Yield Curve Volatility and Macro Risks

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Abstract

How important are macro risks for explaining variation in bond yields? We show that the macroeconomic impact on yield curve volatility varies substantially over time. For this purpose, we introduce a novel no-arbitrage macro-finance term structure model with multivariate GARCH volatility. Our model is tractable and captures empirical measures of volatility in U.S. Treasury bond yields between 1971 and 2019 closely. We find that the fraction of yield curve variation due to macro risks ranges between 0 and 56 pct with large month-to-month changes. Macro risks explain most variation in expansions and primarily affect bond yields through expectations to future short rates. Also, we show that the importance of macro risks ceased during the Great Moderation but regained explanatory power after the Great Recession. Finally, investors are willing to pay large premia for hedging uncertainty related to macro risks.

Keywords: Yield Curve, macro risks, macro-finance, conditional volatility, multivariate GARCH.

JEL classification: C32, C51, E43, G12.

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1 Introduction

Policy makers and risk managers pay close attention to the joint behavior of the yield curve and macroeconomic risks. A central theme is to understand the extent to which macro fundamentals drive movements in nominal interest rates. Indeed, the literature offers numerous attempts to uncover this relation. The seminal paper by Ang and Piazzesi (2003) shows that inflation and real activity account for 2 to 85 pct of the variation in bond yields depending on model specification and maturity. Ang, Bekaert, and Wei (2008) find that expected inflation explains 80 pct of yield curve variation across all maturities. However, this conclusion is contradicted by Duffee (2018), who estimates the ratio of variation in expected inflation to yield variation to 10 to 20 pct. In sum, the existing literature fails to establish a consensus on the role of macro risks in the yield curve.

This paper seeks to unify these views and establish new knowledge on the linkages between the macroeconomy and the yield curve. Our main argument is that the impact of macro risks on bond yields varies over time. In fact, we show that simple regressions of U.S. Treasury bond yields on macro risks using a rolling-window sample result in highly time-varying R-squared values that range between 0 and 80 pct. Our hypothesis is that the time-variation in the explanatory power of macro risks arises because the amount of variation in yields to be explained is time-varying. Indeed, estimates of yield curve volatility are time-varying and it is a widespread result that macro shocks generate jumps and volatility in bond yields (Andersen, Bollerslev, Diebold, and Vega, 2007, Bollerslev, Cai, and Song, 2000, Fleming and Remolona, 1999, Johannes, 2004, Piazzesi, 2005, Pierluigi Balduzzi and Green, 2001).

In order to understand this seemingly time-varying link between the macroeconomy and the Treasury bond market, we build a macro-finance term structure model with time-varying conditional second moments. Following standard practice, we model bond yields as affine functions of a few risk factors imposing restrictions that preclude arbitrage opportunities. Bond market investors face two types of risks: (i) macro risks that are given by observed variables measuring inflation and economic activity, and (ii) non-macro risks that are latent factors extracted from the residual variation in yields after accounting for macro risks. These risk factors are modeled jointly, which allows for interdependence between the macroeconomy and financial markets.
Compared to the previous macro-finance literature, we devise two new components. First, the joint dynamics follow a VAR model with multivariate GARCH volatility. Specifically, we find that the Baba-Engle-Kraft-Kroner (BEKK) model proposed by Engle and Kroner (1995) provides a good characterization of the dynamics of yield curve volatility as the model can match empirical volatility measures closely. Our second modeling contribution is an application of the second-order pricing kernel of Monfort and Pegoraro (2012). We show that this pricing kernel can be structurally justified by the marginal rate of substitution of a representative household with recursive preferences. By allowing for second-order terms in the pricing kernel, we enable investors to price both level- and variance-covariance-based risks, which introduces differences between the physical and risk-neutral conditional second moments. This feature is important for the ability of the model to simultaneously match the cross-section of bond yields and conditional volatility of the yield curve. Intuitively, while the parameters governing the price of level-based risk are estimated to match the yield curve, the parameters of the variance-covariance-based risk price are free to capture yield curve volatility. In contrast, canonical affine term structure models only allow for a price of level-based risk leaving no additional freedom for second moments. Indeed, the literature has shown that these models struggle to match volatility with low and often negative correlation between predicted and realized variances (Christensen, Lopez, and Rudebusch, 2014, Collin-Dufresne, Goldstein, and Jones, 2009, Jacobs and Karoui, 2009).

Our model admits closed-form recursive solutions for no-arbitrage bond yields. We build upon the results in Joslin, Singleton, and Zhu (2011) and show that the model can be rotated into a unique model with observable risk factors only. The important work in Joslin, Singleton, and Zhu (2011) is valid for Gaussian affine term structure models (GATSMs) only. In this paper, we provide a generalization of their results that are valid for our model with time-varying second moments. The result hinges on the fact that our model is invariant to affine transformations such that the latent non-macro factors can be rotated into observed yield portfolios. In addition, the parameters that govern the model dynamics are distinct from those that price the yield curve. Therefore, the

1Roussellet (2017) also implement a term structure model with second-order pricing kernels in order to model double-sided inflation fears.
2Ghysels, Le, Park, and Zhu (2014) also provide a generalization of Joslin, Singleton, and Zhu (2011). Their generalization holds for a model that differs from ours in two ways. First, they model ARCH-in-mean in spirit of Engle, Lilien, and Robins (1987) using a GARCH-type process with long- and short-run components. Second, their model only contains latent risk factors, whereas we allow for both latent and observed risks.
physical dynamics of our term structure model can be estimated separately from the pricing dynamics. Our model also facilitates closed-form computation of term premia and state-dependent impulse response functions.

We estimate the model on U.S. Treasury bond yields sampled monthly between September 1971 and June 2019. The macro risks are captured by CPI inflation and the unemployment gap. Our empirical results provide new insights about the role of macro risks in the yield curve. First, we characterize the ratio of variation explained by macro risks to total variation in bond yields. These variance ratios are highly time-varying with a range between 0 and 54 pct for the 10-year maturity. The ranges are similar across all maturities. Our estimated macro variance ratios are volatile and there are occasional large month-to-month changes, which indicate that some of the time-variation is driven by announcement effects. Focusing on the underlying trend of the variance ratios, we find that macro risks explain most variation in expansions and early recoveries. This result continues to hold even at long (10-year) forecasting horizons. Assuming that monetary policy rate follows a Taylor rule, these findings are consistent with the view that monetary policy is most effective in expansions (Angrist, Jordá, and Kuersteiner, 2017, Barnichon and Matthes, 2018, Tenreyro and Thwaites, 2016). Our results also indicate that non-macro risks, e.g., factors that relate to financial distress, are driving movements in the yield curve during recessions. Finally, we uncover a U-shaped pattern of the variance ratios associated with the short end of the yield curve: macro variance ratios were large during the 1970s and 1980s, decreased during the Great Moderation, and has increased again following the Great Recession. The recent revival of macro risks as important determinants of the yield curve can be attributed to the unemployment gap.

Policy makers rely on decompositions of the yield curve into expected future short rates and term premia to understand the transmission mechanism and hence the effectiveness of monetary policy. Moreover, Bernanke (2006), Kohn (2005), and Yellen (2014) have argued for the importance of understanding the sources of movements in term premia. We therefore use our model to evaluate the role of macro risks in respectively short-rate expectations and term premia. We find that macro risks explain more variation in each of these components compared to the total yield. Specifically, the variance ratio measuring the macro contribution to variation in expected future short rates range between 0.35 and 76 pct for the 10-year maturity. The range is similar for the macro contribution to variation in term premia. These ratios exhibit clear trends throughout our sample, where macro risks explain increasingly more variation in short-rate expectations, but less
variation in term premia. Along this line, our results show that, on average, macro risks primarily affect bond yields through short-rate expectations. One straightforward interpretation of this result is that macro risks impact the yield curve through the Taylor rule along with the expectations hypothesis. In particular, monetary policy makers adjust the short rate based on inflation and unemployment gap, which transmits to longer-term yields through the expectations that investors form about the future path of monetary policy. It is a novel finding that these traditional ways of understanding yield curve movements are becoming increasingly important channels through which macro risks impact the yield curve.

Next, we analyze how shocks to macro risks impact the yield curve both in terms of levels and volatility. In our model, these impulse responses are state-dependent. We find that the size of the impact of macro shocks on yield levels are state-dependent, but that the sign of the impact and the shape of the adjustment path are constant over time. In contrast, macro shocks can impact the volatility of yields both positively and negatively depending on the state. Thus, it is difficult to predict how macro shocks affect the amount of variation in the yield curve.

Finally, we show that variance risk premia are strongly related to uncertainty about macro risks. Bond market investors are willing to pay large premia to hedge uncertainty and 80 to almost 100 pct of these premia can be attributed macro uncertainty. Surprisingly, the variance risk premium related to non-macro risks is small. These results complement Cieslak and Povala (2016), who find that variance risk premia in Treasury bonds are particularly related to uncertainty about future monetary policy.

The remainder of the paper is structured as follows. We review the related literature in Section 2. In Section 3, we present an empirical exercise that motivate time-varying macro contributions to yield curve variation. Section 4 presents our macro-finance term structure model. In section 5, we provide a tractable approach to estimating our model. Data and estimation results are presented in Section 6. Finally, we present our empirical results, analyzing the time-varying role of macro risks in the yield curve, in Section 7. Conclusions follow in Section 9.

2 Literature Review

A large literature has implemented term structure models with macro risks, see, e.g., Ang, Piazzesi, and Wei (2006); Duffee (2006); Hördahl, Tristani, and Vestin (2006); Diebold,
Piazzesi, and Rudebusch (2005); Rudebusch and Wu (2008); Kozicki and Tinsley (2001); Dewachter and Lyrio (2006); and Moench (2008). Specifically, Ang and Piazzesi (2003), Ang, Bekaert, and Wei (2008), Bikbov and Chernov (2010), Diebold, Rudebusch, and Aruoba (2006), Duffee (2018), and Feunou, Fontaine, and Roussellet (2019) quantify the extent to which yield curve variation can be explained by various sources of macro information. We tabulate the conclusions of these papers in Appendix A. In short, the results of the current literature are highly different and there is little agreement about what role, if any, that macro risks play in the yield curve. In fact, Joslin, Priebsch, and Singleton (2014) and Bauer and Rudebusch (2019) argue that macro risks have predictive power for future yields, they do not directly explain bond yields, i.e., macro risks are unspanned. In this paper, we show that the time-variation in the macro contribution to yield curve variation can explain some of the disagreement in the literature.

The common approach in macro-finance term structure modeling involves variations of the Gaussian affine term structure model (GATSM) in which conditional volatilities are constant. The literature comprises only few attempts to model yield curve volatility jointly with macro risks. In the macro literature, Cogley and Sargent (2001,2005), Primiceri (2005), Sims and Zha (2006), and Muntaz and Zanetti (2013) study structural VARs with stochastic volatility. Creal and Wu (2017) build upon this literature and examine the relationship between macro shocks and uncertainty about monetary policy and risk premia using a no-arbitrage term structure model based on a VAR with stochastic volatility. Campbell, Sunderam, and Viceira (2017) estimate a model with GARCH-type volatility of nominal and real yields and inflation in order to capture the time-varying covariance between bond yields and stock returns. Haubrich, Pennacchi, and Ritchken (2012) model the nominal and real yield curves by introducing volatility factors that determine risk premia. In contrast to these papers, we emphasize that macro risks contain information about yield curve variation and we provide a characterization of the time-variation in this link. Finally, Koeda and Kato (2015) study the role of uncertainty in bond risk premia, where uncertainty is modeled endogenously by independent univariate GARCH-type models similar to Heston and Nandi (2003). Our model is more general than their approach in two ways: first, we model the dynamics of the full conditional variance-covariance matrix. Second, we allow for latent yield curve factors where Koeda and Kato (2015) assume that these are observable. In contrast to Koeda and Kato (2015), we
find that macro risks affect the yield curve primarily through the short-rate-expectations channel rather than the term-premia channel.

3 Empirical Motivation

We argue throughout this paper that the macro contribution to bond yield variation is time-varying. In result, the explanatory power of macro risks for bond yields must vary over time as well. Also, this feature implies that the ability of macro risks to predict future excess returns is time-varying. We begin by providing a simple empirical exercise that strongly confirms these claims.

We use monthly data from September 1971 to June 2019. The bond yields, provided by Gürkaynak, Sack, and Wright (2007), range in maturity from 3 months to 10 years. For the macro risks, we use year-over-year growth in the CPI index (INF) and the unemployment gap (UGAP). A detailed data description is provided in Section 6.1.

First, we consider explanatory regressions of the type

$$Y_{t,10y} = \delta_0 + \delta_1 \text{INF}_t + \delta_2 \text{UGAP}_t,$$

where $Y_{t,10y}$ is the 10-year Treasury bond yield. We focus on the 10-year maturity, but the results discussed below hold for the entire term structure. We estimate the regression equation over rolling windows of 10 and 30 years of monthly data. The resulting R-squared values are shown in the left chart in Figure 2.

Using 10-years rolling windows, the explanatory power of macro risks is extremely time-varying with R-squared values between 0 and 80 pct. We also consider 30-year rolling windows, which corresponds to the sample size obtained from using data from 1990. In this case, the explanatory power of macro risks remain time-varying with a range between 0 and 50 pct. Therefore, when evaluating the importance of macro risks for explaining movements in the yield curve, the choice of sampling period appears to be crucial.

A large literature has documented that macro fundamentals are important for predicting future excess returns (Bauer and Rudebusch, 2019, Cieslak and Povala, 2015, Jørgensen, 2017, Joslin, Priebsch, and Singleton, 2014). In particular, the R-squared value from regressing excess holding period returns on macro risks and principal components of the yield curve is typically significantly higher than that obtained by using the principal components as regressors alone. We report the differences between R-squared

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3 Many empirical applications of macro-finance term structure models start the sample in 1990, see, e.g., Bauer, Rudebusch, and Wu (2012), Wright (2011), and Wu and Xia (2016).
values obtained with and without macro risks using one-year holding period excess returns for the 10-year maturity\textsuperscript{4} over rolling windows in the right chart in Figure 2. We confirm that the time-varying role of macro risks also transmits into the predictive power of macro risks. Specifically, macro risks predict between 0 and 40 percentage points more variation in excess returns than yield curve information alone using a 10-year rolling window. For the 30-year window, the additional R-squared values from including macro risks are between 5 and 25 percentage points.

These simple exercises reveal that there are large variations in the ability of macro risks to explain and predict variation in the yield curve. In order to understand these time-variations, we build a model that can generate this property of the data.

4 Model

We model the Treasury bond market by two sources of risks: (i) observed macro variables and (ii) latent risk factors that capture yield curve movements that cannot be attributed to macro fundamentals (non-macro risks). Relative to standard macro-finance term structure models,\textsuperscript{5} we introduce two innovations: first, the joint dynamics of the risk factors exhibit multivariate GARCH volatility and second, bond market investors can require a premium for exposure to conditional volatility.

4.1 Physical Dynamics

The yield curve is driven by a $n_X$-dimensional state vector, $X_t$, consisting of $n_x$ latent non-macro risks, $x_t$, and $n_m$ observed macro risks, $m_t$. That is, $X_t = [x_t', m_t']'$. Let $\mathcal{F}_t$ be the filtration given by $(X_t, X_{t-1}, \ldots)$. We specify the conditional mean and variance-covariance matrix separately in the following.

**Conditional Mean**

The joint dynamics of the macro and non-macro risk factors are modeled by a VAR model, allowing for some lag length $L$ of the macro risks in the equation for $m_t$. The The

\textsuperscript{4}The one-year excess holding period return of the 10-year bond is defined by $RX_{t,n} = -(n+1)/12Y_{t,n+1} + n/12Y_{t,n} - 12Y_{t,12}$.

\textsuperscript{5}The macro-finance term structure literature is dominated by the Gaussian affine term structure model (GATSM), which is based on VAR dynamic with constant conditional volatility.
equations for $x_t$ and $m_t$ are given by

$$x_t = \mu_x + \Phi_x^{(1)} x_{t-1} + \Phi_x^{(1)} m_{t-1} + \varepsilon_{x,t},$$  \hspace{1cm} (1)

$$m_t = \mu_m + \Phi_m^{(1)} x_{t-1} + \Phi_m^{(1)} m_{t-1} + \Phi_m^{(2)} m_{t-2} + \ldots + \Phi_m^{(L)} m_{t-L} + \varepsilon_{m,t}. \hspace{1cm} (2)$$

Equivalently, we write

$$X_t = \mu_X + \Phi_X^{(1)} X_{t-1} + \ldots + \Phi_X^{(L)} X_{t-L} + \varepsilon_t$$  \hspace{1cm} (3)

with appropriate zero restrictions of $\Phi_X^{(l)}$ for $l = 2, \ldots, L$. The errors, $\varepsilon_{x,t}$ and $\varepsilon_{m,t}$, are conditionally Gaussian given $\mathcal{F}_{t-1}$ with mean zero and some conditional variance-covariance matrix, $V_t$.

**Conditional Variance-Covariance Matrix**

We model the full variance-covariance matrix, $V_t$, allowing for time-varying conditional covariances between the non-macro and macro risk factors. In specifying the dynamics of $V_t$, we emphasize a model in the GARCH framework that (i) is sufficiently flexible; (ii) warrants a symmetric positive definite variance-covariance matrix; and (iii) ensures that the model is invariant to affine transformations. The final item is crucial for the validity of the estimation approach that we propose in Section 5. The BEKK-GARCH model of Engle and Kroner (1995) satisfies these conditions. The volatility dynamics of the model are given by

$$V_t = \Sigma_X \Sigma_X' + \sum_{k=1}^{K} A_X^{(k)} \varepsilon_{t-1} \varepsilon_{t-1}' A_X^{(k)'} + \sum_{k=1}^{K} B_X^{(k)} V_{t-1} B_X^{(k)'}.$$  \hspace{1cm} (4)

If $\Sigma_X$ is lower triangular such that $\Sigma_X \Sigma_X'$ is positive definite, then $V_t$ is also positive definite. Allowing for multiple components, $K > 1$, allows for rich dynamics of conditional variances and covariances which is typically needed empirically.

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6Note that $V_t$ denote the conditional volatility of a process at time $t$ that is adapted to $\mathcal{F}_{t-1}$. This choice follows the GARCH literature, but differs from the literature on stochastic volatility.

7Many GARCH models satisfy this requirement including realized GARCH models discussed in Hansen, Huang, and Shek (2011), factor GARCH models in Diebold and Nerlove (1989) and Engle, Ng, and Rothschild (1990), and certain specifications of the double autoregressive (DAR) model in Nielsen and Rahbek (2014). We rule out GARCH models with asymmetric components, the vech representation of multivariate GARCH models in Bollerslev, Engle, and Wooldridge (1988), and the dynamic conditional correlations (DCC) model of Engle (2002).

8To illustrate this notion, consider a bivariate model with $K = 1$ in which $A_X^{(1)}$ and $B_X^{(1)}$ are diagonal matrices with elements $a_{ii}$ and $b_{ii}$, $i = 1, 2$. Then,

$$V_t = \Sigma_X \Sigma_X' + \begin{bmatrix} a_{11}^2 \varepsilon_{1,t-1}^2 & a_{11} a_{22} \varepsilon_{1,t-1} \varepsilon_{2,t-1} \\ a_{11} a_{22} \varepsilon_{1,t-1} \varepsilon_{2,t-1} & a_{22}^2 \varepsilon_{2,t-1}^2 \end{bmatrix} + \begin{bmatrix} b_{11} V_{11,t-1} & b_{11} b_{22} V_{12,t-1} \\ b_{11} b_{22} V_{12,t-1} & b_{22}^2 V_{22,t-1} \end{bmatrix}.$$
The full BEKK-GARCH model, i.e., when $A^{(k)}_X$ and $B^{(k)}_X$ for some $k$ are fully parameterized matrices, suffers from a number of econometric problems including curse of dimensionality and lack of asymptotic properties (Chang and McAleer, 2019). We therefore impose diagonal restrictions on $A^{(k)}_X$ and $B^{(k)}_X$ for all $k = 1, \ldots, K$.\(^9\)

### 4.2 Pricing Kernel

We relate the risk factors to no-arbitrage bond yields by specifying the pricing kernel. Letting $P_{t,n}$ the price of a $n$-period zero-coupon bond, the one-period pricing kernel, $\mathcal{M}_{t+1,t}$, is defined by

$$P_{t,n} = \mathbb{E}_t (\mathcal{M}_{t+1,t} P_{t+1,n-1}),$$

where $\mathcal{M}_{t+1,t}$ is assumed to depend on $X_t$ and $\mathbb{E}_t (\cdot)$ denotes the conditional expectation given $\mathcal{F}_t$ under physical probabilities. Typically, $\mathcal{M}_{t+1,t}$ is specified as an exponential-affine function, which implicitly assumes that investors only care about the conditional mean of $X_t$ when pricing bonds, see, e.g., Le, Singleton, and Dai (2010). However, recent work suggests that more flexible specifications are crucial for capturing conditional volatility of the yield curve (Creal and Wu, 2017, Ghysels, Le, Park, and Zhu, 2014, Joslin and Konchitchki, 2018). Based on these experiences, we apply the exponential-quadratic kernel in Monfort and Pegoraro (2012), which is given by

$$\mathcal{M}_{t+1,t} = \exp \left( -\kappa_X t + \xi_t X_{t+1} + \xi_t' X_{t+1} + \xi_t' X_{t+1} \right) \mathbb{E}_t \left[ \exp \left( \xi_t X_{t+1} + \xi_t' X_{t+1} \right) \right].$$

(5)

Our pricing kernel contains three components. The first component is a time discount determined by the one-period interest rate, $r_t$. We assume a short-rate model that is affine in $X_t$: \(^{(6)}\)

$$r_t = \alpha_X + \beta'_x x_t + \beta'_m m_t = \alpha_X + \beta'_X X_t.$$  \(6\)

The remaining components are compensations for mean-based risk, $\xi_t X_{t+1}$, and variance-covariance-based risk, $X_{t+1} \Xi_t X_{t+1}$. Thus, $\xi_t$ and $\Xi_t$ can be interpreted as the market prices of risks associated with the conditional mean and variance-covariance matrix of

\(^{The parameters $a_{ii}$ and $b_{ii}$ are the ARCH and GARCH effects related to variable $i$. However, these same parameters also determine the conditional covariance between the variables. Thus, there is a tension between modeling conditional variances and covariances simultaneously. This problem is also present when $A^{(1)}_X$ and $B^{(1)}_X$ are full matrices.\(^ {9}\)McAleer, Chan, Hoti, and Lieberman (2008) establish consistency and asymptotic normality of the quasi-maximum likelihood estimates of the parameters of the diagonal BEKK-GARCH model.
If $\Xi_t = 0$, the pricing kernel in (5) reduces to the standard exponential-affine kernel. In our conditionally Gaussian economy, the market prices of risks are given by

$$\xi_t = (V_t^Q)^{-1}\mathbb{E}_t^Q(X_{t+1}) - V_t^{-1}\mathbb{E}_t(X_{t+1}),$$

$$\Xi_t = \frac{1}{2} (V_t^{-1} - (V_t^Q)^{-1}),$$

where $\mathbb{E}_t^Q$ and $V_t^Q$ are the conditional mean and variance-covariance matrix given $\mathcal{F}_{t-1}$ under the risk-neutral probability measure. In particular, $X_t|\mathcal{F}_{t-1} \sim \mathcal{N}(\mathbb{E}_t^Q, V_t^Q)$, see Monfort and Pegoraro (2012) for a proof.

It follows from the market prices of risks that if variance-covariance-based risk is priced in the bond market, i.e., when $\Xi_t \neq 0$, then the conditional variance-covariance matrix is different under the physical and risk-neutral probability measures. This discrepancy is feasible in the discrete-time framework in which we are not restricted by Girsanov’s theorem that rules out different volatility specifications across probability measures in continuous-time models. In result, we model a form of unspanned volatility without introducing restrictions as in the traditional USV literature (Collin-Dufresne, Goldstein, and Jones, 2009).

**Structural and Empirical Justification**

In Appendix B, we provide structural justification of the pricing kernel in (5). In particular, we show that the long-run risk model of Bansal and Yaron (2004) results in an exponential-quadratic pricing kernel when solved by the second-order projection developed by Andreasen and Jørgensen (2019). Also, Hansen and Heaton (2008) propose a long-run risk model based on VAR dynamics that results in a second-order pricing kernel when allowing for a time-varying wealth-consumption ratio. Empirically, Roussellet (2017) applies an exponential-quadratic pricing kernel in term structure modeling, which facilitates double-sided inflation fears. In the equity literature, second-order pricing kernels have been considered by Bakshi, Madan, and Panayotov (2010), Christoffersen, Heston, and Jacobs (2013), and Rosenberg and Engle (2002).

### 4.3 Risk-Neutral Dynamics and Bond Pricing

The pricing kernel in (5) results in a high degree of freedom for the specification of risk-neutral dynamics. In particular, both the conditional mean and variance-covariance

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Joslin (2017) argues that traditional USV models restrict the cross-section of yield volatilities in a way that is inconsistent with data.
matrix can be different from those specified under the physical measure in Section 4.1. We utilize this flexibility to specify a model under the risk-neutral measure that is successful at fitting the yield curve while maintaining tractability. A VAR model with constant volatility and of order one satisfy both of these criteria (Joslin, Le, and Singleton, 2013). We also assume that the non-macro and macro risk factors are independent under the risk-neutral measure, which simplifies the estimation procedure. Intuitively, we allow for interdependent dynamics of macro and non-macro risk factors in the actual world, but bond market investors do not price these interactions. In sum, we have assumed the following risk-neutral dynamics of the state vector:

\[
\begin{bmatrix}
x_t \\
m_t
\end{bmatrix}
= \begin{bmatrix}
\mu^Q_x \\
\mu^Q_m
\end{bmatrix} + \begin{bmatrix}
\Phi^Q_x & 0 \\
0 & \Phi^Q_m
\end{bmatrix} \begin{bmatrix}
x_{t-1} \\
m_{t-1}
\end{bmatrix}
+ \begin{bmatrix}
\Sigma^Q_x & 0 \\
0 & \Sigma^Q_m
\end{bmatrix} \begin{bmatrix}
z_{x,t} \\
z_{m,t}
\end{bmatrix},
\]  

(7)

where \(z_{x,t}\) and \(z_{m,t}\) are i.i.d. standard normally distributed errors. We will also make use of the following compact notation:

\[
X_t = \mu^Q_X + \Phi^Q_X X_{t-1} + \Sigma^Q_X z_t,
\]

where \(\Phi_X\) and \(\Sigma_X\) are block-diagonal. The resulting no-arbitrage bond yields are standard and given in the following theorem.

**Theorem 1** Given (6) and (7), no-arbitrage bond yields, defined by \(Y_{t,n} = -n^{-1} \log(P_{t,n})\) are given in closed form by

\[
Y_{t,n} = a_n + b'_{x,n} x_t + b'_{m,n} m_t = b'_{n} X_t,
\]

(8)

where \(a_n = -n^{-1} \tilde{a}_n\) and \(b_{i,n} = -n^{-1} \tilde{b}_{i,n}\) for \(i = \{x, m\}\) are given recursively by

\[
\tilde{a}_{n+1} = -\alpha_X + \tilde{a}_n + \tilde{b}'_{x,n} \mu^Q_x + \tilde{b}'_{m,n} \mu^Q_m + \frac{1}{2} \tilde{b}'_{x,n} \Sigma^Q_x \Sigma^Q_x b_{x,n} + \frac{1}{2} \tilde{b}'_{m,n} \Sigma^Q_m \Sigma^Q_m b_{m,n},
\]

(9)

\[
\tilde{b}'_{x,n+1} = -\beta_x' + \tilde{b}'_{x,n} \Phi^Q_x,
\]

(10)

\[
\tilde{b}'_{m,n+1} = -\beta_m' + \tilde{b}'_{m,n} \Phi^Q_m.
\]

(11)

The recursions are initiated at \(n = 0\) with \(a_0 = 0\), \(b_{x,0} = 0_{n_x \times 1}\), and \(b_{m,0} = 0_{n_m \times 1}\).

**Proof.** Straightforward, see, e.g., Ang and Piazzesi (2003). \(\square\)

The bond pricing equation in (8) clarifies that our model exhibits a form of unspanned volatility because yields do not depend on the conditional covariance-variance matrix of
the state vector, \( V_t \). Conditional variances of yields, however, are given directly in terms of \( V_t \):

\[
\text{Var}_{t-1}(Y_{t,n}) = b_n' V_t b_n.
\]  

(12)

### 4.4 Risk Compensation

Our model distinguishes between two concepts of risk compensation. The first relates to term risk for which term premia are defined by the difference between bond yields and expected future short rates:

\[
\Psi_{t,n} = Y_{t,n} - \Upsilon_{t,n}, \quad \Upsilon_{t,n} = \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{E}_t(r_{t+i}).
\]  

(13)

The following theorem provides term premia and expected future short rates in closed form.

**Theorem 2** Given (3) and the assumptions of Theorem 1, expected future short rates and term premia, respectively, are given by

\[
\Upsilon_{t,n} = a_{n}^{EH} + (b_{n}^{EH})' \mathcal{X}_t,
\]

\[
\Psi_{t,n} = (a_{n} - a_{n}^{EH}) + b_{n}' X_t - (b_{n}^{EH})' \mathcal{X}_t.
\]

where \( \mathcal{X}_t = [X_t, X_{t-1}, \ldots, X_{t-L+1}] \) and

\[
a_{n}^{EH} = \alpha_X + \frac{1}{n} \sum_{i=0}^{n-1} \sum_{j=0}^{i-1} \beta^t_X \Phi^j \mu_X,
\]

\[
b_{n}^{EH} = \frac{1}{n} \sum_{i=0}^{n-1} \beta^m_X \Phi^i.
\]

Here, \( \beta_X = [\beta_X^t, \beta_X^m, 0_{1 \times (p+m)(L-1)}]' \) and \( \mu_X \) and \( \Phi_X \) denote the parameters of the conditional mean associated with the companion form of (3).

**Proof.** Straightforward. □

It follows that

\[
\text{Var}_{t-1}(\Upsilon_{t,n}) = (b_{n}^{EH})' \text{Var}_{t-1}(\mathcal{X}_t) b_{n}^{EH}
\]

\[
\text{Var}_{t-1}(\Psi_{t,n}) = \left( \tilde{b}_n - b_{n}^{EH} \right)' \text{Var}_{t-1}(\mathcal{X}_t) \left( \tilde{b}_n - b_{n}^{EH} \right),
\]

where \( \tilde{b}_n = (b_{n}', 0, 0, \ldots, 0)' \).
The second concept of risk compensation relates to the variance-covariance structure of the yield curve. Similar to Bollerslev, Tauchen, and Zhou (2009), we consider premia for variance risk given by the difference between the expected variance under physical and risk-neutral probabilities:

**Theorem 3** Let the conditional variance-covariance matrix under the physical and risk-neutral measures be given by (4) and

\[
V_Q = \begin{pmatrix}
\Sigma^Q_x (\Sigma^Q_x)' & 0 \\
0 & \Sigma^Q_m (\Sigma^Q_m)'
\end{pmatrix}
\]

Then, the premium for variance risk defined by

\[
\Gamma_{t,n} = E_t (V_{t+2}) - E_Q^t (V_{t+2})
\]

is given by

\[
\Gamma_{t,n} = \sum_{k=1}^{K} \left[ A^{(k)} X V^{(k)} - A^{(k)} V_Q A^{(k)'} \right]
\]

**Proof.** Straightforward.

5 Econometric Method

This section provides a simple approach to estimating our model. We first discuss how we identify latent risk factors that capture non-macro risks. Then, we show that if there exists a linear combination of yields that is observed without measurement errors, then the model can be estimated by separable maximum likelihood problems. The key observations are (i) the latent non-macro risk factors can be rotated into portfolios of yields; and (ii) identifying restrictions can be imposed on the risk-neutral dynamics. Our result corresponds to that developed by Joslin, Singleton, and Zhu (2011), henceforth JSZ, for the GATSM and our proofs make heavily use of their propositions and structure of proofs. Compared to JSZ, however, we allow for time-varying conditional variance in the form given by (4) along with observable and spanned macro risks.

5.1 Identifying Non-Macro Related Risks

We identify latent risk factors, \(x_t\), such that capture risks that are not related to macro fundamentals. To achieve this, we decompose \(a_n\) as defined in Theorem 1 into a component related to the macro risks, \(a_{m,n}\), and another that is orthogonal to macro risks, \(a_{x,n}\). These
are defined by $a_{m,n} = n^{-1}\tilde{a}_{m,n}$, $a_{x,n} = n^{-1}\tilde{a}_{x,n}$ and

$$\tilde{a}_{m,n} = \tilde{a}_{m,n-1} + \tilde{b}_{m,n}^{\mu} + \frac{1}{2} \tilde{b}_{m,n}^{\mu} \Sigma_{m}^{Q} \Sigma_{m}^{Q'} \tilde{b}_{m,n},$$

(14)

$$\tilde{a}_{x,n} = \alpha_{X} + \tilde{a}_{x,n-1} + \tilde{b}_{x,n}^{\mu} + \frac{1}{2} \tilde{b}_{x,n}^{\mu} \Sigma_{x}^{Q} \Sigma_{x}^{Q'} \tilde{b}_{x,n}.$$  

(15)

Then, we define synthetic yields that contain the residual yield variation that the macro risks do not explain by

$$Y_{t,n}^{\perp} = Y_{t,n} - a_{m,n} - b_{m,n}' m_{t}.$$  

(16)

By an application of Theorem 1, these are given endogenously in the model by

$$Y_{t,n}^{\perp} = a_{x,n} + b_{x,n}' x_{t}.$$  

(17)

Thus, the latent non-macro risk factors, $x_{t}$, are extracted from $Y_{t,n}^{\perp}$ rather than $Y_{t,n}$. This is similar to Cieslak and Povala (2015) and Jørgensen (2017).

5.2 Estimation Method

Suppose that we observe yields of bonds with $N$ different maturities given by $n_{1}, \ldots, n_{N}$ periods. Let $Y_{t} = (Y_{t,n_{1}}, \ldots, Y_{t,n_{N}})'$ denote a vector of these observations and let $Y_{t}^{\perp}$ denote the vector of corresponding synthetic yields defined by (16). Analogously, let $a_{x} = (a_{x,n_{1}}, \ldots, a_{x,n_{N}})'$ and $b_{x} = (b_{x,n_{1}}, \ldots, b_{x,n_{N}})'$ be vectors of the bond loadings given by $a_{x,n} = n^{-1}\tilde{a}_{x,n}$ and $b_{x,n} = n^{-1}\tilde{b}_{x,n}$, where $\tilde{a}_{x,n}$ and $\tilde{b}_{x,n}$ are given in (15) and (10), respectively. Define $a_{m}$ and $b_{m}$ along the same lines. We impose the following assumption on the accuracy of yield measurements:

**Assumption 1** Yields, $Y_{t}$, are measured with errors given by $\eta_{t} \sim i.i.d. \ N(0, \sigma^{2}I_{N})$. A part of the measurement error, $\eta_{t}^{\perp} \sim i.i.d. \ N(0, \sigma^{2\perp}I_{N})$, persists in the synthetic yields, $Y_{t}^{\perp}$.

Identifying latent factors from synthetically constructed yields calls for a step-wise approach to estimation. In a first step, we obtain parameters that govern $a_{m,n}$ and $b_{m,n}$, i.e.,

$$\Theta_{m}^{Q} = \{\beta_{m}, \mu_{m}, \Phi_{m}^{Q}, \Sigma_{m}^{Q}\}.$$ 

This allows us to construct $Y_{t,n}^{\perp}$ from (16). Then, using these synthetic yields, we estimate the latent factors and parameters related to $a_{x,n}$ and $b_{x,n}$, i.e.,

$$\Theta_{x}^{Q} = \{\alpha_{X}, \beta_{x}, \mu_{x}, \Phi_{x}^{Q}, \Sigma_{x}^{Q}\},$$ 

along with the joint dynamics of the full state vector $X_{t} = [x_{t}', m_{t}']'$ governed by the parameter vector $\Theta_{X}^{P} = \{\mu_{X}, \Phi_{X}, \Sigma_{X}, A_{X}^{(1)}, \ldots, A_{X}^{(K)}, B_{X}^{(1)}, \ldots, B_{X}^{(K)}\}$.  

15
Step 1: Estimating Synthetic Yields and $\Theta^Q_m$

Since the latent non-macro risk factors capture yield curve variation that is orthogonal to the macro risks, we estimate $\Theta^Q_m$ by maximum likelihood applied to the following model:

$$Y_t = a_m + b'_m m_t + \eta_t, \quad \eta_t \sim \text{i.i.d. } \mathcal{N}(0, \sigma^2).$$

Denoting by $\hat{a}_m$ and $\hat{b}_m$ the loadings computed from the estimate $\hat{\Theta}^Q_m$, we construct synthetic yields by

$$\hat{Y}^\perp_{t,m} = Y_t - \hat{a}_m - \hat{b}'_m m_t.$$

The estimation errors are accounted for by allowing for a measurement error in the observed $Y^\perp_{t,m}$, see Assumption 1.

Step 2: Estimating Latent Non-Macro Risks, $\Theta^Q_x$, and $\Theta^P_X$

The remainder of the estimation problem can be approached by maximum likelihood estimation combined with a Kalman filter and parameter restrictions for econometric identification. This approach is, however, cumbersome because the model encounters non-linearities both in the conditional variance-covariance matrix, $V_t$, and in the loading recursions, $a_x$ and $b_x$. In the remainder of this section, we provide a much simpler approach that separates the likelihood problem such that non-linearities in the physical dynamics are treated separately from the non-linear bond yield equation. Furthermore, our approach avoids Kalman filtering and hence the numerical stability issues that are likely to be associated with filtering methods. For the remainder of this section, we follow the structure of JSZ closely.

5.3 Rotation and Identifying Restrictions

Adopting the JSZ method requires a model that is invariant to affine transformations of the latent factors given by $\rho_t = c_x + C_x x_t$. This is a non-trivial property that fortunately is satisfied by our model specification as shown in Lemma 1 in appendix. Therefore, we can rotate the latent factors into linear combinations, or portfolios, of the synthetic yields, $Y^\perp_{t,n,m}$. This rotated model is unique given a choice of portfolio weights.

**Theorem 4** Any model defined by (3)-(7) and (17) is observationally equivalent to a unique model in which the latent factors are portfolios of synthetic yields given by $\rho_t =$
\( WY_t^{\perp m} \), where the weighting matrix \( W \) has full rank. Let \( P_t = (\rho_t', m_t')' \) denote the resulting observable state vector. Then, this unique model is given by

\[
\begin{align*}
P_t &= \mu_P + \sum_{l=1}^L \Phi_P^{(l)} P_{t-l} + \varepsilon_{P,t}, \quad \varepsilon_{P,t} = V_t^{1/2} z_{P,t}, \\
V_t &= \Sigma_P \Sigma_P' + \sum_{k=1}^K A_P^{(k)} \varepsilon_{t-1}' A_P^{(k)} + \sum_{k=1}^K B_P^{(k)} V_{t-1} B_P^{(k)'}, \\
P_t &= \mu_P^Q + \Phi_P^Q P_{t-1} + \Sigma_P^Q z_{P,t}, \\
r_t &= \alpha_P + \beta'_m m_t + \beta'_\rho \rho_t
\end{align*}
\]

with \( z_{P,t} \sim i.i.d. \mathcal{N}(0, I_{n_X}) \) and \( z_{P,t}^Q \sim i.i.d. \mathcal{N}(0, I_{n_X}) \).

**Proof.** See Appendix C.1. \( \square \)

**Remark 1** Given the rotation in Theorem 4, the model-implied synthetic yields are given by

\[
Y_t^{\perp m} = a_\rho + b_\rho \rho_t,
\]

where \( a_\rho \) and \( b_\rho \) are defined by Theorem 1 given the rotated parameters in (18)-(21).

The next step is to impose parameter restrictions such that the model is identified. In a model with constant conditional variance-covariance matrix under the physical measure, these restrictions can be imposed on the risk-neutral dynamics only (JSZ). We adopt these restrictions as formalized in part (i) of the next theorem. In our model, however, we need additional restrictions for identifying the physical dynamics. These are given by part (ii) of the theorem. To state the theorem, we explicitly express the bond yields loadings \( a \) and \( b \) in terms of the parameters that they depend on, i.e., \( a_x = a(\alpha_x, \beta_x, \mu_x^Q, \Phi_x^Q, \Sigma_x^Q) = (a_{x,n_1}, a_{x,n_2}, \ldots, a_{x,n_N})' \), and \( b_x = b(\beta_x, \Phi_x^Q) = (b_{x,n_1}, b_{x,n_2}, \ldots, b_{x,n_N})' \).

**Theorem 5** The unique model defined by (18)-(21) in Theorem 4 and (17) is identified with the following parametrizations:

(i) The risk-neutral dynamics and the short-rate equation are uniquely parametrized by \( \Theta_{\rho}^Q = \{k_{\infty}^Q, \lambda^Q, \Sigma_{\rho}^Q\} \), where \( \lambda^Q \) contains ordered eigenvalues of the transformed risk-neutral autoregressive coefficient matrix given by \( (Wb_x)^{-1} \Phi_{\rho}^Q (Wb_x) \), c.f., Lemma 1 in appendix. Define \( e_1 = (1, 0, \ldots, 0)' \) and \( J(\lambda^Q) = \text{diag}(J_1^Q, J_2^Q, \ldots, J_D^Q) \), where
each \( J^Q_d, d = 1, 2, \ldots, D \) are Jordan blocks. Also, let \( a_W = Wa(0, \iota, k^Q_{\infty}e_1, J(\lambda^Q), b_W^{-1}\Sigma^Q_{\rho}) \) and \( b_W = Wb(\iota, J(\lambda^Q)) \). In particular,

\[
\begin{align*}
\alpha_p & = -\beta'_p a_W, \\
\beta_p & = (b_W^{-1})', \\
\mu^Q_{\rho} & = k^Q_{\infty}b_We_1 + (I_{n_\rho} - \Phi^Q_{\rho}) a_W, \\
\Phi^Q_{\rho} & = b_W J(\lambda^Q) b_W^{-1},
\end{align*}
\]

where \( \Sigma^Q_{\rho} \) is lower triangular with strictly positive elements on the diagonal.

(ii) For identification of the conditional variance-covariance matrix under the physical measure, we restrict the diagonal of \( \Sigma_P \) to be strictly positive and the first entries of \( A^{(k)}_P \) and \( B^{(k)}_P \) for \( k = 1, \ldots, K \) to be non-negative. Finally, the feasible generalization of the BEKK-GARCH model is bounded by the condition \( K \leq \text{floor}\left(\frac{1}{2}n_X + \frac{1}{2}\right) \) to achieve identification.

**Proof.** See Appendix C.2. \( \square \)

### 5.4 Marginal Likelihood for Synthetic Yields

As in JSZ, we impose the following additional assumption.\(^{11}\)

**Assumption 2** The portfolios of synthetic yields, \( \rho_t \), balance out the measurement errors such that \( \rho_t \) and hence the rotated state vector, \( \mathcal{P}_t \), are observed.

Then, the marginal log-likelihood can be separated by

\[
\log \ell \left( Y_{1:m}^{\perp} | Y_{t-1:m}^{\perp}; \Theta^p_P, \Theta^Q_{\rho} \right) = \log \ell_P \left( \mathcal{P}_t | \mathcal{P}_{t-1}; \Theta^p_P \right) + \log \ell_Y \left( Y_{1:m}^{\perp} | \mathcal{P}_t; \Theta^Q_{\rho} \right).
\]

Notice that the two terms of the log-likelihood contribution do not depend on the same parameters. Therefore, \( \Theta^p_P \) and \( \Theta^Q_{\rho} \) can be estimated by solving two unrelated maximum likelihood problems. The marginal log-likelihood function related to the parameters of the physical dynamics is given by

\[
\begin{align*}
\log \ell_P \left( \mathcal{P}_t | \mathcal{P}_{t-1}; \Theta^p_P \right) & \propto -\frac{1}{2} \log |V_t| - \frac{1}{2} \left( \mathcal{P}_t - \mu_P - \sum_{l=1}^{L} \Phi^{(l)}_P \mathcal{P}_{t-l} \right)' V_t^{-1} \left( \mathcal{P}_t - \mu_P - \sum_{l=1}^{L} \Phi^{(l)}_P \mathcal{P}_{t-l} \right) \\
& \quad \left(23\right)
\end{align*}
\]

\(^{11}\)JSZ shows that the parameter estimates are not seriously affected by this assumption in case of the GATSM.
where we recall that $V_t$ depends on the parameters and data through (19). For the remaining parameters that determine the cross-section of synthetic yields, the log-likelihood contribution is given by

$$\log \ell_Y(Y_t^\perp | P_t; \Theta^Q_\rho) \propto -\frac{1}{2\sigma^2_{\perp m}} (Y_t^\perp - a - bP_t)' (Y_t^\perp - a - bP_t),$$

(24)

where $a = a(\alpha_P, \beta_{\rho}, \mu^Q_\rho, \Sigma^Q_\rho)$ and $b = b(\beta_{\rho}, \Phi^Q_\rho)$ with $\alpha_P, \beta_{\rho}, \mu^Q_\rho$, and $\Phi^Q_\rho$ given as explicit functions of $\Theta^Q_\rho = \{k^Q_\infty, \lambda^Q, \Sigma^Q_\rho\}$ in Theorem 5. It is noteworthy that the log-likelihood function in (24) is identical to that of the GATSM. Thus, this part of the estimation is no more difficult to implement than the GATSM, which is striking because GATSMs are celebrated precisely due to their tractability.

6 Data and Estimation Results

6.1 Data

We apply zero-coupon bond yields of U.S. Treasury bonds provided by Gürkaynak, Sack, and Wright (2007) at the monthly frequency with end-of-month observations. We consider a long sample from September 1971 to June 2019, which contains both periods with extremely volatile yields, e.g., in the beginning of the 1980s, and periods with little volatility yields, e.g., before the outbreak of the Great Recession and during the zero-lower bound regime. We include maturities of 1, 2, 3, 5, 7, and 10 years. To capture the short end of the yield curve, we also use the 3- and 6-month Treasury bill rates from Federal Reserve Economic Data (FRED).

Following the macro-finance literature, we focus on two sources of macro risks: inflation and economic activity. There is no consensus in the literature about which specific variables to use as measures of these risks.\footnote{For example, Ang and Piazzesi (2003) construct factors from principal components of a wide range of macro variables; Joslin, Priebsch, and Singleton (2014) use expected inflation measured from surveys along with the Chicago Fed National Activity Index (CFNAI), which is an estimate of overall economic growth; Bikbov and Chernov (2010) use CPI inflation and the Help Wanted Advertising in Newspapers index.} What is more, economic activity has been defined by both growth and slack measures. Bauer and Rudebusch (2016) argue that these series are widely different and uncorrelated over the business cycle. To be consistent with empirical monetary policy rules, Bauer and Rudebusch (2016) suggest that slack variables are appropriate measures of economic activity in the context of yield curve modeling. Moreover, they show that the unemployment gap is related to the slope of the

\[ \text{19} \]
yield curve. Also, Bean (2005) notes that "an assessment of the slack in the economy is an essential ingredient" in monetary policy decision making.

In light of these experiences, we adopt the unemployment gap (UGAP) as our measure of economic activity. We construct UGAP by the difference between the unemployment rate from the U.S. Bureau of Labor Statistics (BLS) and the estimate of natural unemployment from the Congressional Budget Office (CBO). We measure inflation by year-over-year growth in the headline CPI index from BLS.

The data are exhibited in Figure 1 and descriptive statistics are detailed in Table 2. The yield data are highly persistent for all maturities. They are upward-sloping on average with a decreasing term structure standard deviations. Yield curve levels peak in the beginning of 1980 and reach the zero-lower bound in the aftermath of the Great Recession. The unemployment gap is weakly and negatively correlated with the yield curve, while the correlations between inflation and yields are higher and positive. The data are highly cyclical: yields, particularly for short-term maturities, and inflation decrease in recessions and the unemployment gap increase in recessions. The unemployment gap has decreased steadily during the current expansion.

[Figure 1 and Table 2 around here.]

6.2 Model Specification

Non-Macro Risks: The non-macro risk factors are latent and ensures that the model can match the yield curve data with satisfactory precision. Since Litterman and Scheinkman (1991), the literature widely agrees that three latent factors are sufficient for modeling the yield curve. The macro risks can potentially reduce the number latent factors necessary for capturing the residual variation in the synthetic yields. However, we find that three latent factors remain necessary. One explanation for this is that, as we showed in Section 3 and shall elaborate on further in Section 7, there are periods in our sample in which the macro risks explain practically no variation of the yield curve.

As shown in Theorem 4, the latent factors in our model can be rotated into observed portfolios of synthetic yields, \( \rho_t = WY_t^\perp \), where \( W \) has full rank. We choose \( W \) such that the yield portfolios correspond to the first three principal components of the synthetic yield curve. This choice ensures that the portfolios are orthogonal and thus span the entire three-dimensional subspace.

\(^{13}\)The natural unemployment estimate is available at the quarterly frequency only. We assume that the natural unemployment rate is constant in-between quarters.
Lag Length Under the Physical Measure: The order of the VAR model under the physical measure determines the lag length of the macro risks. We implement our model with an order of $L = 12$ corresponding to an annual lag structure, which is a common choice in the macro-finance term structure literature.\footnote{See, for instance, Ang, Dong, and Piazzesi (2007), Ang and Piazzesi (2003), Bikbov and Chernov (2010), and Jardet, Monfort, and Pegoraro (2013).}

Generality of the Conditional Variance-Covariance Matrix Under the Physical Measure: With three non-macro risks and two macro risks, we have $N_X = 5$ risk factors in total, which identifies up to $K = 3$ components in the conditional variance-covariance matrix. As already discussed, $K = 1$ does not give the model sufficient freedom to capture both conditional variances and covariances. In contrast, we find that the model can match empirical proxies of yield curve volatility closely with $K = 2$, see Section 6.3. Given this result, and to avoid the curse of dimensionality in the number of parameters, we do not consider the option of using $K = 3$.

6.3 Estimation Results

Estimated Parameters and Non-Macro Risks
Tables 3 and 4 report estimates of parameters related to the conditional mean and variance-covariance matrix under the physical measure. Parameter estimates related to bond pricing are shown in Table 5. All parameters are highly significant by conventional critical values.

The model is highly persistent under both the physical and risk-neutral measures with maximum eigenvalues of the autoregressive coefficients of respectively 0.995 and 0.997. The conditional variance-covariance matrix of the physical dynamics is persistent as well as captured by the estimated values of $B^{(1)}_P$ and $B^{(2)}_P$. However, the conditional variance-covariance specification remains stationary because

$$\max \left[ \sum_{k=1}^K (A^{(k)}_P \circ A^{(k)}_P + B^{(k)}_P \circ B^{(k)}_P) \right] = 0.981 < 1,$$

where $\circ$ denotes element-wise Hadamard product.\footnote{The stationarity condition for the BEKK-GARCH model is given in Engle and Kroner (1995).}

The estimated latent non-macro risks are exhibited in Figure 3 along with the factor loadings $b_{x,n}$ across maturities $n$. The factor loadings clearly shows that the non-macro risks capture respectively level, slope, and curvature risks of the synthetic yields, $Y_t^{\perp,m}$. This is consistent with the finance term structure literature, which agrees that latent
factors capture the level, slope, and curvature of the yield curve. However, as illustrated in the figure, the estimated non-macro risks are highly different from the usual latent yield curve factors, especially for the level and slope risks. This result shows that macro risks explain variation related to the level and slope of the yield curve. This is consistent with Rudebusch and Wu (2008), who show that the level and slope risks have important macroeconomic underpinnings.

[Figure 3 around here.]

**Goodness of In-Sample Fit**

The in-sample fit to the yield curve is summarized by root mean squared errors (RMSEs) in Table 6. The model fits the data well with an average RMSE across maturities of 8 basis points. This result is non-surprising given that the risk-neutral dynamics of our model correspond to the GATSM, which is known to have good in-sample fitting properties.

[Table 6 around here.]

**Matching Conditional Volatilities**

The reliability of our analysis of yield curve variation depends on the ability of our model to match conditional volatilities of bond yields. As conditional volatilities are unobserved, we apply two proxies for evaluating our model. First, we construct a realized variance-covariance matrix using daily bond yield data available from FRED and Gürkaynak, Sack, and Wright (2007). This measure is likely to be a good proxy of conditional volatility, because realized volatility is a consistent estimator of integrated volatility (Barndorff-Nielsen and Shephard, 2004), and conditional volatility can be viewed as a noisy measure of integrated volatility. As a second proxy, we use rolling conditional volatilities constructed using daily data with a 6-month look-back.

Figure 4 shows model-implied conditional variances and covariances along with the corresponding realized and rolling measures for the 1-, 5-, and 10-year maturities.

---

16We construct the realized variance-covariance matrix as follows. Let $y_{t,s,n}$ denote the $n$-maturity yield on day $s$ in month $t$ and let $S$ denote the total number of daily observations in month $t$. Then, we define the realized covariance between the $n_1$- and $n_2$-period bond yields observed by the end of month $t$ by

$$RV_{t,n_1,n_2} = \sum_{s=1}^{S} (y_{t,s,n_1} - y_{t,s-1,n_1})(y_{t,s,n_2} - y_{t,s-1,n_2}).$$

Our model captures these empirical measures closely both in the highly volatile period in the beginning of the 1980s and in periods with low volatility, e.g., as observed during the zero-lower bound regime. The model does, however, not reproduce the high degree of variability observed in the empirical volatility proxies. In particular, the empirical volatility proxies exhibit bursts that are overlooked by our model. The most pronounced cases of this occurs in the high-volatility period, but also during the Great Recession. One explanation for these deviations are given by Andersen, Bollerslev, and Meddahi (2005), who note that realized variance is subject to measurement errors that overstate the variability compared to the true integrated variance. Also, Cieslak and Povala (2016) conclude that the positive outliers of realized variance should not be predictable ex ante. Therefore, we do not interpret these deviations from empirical proxies as a sign of misspecification of our model.

Term Structure Decomposition

We are not only interested in the role of macro risks in the yield curve, but also how macro risks shape the decomposition of yields into expected future short rates and term premia, see (13). The estimated decomposition is shown for the 5-year and 10-year maturities in Figure 5.

The expected short rates mimic the yield levels due to the high degree of persistence in the estimated physical dynamics. This implies term premia that are relatively stable. Aligned with a priori intuition, our term premia estimates are highly countercyclical as investors demand increasing compensation for risk in times of crisis. Also, the term premia are generally higher for the 10-year bond compared to the 5-year maturity.

High term premia signal that bond market investors fear risk or that the amount of uncertainty in the market is high. On the other hand, high expected short rates reflect an optimism about future economic conditions. Therefore, we expect that these measures are primarily negatively correlated. Our model provides time-varying correlations of the term structure decomposition, as shown in Figure 6.
Indeed, we estimate negative correlations between term premia and expected future short rates throughout a majority of the sample. The sign, however, tends to reverse during expansions, which may reflect the fact that mid-expansions are typically characterized by a flattening yield curve and low volatilities. The correlation tends to decrease further during recessions, which is consistent with the consensus that monetary policy is most effective in recessions.

7 Time-Varying Role of Macro Risks

This section contains our main results. We begin by showing that the role of macro risks in the yield curve varies over time and we provide insights into the sources of this variation. In particular, we look at variations in expected short rates and term premia. Then, we evaluate how macro shocks affect both levels and volatilities of bond yields. Finally, we relate model-implied variance risk premia to the macroeconomy.

7.1 Macro Ratios of Bond Yield Volatility

We analyze the importance of macro risks for explaining bond yield variation by decomposing the conditional yield volatility into macro and non-macro contributions. For this purpose, we apply a recursive identification scheme assuming that macro risks are slow-moving and do not react to contemporary non-macro shocks. Furthermore, we assume that the unemployment gap responds to inflation shocks with a time lag of one month. These assumptions are standard in the structural VAR literature with Choleski identification schemes (Bernanke, Boivin, and Eliasz, 2005, Christiano, Eichenbaum, and Evans, 1999, Stock and Watson, 2001). Our approach is also standard in the macro-finance term structure literature (Creal and Wu, 2017, Diebold and Li, 2006, Wu and Xia, 2016), in which more sophisticated identification methods are not yet widespread.

Given these assumptions, conditional yield volatility can be decomposed into variation generated by macro and non-macro risks. Moreover, the macro contribution can be decomposed into an inflation and an unemployment-gap component:

\[
\text{Var}_{t-1}(Y_{t,n}) = \text{Var}_{t-1}(\epsilon_{t}^{\text{macro}}) + \text{Var}_{t-1}(\epsilon_{t}^{\text{non-macro}}) = \text{Var}_{t-1}(\epsilon_{t}^{\text{INF}}) + \text{Var}_{t-1}(\epsilon_{t}^{\text{UGAP}}) + \text{Var}_{t-1}(\epsilon_{t}^{\text{non-macro}}).
\]

18This relationship has been established by Norland (2018) using the VIX index as a measure of volatility. While the VIX index relates to equity markets, it is positively correlated with volatility measures related to bond markets, e.g., the Merill-Lynch Option Implied Volatility Estimate (MOVE).
This decomposition implicitly define macro, comprising inflation and unemployment gap, and non-macro variance ratios by

\[
1 = \frac{\text{Var}_{t-1}(\epsilon_t^{\text{macro}})}{\text{Var}_{t-1}(Y_{t,n})} + \frac{\text{Var}_{t-1}(\epsilon_t^{\text{non-macro}})}{\text{Var}_{t-1}(Y_{t,n})} \\
= \frac{\text{Var}_{t-1}(\epsilon_t^{\text{INF}})}{\text{Var}_{t-1}(Y_{t,n})} + \frac{\text{Var}_{t-1}(\epsilon_t^{\text{UGAP}})}{\text{Var}_{t-1}(Y_{t,n})} + \frac{\text{Var}_{t-1}(\epsilon_t^{\text{non-macro}})}{\text{Var}_{t-1}(Y_{t,n})}.
\]

The novelty of our analysis is that the variance ratios are time-varying due to the time-varying conditional volatility. We characterize the time series properties of the term structure of macro variance ratios in Table 7.

Uncovering dynamic aspects of macro variance ratios uncovers new knowledge on the linkages between the yield curve and the macroeconomy. The fraction of bond yield variation explained by macro risks is highly time-varying and volatile. Indeed, standard deviations of the macro variance ratios are almost equal to the average values. For the ten-year maturity, the macro variance ratio ranges between 0 and 54 pct. In other words, in some periods, non-macro risks fully account movements in the ten-year yield, whereas macro risks explain more than half of the variation in other periods. This pattern is consistent across maturities.

The macro variance ratios are autocorrelated, which is consistent with the persistent behavior of yields. However, the autocorrelation is short-lived. The sample kurtosis along with the quantiles suggest that there are outliers in both the upper and lower tails. These are likely to be linked with macroeconomic announcement effects, which are known to impact yield curve variation (Andersen, Bollerslev, Diebold, and Vega, 2007, Bollerslev, Cai, and Song, 2000, Fleming and Remolona, 1999, Johannes, 2004, Piazzesi, 2005, Pierluigi Balduzzi and Green, 2001).

We plot the macro variance ratios for the 3-month and 10-year maturities in the left panel of Figure 7.

These charts confirm the properties described above. To abstract from some of the noise in the macro variance ratios, the right panel shows the 12-month moving average of these series. Even when smoothed over annual periods, the time-variation remains with large spikes that tend to occur in periods of expansion or early recovery. If monetary policy
makers follow a Taylor rule, this result is consistent with the widespread finding that an accommodative monetary policy is more effective than a contractive policy (Angrist, Jordá, and Kuersteiner, 2017, Barnichon and Matthes, 2018, Tenreyro and Thwaites, 2016). We observe a U-shaped pattern of the variance ratios related to the 3-month yield. In particular, macro risks were important predictors of short-term yields in the 1970s and 1980s, but ceased to be relevant during the Great Moderation. These results are highly intuitive and non-surprising as the stabilization of macro fundamentals reduce the extent to which these can explain volatility in financial markets. In the aftermath of the Great Recession, however, the link between macro risks and short-term yields has strengthened.

To understand this recent trend, the right panel of Figure 7 shows the decomposition into risks related to inflation and the unemployment gap. Inflation risk has the largest impact on yield curve movements and generates the majority of the spikes in the total macro variation ratios. The unemployment gap, however, is becoming increasingly important for explaining variation in the short end of the yield curve. In particular, the recent strengthening of the macro-finance linkage in short-term yields can be attributed to the unemployment gap. This trend coincides with the unemployment rate slowly recovering from the Great Recession.

Intuitively, there must be stronger link between the yield curve and macro fundamentals over longer forecasting horizons. To test this hypothesis, we show the 12-month and 10-year ahead variance ratios, defined by

$$1 = \frac{\text{Var}_{t-h}(\epsilon_t^{\text{INF}})}{\text{Var}_{t-h}(Y_{t,n})} + \frac{\text{Var}_{t-h}(\epsilon_t^{\text{UGAP}})}{\text{Var}_{t-h}(Y_{t,n})} + \frac{\text{Var}_{t-1}(\epsilon_t^{\text{non-macro}})}{\text{Var}_{t-h}(Y_{t,n})}.$$ 

Figure 8 shows these variance ratios for forecasting horizons of $h = \{12, 120\}$.

Indeed, the levels of macro variance ratios are increasing with the horizon and stabilize around 40% across all maturities at the 10-year forecasting horizon. At long forecasting horizons, inflation and the unemployment gap determine equal amounts of the yield curve variation. Due to the persistence of the conditional variance-covariance dynamics, the time-variation persists even for the 10-year horizon. The time-variation is particularly pronounced during deep recessions, where yield curve movements are dominated by non-macro risks.
7.2 Insights from the Term Structure Decomposition

The link between yield curve variation and macro risks can be generated through two channels, namely expected short rates and term premia, c.f., the term structure decomposition in (13). Figure 9 illustrates macro variance ratios relating to this decomposition, while Tables 8 and 9 exhibit descriptive statistics.

In the upper panel of Figure 9, we present the contribution of macro risks to variation in expected future short rates. The lower panel shows macro variance ratios for variation in the term premium. We focus on the long end of the yield curve as term premia are negligible for short-term yields. The macro variance ratios related to the decomposition unfold important patterns that are not detectable from the analysis of total yield variation.

First, macro risks explain more variation of the path of expected future short rates and term premia than in the current yield curve. The range of macro variance ratios for the 10-year short-rate expectations is 0.4 to 76 pct with an average and median of 42 pct. In addition, the standard deviation and kurtosis are smaller than those of the total bond yields. Similar properties are observed for macro variance ratios associated with term premia. On average, however, we find that macro risks explain more variation in expected short rates than in term premia. In result, macro risks seem to be related to the yield curve through the expectations hypothesis, i.e., the transmission of monetary policy from short- to long-term yields.

Second, there is a clear trend in the macro variance ratios that is upward-sloping throughout our sample in case of the short-rate expectations, and downward-sloping for the term premia. Thus, the transmission mechanism is becoming an increasingly important channel for how macro risks shape the yield curve. Due to these trends, an unconditional approach to evaluating the importance of macro risks in the yield curve will be misleading with serious biases toward the beginning and end of the sample.

Third, the increase in importance of the unemployment gap after the Great Recession stems from the short-rate expectations rather than term premia. Interestingly, this shift coincide with the Federal Reserve adopting an official inflation target. Our results could indicate that this shift in policy lowered the uncertainty about long-term ahead inflation, hence, the extent to which inflation risk affects long-horizon short-rate expectations. This mechanism is not trivial due to the dual mandate of the Federal Reserve that involve both the inflation target and maintaining minimum stable unemployment.
Finally, at the outbreak of the Great Recession, the macro variance ratio of the short-rate expectations dips from approximately 70 to 0 pct. This extreme decrease begins by the default of Bear Stearns in March 2008 and also subsumes the default of Lehman Brothers in September 2008. It is intuitively appeal that macro fundamentals did not play an important role for expectations in financial markets during this period of high financial distress.

7.3 The Impact of Macro Shocks

We have shown so far that macro risks explain variation in the yield curve and that the amount of variation explained varies greatly over time. To shed further light on this phenomenon, we ask, how do bond yields respond to macro shocks? We consider the response of both the level and volatility of bond yields.

Impulse Responses of Yield Levels

Figure 10 depicts impulse response functions of the 3-month and 10-year yields following a one standard deviation shock to respectively inflation and the unemployment gap. Due to the time-varying conditional variance-covariance matrix, there is an impulse response function for each month in our sample. To summarize the overall distribution, we show the median along with the 10, 25, 75, and 90 pct quantiles.

Inflation shocks increase yields levels, while shocks to the unemployment gap decrease yields. These responses are consistent with the Taylor rule and the transmission of monetary policy from short- to long-term yields. For a small part of the sample, the impact of a shock to inflation is negative. The adjustment paths are humped-shaped and the impact of the shocks generally persists for many periods, which reflects the high persistency of yields. The quantiles of the impulses responses are parallel implying that the amount of volatility affects the level, but not the shape, of the responses.

Impulse Responses of Yield Volatilities

To analyze how yield volatilities respond to macro shocks, we apply the concept of conditional moment profiles defined by Gallant, Rossi, and Tauchen (1993).\textsuperscript{19} In particular, we

consider differences between volatility forecasts given the past with an additional shock \( e \) versus the actual past:

\[
\text{Var} \left( Y_{t+h} | \varepsilon_t + e, V_t \right) - \text{Var} \left( Y_{t+h} | \varepsilon_t, V_t \right)
\]

for \( h = 1, 2, \ldots \). Figure 11 summarizes conditional moment profiles for a one standard deviation shock to respectively inflation (upper panel) and unemployment gap (lower panel). The figure report results for volatilities of the 3-month and 10-year yields.

The median one standard deviation inflation shock raises inflation uncertainty by 25 annual basis points. Over time, this shock translates into a slight increase in yield variances of a few basis points that persists due to the highly persistent variance-covariance dynamics. Inflation shocks can, however, generate much larger effects. At the 90 pct quantile of the distribution of conditional moment profiles, a one standard deviation inflation shock increases inflation uncertainty about 120 bps, which has an immediate positive impact on the yield curve variances of 10 to 15 bps. At the short end of the yield curve, the inflation shock causes further increases in the conditional variance which peaks at 25 bps after about 5 years. The impact of shocks to the unemployment gap is much lower and dies out at a faster rate.

The conditional moment profiles show that macro shocks can have both positive and negative effects on yield curve volatility. Strikingly, the impact of unemployment-gap shocks is almost symmetric around zero. In case of inflation shocks, the impact is mostly positive, but the distribution is extremely wide with differences between the 90 pct and 10 pct quantiles as high as 50 bps. Thus, it is difficult to predict the effect of macro shocks on yield curve volatility.

**7.4 Variance Risk Premia**

Finally, we analyze how macro risks impact variance risk premia, which are given endogenously by our model as detailed in Theorem 3. Although our model results in time-varying variance risk premia, we abstract from analyzing these dynamics because they, by construction of our model, are given solely by the specification of volatility under the physical measure. Instead, we focus on the average variance risk premia over time.

Following Carr and Wu (2009), we report average variance risk premia as ratios of expected variance under the physical measure, i.e., the sample average of \( \Gamma_{t,n} \) as a fraction of the sample average of \( E_t(V_{t+2}) \), see the left chart in Figure 12.
As expected, the variance risk premia are negative across all maturities, which is consistent with the stylized fact that implied volatilities exceed realized volatilities. The sizes of are variance risk premia range between 30 to 70 pct of the expected variance. This ratio decreases in magnitude across maturities with a small hump at the very short end of the term structure. Thus, bond market investors pay a premium for protection against variance risk, particularly in the short end of the yield curve. These result are aligned with existing estimates of variance risk premia in Treasury bond markets (Choi, Mueller, and Vedolin, 2017, Trolle, 2009, Trolle and Schwartz, 2015).

The right chart in Figure 12 decomposes the variance risk premia into macro- and non-macro contributions. A majority of the variance risk premia, between 80 and near 100 pct, are related to macro uncertainty. This share is increasing in maturities as the size of the variance risk premium decreases. Thus, bond market investors demand large compensations for being exposed to macro uncertainty, whereas uncertainty related to non-macro risks is less important for variance risk premia. This finding is consistent with Cieslak and Povala (2016), who find that variance risk premia in Treasury bonds are particularly related to uncertainty about future monetary policy.

8 Conclusions

Using a novel macro-finance term structure model with time-varying second moments, we established that the role of macro risks in the U.S. Treasury yield curve varies over time. In particular, macro variance ratios range between 0 and 56 pct with large month-to-month changes. Macro risks explain most variation during expansions and early recessions. Moreover, we discovered a U-shaped trend in macro variance ratios, implying that the macro risks were important yield curve factors in the 1970s and 1980s, became less relevant during the Great Moderation, but has regained importance after the Great Recession. Macro risks primarily impact Treasury bond markets through the expectations that investors form about the future path of monetary policy. Finally, we showed that model-implied variance risk premia can be almost fully attributed to uncertainty about macro risks.

Our empirical results imply that non-macro risks account for significant shares of bond yield variation, especially during recessions. A natural way forward is to identify
these risks, e.g., by measures of financial distress or risk aversion, which are likely to affect bond yields in recessions.
References


## A Overview of the Literature

**Table 1:** Overview of the Literature

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<thead>
<tr>
<th>Maturity</th>
<th>Variance ratio</th>
</tr>
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<tbody>
<tr>
<td><strong>Ang and Piazzesi (2003)</strong></td>
<td></td>
</tr>
<tr>
<td>Inflation</td>
<td></td>
</tr>
<tr>
<td>1-month</td>
<td>67 pct</td>
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<tr>
<td>1-year</td>
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<tr>
<td>5-year</td>
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<td>Real activity</td>
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<td>1-month</td>
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<tr>
<td>1-year</td>
<td>11 pct</td>
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<tr>
<td>5-year</td>
<td>1 pct</td>
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<td><strong>Ang, Bekaert, and Wei (2008)</strong></td>
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<td>Expected inflation</td>
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<td>3-month</td>
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<tr>
<td>5-year</td>
<td>81 pct</td>
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<td><strong>Diebold, Rudebusch, and Aruoba (2006)</strong></td>
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<td>Inflation</td>
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<td>1-year</td>
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<tr>
<td>5-year</td>
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<td>Capital Utilization</td>
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<td>5-year</td>
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<th>Variance ratio</th>
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<td>Short-horizon expected inflation</td>
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<td>1-year</td>
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<td><strong>Feunou, Fontaine, and Roussellet (2019)</strong></td>
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<td>Macro news</td>
<td>3-month</td>
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<td></td>
<td>5-year</td>
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<td>3-month</td>
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</tr>
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B Structural Justification of Second-Order Pricing Kernels

We provide two examples from the equilibrium asset pricing literature that results in second-order pricing kernels. In both examples, the pricing kernel is determined by the marginal rate of substitution of a representative household with Epstein and Zin (1989), Kreps and Porteus (1978) and Weil (1990) recursive preferences.

B.1 Long-Run Risk Model with Second-Order Projection Solution

Consider an indirect utility function given by

$$U_t = \frac{1}{1 - \frac{1}{\psi}} C_t^{1-\frac{1}{\psi}} + \beta \left( \mathbb{E}_t \left[ U_{t+1}^{1-\alpha} \right] \right)^{\frac{1}{1-\alpha}}$$

The pricing kernel is given by

$$M_{t+1,1} = \beta \exp(-r_t) \left( \frac{C_{t+1}}{C_t} \right)^{-\frac{1}{\psi}} \left( \frac{U_{t+1}}{\mathbb{E}_t (U_{t+1}^{1-\alpha})} \right)^{-\alpha}$$

Bansal and Yaron (2004) specify a model with a long-run growth factor $g_t$ and a volatility factor $\sigma_t$. The state vector also includes log consumption, $c_t = \log C_t$. The state dynamics are:

$$\Delta c_{t+1} = \mu_c + x_t + \sigma_c \sigma_{t} z_{c,t+1},$$

$$g_{t+1} = \rho_g g_t + \sigma_g \sigma_{t} z_{g,t+1},$$

$$\sigma_{t+1}^2 = 1 - \rho_\sigma + \rho_\sigma \sigma_t^2 + \sigma_\sigma \varepsilon_{\sigma,t+1}$$

where $z_{c,t+1} \sim \text{i.i.d. } \mathcal{N}(0,1)$, $z_{g,t+1} \sim \text{i.i.d. } \mathcal{N}(0,1)$, $z_{\sigma,t+1} \sim \text{i.i.d. } \mathcal{N}(0,1)$.

Bansal and Yaron (2004) propose an analytical solution using a log-linearization. This approximation results in an exponential-linear pricing kernel. However, Pohl, Schmedders, and Wilms (2018) argue that a first-order approximation is likely to generate large numerical errors. In response, Andreasen and Jørgensen (2019) solve the model using a second-order projection in which

$$u_t = \gamma_0^u + \gamma_X^u X_t + X_t' \gamma_{XX}^u X_t,$$

$$\tilde{u}_t = \gamma_0^\tilde{u} + \gamma_X^\tilde{u} X_t + X_t' \gamma_{XX}^\tilde{u} X_t$$
where \( u_t = \log \mathcal{U}_t \), \( \tilde{u}_t = \log \mathbb{E}_t \left( \mathcal{U}_{t+1}^{1-\rho} \right) \), and \( X_t = \left[ c'_t, g'_t, \sigma_t^2 \right]' \). The implied log pricing kernel is then given by

\[
\log \mathcal{M}_{t+1,1} = \log \beta - r_t - \frac{1}{\psi} \Delta c_{t+1} - \alpha u_{t+1} + \alpha \tilde{u}_{t+1} \\
\propto \xi' X_{t+1} + X'_{t+1} \Xi X_{t+1}
\]

with \( \xi = -\frac{1}{\psi} - \alpha \gamma X + \alpha \tilde{\gamma} X \) and \( \Xi = \gamma^2 X X + \tilde{\gamma}^2 X X \). This pricing kernel is a special case of (5) with constant market prices of risks.

### B.2 Long-Run Risk Model with Time-Varying Wealth-Consumption Ratio

Hansen and Heaton (2008) and Hansen, Heaton, Lee, and Roussanov (2007) consider a long-run risk model that focuses on the intertemporal composition of far-ahead risk prices. Indirect utility is specified by

\[
U_t = \left\{ (1 - \beta) C_{t+1}^{1-\rho} + \beta \left[ \mathbb{E}_t \left( \mathcal{U}_{t+1}^{1-\gamma} \right) \right]^{1-\rho} \right\}^{1/\rho}.
\]

The log-consumption dynamics is defined by

\[
\mathcal{M}_{t+1,1} \Delta c_{t+1} = \mu c + \rho c x_t + \sigma c z_{t+1},
\]

where \( x_t \) is the state vector, which follows a first-order VAR:

\[
x_{t+1} = \Phi x_t + \Sigma z_{t+1},
\]

where \( z_t \sim \text{i.i.d. } \mathcal{N}(0, I) \). Indirect utility is related to the ratio of wealth to consumption:

\[
\frac{W_t}{C_t} = \frac{1}{1 - \beta} \left( \frac{U_t}{C_t} \right)^{1-\rho}. \]

When the wealth-consumption ratio is constant, \( \rho = 1 \), the model implies a linear-exponential pricing kernel given by

\[
\log \mathcal{M}_{t+1,1} = \mu_M + \rho_M c x_t + \sigma_M z_{t+1}.
\]

In the more realistic case of \( \rho \neq 1 \), Hansen and Heaton (2008) derives an approximate solution by expanding around the case of \( \rho = 1 \):

\[
\log \mathcal{M}_{t+1,1} \approx \log \mathcal{M}_{t+1,1|\rho=1} + (\rho - 1) \left[ \frac{1}{2} z'_{t+1} \Gamma_1 z_{t+1} + z'_{t+1} \Gamma_1 x_t + \varrho_0 + \varrho_M x_t + \varrho_M z_{t+1} \right].
\]

Thus, by allowing for a time-varying wealth-consumption ratio, the pricing kernel becomes quadratic as in (5) with risk prices given by \( \xi_t = \varrho_M + \Gamma_1 x_t \) and \( \Xi = \frac{1}{2} \Gamma_1 \).
C Proofs

C.1 Proof of Theorem 4

The proof follows the steps in Joslin, Singleton, and Zhu (2011), henceforth JSZ, closely. We rely on invariant affine transformations of the state vector $X_t$ given by $P_t = c + CX_t$, where

$$c = \begin{bmatrix} c_x \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} C_x & 0 \\ 0 & I_{n,m} \end{bmatrix}.$$  \hfill (25)

The following lemma gives the model parameters resulting from this rotation.

**Lemma 1** Consider the invariant transformation $P_t = c + CX_t$ restricted by (25). Applying this transformation to the model in (3)-(7) gives the observationally equivalent model with state vector $P_t = (\rho_t, m_t)'$ and parameters

$$\mu_P = c + C\mu_X - \sum_{l=1}^{L} C\Phi_X^{(l)} C^{-1} c,$$

$$\Phi_P^{(l)} = C\Phi_X^{(l)} C^{-1}, \quad l = 1, \ldots, L$$

$$\Sigma_P = C\Sigma_X,$$

$$A_P^{(k)} = CA_X^{(k)} C^{-1}, \quad k = 1, \ldots, K$$

$$B_P^{(k)} = CB_X^{(k)} C^{-1}, \quad k = 1, \ldots, K$$

$$\alpha_P = \alpha_X - \beta_x C^{-1} c_x,$$

$$\beta_P' = \beta_x' C^{-1},$$

$$\mu_P^Q = c + C\mu_X^Q - C\Phi_X^Q C^{-1} c,$$

$$\Phi_P^Q = C\Phi_X^Q C^{-1},$$

$$\Sigma_P^Q = C\Sigma_X^Q.$$  

*Proof.* Straightforward. \hfill \square

To prove uniqueness, we apply the following lemma, which is identical to Proposition 1 in JSZ. Although our model differs from the class of Gaussian affine term structure models considered in JSZ, the lemma remains valid because our model involves Gaussian-affine risk neutral dynamics, which is the only part of the model invoked in the proof of the result.
Lemma 2 Every model defined by (3)-(7) and (17) is observationally equivalent to a model in real ordered Jordan form with \( r_t = \iota' x_t \), where \( \iota \) is a vector of ones. The parameters determining the \( Q \)-dynamics of this model are given by \( \mu^Q_x = (k^Q_\infty, 0, \ldots, 0)' \), \( \Sigma^Q_x \) is lower triangular with positive diagonal, and \( \Phi^Q_x = \text{diag}(J^Q_1, J^Q_2, \ldots, J^Q_D) \), where for \( d = 1, \ldots, D \),

\[
J^Q_d = \begin{bmatrix}
\lambda^Q_d & 1 & \cdots & 0 \\
0 & \lambda^Q_d & \cdots & 0 \\
\vdots & \vdots & \ddots & 1 \\
0 & \cdots & 0 & \lambda^Q_d \\
\end{bmatrix}.
\]

Proof. See JSZ.

Given these lemmas, we are ready to prove Theorem 4. The portfolio of synthetic yields given by \( \rho_t = WY^{\perp m}_t \) is an affine transformation of the latent non-macro risk factors because

\[
\rho_t = WY^{\perp m}_t = Wa_x + Wb_x x_t = a_W + b_W x_t.
\]

Thus, \( \mathcal{P}_t = c + CX_t \) with

\[
c = \begin{bmatrix} a_W \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} b_W & 0 \\ 0 & I_{n_m} \end{bmatrix}.
\]

By an application of Lemma 1, the model can be rotated into one with state vector given by \( \mathcal{P}_t \). We prove by contradiction that this observable model is unique. Suppose there exists two models with state vector \( \mathcal{P}_t \) parametrized by respectively \( \Theta_1 \) and \( \Theta_2 \). By Lemma 2, each model is observationally equivalent to the model in real ordered Jordan form with \( r_t = \iota' x_t \), whose parametrization we denote by \( \Theta_J \). It follows that \( \Theta_1 = \Theta_J = \Theta_2 \).

C.2 Proof of Theorem 5

By Lemma 2, any model defined by (3)-(7) and (17) can be re-written in real ordered Jordan form and the rotation only affects parameters of the \( Q \)-dynamics and the short-rate equation. These rotated parameters are given by

\[
\left\{ \alpha_X, \beta_x, \mu^Q_x, \Phi^Q_x, \Sigma^Q_x \right\} = \left\{ 0, \iota, k^Q_\infty e_1, J(\lambda^Q), \Sigma^Q_x \right\}.
\]
where \(e_1\) is a vector of zeros except with the first entry equal to one, and \(\Sigma_Q\) lower triangular with strictly positive diagonal. Given the model in Jordan form from Lemma 2, we apply the invariant transformation \(P_t = c + CX_t\) with \(c\) and \(C\) now given by (26). By an application of Lemma 1, the parameters of the rotated model are given by \(\Theta^\rho_P = \{k^\rho_\infty, \lambda^\rho_Q, \Sigma^\rho_P\}\) and

\[
\Theta^\rho_P = \left\{ \mu_P, \{\Phi_{P}^{(l)}\}_{l=1}^L, \Sigma_P, \{A_{P}^{(k)}, B_{P}^{(k)}\}_{k=1}^K \right\} = \left\{ c + C \mu_X - \sum_{l=1}^L C \Phi_{P}^{(l)} C^{-1} c, \{C \Phi_{X}^{(l)} C^{-1}\}_{l=1}^L, C \Sigma_X, \{C A_{P}^{(k)} C^{-1}, C B_{P}^{(k)} C^{-1}\}_{k=1}^K \right\}.
\]

Since \(\Theta^\rho_X = \{ \mu_X, \{\Phi_{X}^{(l)}\}_{l=1}^L, \Sigma_X, \{A_{X}^{(k)}, B_{X}^{(k)}\}_{k=1}^K \}\) is not involved in the rotation into the Jordan form, restricting \(\Theta^\rho_P\) is not necessary to preclude state vector rotations given \(\Theta^\rho_Q = \{k^\rho_\infty, \lambda^\rho_Q, \Sigma^\rho_P\}\).

Few restrictions are, however, necessary to identify parameters of the conditional mean under the \(\mathbb{P}\)-measure. To ensure a unique Cholesky decomposition, \(\Sigma_P \Sigma_P'\), we restrict the diagonal of the lower triangular matrix \(\Sigma_P\) to be strictly positive. Also, since \(A_{P}^{(k)}\) and \(B_{P}^{(k)}\) enter the model in quadratic form, we require the first elements in these matrices to be non-negative. The following lemma shows that these restrictions are sufficient for identifying the diagonal BEKK-GARCH model for a sufficiently small \(K\).

**Lemma 3** Consider the variance specification given by

\[
V_t = \Sigma_P \Sigma_P' + \sum_{k=1}^K A_{P}^{(k)} \varepsilon_{t-k} \varepsilon_{t-k}' A_{P}^{(k)'},
\]

where \(\Sigma_P\) is lower triangular with strictly positive elements on the diagonal. Let \(A_{P}^{(k)}\) and \(B_{P}^{(k)}\) are diagonal matrices for \(k = 1, \ldots, K\). Then, a sufficient condition for the parameters to be identified is \(K \leq \text{floor}(\frac{1}{2} n_X + \frac{1}{2})\), where \(n_X\) is the dimension of \(V_t\).

**Proof.** We conduct the proof a trivariate model, but the arguments hold for any dimension. We assume that \(K = \text{floor}(\frac{1}{2} n_X + \frac{1}{2}) = 2\) and show that this degree of generalization gives the maximum number of identified parameters. The ARCH and GARCH terms of the BEKK-GARCH model can be treated separately. Furthermore, the arguments for the two terms are identical. We focus on the ARCH term below. Let

\[
A_{P}^{(1)} = \begin{bmatrix}
a_{1,1} & 0 & 0 \\
0 & a_{1,2} & 0 \\
0 & 0 & a_{1,3}
\end{bmatrix} \quad \text{and} \quad A_{P}^{(2)} = \begin{bmatrix}
a_{2,1} & 0 & 0 \\
0 & a_{2,2} & 0 \\
0 & 0 & a_{2,3}
\end{bmatrix}.
\]
Then,
\[
A_p^{(1)} \varepsilon_t \varepsilon_t' A_p^{(1)'} + A_p^{(2)} \varepsilon_t \varepsilon_t' A_p^{(2)'} =
\begin{bmatrix}
(a_{1,1}^2 + a_{2,1}^2) \varepsilon_{1,t}^2 & (a_{1,1}a_{1,2} + a_{2,1}a_{2,2}) \varepsilon_{1,t} \varepsilon_{2,t} & (a_{1,1}a_{1,3} + a_{2,1}a_{2,3}) \varepsilon_{1,t} \varepsilon_{3,t} \\
(a_{1,1}a_{1,2} + a_{2,1}a_{2,2}) \varepsilon_{1,t} \varepsilon_{2,t} & (a_{1,2}^2 + a_{2,2}^2) \varepsilon_{2,t}^2 & (a_{1,2}a_{1,3} + a_{2,2}a_{2,3}) \varepsilon_{2,t} \varepsilon_{3,t} \\
(a_{1,1}a_{1,3} + a_{2,1}a_{2,3}) \varepsilon_{1,t} \varepsilon_{3,t} & (a_{1,2}a_{1,3} + a_{2,2}a_{2,3}) \varepsilon_{2,t} \varepsilon_{3,t} & (a_{1,3}^2 + a_{2,3}^2) \varepsilon_{3,t}^2
\end{bmatrix}
\]

Suppose that \(a_{2,1}, a_{2,2},\) and \(a_{2,3}\) are identified. Then, \(a_{1,1}, a_{1,2},\) and \(a_{1,3}\) follows from the coefficients on respectively \(\varepsilon_{1,t}^2, \varepsilon_{2,t}^2,\) and \(\varepsilon_{3,t}^2\). This leaves the following three equations to show that \(a_{2,1}, a_{2,2},\) and \(a_{2,3}\) are indeed identified:

\[
\begin{align*}
a_{2,1}a_{2,2} &= c_1 \\
a_{2,1}a_{2,3} &= c_2 \\
a_{2,2}a_{2,3} &= c_3.
\end{align*}
\]

This system is obviously identified, which completes are argument that \(K = \text{floor} \left( \frac{1}{2} n_X + \frac{1}{2} \right) \) gives an identified model. Naturally, it follows that \(K < \text{floor} \left( \frac{1}{2} n_X + \frac{1}{2} \right) \). Now, suppose that \(K\) exceeds \(\text{floor} \left( \frac{1}{2} n_X + \frac{1}{2} \right) = 2\), say \(K = 3\). For the ARCH equation, this leaves \(n_X(n_X + 1)/2 = 6\) equations to identify \(Kn_X = 9\) parameters, which is not feasible. □
**Figure 1:** Bond Yield and Macro Data

**Notes:** The left panel exhibits the bond yield data for selected maturities of 3 month and 1, 3, 5, 7, and 10 years. The data are from Gürkaynak, Sack, and Wright (2007). The right panel presents the macro data given by year-over-year growth in the CPI index and the unemployment gap. The unemployment gap is constructed by subtracting the natural unemployment rate estimated by the Congressional Budget Office (CBO) from the unemployment rate. Inflation and unemployment rates are provided by the Bureau of Labor Statistics (BLS). The data are sampled monthly from September 1971 to June 2019. Shaded areas represent recessions as defined by the NBER.
Figure 2: R-Squared Values from Rolling Regressions

Notes: The figure depicts R-squared values from regressions over rolling windows corresponding to 10- and 30-years of monthly data. Dotted lines are averages of the R-squared values. The left chart shows R-squared values from explanatory regressions of the type $Y_{t,10y} = \delta_0 + \delta_1 INFT + \delta_2 UGAPT$, where $Y_{t,10y}$ is the 10-year Treasury bond yield, INF is CPI inflation and UGAP is the unemployment gap. The right chart relates to predictive regressions of the type $RX_{t,10y} = \delta_0 + \delta_1 INFT + \delta_2 UGAPT + \delta'_3 PC3T$, where $RX_{t,10y}$ denotes the one-year excess holding period return of the 10-year bond defined by $RX_{t,n} = -(n + 1)/12Y_{t,n+1} + n/12Y_{t,n} - 12Y_{t,12}$. PC3t denotes the first three principal components of the yield curve. In particular, the chart presents the additional R-squared values obtained from including the macro risks in this regression compared to using PC3 only.
Figure 3: Estimated Non-Macro Risks and Loadings

Notes: The figure presents the non-macro risks estimated by the first three principal components of the synthetic yields that capture the residual variation that is not explained by macro risks (solid lines). We compare these with the principal components of the observed yield curve (dotted lines). In the lower right chart, we show the bond loadings for each non-macro risk factor across the maturity spectrum.
Figure 4: Matching Conditional Yield Volatilities

Notes: We compare the model-implied conditional variance and covariance estimates to empirical measures given by (i) realized variances constructed from daily data and (ii) rolling variances of daily data with a 6-month look-back. We present results for the 3-month, 5-year, and 10-year volatilities along with their covariances.
Notes: The figure exhibits decompositions of the 5- and 10-year model-implied yields into term premia and expected future short rates. Shaded areas represent recessions as defined by NBER.
Figure 6: Correlations Between Term Premia and Expected Future Short Rates

Notes: The figure shows conditional correlations between term premia and expected future short rates of the 5- and 10-year model-implied yields. Shaded areas represent recessions as defined by NBER.
Figure 7: Conditional Variance Ratios of Bond Yields

Notes: The left panels show conditional variance ratios that measure the contribution of macro risks to total yield variation. In the right panels, these ratios are smoothed by a 12-month moving average and decomposed into contributions from inflation and unemployment gap. Results are shown for the 3-month (upper panels) and 10-year (lower panels) maturities. Shaded areas represent recessions as defined by NBER.
Figure 8: Long-Horizon Conditional Variance Ratios of Bond Yields

Notes: The figure shows conditional variance ratios for 12-month (left panels) and 10-year (right panels) forecasting horizons. We plot both the total contribution of macro risks and the individual contributions by inflation and unemployment gap. Results are shown for the 3-month (upper panels) and 10-year (lower panels) maturities. Shaded areas represent recessions as defined by NBER.
Figure 9: Conditional Macro Variance Ratios for Term Structure Decomposition

Notes: The left panels show conditional variance ratios that measure the contribution of macro risks to total variation in 10-year short-rate expectations (upper panel) and term premia (lower panel). In the right panels, these ratios are smoothed by a 12-month moving average and decomposed into contributions from inflation and unemployment gap. Shaded areas represent recessions as defined by NBER.
Notes: The figure summarizes the distributions of impulse response functions by the median and 10, 25, 75, and 90 pct quantiles. The left panels show how a one standard deviation shock to inflation impacts inflation (upper panel), the 3-month yield (mid panel), and the 10-year yield (lower panel). The right panel consider a one standard deviation shock to the unemployment gap and the response to unemployment gap (upper), the 3-month yield (mid panel), and the 10-year yield (lower panel).
Figure 11: Conditional Moment Profiles

Notes: The figure summarizes the distributions of conditional moment profiles by the median and 10, 25, 75, and 90 pct quantiles. The left panels show how a one standard deviation shock to inflation impacts the conditional variances of inflation (upper panel), the 3-month yield (mid panel), and the 10-year yield (lower panel). The right panel consider a one standard deviation shock to the unemployment gap and the responses to conditional variances of unemployment gap (upper), the 3-month yield (mid panel), and the 10-year yield (lower panel).
**Figure 12:** Variance Risk Premia

*Notes:* The right panel presents the ratio of average variance risk premia to the average conditionally expected variance under the physical measure. The right panel decomposes the average variance risk premia into macro- and non-macro risks. Results are shown across maturities.
## Table 2: Descriptive Statistics

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**Cross-correlations**

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**Note:** Descriptive statistics of bond yield data and macro data in pct p.a.. The yield data consists of maturities between 3 months and 10 years and are from Gurkaynak, Sack, and Wright (2007). The macro data are given by year-over-year growth in the CPI index (INF) and the unemployment gap (UGAP). The unemployment gap is constructed by subtracting the natural unemployment rate estimated by the Congressional Budget Office (CBO) from the unemployment rate. Inflation and unemployment rates are provided by the Bureau of Labor Statistics (BLS). The data are sampled monthly from September 1971 to June 2019. ACF(k) denotes the sample autocorrelation with lag k months.
Table 3: $P$-Parameter Estimates: Conditional Mean

[TBA]

Table 4: $P$-Parameter Estimates: Conditional Variance

[TBA]

Table 5: $Q$-Parameter Estimates:

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*Notes:* Root mean squared errors of model-implied against observed bond yields for each maturity and for the average across maturities. Reported in basis points per annum.
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*Note:* Descriptive statistics of conditional variance ratios that measure the contribution of macro risks to total variation in yields across maturities. Statistics are reported for ratios in pct. ACF(k) denotes the sample autocorrelation with lag k months.
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<td>15.16</td>
<td>25.34</td>
<td>33.78</td>
<td>36.87</td>
<td>40.32</td>
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<td>53.82</td>
<td>56.85</td>
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<td>0.76</td>
<td>0.83</td>
<td>0.82</td>
<td>0.82</td>
<td>0.82</td>
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<td>0.67</td>
<td>0.76</td>
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<td>0.74</td>
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<td>0.6</td>
<td>0.59</td>
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</tr>
<tr>
<td>ACF(12)</td>
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<td>0.51</td>
<td>0.46</td>
<td>0.45</td>
<td>0.43</td>
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<td>0.43</td>
</tr>
</tbody>
</table>

*Note:* Descriptive statistics of conditional variance ratios that measure the contribution of macro risks to total variation in short-rate expectations across maturities. Statistics are reported for ratios in pct. ACF(k) denotes the sample autocorrelation with lag k months.
Table 9: Descriptive Statistics of Term Premia

<table>
<thead>
<tr>
<th></th>
<th>3M</th>
<th>6M</th>
<th>1Y</th>
<th>2Y</th>
<th>3Y</th>
<th>5Y</th>
<th>7Y</th>
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<tbody>
<tr>
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<td>38.52</td>
<td>34.31</td>
<td>37.32</td>
<td>40.51</td>
<td>39.98</td>
<td>33.87</td>
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<td>14.66</td>
<td>15.09</td>
<td>15.32</td>
<td>14.98</td>
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<td>0.41</td>
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<td>0.25</td>
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<td>0.87</td>
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<td>2.82</td>
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<td>3.17</td>
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<td>Min</td>
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<td>0.21</td>
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<tr>
<td>Max</td>
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<td>84.15</td>
<td>83.69</td>
<td>86.63</td>
<td>87.15</td>
<td>83.06</td>
<td>76.27</td>
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</tr>
<tr>
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<td>5.19</td>
<td>5.36</td>
<td>5.53</td>
<td>5.68</td>
<td>5.45</td>
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<tr>
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<td>17.3</td>
<td>15.44</td>
<td>18.82</td>
<td>21.28</td>
<td>20.42</td>
<td>15.00</td>
<td>9.97</td>
<td>6.81</td>
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<td>Quantile (0.25)</td>
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<td>26.79</td>
<td>29.9</td>
<td>29.39</td>
<td>23.32</td>
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<td>Quantile (0.75)</td>
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<td>59.79</td>
<td>54.56</td>
<td>46.75</td>
<td>38.72</td>
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<tr>
<td>ACF(1)</td>
<td>0.71</td>
<td>0.71</td>
<td>0.67</td>
<td>0.64</td>
<td>0.64</td>
<td>0.66</td>
<td>0.68</td>
<td>0.69</td>
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<tr>
<td>ACF(2)</td>
<td>0.54</td>
<td>0.54</td>
<td>0.47</td>
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<td>0.43</td>
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<td>0.05</td>
<td>0.08</td>
<td>0.09</td>
<td>0.10</td>
<td>0.12</td>
</tr>
</tbody>
</table>

Note: Descriptive statistics of conditional variance ratios that measure the contribution of macro risks to total variation in term premia across maturities. Statistics are reported for ratios in pct. ACF(k) denotes the sample autocorrelation with lag k months.