[
m^* = c^* = 1
\]

The corresponding Jacobian matrix with \( \beta_1 = -\beta_2 = \beta \) can be derived as
\[
J = \begin{pmatrix}
1 & 0 & B & -B & C \\
0 & \varphi & 0 & 0 & 0 \\
0 & 0 & \varphi + \beta \delta B & -\beta \delta B & \beta \delta C \\
0 & 0 & -\beta \delta B & \varphi + \beta \delta B & \beta \delta C \\
0 & 0 & D & -D & E
\end{pmatrix}
\]

where
\[
\delta = \mu - \bar{v}, B = \frac{1}{3} \gamma \rho \beta (\mu - \bar{v}), C = \frac{1}{3} \gamma \beta (\mu - \bar{v}) (h1 + h2) \tanh(\mu - \bar{v})
\]
\[
D = -\frac{1}{9} \rho \eta_2 [\gamma \beta (\mu - \bar{v}) - 3\varphi \tanh(\kappa (\mu - \bar{v}))],
\]
\[
E = \eta_1 - \frac{1}{9} \eta_2 \gamma \beta (\mu - \bar{v}) (h1 + h2) \tanh(\kappa (\mu - \bar{v}))
\]

There are five eigenvalues for this Jacobian matrix, which are
\[
\lambda_1 = 1, \lambda_2 = \varphi, \lambda_3 = \varphi + \beta \delta B, \lambda_4 = \frac{\varphi^2 + 2\varphi \beta \delta B}{\varphi + \beta \delta B},
\]
\[
\lambda_5 = E - \frac{2 \beta \delta CD}{\varphi} + \frac{4 \beta \delta CD(\varphi + \beta \delta B)}{\varphi^2 + 2\varphi \beta \delta B} - \frac{2 \varphi \beta \delta CD(\varphi + \beta \delta B)}{(\varphi^2 + 2\varphi \beta \delta B)(\varphi + \beta \delta B)}
\]

To make the system locally stable, we need to have \(|\lambda_3|, |\lambda_4|, |\lambda_5| < 1\). For a special case \( \mu = \bar{v} \),
the Jacobian matrix could be simplified to
\[
J = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & \varphi & 0 & 0 & 0 \\
0 & 0 & \varphi & 0 & 0 \\
0 & 0 & 0 & \varphi & 0 \\
0 & 0 & D & 0 & \eta_1 \\
\end{pmatrix}
\]

The corresponding five eigenvalues for this diagonal matrix are

\[
\lambda_1 = 1, \lambda_2 = \varphi, \lambda_3 = \varphi, \lambda_4 = \varphi, \lambda_5 = \eta_1
\]

As \(0 < \varphi < 1\) and \(0 < \eta_1 < 1\), the system is always stable for this special case.

If \(\beta_1 \neq -\beta_2\), both FSS and NFSS are possible, and steady state \((p^*, u_f^*, u_{mo}^*, u_{co}^*, S^*)\) of the deterministic system (3.4) in one regime case must satisfy the following equations:

\[
\omega_f^* A^*(\mu - p^*) + (\omega_{mo}^* \beta_1 m^* + \omega_{co}^* \beta_2 c^*) (p^* - \bar{v}) = 0
\]

\[
\omega_f^* = \omega_{mo}^* = \omega_{co}^* = \frac{1}{3}
\]

\[
u_f^* = u_{mo}^* = u_{co}^* = 0
\]

\[
S^* = \frac{\eta_2}{3(1 - \eta_1)} \tanh(\kappa(\mu - p^*))
\]

\[
m^* = 1 + \tanh(\kappa(p^* - \bar{v})) * h_1 * S^*
\]

\[
c^* = 1 - \tanh(\kappa(p^* - \bar{v})) * h_1 * S^*
\]

For fundamental steady state, we have \(p^* = \mu = \bar{v}\), and Jacobian matrix can be derived as

\[
J = \begin{pmatrix}
1 + \frac{1}{3} \gamma (\beta_1 + \beta_2), & 0 & 0 & 0 & 0 \\
0 & \varphi & 0 & 0 & 0 \\
0 & 0 & \varphi & 0 & 0 \\
0 & 0 & 0 & \varphi & 0 \\
-\frac{1}{3} \eta_2 & 0 & 0 & \eta_1
\end{pmatrix}
\]
Five eigenvalues for this Jacobian matrix will be

\[ \lambda_1 = 1 + \frac{1}{3}\gamma(\beta_1 + \beta_2), \lambda_2 = \varphi, \lambda_3 = \varphi, \lambda_4 = \varphi, \lambda_5 = \eta_1 \]

We can have a locally stable system provided that \(-6 < \gamma(\beta_1 + \beta_2) < 0\).

For NFSS, we focus on the features of the equilibria. There are two types of steady states, positive and negative sentiment steady states. From the sentiment function, it is clear that when \(p^* < \mu\), \(S^* > 0\) and vice versa. But \(p^*\) must satisfy the first equation of system (3.4).

\[ A^*(\mu - p^*) + (\beta_1 + \beta_2)(p^* - \bar{v}) + (\beta_1 h_1 - \beta_2 h_2)[S^* \tanh(\kappa(p^* - \bar{v}))](p^* - \bar{v}) = 0 \]

If \(p^* < \mu\), we have \(S^*\) and the first and third term of this equation are positive, so the second term must be negative. When \(\beta_1 > -\beta_2\) (\(\beta_1 < -\beta_2\)), we have \(p^* < \bar{v}\) (\(p^* > \bar{v}\)). For negative sentiment non-fundamental steady state \(S^* < 0\), we have \(p^* > \mu\) and \(p^* > \bar{v}\) (\(p^* < \bar{v}\)) when \(\beta_1 > -\beta_2\) (\(\beta_1 < -\beta_2\)).
Appendix F
FIGURE A2. The dynamics of the deterministic models with sentiment (left) and without sentiment (right). The panels show, from top to bottom, the evolution of the stock prices, the returns, the histogram of returns overlayed by normal curve and the autocorrelation function of raw returns (red line) together with the autocorrelation function of absolute returns (blue line), respectively. The simulation run is based on 17000 observations.
### Table F1. Means of some statistics with different sentiment sensitivity levels for 1000 simulations

<table>
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<tr>
<th></th>
<th>h=0</th>
<th>h=0.2</th>
<th>h=0.4</th>
<th>h=0.6</th>
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<th>h=1</th>
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<td>2.075</td>
<td>3.396</td>
<td>4.615</td>
<td>4.730</td>
<td>5.804</td>
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<td>skewness</td>
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<td>-0.015</td>
<td>-0.026</td>
<td>-0.036</td>
<td>-0.037</td>
<td>-0.040</td>
</tr>
<tr>
<td>AC $r_1$</td>
<td>0.010</td>
<td>0.041</td>
<td>0.166</td>
<td>0.219</td>
<td>0.189</td>
<td>0.166</td>
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<tr>
<td>AC $r_5$</td>
<td>-0.006</td>
<td>-0.004</td>
<td>0.004</td>
<td>0.001</td>
<td>0.007</td>
<td>0.004</td>
</tr>
<tr>
<td>AC $r_{10}$</td>
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<td>-0.004</td>
<td>-0.002</td>
<td>-0.003</td>
<td>-0.002</td>
<td>-0.003</td>
</tr>
<tr>
<td>AC $r_{20}$</td>
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<td>-0.002</td>
<td>-0.002</td>
<td>-0.001</td>
<td>-0.001</td>
<td>-0.001</td>
</tr>
<tr>
<td>AC $</td>
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<td>$</td>
<td>-0.014</td>
<td>-0.003</td>
<td>0.172</td>
<td>0.344</td>
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<tr>
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<td>0.075</td>
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<td>217.539</td>
<td>270.436</td>
<td>291.671</td>
<td>339.284</td>
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<td>magnitude of crisis (%)</td>
<td>14.703</td>
<td>17.141</td>
<td>30.961</td>
<td>38.443</td>
<td>39.213</td>
<td>43.804</td>
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References


