High-Frequency Trading, Endogenous Capital Commitment and Market Quality

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July 30, 2019

Abstract

I study the market quality implications of the competition between traditional market making and high-frequency trading. A long-run traditional market maker responds to the competition from high-frequency traders by reducing both the spread and the amount of capital committed in market making. While a lower spread is beneficial, lower capital commitment deteriorates market quality. Specifically, the market’s ability to satisfy large demand is impaired. I consider both price and quantity effects of high-frequency trading to obtain a more integrated characterization on how high-frequency trading changes market quality. I further use this framework to analyze how market quality is affected by different regulatory measures and argue that the spread is not necessarily a good measure of market liquidity.

1 Introduction

Over the past decade, high-frequency trading has become increasingly prevalent and is a continuing feature of financial markets worldwide. According to O’Hara (2015), high-frequency traders (henceforth HFTs) contribute more than half of the market trading volume. This growing trend of high-frequency trading has lead to a important

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policy debate on the proper regulatory measures to adapt to this change. Clearly, policy makers have yet to reach a consensus as different countries are still implementing regulations aiming at opposite directions.\footnote{For a comprehensive survey of the global high-frequency trading regulation environment, see Bell and Searles (2014)} Most European countries have carried out strict rules to reduce high-frequency trading and “level the playing field” while some Asian countries such as Japan and Singapore embrace high-frequency trading by providing systematic supports including introducing co-location service and rebating high-frequency trades.

Economists seem to have a more positive attitude toward high-frequency trading. Extant empirical research has documented that the presence of HFTs lead to lower spreads and some papers in the literature take this as a direct evidence that high-frequency trading improves market quality.\footnote{See Hendershott, Jones, and Menkveld (2011), Boehmer, Fong, and Wu (2018), Brogaard, Hendershott, and Riordan (2014), Hendershott, Jones, and Menkveld (2011), Boehmer, Fong, and Wu (2018), Hendershott and Riordan (2013), Hasbrouck and Saar (2013), Brogaard, Hagströmer, Nordén, and Riordan (2015), Conrad, Wahal, and Xiang (2015) and Conrad and Wahal (2018), among others.} However, this is only a partial characterization of the change brought by high-frequency trading. Other than inducing competition in pricing, the presence of HFTs also affects existing market makers’ incentive to commit capital in market making. Specifically, traditional market makers would reduce his capacity in absorbing market imbalance since the competition from HFTs makes market making less profitable. On the other hand, since HFTs usually do not take inventory, their ability to provide liquidity are constrained by market conditions and thus insufficient to fill the gap left by market makers. The decrease of capital commitment deteriorates market quality. Indeed, O’Hara, Yao, and Ye (2014) and Korajczyk and Murphy (2019) show that the average order size becomes smaller and investors have difficulties executing large orders.

My model differs from existing theory the has focused on the price discovery channel of high-frequency trading in two important ways.\footnote{For examples, see Goettler, Parlour, and Rajan (2009), Budish, Cramton, and Shim (2015), Biais, Foucault, and Moinas (2015) and Foucault, Hombert, and Roșu (2016), etc.} First, my model explicitly considers the market maker’s capital commitment decision, which has a critical implication in market quality. Second, the HFT and the market maker face different constrains as liquidity suppliers. Since the HFT does not take inventory and relies on her superior technology to front-run other investors, the HFT’s entry decision and
her liquidity supplied (extensive and intensive margin) depend on exogenous market conditions. On the contrary, a traditional market maker has an affirmative obligation to provide liquidity. Moreover, since the traditional market maker is unable to predict orders as well as the HFT does and has to provide liquidity, the market maker will commit capital to market making. In this paper, I analyze the steady state market quality implications when the market maker optimally determines his spread and capital commitment while facing the potential competition from high-frequency trading.

Specifically, I consider a market where the market maker and the HFT compete to sell shares in each period to a potential buyer with a random demand and valuation.\(^4\) The market maker is a traditional liquidity provider such as a designated market maker or a specialist. He contracts with the exchange to provide liquidity and is obliged to post quotes in the market. In return, the market maker enjoys lower transaction fees and the exchange rebates the market maker for providing liquidity.\(^5\) When no HFT exists, the market maker is a monopolist due to the market power he enjoys from advantageous terms provided by the exchange. This differs from the competitive market making assumption in Kyle (1985) and the literature in market micro-structure. Since the demand and the supply for shares may not arrive the market simultaneously, the market maker needs to commit capital for inventory. On the other hand, the market maker can spend his net worth on an outside option. The market maker endogenously determines his capital commitment level corresponding to the trade off between market making and taking the outside option.

The HFT in my model also acts as a liquidity supplier. She makes a profit by anticipating the arrival of buying orders. If the HFT detects a buying order, she tries to quickly buy cheaper shares from other exchanges and sells to the buyer with a slightly higher price. The HFT’s presence in the market and the amount of shares supplied by the HFT strongly depend on the market conditions. In my first model, I assume that the HFT enters the market with an exogenous probability \(\pi\) with fixed shareholding \(q_h\) to capture this feature. The competition from HFTs affects the market maker’s pricing and capital commitment decisions. From the pricing perspective, the market maker may tighten the spread to compete with the HFT. This reduces

\(^4\)This model has similarities to Kreps and Scheinkman (1983).
\(^5\)See NYSE Group (2019) and NYSE Rule 104 for an example of obligations and benefits to be a designated market maker in NYSE
the cost of buyers and benefits market liquidity. Moreover, the marginal benefit of capital commitment decreases and providing liquidity becomes less attractive. The market maker will reduce his capital commitment in market making. This reduces the market’s capacity to meet large demand from the buyer and effectively makes the market shallower.⁶

Possessing a superior trading technology relative to market maker, in the baseline model, the HFT observes the market maker’s shareholding and spread before making the price decision. Hench, the market maker faces a trade-off. By setting a wide spread, he would achieve high expected payoff when the HFT does not enter but only receives the residual demand if the HFT enters. In contrast, by setting a tight spread, he gives up some profit when HFT does not enter. Yet a tight spread makes undercutting not profitable and protects the market maker when the HFT enters. In the steady state, the market maker posts a wide (tight) spread if the HFT’s entry probability is low (high). Market quality is improved through the price effect when the HFT’s entry probability is high. On the other hand, compared to the monopolistic scenario, competition leads to a lower marginal value to allocating capital to market making. Thus, the market maker’s steady state capital commitment is (weakly) decreasing in the HFT’s entry probability. This deteriorates market quality since larger orders may be unfilled. I further analyze how liquidity changes with the HFT’s entry probability. Here liquidity is defined to be the expected volume traded by the buyer each period, which is related to fill rates. Liquidity is increasing in the HFT’s entry probability when the entry probability is high. When the HFT’s entry probability is low, the change in liquidity is ambiguous. Under mild assumptions there always exists a region where liquidity is decreasing in the HFT’s entry probability.

I also analyze the situation where the market maker and the HFT’s trading technologies are “head to head”; i.e., the HFT and the market maker plays a simultaneous pricing game. This corresponds to scenarios where HFTs become market makers⁷ or regulations slow HFTs down. In the equilibrium, the market maker and the HFT both use mixed strategies in posting spreads. The market maker’s expected payoff is identical to that in the sequential pricing setting. When the HFT’s entry probability is low, her expected profit is lower under this situation. This is in line with

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⁶In my model, the buyer leaves the market with partially fulfilled order. In practice, it can be the case that the buyer turns to other liquidity providers and purchase the remaining shares with higher price. This is equivalent to a shallower market.

⁷This effect is documented in Brogaard, Hagström, Nordén, and Riordan (2015)
the evidence in Baron, Brogaard, Hagstr¨ omer, and Kirilenko (2018) that faster HFTs achieve higher payoffs. Thus, my model implies that the HFT has an incentive to acquire superior trading technology. However, this incentive can be detrimental to market quality. Specifically, when the HFT’s entry probability is low, the model predicts that the liquidity is higher when the market maker and the HFT post spreads simultaneously.

Thus, regardless of the relative trading speeds of the market maker and the HFT, two regimes exist depending on the HFT’s entry probability. When the HFT’s entry probability is low, the market maker commits less capital facing the potential competition from the HFT. Higher HFT entry probability may not lead to better market quality due to this effect. When the HFT’s entry probability is high, the market maker posts lower spread and his expected payoff is not affected by the HFT’s entry. In this region, higher HFT entry probability leads to better market quality. Moreover, when the HFT’s entry probability is low, leveling the trading technologies of the market maker and the HFT is beneficial to the market. When setting spread simultaneously, the HFT cannot undercut the market maker at the wide spread anymore and this induces lower market price of shares.

I further consider two extensions. In the first extension, the HFT can choose whether to participate in high-frequency trading and is subject to a fixed participation cost. This can be interpreted as a lump-sum tax on high-frequency trading. The market maker in this situation enjoys an additional strategic advantage. When the participation cost is high, market maker can sell at a high price without worrying being undercut by the HFT since the HFT’s expected profit from undercutting cannot cover the participation cost. Effects of the participation cost on market quality depend on its magnitude. When the cost is low, market quality is the same as in the baseline model. When the cost is high, the HFT may not participate in high-frequency trading. On the other hand, the market maker’s spread and capital commitment is increasing with the participation cost. As the cost increases, the market converges to the monopolistic market. The overall effect of the participation cost on market quality is ambiguous. Yet it is certain that the effect is negative when the cost is too high. In the second extension, the HFT has a superior technology and can purchase shares from the market market and re-supply them to the market at a higher price. When the HFT’s entry probability is high, the market maker always sets a low price to induce flipping. If we count the HFT also as a buyer, market quality
appears to be very good since the expected trading volume is large and the average spreads are low. However, it is not a faithful characterization of the market for two reasons. First, most of the low price shares are purchased by the HFT rather than the buyer with higher valuation. Second, the trading volume is “double-counted”; the actually volume sold to the buyer is much lower. This extension demonstrates the importance to separate trades between liquidity suppliers in order to achieve a faithful characterization of market quality.

My model contributes to the theoretical literature on high-frequency trading by exploring how high-frequency trading effects market quality via the capital commitment channel of the market maker; Competition from the HFT leads the market maker to lower his capital commitment. This effect dampens the price benefit brought by competition, and, if large enough, the presence of a potential HFT might even deteriorate market quality. Ait-Sahalia and Sağlam (2017a) and Han, Khapko, and Kyle (2014) also consider market quality implications with competition between the HFT and traditional market makers. However, in these papers, the size of orders is fixed. This assumption constrains these models’ abilities to capture how capital commitment of the market maker affects market quality. In my model, it is possible that a market with wide spread has better quality than a market with tight spread. The reason is that in the latter market, the market maker commits much less capital in market making.

The implications of my model are consistent with following empirical findings in the literature: (1) High-frequency trading leads to lower average spreads in the market; (2) the average trade size becomes smaller; (3) market makers commit less capital in market making; (4) Large orders might face higher trading costs with the presence of HFTs. Moreover, my model provides new insights for future empirical studies. First, the price information alone does not provide a complete characterization of market quality. The volume information is equally important. Second, some variables (liquidity, average spread, etc.) reflecting market quality may not change monotonically with increasing HFT presence. In this sense, the current empirical evidence may fail to deliver an accurate prediction regarding the welfare implications of HFTs. Third, when the HFT can flip orders, it is important to differentiate trades

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8In Ait-Sahalia and Sağlam (2017a), the HFT, as a long run market maker, also holds inventory. However, since the supply is fixed to 1, the inventory does not have a quantity effect. Instead, it has a price effect due to the inventory aversion assumption.
between liquidity providers and trades from liquidity providers to other investors. Otherwise, the data cannot faithfully reflect market quality since HFTs would exploit most of the cheaper orders with superior trading technology.

This paper also generates important insights for HFT regulations. Three types of regulatory measures are considered. First, the policy maker can change the HFT’s entry probability by measures such as banning co-location (decreasing entry probability), improving exchange server’s processing capacity (increasing entry probability), etc. The model suggests that if HFTs are already prevalent in the market (high entry probability), encouraging HFT entry benefits liquidity. Since HFTs act as suppliers for the residual demand, more HFT presence leads to higher market capacity to process large orders. On the other hand, when the HFT’s entry probability is low, more HFT presence drives out market maker’s capital and has ambiguous effects on market quality. I also consider regulations that level the relative trading speed between traditional market makers and HFTs. Exchange policies encouraging HFTs to become designated market makers\footnote{Actually, two out of four largest designated market makers of the NYSE are considered as high-frequency trading firms. In the context of my model, they should be considered as fast traditional market makers.} or setting holding times on all orders\footnote{For example, IEX is applying this measure to prevent high-frequency arbitrage. See Lewis (2014) for more details.} have this effect. This model predicts that when the HFT’s entry probability is low, these measures improves market quality. When the HFT’s entry probability is high, these measures benefits mid-valuation buyers yet hurts low-valuation buyers. Finally, I consider a lump-sum high-frequency tax. For instance, European Union’s tax on both executed and canceled orders fits into this category. This model suggests that a low tax does not effect the market quality while a high tax increases market maker’s capital commitment but also drives up the spread.

The rest of the paper is organized as follows. Section 2 reviews related literature. Section 3 presents the model setting. Section 4 analyzes the baseline model. Section 5 considers the costly participation extension. Section 6 considers the flipping extension. Section 8 concludes.
2 Related Literature

2.1 HFT Behavior

An existing theory literature analyzes how high-frequency trading effects market quality from the information perspective.\textsuperscript{11} Han, Khapko, and Kyle (2014) demonstrate how adverse selection problem arising from fast order cancellation leads to wide spreads when the HFT enters the market with probability between 0 and 1. Budish, Cramton, and Shim (2015) show how mechanical arbitrage in high-frequency time horizon hurts liquidity and propose frequent batch auctions mechanism as a solution. Biais, Foucault, and Moinas (2015) endogenize investment decisions on fast trading technology and show that equilibrium investment level on fast trading is higher than the social optimal level because high-frequency trading has a negative externality. Foucault, Hombert, and Roșu (2016) analyzes news trading by fast speculators and its implications to trading volume and asset price. Ait-Sahalia and Sağlam (2017a) and Ait-Sahalia and Sağlam (2017b) analyze high-frequency market making and show that the faster market maker provides more liquidity. My model differs from the existing literature by explicitly considering the market maker’s capital commitment decision facing competition from HFT and its implications on market quality.

Many empirical papers test whether high-frequency trading’s impact on liquidity. Research generally documents an increase in liquidity with high-frequency trading. For instance, Hendershott, Jones, and Menkveld (2011), Hendershott and Riordan (2013), Hasbrouck and Saar (2013), Conrad, Wahal, and Xiang (2015) and Conrad and Wahal (2018),\textsuperscript{12} using spread as a proxy for liquidity, conclude that liquidity is improved by high-frequency trading. Brogaard, Hendershott, and Riordan (2014), using order flow data, conclude that HFT is liquidity improving around macroeconomic news since liquidity supply is greater than liquidity demand. Boehmer, Fong, and Wu (2018) using execution shortfalls as a proxy, reach the similar conclusion. My model does not contradict these evidences. However, it does suggest that some important quantity aspects of market quality cannot be captured by these proxies. Specifically, spread measures might not capture the quantity information related to

\textsuperscript{11} For a comprehensive survey, see Menkveld (2016).
\textsuperscript{12} Hasbrouck and Saar (2013) also examines number of shares displayed on the order book as a proxy for depth. One concern is that since HFTs can cancel orders with fast speed, this NearDepth might not able to capture real market depth.
the market maker’s capital commitment. The execution shortfall can better capture the price change facing large demand. Yet even the execution shortfall does not incorporate information about unexecuted and canceled orders. Moreover, order flow as a proxy of liquidity often includes trades between HFTs. This might lead to an over estimate of market quality. The extension on flipping directly addresses this concern. Recently, Korajczyk and Murphy (2018) and Korajczyk and Murphy (2019) document that less high-frequency trading is associated with higher transaction costs for small trades and lower transaction costs for large trades. This finding is in line with predictions of this model.

Some empirical papers focus on characteristics of traditional market makers and HFTs. Kirilenko, Kyle, Samadi, and Tuzun (2017) document that, different from traditional market makers, HFTs behaviors during the flash crash are more consistent with the latency arbitrage theory. Hirschey (2018) shows that HFTs can anticipate and trade ahead of other investors’ order flow. Baron, Brogaard, Hagströmer, and Kirilenko (2018) find that faster HFTs gain higher payoffs. This is in line with the prediction of my model that small HFTs has incentive to upgrade trading technology to be able to undercut the market maker. Van Kervel and Menkveld (2019) document that HFTs initially lean against institutional orders but eventually trade along long-lasting orders since they are likely to be information-motivated. Clark-Joseph, Ye, and Zi (2017) use data of two trading halts to show that designated market makers’ participation has important liquidity implications. This clearly shows that designated market makers and voluntary liquidity providers (HFTs) operate on different business models. Bessembinder, Hao, and Zheng (2019) also highlight the importance of designated market makers by showing that an improving of contract terms for designate market makers in NYSE improves market quality. This is consistent to the prediction of my model. If the market maker receives extra rebate on each share, he will commit more capital in market making and posts a lower spread.\footnote{This finding is consistent with my assumption that the HFT acts as a liquidity provider. However, my model is silent on the HFT trading alone the information-motivated orders since my model does not consider informed trading.}

\footnote{Bessembinder, Hao, and Zheng (2019) also document the spillover effect in market quality improvement because of the strategic complementary effect in market making. My model is silent on this aspect because I assume a deep inter-dealer market.}
2.2 Capital Constraint and Capital Commitment

Many models explore the link between capital constraints of intermediaries and liquidity provision. Kyle and Xiong (2001) describe the situation that when convergence traders lose capital, their liquidation leads to excess volatility and more correlation among different markets. Gromb and Vayanos (2002) show that constrained arbitrageurs might provide too much or too little liquidity compared to the social optimal level, depending on their initial investment positions. Weill (2007) and Brunnermeier and Pedersen (2008) both demonstrate that insufficient capital of the market maker would lead to lower liquidity provision than the optimal level. In Weill (2007), lack of capital prevents the market maker to absorb enough order imbalance when the economy is recovering from a negative shock. In Brunnermeier and Pedersen (2008), traders’ lack of funding and market liquidity deterioration reinforce each other and let to “liquidity spiral”. My paper contributes to this strand of literature by showing that, even when the market maker is not constrained, his capital commitment decision plays an important role to market quality when facing competition from high-frequency trading.

A relatively small empirical literature examines the capital commitment of market makers. Hameed, Kang, and Viswanathan (2010) show that negative market return decreases liquidity asymmetrically. The authors attribute the decrease to the market maker’s capital constraint. Comerton-Forde, Hendershott, Jones, Moulton, and Seasholes (2010) find a similar result using data on NYSE specialist positions and revenues. Bessembinder, Jacobsen, Maxwell, and Venkataraman (2018) document that capital commitments of corporate bond dealers are decreasing overtime, specifically in markets with more electronically facilitated trades. The authors interpret this as a result of electronic trading reducing search cost and required capital. This model suggests an alternative explanation. The decrease of capital commitment might due to the growing entry of HFTs facilitated by electronic trading.

3 Model Setting

3.1 The Setup

The game has infinite periods with three kinds of players: a long-run market maker, a short-run HFT and a short-run buyer. The market maker has a discount rate $\delta$
and starts the game with net worth $w_0$. In each period, the market maker can either pay dividend or acquire shares from a inter-dealer market with price 1 for market making. The market maker maximizes discounted dividend payout, $E_0(\sum_{t=0}^{\infty} \delta^t d_t)$. In each period, a short-run HFT enters market with probability $\pi$. If enters, she holds $q_h$ shares and aims at maximizing expected profit. The market maker and the HFT are both sellers and compete to provide liquidity for a short-run buyer with liquidity or hedging need. The buyer has a random demand $q_b$ and a random valuation $v > 1$; i.e., his is willing to pay a premium for each share up to his demand $q_b$.

The sequence of events in a single period is specified as follow. Let $w_t$ be the market maker’s net worth at the beginning of period $t$. The market maker first chooses dividend payout $d_t$. He then commits the remaining net worth $w_t - d_t$ to purchase $q_{m,t} = w_t - d_t$ shares from the inter-dealer market at the price $p = 1$. The market maker then posts a spread $x_{m,t}$, committing to sell all shares at the ask price $1 + x_{m,t}$. After market maker sets his spread, a short-run HFT with $q_h$ shares enters the market with probability $\pi$. If the HFT possesses a superior trading technology and trades faster than the market maker, she observes the market maker’s shareholding $q_{m,t}$ and $x_{m,t}$ before setting her own spread $x_h$ (the sequential pricing game). If the market maker and the HFT have the same trading technology, the HFT only observes $q_{m,t}$ (the simultaneous pricing game). After the market maker and the HFT determine their spreads, a short-run buyer arrives with demand $q_b$ and valuation $v > 1$; i.e., the buyer is willing to pay $v$ for each share up to $q_b$ shares. If the market maker and the HFT set the same price and is lower than $v$, the buyer would buy from the HFT first. After the buyer finish buying shares, the market maker and the HFT (if enters) may sell the remaining shares at the fair price $p = 1$ back to the inter-dealer market. This concludes a period.

I make the following distribution assumptions: $v - 1$ follows a distribution with CDF $F$ supported on $[0, \hat{x}]$. $q_b$ follows a distribution with CDF $G$ with a positive support. $F$ and $G$ are independent and continuously differentiable. I further assume that $F$ has non-decreasing hazard rate; i.e., $\frac{f(x)}{1-F(x)}$ is non-decreasing, or equivalently, $f$ is log-concave.

\footnote{Another interpretation can be that the market maker put the rest of capital into a margin account to cover the cost of potential short selling.}

\footnote{It is without loss of generality to assume that the market maker commits all remaining net worth in market making. If he chooses to commit less, he may raise his dividend payout to achieve a higher payoff.}
Three specific assumptions are worth more discussion. First, for a fixed buyer, his demand $q_b$ is inelastic with respect to a price lower than $v$. In a financial market, this is the same as posting a limit order with quantity $q_b$ and price $v$. Moreover, $v$ is observed by the market maker and the HFT when setting spreads. Higher spreads reduce the purchasing probability. Thus, the demand curve is downward sloping in expectation. Second, the HFT is modeled as a short-run player with an exogenous entry probability $\pi$ and a fixed shareholding $q_h$. This assumption by no means suggests that the HFT is not a long term market participant. Instead, it characterizes two realistic features of high-frequency trading: The HFT’s entry decision and shareholding heavily depend on exogenous market conditions; moreover, the HFT focuses on short term trading and only carries positions for a short period of time. Third, this model only considers a one-sided market; i.e., liquidity providers in this model only sell shares to other investors. This is without loss of generality given that the market maker can adjust his position with no cost at the inter-dealer market. If I consider a two-sided market where only one buyer or seller enters the market each period, the qualitative predictions on market quality are essentially the same.

### 3.2 Liquidity

In this section, I formally define liquidity as an important indicator of market quality. Let ex ante liquidity $L_t$ be the expected shares sold to the buyer at period $t$. In this paper, I focus on steady state equilibria, in which the market maker’s pricing and capital commitment decisions are time invariant. In this case, the time subscript can be dropped since $L_t$ is time invariant. Moreover, to achieve a more specific characterization on market quality for buyers with various valuations, define $L(v)$ to be the expected volume sold to the buyer given the buyer’s valuation for each share is $v$. Notice that given $v$, the amount of shares available to the buyer is fixed. Thus, $L(v)$ captures the market’s ability to fill large orders at price $v$. In other word, the
fill rate at valuation $v$ can be defined as $\frac{L(v)}{E_G(q_v)}$.

Several features of this definition worth discussion. First, $L$ incorporates both price and quantity information of the market. If the spread is high, even with a large supply of shares in the market, liquidity might be low since only a buyer with high valuation will purchase shares. Similarly, a low spread alone cannot guarantee high liquidity. If the supply is low, the buyer’s demand may not be completely satisfied, which leads to low liquidity. Second, this measure of liquidity is closely related to welfare. Since the buyer has a higher valuation for each share than the market maker and the HFT, holding everything else equal, higher liquidity indicates larger welfare. Note that this measure differs from buyer’s surplus by ignoring the buyer’s specific valuation as long as shares are sold. Yet since buyers’ valuations are hard to observe in practice, this measure is superior than buyer’s surplus in measuring market quality in two aspects: (1) Liquidity, as expected volume sold to the buyer, is easier to observe by empirical analysis; (2) liquidity is more robust measure for welfare since it does not depend on specific estimates of buyers’ valuations. Third, this measure excludes volume traded in the inter-dealer market because only shares sold to the buyer are welfare improving in this model. Moreover, trades in the inter-dealer market might happen in a different time and location. Thus, from an empirical perspective, these trades are less likely to be documented in the observed trading data.

3.3 Equilibrium Definition

Two facts suggest that the market maker’s net worth should be considered as the state variable of the equilibrium. First, the market maker is only constrained by his net worth. Second, as a short-run player, the HFT has no incentive to relate her action to the history of the game. Formally, an equilibrium is defined as follows:

**Definition 1** Consider a infinite horizon game $(w_0, q_h, \pi)$ where the market maker starts with net worth $w_0$ and the HFT enters the market with probability $\pi$ holding $q_h$ shares.

1. An equilibrium in a sequential pricing game is a triple $(d(w), x_m(w-d(w)), x_h(d, x_m))$ such that: (i) Given $d$ and $x_m$, $x_h(d, x_m)$ maximizes the expected payoff of HFT. (ii) Given $x_h(d, x_m)$, $\{d_t = d(w_t)\}_{t=0}^{\infty}$ and $\{x_{m,t} = x_m(w_t - d(w_t))\}_{t=0}^{\infty}$ maximize
\[ E_0(\sum_{t=0}^{\infty} \delta^t d_t). \]  

2. An equilibrium in a simultaneous pricing game is a triple \((d(w), x_m(w-d(w)), x_d(d))\) such that: (i) Given \(d\), \(x_d(d)\) maximizes the expected payoff of the HFT. (ii) Given \(x_d(d)\), \(\{d_t = d(w_t)\}_{t=0}^{\infty}\) and \(\{x_{m,t} = x(w_t-d(w_t))\}_{t=0}^{\infty}\) maximize \(E_0(\sum_{t=0}^{\infty} \delta^t d_t)\).

I focus on the steady state capital commitment and spread to characterize the long term market quality. The formal definition of a steady state equilibrium is as follows:

**Definition 2** An equilibrium is a steady state equilibrium if there exist a \(q_m > 0\) such that \(d_t = w_t - q_m\) for all \(t\).

Intuitively, in a steady state equilibrium, the market maker’s capital commitment is a constant over time. Since the focus of this paper is on capital commitment rather than capital constraint, I assume that the market maker always starts the game with a sufficiently large net worth \(w_0\). Under this situation, an steady state equilibrium always exists.

### 4 Baseline Models

#### 4.1 Benchmark Case with No HFT

First consider the situation where no HFT exists (or equivalently, \(\pi = 0\)). The market maker’s value function satisfies the following equation:

\[
V(w) = \max_{d, x_m} d + \delta F(x_m)V(w - d) + \delta(1 - F(x_m))[\int_0^{w-d} V(w - d + x_m q)g(q) dq + (1 - G(w-d))V((1 + x_m)(w - d))] \tag{1}
\]

with the constraint

\[
0 \leq d \leq w \tag{2}.
\]

I show that there exists a steady state capital commitment \(\bar{q}\) and a steady state spread \(x^*\). In other words, the market maker pays dividend \(w_0 - \bar{q}\) at period 0 and

\[17\]Notice that the distribution of \(w_{t+1}\) can be uniquely determined by \(w_t\) and the equilibrium. Since \(w_0\) is given, the dynamic of \(w_t\) is well-defined.
supply $\bar{q}$ shares to the market with spread $x^*$. In following periods, the market maker pays his profit as dividend and maintains the same shareholding and spread. The following theorem formalizes this statement:

**Theorem 1** Let $k(s) = E_G(\min(q_b, s))$ be the effective supply when $s$ shares are supplied to the market.

1. The market maker’s optimal strategy is $d_t = w_t - \bar{q}$ and $x = x^*$ for all $t$.
   
   $$x^* = \arg\max_x (1 - F(x))x$$ and $\bar{q}$ satisfies $\frac{1}{1-δ}(1 - F(x^*))x^*(1 - G(\bar{q})) = 1$.\(^{19}\)

   The market maker’s expected payoff is $V(w_0) = \frac{δ}{1-δ}(1 - F(x^*))x^*k(\bar{q}) + (w_0 - \bar{q})$.

2. Liquidity at the steady state is $L = (1 - F(x^*))k(\bar{q})$ and the average spread is $x^*$.

**Proof.** See Appendix.  

This theorem has a clear economic interpretation. Facing a random demand, the market maker’s capital commitment has decreasing marginal value since at any spread, each additional share is less likely to be sold. On the other hand, the marginal value of dividend payout is constant. Since the market maker is discounting future dividend payment, he only commits capital up to a point where the marginal value of holding shares equals the marginal value of paying dividend. At the steady state, the market maker maintains his shareholding in each period and pays out the profit. This incentivizes him to set a spread to maximize the expected profit, which makes him act like a short-run monopolist.

The market with no HFT serves as a benchmark for comparing the market quality. In this market, liquidity $L = (1 - F(x^*))k(\bar{q})$ and the average spread is $x^*$. Specifically, in the steady state, this market has high share supply at a high price. To see this, notice that $x^*$ is the highest possible spread a liquidity supplier would set. If the spread is higher than $x^*$, in the steady state, the loss from selling with lower probability dominates the benefit from selling at a higher price. On the other hand, facing no competition, the market maker supplies $\bar{q}$ shares to the market. When the HFT might enter the market, the market maker’s equilibrium capital commitment is always lower than $\bar{q}$.

The effective share supply $k(s)$ plays an important role in the analysis. This function measures expected shares sold given the buyer is willing to buy and $s$ shares sold to the market.

\(^{18}\)Note that the fair price of each share is 1.

\(^{19}\)If no such $\bar{q}$ exists, the optimal strategy is to liquidate ($d = w_0$) at $t = 0$. 

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are available in the market. Under any (non-degenerate) random demand \( q \), this function is strictly concave, which leads to the decreasing marginal value of capital commitment. To see how the randomness of buyer’s demand plays a role, notice that if the buyer’s demand \( q \) is deterministic and the market maker is sufficiently patient, his capital commitment equals to \( q \) in the steady state. This implies any buyer with a valuation higher than \( 1 + x^* \) can always fully fulfills his orders. Moreover, the concavity of \( k(s) \) implies that a mean-preserved spread of \( s \) leads to a lower effective supply. Thus, the uncertainty of the HFT’s presence in the market is detrimental to liquidity.

### 4.2 Sequential Pricing Case (High Tech HFT)

In this section, I consider the situation when the HFT has a superior trading technology comparing to the market maker. In this case, the HFT can observe the market maker’s shareholding \( q_m \) and spread \( x_m \) before posting her spread \( x_h \). I focus on the steady state where the market maker’s capital commitment and spread are constant over time.

To characterize the steady state, it is helpful to first consider a one-shot game with a fixed market maker’s shareholding. The reason is clear: In the steady state, the market maker’s shareholding \( q_{m,t} \) is constant and he posts the spread to maximize expected profit. Thus, his pricing strategy coincides to the pricing strategy of an expected profit maximizer in a one-shot game.

In a sequential one-shot game \((q_m, q_h, \pi)\), the market maker has shareholding \( q_m \) and the HFT enters the market with probability \( \pi \) with shareholding \( q_h \). The market maker sets spread \( x_m \) first and the HFT, if enters, sets spread \( x_h \) after observing \( x_m \). Each player aims at maximizing his/her expected profit and can sell shares back to the inter-dealer market at the end of the period at price 1. To simplify the notation, define \( a(x) = \frac{(1-F(x))x}{(1-F(x^*))x^*} \leq 1 \).

**Definition 3** An equilibrium of a one-shot sequential pricing game \((q_m, q_h, \pi)\) is a pair \((x_m, x_h(x_m))\). Given the market maker’s spread \( x_m \), the HFT’s spread \( x_h(x_m) \) maximizes her expected payoff. Given the HFT posting her spread according to \( x_h(x_m) \), the market maker’s spread \( x_m \) maximizes his expected payoff.

Consider the HFT’s pricing problem after observing \( x_m \). If the HFT sets \( x_h \leq x_m \), her shares will be sold before the market maker’s but at a lower price. If the HFT
sets \( x_h > x_m \), she can sell at a higher price but she only receives the residual demand. Since \((1 - F(x))x\), the expected marginal value of providing one share within the buyer’s demand \( q \), is increasing in \( x \) for \( x < x^* \), the HFT would either undercut the market maker as the same spread, or sell shares at the spread \( x^* \).

**Lemma 1** Given the market maker’s shareholding \( q_m \) and spread \( x_m \), the HFT’s best response is either \( x_h = x_m \) or \( x_h = x^* \).

**Proof.** See Appendix.

Next consider the market maker’s pricing problem. If the market maker expects the HFT to undercut him upon entering, it is optimal for the market maker to set the monopolistic spread \( x^* \). Otherwise, the market maker may set a tight spread such that the HFT is indifferent between undercutting him and setting \( x_h = x^* \). All other pricing strategies are dominated by either of the two strategies. Formally, the market maker’s pricing problem can be characterized by the following lemma:

**Lemma 2** For any \( q_m \), the market maker’s optimal spread is either \( x_m = x^* \) or \( x_m = \bar{x} < x^* \). \( \bar{x} \) is pinned down by the HFT’s indifference condition \( a(x)k(q_h) = k(q_m + q_h) - k(q_m) \).

**Proof.** See Appendix.

By Lemma 1 and 2, it is sufficient to compare the market maker’s expected profit at \( x_m = \bar{x} \) and \( x_m = x^* \) to pin down the equilibrium pricing strategies of both players.

**Proposition 1** If \( k(q_m) > \pi k(q_h) \), the unique equilibrium strategy is \( x_m = x_h = x^* \). If \( k(q_m) < \pi k(q_h) \), the unique equilibrium strategy is \( x_m = \bar{x}, x_h = x^* \). When \( k(q_m) = \pi k(q_h) \), two equilibria both exist.

**Proof.** See appendix.

Note that from Lemma 1 and 2 the market maker has two possible pricing strategies against the potential HFT. The market maker can use the wide spread strategy by setting the monopolistic spread \( x^* \). This strategy yields high expect profit when the HFT does not enter the market. When the HFT enters, however, the market maker will be undercut and only receives the residual demand. The effectiveness of this strategy depends on the HFT’s entry probability \( \pi \) and shareholding \( q_h \). Otherwise, the market maker may use the tight spread strategy by setting spread \( \bar{x}(q_m) \),
receiving lower expected profit when the HFT does not enter. With a tight spread, it is not profitable for the HFT to undercut the market maker upon entry. If the market maker adopts this strategy, the buyer always buys shares from the market maker first and the HFT’s entry does not affect the market maker’s expected profit. Another observation is that the HFT always posts spread $x_h = x^*$ in the equilibrium. However, this does not imply that the HFT always sells shares at a higher price than the market maker. With the technology advantage, the HFT only needs to reduce her spread by a very small amount to undercut the market maker. On the other hand, a large price reduction is needed for the market maker if he tries to prevent himself from being undercut by the HFT.

4.2.1 Steady State Characterization

In this section, I solve for the steady state equilibrium of the infinite period game. Let $M(q)$ be the market maker’s expected profit in a one-shot game with $q_m = q$. Let functions $\hat{x}_m(q)$ and $\hat{x}_h(q)$ correspond to the market maker and the HFT’s equilibrium spreads in this game. If the infinite period game reaches the steady state at $t = 0$ and the market maker’s steady state capital commitment is $q$, his payoff equals to $\delta_1 - \delta M(q) + (w_0 - q)$. $\delta M(q)$ is the expected payoff from a perpetuity paying out the market maker’s profit starting from period 1. $w_0 - q$ is his dividend payout at period 0 to reach the steady state (note that the market maker buys shares at price $p = 1$). An obvious candidate of the market maker’s steady state capital commitment is $q_m = \text{argmax}_{q \in [0, \bar{q}]} \frac{\delta}{1-\delta} M(q) + (w_0 - q)$. The following theorem shows that this is indeed a steady state equilibrium.

**Theorem 2** Let $q_m = \text{argmax}_{q \in [0, \bar{q}]} \frac{\delta}{1-\delta} M(q) + (w_0 - q)$.

1. $d_t = w_t - q_m$, $x_m = \hat{x}_m(w_t - d_t)$, $x_h = \hat{x}_h(w_t - d_t)$ consists a steady state equilibrium. The market maker’s expected payoff in the equilibrium is $V(w_0) = \frac{\delta}{1-\delta} M(q_m) + (w_0 - q_m)$.

2. If the market maker uses the wide spread strategy in the equilibrium, steady state liquidity is $L = (1 - F(x^*))[\pi k(q_m + q_h) + (1 - \pi)k(q_m)]$. The average spread is $x^*$.

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20I suppress the dependency of these functions on $q_h$ because $q_h$ in the baseline model is common knowledge and taken as given.
3. If the market maker uses the tight spread strategy in the equilibrium, steady state
liquidity is 

\[ L = (1 - F(x_m))k(q_m) + \pi(F(x^*) - F(x_m))(k(q_m + q_h) - k(q_m)) \]. The
average spread is lower than \( x^* \).

**Proof.** See appendix. ■

We now discuss some important corollaries.

**Corollary 1** For \( \pi > 0 \), the market maker’s steady state capital commitment \( q_m < \bar{q} \).

This corollary shows that the market maker commits less capital in market making
facing competition from the HFT. With potential HFT entry, the marginal value of
the market maker’s capital commitment is smaller. This is either because the market
maker only receives the residual demand with the wide spread strategy or the market
maker’s spread becomes lower with the tight spread strategy. This leads to lower
capital commitment.

**Corollary 2** If \( \bar{q} > 0 \), \( q_m > 0 \). In other words, the market maker never exit the
market in the steady state equilibrium. Moreover, \( q_m \) satisfies the following condition:

1. If the market maker uses the wide spread strategy, \( q_m \) satisfies

\[ \frac{\delta}{1 - \delta}(1 - F(x^*))x^*[(1 - \pi)(1 - G(q_m)) + \pi(1 - G(q_m + q_h))] = 1. \]

2. If the market maker uses the tight spread strategy, \( q_m \) satisfies

\[ \frac{\delta}{1 - \delta}(1 - F(x))x(1 - G(q_m)) > 1. \]

**Proof.** See appendix. ■

This corollary specifies conditions satisfied by \( q_m \) in a steady state equilibrium.
These conditions can be derived directly from first order conditions of the market
maker and are useful for comparative statics with respect to \( \pi \). Moreover, the second
part of this corollary implies the connection between the market maker’s steady state
capital commitment and corresponding tight spread. Specifically, from HFT’s indif-
ferent condition, higher capital commitment leads to a lower tight spread. Thus, if
the market maker uses the tight spread strategy in the steady state equilibrium, the
marginal benefit of increasing capital commitment holding the spread fixed is larger
than 1.
4.2.2 Comparative Statics on HFT’s Entry Probability

In this section, I analyze how the steady state equilibrium and market quality change with $\pi$, the HFT’s exogenous entry probability. Higher $\pi$ indicates a more fierce competition between the market maker and the HFT. The market maker would adjust his capital commitment and pricing, which leads to changes in market quality.

First, consider the market maker’s pricing decision. A key observation is that, the HFT’s pricing decision does not depend on $\pi$. Thus, fix the market maker’s capital commitment $q_m$, the tight spread $x$, does not depend on $\pi$. Thus, regardless of $\pi$, the market maker’s candidates for the optimal spread are the same, namely, the wide spread $x^*$ and the tight spread $x$. Furthermore, when posting spread $x^*$, the market maker’s expected payoff is decreasing in $\pi$ since he only receives the residual demand when the HFT enters. On the contrary, his expected payoff does not depend on $\pi$ if he sets the tight spread $x$. Consequently, the tight spread strategy becomes more attractive as $\pi$ increases. Formally, the comparative statics for one-shot games can be characterized by the following proposition:

**Proposition 2** Consider two one-shot games $(q_m, q_h, \pi_1)$ and $(q_m, q_h, \pi_2)$ with $\pi_2 > \pi_1$.

1. If the market maker’s equilibrium strategy in game $(q_m, q_h, \pi_1)$ is the tight spread strategy, then his equilibrium strategy is also the tight spread strategy in game $(q_m, q_h, \pi_2)$. His expected profits in two games are the same.

2. If the market maker’s equilibrium strategy in game $(q_m, q_h, \pi_2)$ is the wide spread strategy, then his equilibrium strategy is also the wide spread strategy in game $(q_m, q_h, \pi_1)$. His expected payoff is higher in game $(q_m, q_h, \pi_1)$.

**Proof.** Since $q_m$ and $q_h$ are fixed, the equilibrium strategy choices are directly implied by proposition 1.

Note that the tight spread $x$ is determining by the equation $k(q_h + q_m) - k(q_m) = a(x)k(q_h)$, which does not depend on $\pi$. Thus, the market maker’s expected payoff when adopting the tight spread strategy, $(1 - F(x|x))k(q_m)$, do not depend on $\pi$.

The market maker’s expected net profit of adopting the wide spread strategy is $(1 - F(x^*|x^*)[\pi(k(q_h + q_m) - k(q_h)) + (1 - \pi)k(q_m)])$. This quantity is decreasing in $\pi$ since $k(q_h + q_m) < k(q_h) + k(q_m)$. ■
Now consider the market maker’s capital commitment problem in the infinite period game. By Proposition 2, if the market maker uses the tight spread strategy in the steady state, when \( \pi \) changes, stick to the same policy leads to the same payoff.\(^{21}\) Consider two markets with different HFT entry probabilities; if the market maker uses the tight spread strategy in both steady states, then the market maker’s pricing and capital commitment decisions are identical and he enjoys the same payoff. Furthermore, by Corollary 2, if the market maker sticks to the wide spread strategy when the HFT’s entry probability increases, in the steady state he commits less capital and achieves lower expected payoff. Combining these analyses leads to the following result:

**Theorem 3** There exists \( \hat{\pi} \in (0, 1] \) such that in the steady state equilibrium, the market maker posts the wide spread \( x^* \) when \( \pi < \hat{\pi} \) and posts the tight spread \( \underline{x} \) when \( \pi > \hat{\pi} \). Define \( (0, \hat{\pi}) \) to be the wide spread region and \( (\hat{\pi}, 1] \) to be the tight spread region.

1. The market maker’s expected payoff \( V(w_0) \) and equilibrium capital commitment \( q_m \) is decreasing in \( \pi \) in the wide spread region. Liquidity \( L \)’s change with respect to \( \pi \) is ambiguous.

2. The market maker’s expected payoff \( V(w_0) \) and equilibrium capital commitment \( q_m \) remain constant in the tight spread region. Liquidity \( L \) is increasing in \( \pi \) in the tight spread region.

3. The market maker’s equilibrium capital commitment is smaller in the tight spread region comparing to any equilibrium capital commitment in the wide spread region.

4. In the wide spread region, the average spread is \( x^* \). In the tight spread region, the average spread is below \( x^* \) and increasing in \( \pi \).

**Proof.** See appendix. \( \blacksquare \)

This theorem documents important features of the steady state and market quality depending on the HFT’s entry probability \( \pi \). In particular, the steady state equilibrium can be categorized into two regimes accordingly to the magnitude of \( \pi \). When

\(^{21}\)Note that the policy itself may not be the optimal policy under a different \( \pi \).
the HFT’s entry probability is low (wide spread region), the market maker sticks to setting the monopolistic spread. He responds to the competition by committing less capital to the market. In this region, the competition does not benefit low evaluation buyers since shares are still selling at the monopolistic price. Instead, when the HFT does enter the market, she improves the market quality by increasing the market’s capacity to satisfy buyers with valuations higher than the monopolistic price. On the other hand, when the HFT does not enter, the market’s capacity to satisfy large demands is decreasing in \( \pi \) since the market maker’s steady state capital commitment becomes smaller.

When the HFT’s entry probability is high (tight spread region), the market maker sets a tight spread at the steady state. Low valuation buyers benefit from the competition since the market maker is supplying shares at a lower price. However, to deter the HFT from undercutting, the market maker keeps his capital commitment at a low level. This impairs the market’s capacity to satisfy large demands. This can be interpreted as the market becoming shallower. Indeed, though buyers have access to cheaper shares, the supply of these shares are limited. With a large demand, either the price of a share would jump to the monopolistic price when the HFT is present or not enough shares exist to fulfill the order.\(^{22}\) Moreover, in this region, higher HFT entry probability improves steady state market quality since the market maker does not change his capital commitment and pricing decisions. Specifically, higher HFT entry probability increase the market’s capacity to satisfy high valuation buyers, who have valuations higher than the monopolistic price.

Importantly, this theorem demonstrates why the average spread and the implementation shortfall fail to fully characterize market quality. I focus on the average spread in the following discussion because with only the market maker and the HFT as liquidity providers in my model, the implementation shortfall and the average spread are identical. In the wide spread region, although liquidity (and buyer welfare) is changing with \( \pi \) in the steady state, the average spread remains the same since both the market maker and the HFT set the monopolistic spread \( x^* \). In the tight spread region, market quality is clearly improving with higher \( \pi \) since the market maker’s equilibrium strategy is the same regardless of \( \pi \). Yet the average spread is also increasing. The reason is that in this region, the HFT’s spread is higher. With higher

\(^{22}\)In this model, I do not consider other liquidity providers. Yet in reality it can be the case that the rest of the order are fulfilled by other suppliers at a higher unit price.
HFT entry probability, a larger proportion of shares are sold at a higher price. This drives up the average spread.

Next I conduct a more detailed analysis on how market quality changes with $\pi$ in the wide spread region using liquidity $L$ as a proxy of market quality. In particular, more competition is not always beneficial to the market in the sense that, under mild assumptions, there always exists a region where $L$ is decreasing in $\pi$.

**Proposition 3** Suppose the wide spread region is $[0, 1]$; i.e., the market maker uses the wide spread strategy when $\pi = 1$. Then either there exists a region where $L$ is decreasing in $\pi$ or $L$ is constant over $[0, 1]$.

The reason behind this result is simple. If the market maker uses the wide spread strategy at $\pi = 1$, then $q_m + q_h = \bar{q}$. Thus, the market is identical to the monopolistic market and has the same liquidity. Then by continuity, if $L$ is not constant in $\pi$, there exists a decreasing region. An important observation is that when the HFT’s shareholding $q_h$ is small enough, this assumption with respect to the wide spread region is satisfied.

**Corollary 3** For small enough $q_h$, if $L$ is not constant over $\pi \in [0, 1]$, then there exists a region where $L$ is decreasing in $\pi$.

Another assumption considers the property of $G$, the distribution of the buyer’s demand $q$. Specifically, if $G$ has increasing hazard rate (or equivalently, $g$ is log-concave), there always exists a region where $L$ is decreasing in $\pi$. Moreover, as some readers might have conjectured, if $G$ follows the exponential distribution (which has a constant hazard rate), liquidity remains constant over the wide spread region.

**Proposition 4** If $G$ follows an exponential distribution, liquidity is a constant with respect to $\pi$ in the wide spread region.

**Proof.** See appendix

**Theorem 4** If $G$ has increasing hazard rate, then for any $q_h > 0$, there exists a $\hat{\pi}(q_h) > 0$ such that liquidity is decreasing with respect to $\pi$ in $[0, \hat{\pi}(q_h)]$.

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23 Many distributions satisfies this property including uniform distribution, gamma distribution with $\alpha > 1$, truncated normal distribution, etc.
Proof. See appendix.

The following corollary considering the monotonicity of $L$ over $\pi$ can be derived by combining results above with the fact that $L$ is increasing with respect to $\pi$ in the tight spread region.

**Corollary 4** Under two sets of assumptions, $L$ is non-monotonic with respect to $\pi$ on $[0, 1]$:

1. Suppose $G$ has increasing hazard rate, then for any $q_h > 0$, liquidity is non-monotonic with respect to $\pi$ on $[0, 1]$.

2. For small enough $q_h$, if $L$ is not a constant, it is non-monotonic with respect to $\pi$ on $[0, 1]$.

Proof. For the first statement, it is suffice to consider the situation where $\pi = 1$ is in the tight spread region. Since liquidity is increasing with $\pi$ in the tight spread region, there exists $\tilde{\pi}$ such that liquidity is increasing for $\pi \in [\tilde{\pi}, 1]$. The second statement is obvious.

This corollary, albeit simple, bears important implications on empirical analysis of HFT’s impact over market quality and the policy debate of HFT regulation. If liquidity is not changing monotonically with respect to HFT entry, regression analysis may deliver erroneous inference on welfare effects of high-frequency trading. Moreover, regulation corresponding to high-frequency trading would change the market condition facing by HFTs drastically. This result suggests that a better understanding of the high-frequency trading market micro-structure is required to determine welfare implications of regulations.

### 4.3 Simultaneous Pricing Game (Head to Head HFT)

In this section I analyze the situation when the market maker and the HFT have similar trading technologies. In this case, the HFT only observes $q_m$ (but not $x_m$) before setting her spread. It is equivalent to a game where the the HFT and the market maker determine their spreads simultaneously. This corresponds to two real world scenarios. One is that some HFTs become designated market makers, as observed in practice. With a better technology, the market maker can flicker quotes fast enough to avoid being detected by the HFT. Another possibility is that the HFT is constrained
by exchange policies or regulation requirements such that she can no longer observe the price information ahead of other traders.

We first analyze a one-shot simultaneous pricing game \((q_m, q_h, \pi)\). In this game, the market maker’s shareholding is \(q_m\); the HFT’s shareholding is \(q_h\) and enters the market with probability \(\pi\). Similar to the sequential pricing game, the buyer would purchase shares from the HFT first if the HFT and the market maker post the same spread.\(^ {24}\)

**Definition 4** An equilibrium of a one-shot simultaneous pricing game \((q_m, q_h, \pi)\) is a pair of cumulative distribution function \((H_m, H_h)\) such that \(x_m\) follows CDF \(H_m\) and \(x_h\) follows CDF \(H_m\). Let the support of \(x_m\) (\(x_h\)) be a measurable set \(X_m\) (\(X_h\)). The equilibrium satisfies following conditions:

1. Given that the HFT posts spreads according to CDF \(H_h\), the market maker posting spreads according to CDF \(H_m\) maximizes his expected payoff.
2. Given that the market maker posts spreads according to CDF \(H_m\), the HFT posting spreads according to CDF \(H_h\) maximizes her expected payoff.
3. Given \(H_h\), any \(x_m \in X_m\) yields the same expected payoff for the market maker; this expected payoff is weakly higher than the expected payoff by posting a spread \(x_m \notin X_m\).
4. Given \(H_m\), any \(x_h \in X_h\) yields the same expected payoff for the market maker; this expected payoff is weakly higher than the expected payoff by posting a spread \(x_h \notin X_h\).

The following proposition characterizes candidates of an equilibrium.

**Proposition 5** No pure strategy equilibrium exists. Let the infimum of \(X_m(X_h)\) be \(\underline{x}_m(\underline{x}_h)\) and the supremum of \(X_m(X_h)\) be \(\bar{x}_m(\bar{x}_h)\). In any mixed strategy equilibrium, \(\underline{x}_m = \underline{x}_h = \underline{x}, \bar{x}_m = \bar{x}_h = x^\ast\). \(X_m\) and \(X_h\) are dense in \([\underline{x}, x^\ast]\). \(\not\exists x_m(x_h) \in [\underline{x}, x^\ast]\) such that \(x_m(x_h)\) is posted with positive probability in equilibrium.

**Proof.** See appendix. ■

\(^{24}\)The only purpose of this assumption is to make the simultaneous pricing case comparable to the sequential pricing case. The specific tie-breaking rule does not matter.
By Proposition 5, without loss of generality, I consider an equilibrium where \( X_m \) and \( X_h \) are intervals. The equilibrium strategy can be pinned down by the market maker and the HFT’s indifference conditions.

**Proposition 6** Consider a one-shot game \((q_m, q_h, \pi)\). If \( X_m \) and \( X_h \) are both intervals, there exists a unique equilibrium satisfies following conditions:

1. If \( k(q_m) \geq \pi k(q_h) \), in the equilibrium the market maker posts spread \( x_m = x^* \) with positive probability \( \tilde{P}_m = 1 - \frac{\pi k(q_m)}{k(q_m)} \).

   \( x \) is uniquely determined by
   \[
   (1 - \pi) k(q_m) + \pi (k(q_m + q_h) - k(q_h)) = a(x) k(q_m) .
   \] (3)

   The market maker’s mixed strategy satisfies
   \[
   H_m(x) = (1 - \frac{a(x)}{a(x)}) \cdot \frac{k(q_h)}{k(q_m) + k(q_h) - k(q_m + q_h)} \quad \forall x \in [x, x^*) .
   \] (4)

   \( H_m \) satisfies \( H_m(x) = 0, \lim_{x \to x^*} H_m(x) = 1 - \tilde{P}_m \).

   The HFT’s mixed strategy satisfies
   \[
   H_h(x) = \frac{1}{\pi} \left(1 - \frac{a(x)}{a(x)}\right) \cdot \frac{k(q_m)}{k(q_m) + k(q_h) - k(q_m + q_h)} \quad \forall x \in [x, x^*) .
   \] (5)

   \( H_h \) satisfies \( H_h(x) = 0, \lim_{x \to x^*} H_h(x) = 1 \).

2. If \( k(q_m) \leq \pi k(q_h) \), in the equilibrium the HFT posts spread \( x_h = x^* \) with positive probability \( \tilde{P}_h = 1 - \frac{k(q_m)}{\pi k(q_h)} \).

   \( x \) is uniquely determined by
   \[
   k(q_m + q_h) - k(q_m) = a(x) k(q_h) .
   \] (6)

   \( H_m \) satisfies Equation (4). Moreover, \( H_m(x) = 0, \lim_{x \to x^*} H_m(x) = 1 \).

   \( H_h \) satisfies Equation (5). Moreover, \( H_h(x) = 0, \lim_{x \to x^*} H_h(x) = 1 - \tilde{P}_h \).
Proof. By Proposition 5, \( X_m \) and \( X_h \) are dense in \([x, x^*]\). Thus, in any ”regular” equilibrium, \((x, x^*) \in X_m; (x, x^*) \in X_h\). Then the uniqueness naturally follows from the equilibrium construction.

I only prove the first part of the theorem here since the calculation for the second part is similar. The only difference is that \( x^* \) is not in the support of \( X_m \) since the payoff of posting \( x^* \) is strictly lower than posting \( x^* - \epsilon \) for a small \( \epsilon \).

The HFT’s indifference condition implies

\[
(1 - \bar{P}_m)(k(q_m + q_h) - k(q_m)) + \bar{P}_m k(q_h) = a(x)k(q_h) .
\]

The market maker’s indifference condition implies

\[
(1 - \pi)k(q_m) + \pi(k(q_m + q_h) - k(q_h)) = a(x)k(q_m) .
\]

By equation (7) and (8),

\[
\bar{P}_m = \frac{a(x)k(q_h) + k(q_m) - k(q_m + q_h)}{k(q_h) + k(q_m) - k(q_m + q_h)} = 1 - \frac{\pi k(q_h)}{k(q_m)} .
\]

\( \bar{P}_m \) can be pinned down by the HFT’s indifference condition:

\[
a(x)[H_m(x)(k(q_m + q_h) - k(q_m)) + (1 - H_m(x))k(q_h)] = a(x)k(q_h) \quad \forall x \in [x, x^*] .
\]

\( H_h \) can be pinned down by the market maker’s indifference condition:

\[
a(x)\{(1-\pi)k(q_m) + \pi[H_h(x)(k(q_m + q_h) - k(q_h)) + (1-H_h(x))k(q_m)]\} = a(x)k(q_m) \quad \forall x \in [x, x^*] .
\]

Notice that \( a(x) \) is increasing with \( x \) for \( x \in [0, x^*] \) and \( k(q_m + q_h) < k(q_h) + k(q_m) \). Thus, existence and uniqueness of \( H_m \) and \( H_h \) is guaranteed by the intermediate value theorem. For the market maker (HFT), the indifference condition guarantees any strategy in support \( X_m \) (\( X_h \)) yields the same expect profit. From the proof of Proposition 5, no player has incentive to deviate to a spread smaller than \( x \) or larger than \( x^* \).

Corollary 5 The market maker’s expected profits in a one-shot game \((q_m, q_h, \pi)\) are the same under sequential and simultaneous pricing.
Proof. If \( k(q_m) > \pi k(q_h) \), the market maker would use the wide spread strategy in the sequential pricing game with expected profit

\[
(1 - F(x^*))x^*[(1 - \pi)k(q_m) + \pi(k(q_m + q_h) - k(q_h))].
\]

This coincides the expected profit in the simultaneous pricing game when \( k(q_m) > \pi k(q_h) \).

If \( k(q_m) < \pi k(q_h) \), in the sequential pricing game, the market maker would use the tight spread strategy to get

\[
(1 - F(\bar{x}))xk(q_m)
\]
where the tight spread \( \bar{x} \) is determined by \( k(q_m + q_h) - k(q_m) = a(\bar{x})k(q_h) \). This coincides the expected profit in the simultaneous pricing game when \( k(q_m) < \pi k(q_h) \).

\[\blacksquare\]

4.3.1 Steady State Characterization

The following theorem shows how to relate equilibria in one-shot games to the steady state equilibrium of the infinite period game. Moreover, since for any one-shot game, the market maker’s expected payoffs are the same under the sequential and the simultaneous pricing setting, I can compare the market maker and the HFT’s expected payoffs.

**Theorem 5** Let \( q_m = \arg\max_{q \in [0, \bar{q}]} \frac{\delta}{1 - \delta} M(q) + (w_0 - q) \).

1. \( d_t = w_t - q_m \); \( x_m(q_m) \) and \( x_h(q_m) \) follows the mixed strategy defined above consists a steady state equilibrium.\(^{25}\) In this equilibrium, the market maker’s expected payoff is \( V_m(w_0) = \frac{\delta}{1 - \delta} M(q_m) + (w_0 - q_m) \).

2. The market maker’s expected payoffs and steady state capital commitments are the same in both sequential pricing and simultaneous pricing games.

3. The HFT is strictly better off in the sequential pricing game if \( \pi \) is in the wide spread region. The HFT’s expected payoffs are the same under both settings if \( \pi \) is in the tight spread region.

\(^{25}\)This equilibrium can be micro-founded by considering a model where the HFT does not observe \( \delta \) and the market makers signals \( \delta \) with capital commitment. Then there exists a perfect Bayesian equilibrium that shares the same on path property as this steady state equilibrium.
4. In a simultaneous pricing game, the steady state liquidity is
\[ L = (1 - F(x^*)) \pi k(q_m + q_h) + (1 - \pi)k(q_m) \] \[ + \pi \int_{x^*}^{x_h} [H_m(z)H_h(z)k(q_m + q_h) + (1 - H_m(z))H_h(z)k(q_h) + H_m(z)(1 - H_h(z))k(q_m)f(z)dz + (1 - \pi)\int_{x^*}^{x_h} H_m(z)k(q_m)f(z)dz]. \]

Proof. See appendix. ■

It is informative to compare steady state market qualities in sequential and simultaneous pricing games. Since the market maker’s equilibrium capital commitments and expected payoffs are the same under two settings, pricing decisions of the market maker and the HFT lead to the difference in market quality. As in the sequential pricing game, define \([0, \hat{\pi}]\) to be the wide spread region and \([\hat{\pi}, 1]\) the tight spread region.\(^{26}\) First consider the wide spread region. In the sequential pricing game, shares are supplied with monopolistic spread \(x^*\) while in the simultaneous pricing game, shares are supplied with lower spreads for a positive probability. Thus, in the wide spread region, liquidity is higher in the simultaneous pricing game. On the other hand, the HFT’s expected profit is lower. In the sequential pricing game, she is undercutting the market maker with spread \(x_h = x^*\) while in the simultaneous pricing game, her expected payoff equals to the expected payoff of undercutting the market maker with a spread smaller than \(x^*\). Thus, if the HFT in the simultaneous pricing game has an option to upgrade her trading technology with a low cost, the HFT would pay the cost to play a sequential game with the market maker. On the other hand, in a sequential pricing game, the market maker has no incentive to upgrade his technology to catch up with the HFT. This partially justifies the observation that the HFT has superior technology than other market participants. Yet this incentive is detrimental to liquidity.

In the tight spread region, liquidity comparison between two type of games is ambiguous. In the sequential pricing game, more shares are supplied at a low spread. This is because the market maker fixes a tight spread. Yet no share is supplied with a spread between \(\bar{x}\) and \(x^*\). For buyers with valuations between \(1 + \bar{x}\) and \(1 + x^*\), only the market maker’s supply is available. In a simultaneous pricing game, a buyer with valuation \(1 + \bar{x}\) will not buy any share. Yet for a buyer with valuation slightly lower than \(1 + x^*\), in expectation he would be able to purchase more shares in a simultaneous

\(^{26}\)Note that in the simultaneous pricing game the market maker’s equilibrium spread in the tight spread region is indeed "tighter" than his spread in the wide spread region in a probabilistic sense. In the wide spread region, the market maker is posting the monopolistic spread \(x^*\) with positive probability while in the tight spread region the market maker is posting \(x^*\) with zero probability.
pricing game. The liquidity difference between two settings remains the same in the

tight spread region. This provides a convenient way for liquidity comparison.

**Proposition 7** Denote the steady state liquidity in the sequential pricing game and
the simultaneous pricing game be $L_{se}$ and $L_{sim}$.

1. $L_{sim} > L_{se}$ if $\pi$ is in the wide spread region.

2. $L_{sim} - L_{se}$ is constant for any $\pi$ in the tight spread region.

3. $L_{sim}$ is increasing in $\pi$ in the tight spread region.

**Proof.** See Appendix. ■

### 4.4 Numerical Examples

In this section, I present numerical examples to visualize results in previous sections.
In all examples, the buyer’s valuation $v$ follows a uniform distribution. The difference
lays in the distribution of the buyer’s demand $q$ and the magnitude of the HFT’s
shareholding $q_h$.

![Figure 2: Uniform Demand with Small HFT](image)

(a) Liquidity  
(b) Capital Commitment

Figure 2: Uniform Demand with Small HFT

Figure 2 depicts liquidity and the market maker’s equilibrium capital commitment
under different HFT entry probabilities when the buyer’s demand follows a uniform
distribution and the HFT’s shareholding is small. With small $q_h$, even when $\pi = 1$, the
market maker still sets a wide spread in the equilibrium; i.e., the wide spread region is
[0, 1] in this example. As shown in Figure 2b, the market maker’s equilibrium capital commitment is decreasing continuously with $\pi$ since no regime change happens.

The blue line in Figure 2a shows how steady state liquidity changes with $\pi$ in the sequential pricing game. There exists a region where liquidity is decreasing in $\pi$. In this case it is $\pi \in [0, \frac{1}{2}]$. The red line in Figure 2a shows how liquidity changes with $\pi$ in the simultaneous pricing game. As predicted by Proposition 7, liquidity in the simultaneous pricing game is higher.

![Figure 3: Uniform Demand with Large HFT](image)

Figure 3 shows liquidity and the market maker’s capital commitment when the demand follows a uniform distribution with large $q_h$. When $\pi$ is large, the market maker would use the tight spread strategy in the equilibrium. This leads to the liquidity jump in Figure 3a and the capital commitment jump in Figure 3b. Moreover, since the market maker secures his payoff against the HFT entry in the tight spread region, the equilibrium capital commitment is not changing in $\pi$. Moreover, although the average spread is lower in the tight spread region, liquidity is not necessarily higher than liquidity in a monopolistic market since the market maker commits less capital in market making.

Figure 4 shows liquidity and the market maker’s capital commitment when the buyer’s demand follows an exponential distribution. This serves as a robustness check and demonstrates that the comparative statics is similar. The difference is that liquidity remains constant in the wide spread region in the sequential pricing game. This follows from the constant hazard rate property of exponential distribution.
In this section, I consider an extension where the HFT can choose between paying a fixed cost $C$ to participate in high-frequency trading or opt out. Specifically, after observing the market maker’s capital commitment $q_m$ and spread $x_m$ (or capital commitment $q_m$ only in the simultaneous pricing game), the HFT chooses whether to participate in high-frequency trading. If the HFT does not participate, she leaves the game with zero net profit. If the HFT participates, she will successfully enter the market with probability $\pi$. Regardless of whether she enters the market or not, she pays a fixed cost $C$ at the end of the period.\footnote{By assuming the cost is paid at end of the period, I make it easier to compare this setting with the no cost scenario. Change of the payment timing does not affect the qualitative results of this model.}

Another way to model costly entry is to assume that the HFT only pays the cost $C$ upon successfully entering the market. Yet the assumption that the HFT always pays the cost is more in line with the regulatory measures taken in practice. For instance, the German High Frequency Trading Act of 2013 requires exchanges charge excessive system usage fees, including order amendments and order cancellations.\footnote{France and EU also have similar requirements on charging order cancellation fee.} This act also requires algorithm generated orders to be tagged.\footnote{For concrete examples of exchange policies complying this regulation, see Eurex. (2016) and Eurex. (2019).}

With these regulations, an HFT bears high-frequency trading cost no matter she enters the market or not.
5.1 Sequential Pricing

In the sequential pricing game, the HFT observes the market maker’s shareholding $q_m$ and spread $x_m$ before making the entry decision. Consider a one-shot game with high frequency trading cost $C$. Since the HFT observes the market maker’s shareholding and spread before posting her spread, I focus on the pure strategy equilibrium.

Definition 5 An equilibrium of a one-shot sequential pricing game $(q_m, q_h, \pi, C)$ is a triple $(x_m, \eta(x_m), x_h(x_m))$. $\eta = 0$ or 1. $\eta = 1$ indicates that the HFT participates in high-frequency trading. $\eta = 0$ indicates that the HFT does not participate. The equilibrium satisfies:

1. Given the market maker posts a spread $x_m$, the HFT posting the spread $x_h(x_m)$ maximizes her expected payoff. $\eta = 1$ if and only if the HFT’s expected payoff is greater than $C$.

2. Given the HFT posts spreads according to $x_h(x_m)$ and make entry decisions according to $\eta$, the market maker’s spread $x_m$ maximizes his expected payoff.

It is useful to consider the relation between a one-shot game with a positive participation cost ($C > 0$) and a similar game with no participation cost ($C = 0$). If the HFT enters that market, her optimal pricing strategies in two games are the same. Thus, the market maker would use the same pricing strategy if the HFT would participate in both games in the equilibrium. On the other hand, with a positive participation cost, the HFT takes her entry probability $\pi$ into account. The HFT will lose money if she chooses to participate in high-frequency trading but cannot enter the market. This gives the market maker an additional strategic advantage by improving the tight spread the market can post. If the market maker posts spread $x_m \leq x^*$ such that $\pi(1 - F(x_m))x_m k(q_h) \leq C$, the HFT would not undercut the market maker because doing so cannot cover the participation cost. Thus, when the participation cost is high enough, the market maker can prevent the HFT’s undercutting by posting a higher tight spread comparing to its counterpart in the game with no participation cost. Moreover, when facing the tight spread $x$, the HFT is indifferent between setting the wide spread $x^*$ and undercutting the market maker with spread $x$. Thus, in a game with positive participation cost, when the market maker optimally sets a higher tight spread, it must be the case that participating and setting the wide spread cannot
cover the cost either. Thus, when the participation cost is high, by setting a tight spread, the market maker effectively deters the HFT from participating.

Define the spread \( x^a \) satisfying \( \pi(1 - F(x^a))x^a k(q_h) = C \) to be the aggressive tight spread and the spread \( x \) satisfying \( a(x)k(q_h) = k(q_m + q_h) - k(q_m) \) to be the defensive tight spread. Given the equilibrium strategy in a one-shot game with \( C = 0 \), the only additional decision for the market maker to make in a similar game with \( C > 0 \) is whether to post the aggressive tight spread \( x^a \) to deter the HFT from undercutting.

As discussed above, this strategy becomes more profitable with higher entry cost \( C \). Formally, the market maker and the HFT’s pricing decisions in a one-shot game \((q_m, q_h, \pi, C)\) can be characterized as follows:

**Proposition 8** Consider a one-shot game \((q_m, q_h, \pi, C)\). Let \( \bar{C}(\pi) = \pi(1 - F(x^a))x^a k(q_h) \). If \( C \geq \bar{C} \) the market maker posts \( x_m = x^a \) and the HFT does not participate in high-frequency trading \((\eta = 0)\). For \( C < \bar{C} \):

1. If (i) \( k(q_m) < \pi k(q_h) \) and \( C > \pi(1 - F(x^a))x^a[k(q_m + q_h) - k(q_m)] \) or (ii) \( k(q_m) > \pi k(q_h) \) and \( C > \frac{\pi k(q_h)}{k(q_m)}(1 - F(x^a))x^a[\pi(k(q_m + q_h) - k(q_h)) + (1 - \pi)k(q_m)] \) the market maker posts the aggressive tight spread and the HFT does not participate in high-frequency trading \((\eta = 0)\).

2. If \( k(q_m) < \pi k(q_h) \) and \( C \leq \pi(1 - F(x^a))x^a[k(q_m + q_h) - k(q_m)] \), the market maker posts the defensive tight spread and the HFT participates \((\eta = 1)\). Upon a successful entry, the HFT posts \( x_h = x^* \) to supply the residual demand.

3. If \( k(q_m) > \pi k(q_h) \) and \( C \leq \frac{\pi k(q_h)}{k(q_m)}(1 - F(x^a))x^a[\pi(k(q_m + q_h) - k(q_h)) + (1 - \pi)k(q_m)] \), the market maker posts the wide spread and the HFT participates \((\eta = 1)\). Upon a successful entry, the HFT posts \( x_h = x^* \) to undercut the market maker.

**Proof.** See appendix. ■

Now I consider the steady state implications in the infinite period game. The same analysis as the game with no participation cost guarantees the existence of a steady state equilibrium. The following result considers the comparative statics on \( C \).

**Theorem 6** There exists \( \hat{C}(\pi, q_h) \in (0, \bar{C}) \) such that:
1. For \(0 < C \leq \hat{C}\), the steady state equilibrium is the same as the steady state equilibrium with no participation cost \((C = 0)\).

2. For \(\hat{C} < C \leq \bar{C}\), the market maker posts the aggressive tight spread \(x_m = x^a\) and sets the equilibrium capital commitment to satisfy \(\delta I_{-1}(1 - F(x_m))x_m(1 - G(q_m)) = 1\). The HFT does not participate in high-frequency trading.

3. For \(C > \bar{C}\), the steady state equilibrium is the same as the monopolistic steady state equilibrium. The HFT does not participate in high-frequency trading.

This result is intuitive. When the participation cost is low, it is not profitable for the market maker to deterring the HFT from undercutting with the aggressive tight spread strategy.\(^{30}\) In this case, the HFT’s expected payoff is larger than the participation cost \(C\). Thus, the HFT always participates and the steady state equilibrium is the same as the equilibrium with no participation cost. If the participation cost is high enough, the market maker deters the HFT’s undercutting with the aggressive tight spread strategy. Moreover, the market maker optimally commits capital to the level such that the marginal value of capital commitment equals 1. The HFT in this case does not participate in high-frequency trading. Finally, with extremely high participation cost \(C > \bar{C}\), the HFT never breaks even participating in high-frequency trading regardless of the market maker’s spread. Then the market maker becomes a monopolist.

5.2 Simultaneous Pricing

In the simultaneous pricing game, the HFT only observes the market maker’s shareholding before making the entry decision. Consider a one-shot game \((q_m, q_h, \pi, C)\). Similar to the simultaneous pricing game with no participation cost, no pure strategy equilibrium exists. I consider mixed strategy equilibrium.

**Definition 6** An equilibrium of a one-shot simultaneous pricing game \((q_m, q_h, \pi, C)\) is a triple \((H_m, \eta, H_h)\). \(\eta \in [0, 1]\) is the probability that the HFT tries to enter the market. \(x_m\) follows CDF \(H_m\) and \(x_h\) follows CDF \(H_h\). Let the support of \(x_m(x_h)\) be \(X_m(X_h)\). The equilibrium satisfies the following conditions:

\(^{30}\)The market maker may still chooses to deterring the HFT from undercutting with a tight spread strategy as in the baseline model
1. Given that the HFT posts spreads according to CDF $H_h$ and tries to enter according to $\eta$, the market maker posting spreads according to CDF $H_m$ maximizes his expected payoff.

2. Given that the market maker posts spreads according to CDF $H_m$, the HFT posting spreads according to CDF $H_h$ and tries to enter according to $\eta$ maximizes her expected payoff.

3. Given $H_h$ and $\eta$, any $x_m \in X_m$ yields the same expected payoff for the market maker; this expected payoff is weakly higher than the expected payoff by posting a spread $x_m \not\in X_m$.

4. Given $H_m$, any $x_h \in X_h$ yields the same expected payoff for the market maker; this expected payoff is weakly higher than the expected payoff by posting a spread $x_h \not\in X_h$.

To find out the equilibrium pricing strategy of the one-shot game $(q_m, q_h, \pi, C)$, consider $(q_m, q_h, \pi, 0)$, a one-shot game with no participation cost. If the HFT’s expected profit in the equilibrium of game $(q_m, q_h, \pi, 0)$ is greater than $C$, in the game $(q_m, q_h, \pi, C)$, the HFT participates with probability 1 and both players use the same pricing strategy as in game $(q_m, q_h, \pi, 0)$. Conversely, if the HFT’s expected equilibrium profit in game $(q_m, q_h, \pi, 0)$ is lower than $C$, she would mix in participation decision. This mixing has two effects. First, it reduces the expected participation cost. Second, by entering the market with a smaller probability, the HFT improves her strategic position against the market maker in the pricing game. The participating probability $\eta$ can be uniquely determined by the HFT’s indifference condition between participating or not.

**Proposition 9** Consider a one-shot simultaneous pricing game $(q_m, q_h, \pi, C)$. Define $a(x)(\pi)$ as in Proposition 6. That is, if $k(q_m) \geq \pi k(q_h)$, $a(x)(\pi) = 1 - \pi + \pi \frac{k(q_m+q_h)-k(q_h)}{k(q_m)}$; if $k(q_m) < \pi k(q_h)$, $a(x)(\pi) = \frac{k(q_m+q_h)-k(q_m)}{k(q_h)}$.

1. If $\pi(1 - F(x^*))x^*a(x)(\pi)k(q_h) \geq C$, the HFT chooses $\eta = 1$. The equilibrium of game $(q_m, q_h, \pi, C)$ coincides with the equilibrium of game $(q_m, q_h, \pi, 0)$ characterized in Proposition 6.
2. If $\pi(1 - F(x^*))x^*a(x)(\pi)k(q_h) < C$, there exists a unique $\eta \in (0, 1)$ such that $\pi(1 - F(x^*))x^*a(x)(\eta \pi)k(q_h) = C$. In the equilibrium, the HFT attempts to enter the market with probability $\eta$ and receives zero expected payoff if enters. The equilibrium of game $(q_m, q_h, \pi, C)$ coincides the equilibrium of game $(q_m, q_h, \eta \pi, 0)$

**Proof.** See appendix.

An important implication of this proposition is as follows:

**Corollary 6** For any game $(q_m, q_h, \pi, C)$, the market maker’s equilibrium payoffs are the same under both the sequential pricing and the simultaneous pricing settings.

**Proof.** See appendix.

Since the market maker receive the same expected payoffs in game $(q_m, q_h, \pi, C)$ with sequential pricing and simultaneous pricing, the market maker’s steady state capital commitment is the same under two settings:

**Proposition 10** At the steady state, the market maker commits the same amount of capital in both the sequential and the simultaneous pricing game.

### 5.3 Numerical Example

Figure 5 presents a numerical example to illustrate how market quality changes with the HFT’s participation cost. In this example, the HFT’s entry probability is fixed. The buyer’s valuation $v$ follows a uniform distribution while his demand $q_b$ follows an exponential distribution. With no participation cost, the market maker uses the tight spread strategy at the steady state.

The equilibrium can be divided into three regions. With low participation cost, it is still profitable for the HFT to enter with probability 1. Thus, the market is the same as the market with no participation cost. As the participation cost increases, the market maker’s deterring strategy becomes more profitable. Moreover, the marginal value of capital commitment also increases. Thus, the market maker’s capital commitment is increasing with participation cost. One observation is that in the sequential game, the market maker’s spread jumps downward when transiting into the deterring region. It is obvious when the market maker uses the wide spread strategy at $C = 0$. Yet this is also valid under the situation where when the market
maker uses the tight spread strategy at $C = 0$, as depicted in this numerical example. The reason is that when using the defensive tight spread strategy, the spread is decreasing with the market maker’s capital commitment. On the other hand, when deterring the HFT with an aggressive tight spread, this effect does not exist. Thus, when the market maker is indifferent between using the defensive tight spread strategy and using the aggressive tight spread strategy, the aggressive tight spread must be smaller. Finally, with high participation cost, the market becomes monopolistic.

6 Extension: Flipping

In this section, I consider the situation where the HFT can flip orders by first purchasing shares from the market maker and then resupplying them at a higher spread. The
implicit assumptions are that the HFT is not capital constrained and the market maker does not have enough time to buy shares from the inter-dealer market after the HFT purchases shares from him. In this extension, the HFT observes the market maker’s capital commitment and spread before making flipping and pricing decisions. For the ease of notation, in this section, I assume that $G$ has an unbounded support. When $G$ has a bounded support, the notation is more complicated but the general idea is essentially the same.

I first consider the HFT’s flipping and pricing decisions in a one-shot game $(q_m, q_h, \pi)$. Notice that if the HFT flips shares from the market maker, her resupplying spread must be higher than the market maker’s spread. In this case, her optimal spread is $x_h = x^*$. If the market maker holds $q_m$ shares and his spread is $x_m < x^*$, the HFT’s expected payoff when buying $q_f$ shares from the market maker is

$$r(q_f) = (1 - F(x^*))x^*[k(q_m + q_h) - k(q_m - q_f)] - x_m q_f .$$  \hspace{1cm} (12)

The first term of the right hand side is the expected gain from selling $q_h + q_f$ shares at spread $x^*$ when the market maker is left with $q_m - q_f$ shares at a lower spread $x_m$. The second term of the right hand side is the premium paid by the HFT. The HFT pays $1 + x_m$ for each flipped share. If the buyer does not purchase these shares at price $1 + x^*$, the HFT only receives 1 selling each unit to the inter-dealer market. Notice that by purchasing shares from the market maker, the HFT reduces the market maker’s supply and thus the competition. Since the market maker is selling at a lower spread, the more the HFT purchases from the market maker, the easier it is for the HFT to sell shares. In other words, the marginal benefit of purchasing shares is increasing in $q_f$ and the HFT would follow an “all or nothing” strategy. The formal statement is the following proposition:

**Proposition 11** The HFT either purchases the market maker’s entire shareholding $q_m$ or nothing. In other words, $q_f = q_m$ or 0.

**Proof.** Notice that

$$r'(q_f) = (1 - F(x^*))x^*[1 - G(q_m - q_f)] - x_m ,$$

$$r''(q_f) = (1 - F(x^*))x^* g(q_m - q_f) > 0 .$$

31Her shareholding $q_h$ only reflects the exogenous market condition
This implies the maximum is achieved at the boundary $q_f = 0$ or $q_f = q_m$. ■

Consider the market maker’s pricing problem. When his spread is low enough, by Proposition 11, the HFT would purchase all shares from him upon entry. Thus, comparing to the baseline case, the market maker has an additional option to strategically post a low spread $x_m^f$ to induce flipping. $x_m^f$ can be pinned down by considering the HFT’s indifference conditions: Buying all shares from the market maker should be more profitable than buying nothing; buying all shares from the market maker should be more profitable than undercutting the market maker. For the market maker, any spread lower than $x_m^f$ cannot be optimal for him. This can be summarized by the following lemma:

**Lemma 3** $x_m^f$ satisfies

\[
(1 - F(x^*)) x^* k(q_m) \geq x_m^f q_m
\]

and

\[
(1 - F(x^*)) x^* k(q_m + q_h) \geq x_m^f q_m + (1 - F(x_m^f)) x_m^f k(q_h).
\]

At least one inequality is binding. Moreover, if Inequality (14) binds, the flipping strategy dominates the tight spread strategy.

**Proof.** Inequality (13) guarantees that the HFT is better off with $q_f = q_m$ than with $q_f = 0$ when setting $x_h = x^*$. Inequality (14) guarantees that the HFT is better off with $w = q_m$ than undercutting the market maker. Since the market maker is better off choosing the highest possible spread given the HFT is flipping orders, one of the inequalities must be binding.

Moreover, if inequality (13) binds, $x_m^f = \frac{(1 - F(x^*)) x^* k(q_m)}{q_m}$. Otherwise, since $(1 - F(x))x$ is increasing in $[0, x^*]$, there exists a unique $x_m^f \in (0, \frac{(1 - F(x^*)) x^* k(q_m)}{q_m})$ such that

\[
(1 - F(x^*)) x^* k(q_m + q_h) = x_m^f q_m + (1 - F(x_m^f)) x_m^f k(q_h).
\]

If inequality (14) binds,

\[
(1 - F(x^*)) x^* k(q_m) > x_m^f q_m
\]

and

\[
(1 - F(x^*)) x^* k(q_m + q_h) = x_m^f q_m + (1 - F(x_m^f)) x_m^f k(q_h).
\]

Then,

\[
(1 - F(x^*)) x^* [k(q_m + q_h) - k(q_m)] < (1 - F(x_m^f)) x_m^f k(q_h).
\]
Thus,

$$x^f_m > x.$$  \hspace{1cm} (18)

In this case, the tight spread strategy is never optimal because the market maker can raise the spread to \(x^f_m\) to achieve higher expected payoff.

The wide and tight spread strategies are still available to the market maker. Specifically, if the market maker uses a wide spread, his expected payoff is

$$(1 - F(x^*))x^*\left[\pi(k(q_m + q_h) - k(q_h)) + (1 - \pi)k(q_m)\right].$$

If \(x > x^f_m\), the market maker’s expected payoff from the tight spread strategy is

$$(1 - F(x))x^*k(q_m).$$

If the market maker posts \(x^f_m\), his expected payoff is

$$\pi x^f_m q_m + (1 - \pi)(1 - F(x^f_m))x^f_m k(q_m).$$

An important observation is that, if the market maker expects the HFT to flip shares, the market maker’s expected payoff is increasing with \(\pi\). This is because with flipping, the HFT is providing insurance for the market makers. When \(\pi\) is large enough, the market maker would always induce flipping.

**Proposition 12** For any \(q_m\) and \(q_h\), if \(\pi\) is high enough, the market maker sets spread \(x^f_m\) in the equilibrium.

**Proof.** Consider the situation when \(\pi = 1\). If inequality (13) is binding, the market maker’s expected payoff with flipping is \(x^f_m q_m = (1 - F(x^*))x^*k(q_m)\). This is the highest possible payoff. If inequality (14) is binding, by Lemma 3, the tight spread strategy is dominated. Moreover,

$$x^f_m q_m = (1 - F(x^*))x^*k(q_m + q_h) - (1 - F(x^f_m))x^f_m k(q_h) > (1 - F(x^*))x^*k(q_m + q_h) - k(q_h)).$$

Thus, setting \(x_m = x^f_m\) is better than setting \(x_m = x^*\). By continuity, for \(\pi\) large enough, it is always optimal to induce flipping.

Now consider the infinite period game. Although the market maker can be insured by the HFT, he does not have the inventive to increase capital commitment indefi-
ninitely. This is because the expected payoff by committing $q_m$ is upper-bounded by $(1 - F(x^*))x^* k(q_m)$, the monopolistic payoff. For $q_m \to \infty$, $\frac{\delta}{1-\delta} x_M^f \to 0$. This implies an upper-bound exists for the market maker’s capital commitment in the steady state equilibrium. The existence of a steady state equilibrium (when the market maker has a large initial net worth) can be proved in a similar manner.

**Proposition 13** For large enough $w_0$, a steady state equilibrium exists.

**Proof.** The proof is omitted since it is similar to the existence result proved in previous sections.

![Figure 6: Equilibrium Volume and Price with Flipping](image)

(a) Liquidity v.s. Volume  
(b) Buyers’ v.s. Average Spread

Figure 6 presents a numerical example showing the equilibrium liquidity and average spread when the HFT is able to flip orders. When the HFT’s entry probability is low, whether the HFT can flip orders or not does not make a difference in the equilibrium. Since the benefit of inducing flipping cannot cover the cost of setting low spread, the market maker sticks to the equilibrium strategy inducing no flipping.

When the HFT’s entry probability becomes large, the equilibrium enters the flipping region where the market maker sets a low spread to induce flipping. In this region, a large portion of transactions happens between the market maker and the HFT. With the trading technology advantage, the HFT purchases all low price shares when entering the market. The buyer only benefits from the market maker’s low spread when the HFT fails to enter the market. This suggests that it is important to separate trades between liquidity suppliers (the market maker and the HFT) and trades from liquidity suppliers to the buyer. Otherwise, as shown in Figure 6a, the
expected trading volume and the average spread do not accurately reflect the market quality. In Figure 6a, the expected shares sold to the buyer only increase modestly in $\pi$ comparing to the expected trading volume. Moreover, the average spread remains low in the flipping region while the buyer’s average spread is much higher and increasing in $\pi$. This is because the majority of low price shares are purchased by the HFT. When the HFT becomes more likely to enter the market, the buyer becomes less likely to purchase cheap shares. This implies that if the HFT has far superior trading technology than the market maker, the entry of HFT only has limited benefits for the buyer. Furthermore, if we look at the overall trading data, the welfare effect of high-frequency trading will be overestimated.

7 Policy Implications on Market Quality

In this section, I collect results developed in previous sections to discuss the impact of different regulation policies. Taking the baseline model as a starting point, this paper examines three types of regulations on high-frequency trading: changing the HFT’s entry probability $\pi$, leveling the trading technology difference between the HFT and the market maker and imposing a lump sum high-frequency trading cost $C$.

7.1 Altering the HFT’s Entry Probability

In practice, the HFT’s entry probability hinges on the HFT’s ability to detect other investor’s orders and acquire shares in a timely manner. Regulations changing the HFT’s detection and purchasing capacities affect the HFT’s entry probability. For instance, banning co-location or forming a more integrated market would decrease the HFT’s entry probability. Upgrading exchange’s trading system without further restricting high-frequency trading would increase the HFT’s entry probability.

This model predicts that in a market where the HFT’s entry probability is high ($\pi$ in the tight spread region), encouraging the HFT entry is beneficial to market quality. The reason is that the market maker is using the tight spread strategy facing a fierce competition from the HFT. Since the HFT is charging the higher spread to fill the residual demand, increasing HFT entry leads to more liquidity supplied by the HFT without altering the market maker’s incentive to commit capital. On the other hand, in a market with low HFT entry probability, the market maker responses to the
competition by committing less capital in market making. Liquidity would increase only if the benefit from the HFT’s entry dominates the market maker’s incentive in decreasing capital commitment. Moreover, this model predicts that banning high-frequency trading does not necessarily deteriorate liquidity. Yet the spread would become larger due to the lack of competition.

7.2 Leveling the Trading Technology

This type of regulation “levels the playground” by making the market maker’s trading technology comparable to the HFT’s. For instance, the regulator can encourage HFTs to become designated market makers or incentivize existing market makers to upgrade their trading technologies. The batch auction proposed by Budish, Cramton, and Shim (2015) also achieves this goal since the market maker would have a chance to revise his order.

This model predicts that this policy is beneficial when the HFT’s entry probability is low (π in the wide spread region). Without superior technology, the HFT mixes in spreads in equilibrium rather than undercut the market maker at the monopolistic spread. This drives down the average price and improves market quality. When the HFT’s entry probability is high, this model predicts that leveling the trading technology leads to less shares for low valuation buyers (because the market maker now mixes rather than stick to the tight spread) and more shares for high evaluation buyers (because the HFT now mixes rather than stick to the monopolistic spread). The total effect can be ambiguous.

7.3 Imposing High-frequency Trading Participation Tax

The third type of regulation imposes a lump-sum participation cost over high-frequency trading. For example, regulations in France and Germany require a fee to be charged based on both executed and canceled orders. Regulation in Germany further requires all traders to tag algorithm generated orders. These regulations essentially induce a participation cost on high-frequency trading.

In this model, a low participation tax will not change the market quality. If the tax is high, the HFT would (at least partially) exit the market. The market maker’s spread increases with the tax and converges to the monopolistic spread. Moreover, the market maker also commits more capital in market making. The directional change
of liquidity depends on which effect dominates. Yet it is certain that extremely high participation cost always hurts the market.

8 Conclusion

My paper studies how high-frequency trading changes market quality through affecting the traditional market maker’s capital commitment and pricing decisions. I consider a long-run market maker facing competition from possibly entering short-run HFTs in providing liquidity for buyers. In the steady state, the long-run market maker responds to the potential competition from the HFT entry by reducing his spread and committing less capital in market making. Although a higher HFT entry probability leads to lower market maker’s spread in the steady state, the market maker also commits less capital in market making. The latter effect impairs market quality. Thus, when taking the market maker’s capital commitment channel into consideration, high-frequency trading does not necessarily improves market quality though it always reduces the average spread. Moreover, in my model, the trading speed difference between the HFT and the market maker affects market quality. When the HFT’s entry probability is low, “leveling the playground” by making the market maker and the HFT trade at the same speed improves market quality.

I further consider two extensions. The first extension introduces a high-frequency trading participation cost and endogenizes the HFT’s participation choice. When the HFT trades faster than the market maker and the participation cost is low, market quality remains the same. On the other hand, when the cost is high and the HFT trades faster than the market maker, the market maker optimally sets a spread to deter the HFT from entering the market. Moreover, although the HFT does not participate in high-frequency trading after the participation cost passes a certain threshold, the cost level still affects the market quality. The reason is that the market maker’s pricing strategy to deterring participation depends on the cost. When the HFT and the market maker trade at the same speed, the model’s prediction is similar except that the HFT mixes in participation facing a high cost. In the second extension, the HFT can “flip shares” by purchasing shares from the market maker and resupplying them at a higher spread. This behavior effectively provides an insurance for the market maker. Thus, when the HFT’s entry probability is high, the market maker posts a low spread to induce flipping. Yet the buyer may not benefit from the
low price since most of the cheaper shares are acquired by the HFT. This extension demonstrates the importance to exclude the trading between liquidity supplier when evaluating market quality. Otherwise, market quality would be overestimated with an overestimation of the expected trading volume and an underestimation of the average spread.

Finally, I want to emphasize several important insights of this model. First, the price information alone cannot capture all important aspects of market quality; the volume information is equally important. Second, more high-frequency trading does not necessarily improves market quality since it reduces the market maker’s willingness to commit capital in market making. Third, the relative trading speed between the market maker and the HFT affects market quality. When the HFT’s entry probability is low, letting the market maker and the HFT trade at same speed improves market quality. Fourth, it is important to separate the trades among liquidity suppliers to avoid an overestimation of market quality and the high-frequency trading’s welfare effect.

References


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A Base Case Proofs and Claims

A.1 Useful Results

Lemma 4 \((1 - F(x))x\) is unimodal.

Proof. 
\[
((1 - F(x))x)' = 1 - F(x) - xf(x) \\
= \left(\frac{1 - F(x)}{f(x)} - x\right)f(x) \\
\tag{19}
\]

\(f(x) \neq 0\) since the hazard rate is non-decreasing. Moreover, \(\frac{1 - F(x)}{f(x)} - x\) is continuous and decreasing. Easy to see that \((1 - F(x))x\) cannot take maximum at boundaries of the support. Thus, there exists a unique \(x^*\) such that \(1 - F(x^*) - x^*f(x^*)\). Easy to see that for \(x > x^*\), \(((1 - F(x))x)' < 0\); for \(x < x^*\), \(((1 - F(x))x)' > 0\).  

A.2 No HFT

A.2.1 Proof of Theorem 1

Proof. First consider a relaxed problem with \(d \in [-\bar{q}, w]\). Conjecture that the optimal policy is \(d_t = w_t - \bar{q}\) and \(x_t = x^*\) where \(x^* = \arg\max (1 - F(x))x, \forall t\). If
this policy is indeed the optimal policy for this relax problem, then for \( w_0 \geq \bar{q} \), this optimal policy is applicable and thus also optimal for the more constrained original problem. This proposition also implies that the market maker’s payoff is linear in \( w_0 \) with \( w_0 \geq \bar{q} \).

We use a method similar to one-shot deviation principle to establish the optimality of proposed policy. Notice that although the market maker discounts future dividends, the per-period dividend does not necessarily have a uniform bound. Thus, I directly check that this problem is continuous at infinity.

Consider two dividend and pricing policies \( \{d_t, x_t\}_{t=0}^{\infty} \) and \( \{\tilde{d}_t, \tilde{x}_t\}_{t=0}^{\infty} \). \( d_t, x_t, \tilde{d}_t, \tilde{x}_t \) are functions of \( h_t \), the history of the first \( t-1 \) periods. We suppress the dependence for the ease of notation. Consider the case when \( d_t = \tilde{d}_t \) and \( x_t = \tilde{x}_t \) for \( t \leq T \). Define the absolute value of the difference in expected payoffs between two policies to be \( D_T \). We have

\[
D_T = |E_0\left( \sum_{i=0}^{\infty} \delta^i (d_i - \tilde{d}_i) \right)|
\]

\[
\leq |E_0\left( \sum_{i=0}^{\infty} \delta^i c_i \right)| + \delta^{T+1} \frac{1}{1-\delta} \bar{q}
\]

\[
\leq \delta^{T+1} E_0(w_{T+1}) + \sum_{i=0}^{\infty} \delta^i \bar{x} E_G(q) + \delta^{T+1} \frac{1}{1-\delta} \bar{q}
\]

\[
= \delta^{T+1} E_0(w_{T+1}) + \delta^{T+1} \frac{1}{1-\delta} \bar{x} E_G(q) + \delta^{T+1} \frac{1}{1-\delta} \bar{q} .
\]

The first inequality is because the worst dividend plan after period \( T \) is to pay \(-\bar{q}\) for all periods. The second inequality is because for any period \( t \), the expected profit is \((1 - F(x_t)) x_t E_G(\min(q, w_t - d_t))\).\(^3\) This is uniformly bounded by \( \bar{x} E_G(q) \). Thus, in each period, the expected dividend is bounded by \( \bar{x} E_G(q) \) plus part of the market maker’s net worth at \( T + 1 \). Notice that commit more shares cannot improve the expected dividend bound since \( E_G(\min(q, w)) \leq E_G(q) \). Thus, the expected discounted dividend payout is bounded by the case when the market maker pays dividend equal to the entire net worth at \( t = T + 1 \) and pays the upper bound of expected profit in each period.

Notice that \( \delta^{T+1} \frac{1}{1-\delta} \bar{x} E_G(q) \rightarrow 0 \) and \( \delta^{T+1} \frac{1}{1-\delta} \bar{q} \rightarrow 0 \) as \( T \rightarrow \infty \). Moreover,

\(^3\) We define \( h^0 = \emptyset \).
\(^3\) \( E_G \) means \( q \) follows distribution \( G \), I suppress the time notation because demands are i.i.d.
\[ E_t(w_{t+1}) \leq w_t + \bar{x}E_G(q) + \bar{q}. \] This implies that
\[
\delta^{T+1}E_0(w_{T+1}) \leq \delta^{T+1}[w_0 + (T+1)(\bar{x}E_G(q) + \bar{q})].
\] (20)

Thus, \( \delta^{T+1}E_0(w_{T+1}) \to 0 \) as \( T \to \infty \). Thus, for any two policies that differ only after period \( T \), as \( T \to \infty \), \( D_T \to 0 \).

Since this game is continuous at infinity, if there exists a profitable deviation, then there exists a profitable deviation such that the deviating policy is different from the candidate policy for finite periods. Consider a deviation where the deviating policy is different from the candidate policy for \( n \) periods. For \( t \geq n \), the deviating policy switches back to the candidate policy \( \hat{d}_t = w_t - \bar{q} \) and \( \hat{x}_t = x^\ast \). Consider the market maker at time \( t = n \) with net worth \( w_n \). Suppose the deviating policy specifics \( \hat{d}_n = w_n - \hat{w} \) and \( x_n = \hat{x}_n \). Then at period \( n \), the difference between expected payoffs of two policies is
\[
E_n(d_n - \hat{d}_n + \delta d_{n+1} - \delta \hat{d}_{n+1}) = \hat{w} - \bar{q} + \delta(1 - F(x^\ast))x^\ast E_G(min(q, \bar{q}))
- \delta(1 - F(\hat{x}_n))\hat{x}_n E_G(min(q, \hat{w})) - \delta(\hat{w} - \bar{q})
\geq (1 - \delta)(\hat{w} - \bar{q}) + \delta(1 - F(x^\ast))x^\ast[E_G(min(q, \bar{q})) - E_G(min(q, \hat{w}))].
\]
The inequality follows from \( (1 - F(\hat{x}_n))\hat{x}_n \leq (1 - F(x^\ast))x^\ast \).

Define
\[
A(y) = (1 - \delta)(y - \bar{q}) + \delta(1 - F(x^\ast))x^\ast[E_G(min(q, \bar{q})) - E_G(min(q, y))].
\]
Then
\[
A'(y) = 1 - \delta - \delta(1 - F(x^\ast))x^\ast(1 - G(y)),
A''(y) = g(y) > 0.
\]
Since \( A'(y) \) is monotone, \( A'(y) = 0 \) has at most one solution and upon which \( A(y) \) achieves minimum. Note that \( A'(y) = 0 \) implies
\[
\frac{\delta}{1 - \delta}(1 - F(x^\ast))x^\ast(1 - G(y)) = 1.
\]

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Thus, \( A(y) \) achieves minimum at \( y = \bar{q} \) and \( A(\bar{q}) = 0 \). Thus,

\[
E_n(d_n - \hat{d}_n + \delta d_{n+1} - \delta \hat{d}_{n+1}) \geq 0 .
\]

(21)

This implies that if there exists a profitable deviation such that the deviating policy differs from the candidate policy for \( n \) periods, then at period \( n \), the market maker should adopt the candidate policy. Same reasoning then shows that the market maker should adopt the candidate policy at period \( n - 1 \). The backward induction goes back to period 1. Since \( n \) is arbitrary and this problem is continuous at infinity, no profitable deviation exists and the candidate policy is optimal.

\[\blacksquare\]

A.2.2 Existence of Value Function

Proposition 14 There’s a unique \( V \) such that it is continuous and strictly increasing in \( w \).

Proof. We focus on \( V(w) \) for \( w \in [0, \hat{w}] \). Moreover, since \( V(w) = w - \bar{q} + V(\bar{q}) \)

Define operator \( T \) to be

\[
(Tl)(w) = \sup_{d,x} d + \delta \{ F(x)l(w - d)
\]

\[
+ (1 - F(x))\left[ \int_0^{w-d} (l(\min(\bar{q}, w - d + xq)) + \max(0, w - c + xq - \bar{q})g(q)dq +
\]

\[
(1 - G(w - d))(l(\min(\bar{q}, (1 + x)(w - d))) + \max(0, (1 + x)(w - d) - \bar{q}))\right]\}
\]

(22)

satisfying \( c \in [0, w] \).

First check that for large enough \( \bar{K}, l(w) \leq \bar{K} \Rightarrow Tl(w) \leq \bar{K} \). Thus, the value function is bounded and Blackwell condition is applicable. Easy to check \( T \) satisfies monotonicity and discounting.

By contract mapping theorem, operator \( T \) has a unique fixed point \( V \). Easy to see \( T \) maps increasing functions to strictly increasing functions. This implies \( V \) must be increasing. \[\blacksquare\]
A.3 Sequential Pricing

A.3.1 Proof of Lemma 1

**Proof.** If $x_m > x^*$, easy to see the HFT’s optimal strategy is to set $x_h = x^*$. Consider the situation when $x_m \leq x^*$. For $x_h \leq x_m$, the HFT’s expected net profit is

$$(1 - F(x_h))x_h k(q_h),$$

which attains maximum at $x_h = x_m$ by lemma A.1. For $x_h > x_m$, the HFT’s expected net profit is

$$(1 - F(x_h))x_h [k(q_h + q_m) - k(q_m)],$$

which attains maximum at $x_h = x^*$. ■

A.3.2 Proof of Lemma 2

**Proof.** First notice that $x_m > x^*$ cannot be optimal. If $x_m > x^*$, the HFT’s best response is to set $x_h = x^*$ and the market maker’s expected net profit is

$$(1 - F(x_m))x_m [\pi (k(q_h + q_m) - k(q_h)) + (1 - \pi)k(q_m)] < (1 - F(x^*))x^* [\pi (k(q_h + q_m) - k(q_h)) + (1 - \pi)k(q_m)].$$

This implies the market maker will be better off by setting $x_m = x^*$.

Next, there is a unique $x < x^*$ such that if $x_m = x$, the HFT is indifferent between $x_h = x^*$ and $x_h = x_m$. For any $x_m < x^*$, the HFT’s expected net profit with $x_h = x^*$ is

$$(1 - F(x^*))x^* [k(q_h + q_m) - k(q_m)];$$

the HFT’s expected net profit with $x_h = x_m$ is

$$(1 - F(x_m))x_m k(q_h).$$

Since $(1 - F(x))x$ is increasing for $x \in [0, x^*]$ and $k(q_h + q_m) - k(q_m) < k(q_h)$, there exists a unique $x \in (0, x^*)$ such that

$$(1 - F(x^*))x^* [k(q_h + q_m) - k(q_m)] = (1 - F(x))x k(q_h),$$

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or equivalently,
\[ a(x)k(q_h) = k(q_m + q_h) - k(q_m) . \]

Finally, check that any other pricing strategy of the market maker is dominated either by \( x_m = x^* \) or \( x_m = \bar{x} \). If \( x_m \in (\bar{x}, x^*) \), the HFT would set \( x_h = x_m \). The market maker’s expected net profit is
\[
(1-F(x_m))x_m[\pi(k(q_h+q_m)-k(q_h))+(1-\pi)k(q_m)] < (1-F(x^*))x^*[\pi(k(q_h+q_m)-k(q_h))+(1-\pi)k(q_m)] .
\]
Thus, he would be better off switch to \( x_m = x^* \). For \( x_m \in (0, \bar{x}) \), the HFT would set \( x_h = x^* \). The market maker’s expected net profit is \( (1-F(x_m))x_mk(q_m) < (1-F(\bar{x}))\bar{x}k(q_m) \). This suggests that he would be better off to set \( x_m = \bar{x} \). ■

A.3.3 Proof of Proposition 1

**Proof.** For any \( q_m \), the defensive spread can be determined by the equation
\[
a(x(q_m)) = \frac{k(q_m + q_h) - k(q_m)}{k(q_h)} . \tag{23}
\]
The defensive strategy is optimal if
\[
a(x(q_m))k(q_m) \geq [k(q_m + q_h) - k(q_h)]\pi + (1-\pi)k(q_m) . \tag{24}
\]
Subtract \( k(q_m) \) from both sides,
\[
\frac{k(q_m + q_h) - k(q_h) - k(q_m)}{k(q_h)}k(q_m) \geq \pi[k(q_m + q_h) - k(q_h) - k(q_m)] . \tag{25}
\]
Since \( k(q_m + q_h) - k(q_h) - k(q_m) < 0 \) for \( q_m > 0, q_h > 0 \),
\[
\frac{k(q_m)}{k(q_h)} \leq \pi . \tag{26}
\]
■

A.3.4 Proof of Theorem 2

**Proof.** Consider a relaxed problem where \( d_t \in [-\bar{q}, w_t] \). Given HFT’s best response, this problem can be reduced to a decision problem of the market maker. Suppose
the policy proposed in this theorem is not optimal. Using the same argument as in the proof of theorem 1, this game is continuous at infinity. Thus, I can focus on considering a finite period deviation. Consider a better policy with deviation for at most \( n \) periods. At period \( n \), I only need to consider the difference of consumptions in period \( n \) and \( n+1 \). If \( c_n \neq w_n - q_m \), by Proposition 1, the market maker’s optimal strategy is to set \( x_m = \hat{x}_m(q_m) \) and get expected net profit \( M(q_m) \). This is exactly the original policy. Suppose \( d_n = w_n - \hat{w} \). Since the market maker’s maximum expected profit at period \( n \) is \( M(\hat{w}) \),

\[
M(q_m) > w_n - q_m + \delta M(q_m) .
\]

This implies

\[
\frac{\delta}{1 - \delta} M(\hat{w}) - \hat{w} > \frac{\delta}{1 - \delta} M(q_m) - q_m .
\]

Since \( q_m = \arg\max_{w \in [0, \bar{q}]} \frac{\delta}{1 - \delta} M(w) + (w_0 - w) \), if such \( \hat{w} \) exists, it must be \( \hat{w} > \bar{q} \). Since \( M(w) = \max((1 - F(x^*)) x^* [k(w + q_h) - k(q_h)], (1 - F(x(w)) x(w)) k(w)) \) is continuous and differentiable almost everywhere. Easy to see that \( \frac{\delta}{1 - \delta} M'(w) \leq 1 \) for \( w \geq \bar{q} \). Thus, if \( \hat{w} > \bar{q} \),

\[
\frac{\delta}{1 - \delta} M(\hat{w}) - \hat{w} > \frac{\delta}{1 - \delta} M(q_m) - q_m .
\]

This implies that any \( n \) period deviation can be dominated by a \( n - 1 \) period deviation for all \( n \). Repeating this argument implies that no finite period deviation exists and establishes the optimality of the proposed policy. Since \( w_0 > \bar{q} \), the proposed policy is implementable in the original problem and is thus optimal. The HFT’s optimality condition is satisfied since the HFT always plays the best response.

Notice that this proof works as long as \( M(w) \) is continuous, differentiable except at finite number of points and \( \frac{\delta}{1 - \delta} M'(w) \leq 1 \) for \( w \geq \bar{q} \). ■

A.3.5 Proof of Corollary 2

Proof. Two conditions are derived from the first order condition of \( \max_{w \in [0, \bar{q}]} \frac{\delta}{1 - \delta} M(w) + (w_0 - w) \). To see the market maker never fully exit the market, notice that \( x \rightarrow x^* \)
when \( x_m \to 0 \). Since \( \bar{q} > 0 \), \( \frac{\delta}{1-\delta} (1 - F(x^*)x^* > 1 \). Then there always exists a \( q_m > 0 \) such that \( \frac{\delta}{1-\delta} (1 - F(x^*))x^*(1 - G(q_m)) = 1 \).

### A.3.6 Proof of Theorem 3

**Proof.** For the ease of notation, let \( q_m^\pi \) be the equilibrium capital commitment of the market maker when the HFT’s entry probability is \( \pi \). Let \( \bar{x}(q) \) be the tight spread when the market maker’s shareholding is \( q \). Notice that \( x \) does not depend on \( \pi \).

Consider a sequential pricing game with \( \pi = 1 \). If in the steady state equilibrium, the market maker uses the wide spread strategy with shareholding \( q_{m_1} \), then by Theorem 2, for any \( q \in [0, \bar{q}] \),

\[
(1 - F(x^*))x^* (k(q_{m_1} + q_h) - k(q_h)) + (w_0 - q_{m_1}^1) \geq (1 - F(\bar{x}(q))) \bar{x}(q) k(q) + (w_0 - q_{m_1}^1) .
\]

That is, adopting the wide spread strategy with shareholding \( q_{m_1}^1 \) is better than using the tight spread strategy at any level of shareholding. By Proposition 2, the single period payoff for the tight spread strategy is constant regardless of \( \pi \) and the single period payoff for the wide spread strategy is decreasing with \( \pi \). Thus, for \( \pi < 1 \), the market maker’s equilibrium strategy must still be the wide spread strategy. This corresponds to the case where \( \hat{\pi} = 1 \).

If the market maker is using the tight spread strategy at a \( \pi_1 < 1 \), then for \( \pi_2 > \pi_1 \), by a similar argument with Proposition 2, the market maker would still use the tight spread strategy. Moreover, \( q_{m_1}^{\pi_1} = q_{m_2}^{\pi_2} \) and thus \( \bar{x}(q_{m_1}^{\pi_1}) = \bar{x}(q_{m_2}^{\pi_2}) \) and the market maker has the same equilibrium payoff. Denote this equilibrium payoff when the market maker is using a tight spread strategy by \( V_{\pi}^{\text{tight}} \). Define \( V_{\pi}^{\text{wide}} = (1 - F(x^*))x^* [\pi(k(q + q_h) - k(q_h)) + (1 - \pi)k(q)] + (w_0 - q) \) where \( q \) satisfies \( (1 - F(x^*))x^* [1 - \pi G(q + q_h) - (1 - \pi)G(q)] = 1 \). \( V_{\pi}^{\text{wide}} \) is the equilibrium payoff for the market maker if the wide spread strategy is adopted in the equilibrium. \( V_{\pi}^{\text{wide}} \) is continuous and decreasing with respect to \( \pi \). Moreover, \( V_{1}^{\text{wide}} \) goes to the monopolistic payoff. Since \( V_{\pi}^{\text{tight}} > V_{1}^{\text{wide}} \) and \( V_{\pi}^{\text{tight}} \) is bounded away from the monopolistic payoff, there exist \( \hat{\pi} \in (0, 1) \) such that \( V_{\hat{\pi}}^{\text{wide}} = V_{\pi}^{\text{tight}} \). By previous argument, the market maker adopts the wide spread strategy if \( \pi < \hat{\pi} \) and tight spread strategy if \( \pi > \hat{\pi} \).

In the tight spread region, \( L = (1 - F(x_m))k(q_m) + \pi(F(x^*) - F(x_m))(k(q_m + q_h) - k(q_m)) \). Since in the tight spread region, \( q_m \) and \( x_m = \bar{x}(q_m) \) is not changing with respect to \( \pi \), \( L \) is increasing in \( \pi \).
For the third statement, consider a game at \( \pi = \hat{\pi} < 1 \). Two equilibrium shareholdings for market maker, \( w_m^{\text{tight}} \) and \( w_m^{\text{wide}} \) both exist. If the market maker chooses shareholding \( q_m^{\text{tight}} \) (\( q_m^{\text{wide}} \)), he will play the tight (wide) spread strategy in the equilibrium. By Proposition 1, \( k(q_m^{\text{wide}}) \geq \hat{\pi} k(q_h) \geq k(q_m^{\text{tight}}) \). This implies \( q_m^{\text{wide}} \geq q_m^{\text{tight}} \). For \( \pi < \hat{\pi}, q_m > q_m^{\text{wide}} \geq q_m^{\text{tight}} \). This establishes that the market maker always have a higher equilibrium shareholding in the wide spread region.

### A.3.7 Proof of Corollary 3

**Proof.** Fix \( \pi = 1 \). Notice that for any fixed \( q_m > 0 \),

\[
a(x) = \frac{k(q_m + q_h) - k(q_m)}{k(q_h)} \to 1 - G(q_m) < 1 \text{ as } q_h \to 0 .
\]

This implies that the market maker’s payoff by using the tight spread strategy is bounded away from the monopolistic payoff as \( q_h \to 0 \). On the other hand, if the market maker uses the wide spread strategy, easy to see as \( q_h \to 0 \), the expected payoff converges to the monopolistic payoff. Thus, for small enough \( q_h \), the market maker would use the wide spread strategy at the steady state even when \( \pi = 1 \).

### A.3.8 Proof of Proposition 4

**Proof.** For any \( w \geq 0 \), given \( G \) is an exponential distribution, \( k(s) = E_G(\min(q, s)) = E_G(q)G(s) \). By theorem 1, when no HFT exists, the market maker’s capital commitment \( \bar{q} \) satisfies \( \frac{\delta}{1-\delta}(1 - F(x^*)) x^*(1 - G(\bar{q})) = 1 \). By corollary 2, when the market maker posts a wide spread in the equilibrium, his capital commitment satisfies \( \frac{\delta}{1-\delta}(1 - F(x^*)) x^* [(1 - \pi)(1 - G(q_m)) + \pi(1 - G(q_m + q_h))] = 1 \). Thus, \( G(\bar{q}) = \pi G(q_m + q_h) + (1 - \pi)G(q_m) \).

Then,

\[
k(\bar{q}) = E_G(q)G(\bar{q}) = E_G(q)(\pi G(q_m + q_h) + (1 - \pi)G(q_m)) = \pi k(q_m + q_h) + (1 - \pi)k(q_m) . \tag{28}
\]

This implies that liquidity does not depend on \( \pi \) in the wide spread region and is equal to the liquidity in a monopolistic market.
A.3.9 Proof of Theorem 4

**Proof.** Since I take other parameter as fixed and only change $\pi$, in the proof, I represent liquidity by $L(\pi)$ and the market maker’s capital commitment by $q_m(\pi)$ to make their dependences on $\pi$ explicit while suppressing all other dependences.

As $\pi \to 0$, the market maker’s payoff by posting the wide spread converges to the monopolistic payoff. By continuity of the market maker’s payoff, for $\pi$ small enough, the market maker would post a wide spread in the steady state equilibrium. At the wide spread region, the market maker’s capital commitment $q_m(\pi)$ satisfies

$$\frac{\delta}{1 - \delta}(1 - F(x^*))x^*[(1 - \pi)(1 - G(q_m(\pi))) + \pi(1 - G(q_m(\pi) + q_h))] = 1 . \tag{29}$$

Take derivative with respect to $\pi$,

$$G(q_m(\pi)) - G(q_m(\pi) + q_h) - \pi g(q_m(\pi) + q_h)q_m'(\pi) - (1 - \pi)g(q_m(\pi))q_m'(\pi) = 0 . \tag{30}$$

Collecting terms to get

$$q_m'(\pi) = \frac{G(q_m(\pi)) - G(q_m(\pi) + q_h)}{\pi g(q_m(\pi) + q_h) + (1 - \pi)g(q_m(\pi))} . \tag{31}$$

At the wide spread region, $L(\pi) = (1 - F(x^*))[(1 - \pi)k(q_m(\pi)) + \pi k(q_m(\pi) + q_h)].$ Then

$$\frac{1}{1 - F(x^*)}L'(\pi) = k(q_m(\pi) + q_h) - k(q_m(\pi)) + \pi (1 - G(q_m(\pi) + q_h))q_m'(\pi)$$

$$+ (1 - \pi)(1 - G(q_m(\pi)))q_m'(\pi) . \tag{32}$$

Easy to see this function is continuous in $\pi$. Consider $L'(\pi)$ at $\pi = 0$. Since $q_m(0) = \bar{q},$

$$\frac{1}{1 - F(x^*)}L'(0) = k(\bar{q} + q_h) - k(\bar{q}) + (1 - G(\bar{q}))\frac{G(\bar{q}) - G(\bar{q} + q_h)}{g(\bar{q})} . \tag{33}$$

$L'(0) < 0$ if and only if

$$\frac{G(\bar{q} + q_h) - G(\bar{q})}{k(\bar{q} + q_h) - k(\bar{q})} > \frac{g(\bar{q})}{1 - G(\bar{q})} . \tag{34}$$
Use integration by parts,

\[ k(s) = s(1 - G(s)) + \int_0^s qg(q) dq \]

\[ = s(1 - G(s)) + sG(s) - \int_0^s G(q) dq \]

\[ = s - \int_0^s G(q) dq \]

\[ = \int_0^s (1 - G(q)) dq . \]  \hspace{1cm} (35)

Thus, \( L'(0) < 0 \) if and only if

\[ \int_{\bar{q}}^{\bar{q} + q_h} g(q) dq \int_{\bar{q}}^{\bar{q} + x} (1 - G(q)) dq > g(\bar{q}) \frac{1}{1 - G(\bar{q})} . \]  \hspace{1cm} (36)

Let

\[ I(x) = \int_{\bar{q}}^{\bar{q} + x} g(q) dq - \frac{g(\bar{q})}{1 - G(\bar{q})} \int_{\bar{q}}^{\bar{q} + x} (1 - G(q)) dq . \]

Inequality (36) holds if and only if \( I(q_h) > 0 \). Notice that \( I(0) = 0 \). Moreover,

\[ I'(x) = g(\bar{q} + x) - \frac{g(\bar{q})}{1 - G(\bar{q})} (1 - G(\bar{q} + x)) . \]

Since \( \frac{g(x)}{1 - G(x)} \) is increasing, for \( x > 0 \), \( I'(x) > 0 \). Thus, \( I(q_h) > 0 \) and \( L'(0) < 0 \). Then by continuity of \( L'(\pi) \), there exists a small region around 0 such that liquidity is decreasing in \( \pi \).

Notice that the calculation above works for the situation when \( \bar{q} + q_h \) is in the support of \( G \). If \( \bar{q} + q_h \) is not in the support of \( G \), replace \( \bar{q} + q_h \) with the upper-bound of \( G \)'s support yields the same result. ■

A.4 Simultaneous Pricing

A.4.1 Proof of Proposition 5

Proposition 5 can be divided into following claims.

Claim 1 Players never propose a spread greater than \( x^* \)
Proof. If a player propose a spread greater than \( x^* \), regardless of the other player’s strategy, switching to proposing \( x^* \) yields a strictly larger payoff. ■

**Claim 2** Neither players would use pure strategies in an equilibrium.

**Proof.** Suppose the market maker posts spread \( x_m = x \) in an equilibrium. The HFT’s optimal strategy would be posting \( x_h = x^* \), \( x_h = x \) or a mix between these two price. Then the market maker would achieve higher payoff by undercutting the HFT’s lowest possible price for a small enough \( \epsilon \). Contradiction

Suppose the HFT post spread \( x_h = x \) in an equilibrium. Then in an equilibrium the market maker can only post \( x^* \). (Undercutting will lead to no equilibrium because the payoff of the market maker is not continuous at \( x \).) This implies \( x_h \neq x^* \) in the equilibrium. However, if \( x_h < x \), given the market maker is posting \( x_m = x^* \), the HFT would be better off posting \( x_h = x^* \). Contradiction. ■

Suppose there exists a mixed strategy equilibrium. Denote the infimum and supremum of the spread posted by the market maker (HFT) by \( x_m(\bar{x}_h) \) and \( \bar{x}_m(\bar{x}_h) \).

**Claim 3** \( \bar{x}_m = \bar{x}_h \) and neither the market maker nor the HFT would post this spread with positive probability in an equilibrium.

**Proof.** If not, the player with smaller spread lower-bound could raise the lower-bound by a small enough amount to achieve higher payoff. Denote this common lower-bound by \( \bar{x} \). If the HFT posts this spread with positive probability, rather than posting \( \bar{x} \), the market maker would be strictly better off undercutting the HFT for a small amount.

Suppose the market maker posts \( \bar{x} \) with positive probability. Let \( B(x, r) \) be a open ball centered at \( x \) with radius \( r \). First note that \( \forall \epsilon > 0, \exists x_h \in B(\bar{x}, \epsilon) \) such that \( x_h \) is in HFT’s mixed strategy’s support. If not, since \( \bar{x} \) is posted by the HFT with zero probability, the market maker can increase \( \bar{x}_m \) by \( \epsilon \) to achieve higher profit. Then for small enough \( \epsilon \), HFT’s profit of posting \( \bar{x} + \epsilon \) is strictly smaller than posting \( \bar{x} \). Contradiction. ■

**Claim 4** *(No Holes)* \( \exists a, b \in (\bar{x}, \bar{x}_m), a < b \text{ such that } (a, b) \cap X_m = \emptyset \). A similar claim holds for \( X_h \).
Proof. Suppose this claim is false. Without loss of generality, let \((a, b)\) be a maximum interval satisfying the claimed property. That is, \((a, b) \cap X_m = \emptyset\) and for any \(a' < a\) and \(b' > b\), \((a', b) \cap X_m \neq \emptyset\); \((a, b') \cap X_m \neq \emptyset\).

By claim 1, \(\overline{x}_m, \overline{x}_h \leq x^\ast\). Notice that if \((a, b) \not\in X_m\), then \((a, b) \not\in X_h\). This is because if \(x \in (a, b)\) and \(x \in X_h\), the HFT may increase \(x\) by a small amount to increase her payoff.

Then notice that \(a \not\in X_m\). This is because posting \(x_m \in (a, b)\) will achieve a higher payoff given \((a, b) \not\in X_h\). Moreover, \(a \not\in X_h\) by a similar argument.

Given that spread \(a\) is not posted by the HFT and the market maker with positive probability, when \(x_m \to a\) from below, the payoff goes to the payoff of posting \(x_m = a\) by continuity, which is smaller than posting \(x_m \in (a, b)\). Since \((a, b)\) is a maximum interval satisfying \((a, b) \cap X_m = \emptyset\), \(\forall \epsilon > 0, B(a, \epsilon) \cap X_m \neq \emptyset\). This contradicts the equilibrium definition that \(x_m \in X_m\) is a best response to the HFT’s pricing strategy.

Claim 5 \(\overline{x}_m = \overline{x}_h = x^\ast\).

Proof. Suppose that \(\overline{x}_m < \overline{x}_h\). Then \((\overline{x}_m, \overline{x}_h) \cap X = \emptyset\) since posting \(x_h = \overline{x}_h\) yields a higher payoff. This contradicts Claim 4. Similarly, it is impossible that \(\overline{x}_m > \overline{x}_h\). If \(\overline{x}_m = \overline{x}_h < x^\ast\), \(\overline{x}_m \not\in X_m\) since \(x_m = x^\ast\) would yield higher payoff. Since \(\overline{x}_m \not\in X_m\), by the same argument, \(\overline{x}_h \not\in X_h\). However, then by the continuity argument, for small enough \(\epsilon\), \(x_m \in B(\overline{x}_m, \epsilon)\) will be dominated by posting \(x_m = x^\ast\). Contradiction.

Claim 6 \(\forall x \in (\overline{x}, x^\ast) \cap X_m((\overline{x}, x^\ast) \cap X_h), x\) is not proposed by the market maker (HFT) with positive probability in an equilibrium.

Proof. We prove by contradiction. Suppose that the market maker posts spread \(x\) with positive probability. Then by claim 4, \(\forall \epsilon > 0, B(x + \epsilon, \epsilon) \cap X_h \neq \emptyset\). However, by continuity, when \(\epsilon\) is small, the payoff posting that spread is dominated by posting \(x\). Contradiction. If the HFT posts spread \(x\), note that the market maker’s profit when posting a spread approaching \(x\) from the left is larger than the profit when posting a spread approaching \(x\) from the right. This leads to a contradiction.

A.4.2 Proof of theorem 5

Proof. The proof of the first part is the same as the proof of Theorem 2. For the second and the third statement, note that expected payoffs of the market maker are
the same in all one-shot games. Thus, in the equilibrium the market maker commits the same amount of capital to the market. The HFT’s payoffs can be calculated from the corresponding one-shot game.

A.4.3 Proof of proposition 7

Proof. For the first statement, notice that \( L_{se} = (1 - F(x^*))[\pi k(q_m + q_h) + (1 - \pi)k(q_m)] \). Compare this to \( L_{sim} \) in Theorem 5 to reach the conclusion.

Notice that I have shown that \( L_{se} \) is increasing in \( \pi \). Thus, the third statement is merely a corollary of the second statement. If \( \pi \) is in the tight spread region, in equilibrium, \( k(q_m) \leq \pi k(q_h) \) and \( a(x) \) is not changing with \( \pi \). Moreover, \( q_m \) also remains constant with respect to \( \pi \). Then by the market maker’s indifference condition, for all \( x \in (\underline{x}, x^*) \),

\[
a(x)\{ (1 - \pi)k(q_m) + \pi [H_h(x)(k(q_m + q_h) - k(q_h)) + (1 - H_h(x))k(q_m)] \} \tag{37}
\]

is constant for all \( \pi \) in the defensive region. This implies for any given \( x \), \( \pi H_h(x) \) is constant for all \( \pi \) in the tight spread region. This together with Theorem 5 implies that \( L_{sim} - L_{se} \) is constant. It also implies that in the tight spread region, increase in \( \pi \) only benefits buyers with valuations higher than \( 1 + x^* \).

B Extension: Costly Entry

B.1 Sequential Pricing

B.1.1 Proof of Proposition 8

Proof. If \( C \geq \bar{C} = \pi (1 - F(x^*))x^*k(q_h) \), the expected return of the HFT cannot cover the cost even when the HFT undercuts the market maker at spread \( x^* \). Thus, the HFT will not enter the market regardless of the market maker’s spread. In equilibrium, the market maker would choose \( x_m = x^* \).

Now consider the situation where \( C < \bar{C} \). In this case, if the market maker posts the wide spread \( x^* \), the HFT would attempt to enter the market and undercut the market maker upon entry. Moreover, the HFT would not choose to enter and undercut the market maker if the market maker posts the aggressive tight spread \( x \) satisfying \( \pi (1 - F(x))xk(q_h) = C \). If the market maker posts a spread higher than the
aggressive tight spread $x$, the HFT will always enter since she can always undercut the market maker and earn an expected payoff higher than $C$.

If $k(q_m) < \pi k(q_h)$, given the HFT chooses to enter the market, the market maker’s optimal spread is a defensive tight spread satisfying $(1 - F(x)) x k(q_h) = (1 - F(x^*)) x^* [k(q_m + q_h) - k(q_m)]$. Moreover, as long as the HFT does not undercut the market maker, the market maker always prefers to set the spread $x_m$ higher (given $x_m \leq x^*$). Thus, in equilibrium, the market maker will compare the defensive tight spread and the aggressive tight spread and pick the greater one. Specifically, if $C > \pi (1 - F(x^*)) x^* [k(q_m + q_h) - k(q_m)]$, posting the aggressive tight spread is more profitable. Otherwise, posting the defensive spread is more profitable. Furthermore, when facing the defensive tight spread, the HFT is indifferent between posting the monopolistic spread and undercutting the market maker. Then when the market maker posts the aggressive tight spread, upon entering, the HFT is better off undercutting the market maker. This implies that when the market maker posts the aggressive tight spread, the HFT will choose not to try to enter the market. The discussion for $k(q_m) > \pi k(q_h)$ follows the similar logic and is thus omitted.

**B.1.2 Proof of Theorem 6**

**Proof.** Let $x^a$ satisfies $\pi (1 - F(x^a)) x^a k(q_h) = C$ for $C \in [0, \bar{C}]$. Let $q^a_m$ satisfies $\frac{\delta}{1 - \delta} (1 - F(x^a)) x^a (1 - G(q^a_m)) = 1$. This is the equilibrium capital commitment if the market maker uses a deterring entry strategy. The equilibrium payoff is $V_C(w_0) = \frac{\delta}{1 - \delta} (1 - F(x^a)) x^a k(q^a_m) + (w_0 - q^a_m)$. Easy to see that this quantity is increasing in $C$. Easy to see that when $C \geq \hat{C}$, this quantity becomes monopolistic payoff. Let the market maker’s equilibrium payoff when $C = 0$ be $V_0(w_0)$. There exist a unique $\hat{C}$ such that $V_C(w_0) = V_0(w_0)$. Thus, for $C > \hat{C}$, the market maker is using the deterring strategy in the equilibrium.

When the market maker is using a deterring strategy, suppose the HFT chooses to participate, then she optimally set $x_h = x^*$. Since (1) the HFT is not undercutting the market maker, and (2) when the HFT participates, her optimal pricing strategy does not depend on $C$, when $C = 0$, the market maker can use the same equilibrium strategy to achieve a higher expected payoff. Contradiction. Thus, the HFT does not choose to participate.
B.2 Simultaneous Pricing

B.2.1 Proof of Proposition 9

**Proof.** First consider the case where \( C > \bar{C} \). In this case, the HFT’s expect profit can never cover the cost regardless of the market maker’s pricing strategy. Thus, \( \eta = 0 \) and the market maker sets \( x_m = x^* \).

Now consider the situation when \( C \in [0, \bar{C}] \). Suppose the HFT chooses \( \eta = 1 \) and plays a mixed pricing strategy as in a game \( (q_m, q_h, \pi, 0) \). By Proposition 6, the HFT’s expected profit is \( \pi(1 - F(x^*))x^*a(x)(\pi)k(q_h) \). If \( \pi(1 - F(x^*))x^*a(x)(\pi)k(q_h) \geq C \), since \( C \) is paid at the end of the period, the equilibrium characterized by Proposition 6 still holds.

If \( \pi(1 - F(x^*))x^*a(x)(\pi)k(q_h) < C < \bar{C} \), note that \( \eta \neq 0 \) in the equilibrium. This is because if \( \eta = 0 \), the market maker would post \( x_m = x^* \). The HFT has incentive to deviate to \( \eta = 1 \). Thus, I need to consider an equilibrium where the HFT mixes between participating. In other words, \( \eta \in (0, 1) \). \( \eta \) can be pinned down by the indifference condition that the HFT earns zero profit when trying to enter the market.

First consider the situation \( k(q_m) \geq \pi k(q_h) \). By Proposition 6, if the HFT tries to enter with probability \( \eta \), \( \bar{x} \) is determined by

\[
(1 - \eta \pi)k(q_m) + \eta \pi(k(q_m + q_h) - k(q_h)) = a(x)k(q_m). \tag{38}
\]

Notice that \( \bar{x} \) is decreasing in \( \eta \) and \( \bar{x} \to x^* \) as \( \eta \to 0 \). Thus, there exist a unique \( \eta \in (0, 1) \) such that \( \eta \pi(1 - F(x^*))x^*a(x)(\pi)k(q_h) = \eta C \) where \( \bar{x} \) is the lower-bound of the mixed strategy in the game \( (q_m, q_h, \eta \pi, 0) \). If the HFT participates with probability \( \eta \) and posts spread according to \( H_h \) in the game \( (q_m, q_h, \eta \pi, 0) \), the market maker has no incentive to deviate from posting spread according to \( H_m \) in the game \( (q_m, q_h, \eta \pi, 0) \). If the market maker sets price according to \( H_m \), upon entering, the HFT has no incentive to deviate from \( \eta \).

Next consider the situation \( k(q_m) < \pi k(q_h) \). Notice that \( \bar{x} \) remains constant in this region. Let \( \bar{\eta} \) satisfies \( k(q_m) = \bar{\eta} \pi k(q_h) \). By the same argument, there exists a unique \( \eta \in (0, \bar{\eta}) \) such that \( k(q_m) > \pi \eta k(q_h) \) and \( \pi(1 - F(x^*))x^*a(x)(\pi)k(q_h) = C \).
where $x$ is the lower-bound of the mixed strategy in the game $(q_m, q_h, \eta\pi, 0)$. The rest of the verification is the same. □

B.2.2 Proof of Corollary 6

Proof. This proof essentially involves only comparing the market maker’s payoffs under two settings with different parameter values. Fix a game $(q_m, q_h, \pi, C)$. First consider the case when $k(q_m) \geq \pi k(q_h)$. In the one-shot simultaneous pricing game,

$$a(x)(\pi) = 1 - \pi + \frac{k(q_m + q_h) - k(q_h)}{k(q_m)} \pi.$$ 

If $\pi(1 - F(x^*))x^*a(x)(\pi)k(q_h) \geq C$, by Proposition 9, in the simultaneous pricing game, the HFT participates in high-frequency with probability 1. The market maker enjoys the same expected payoff as in the simultaneous one-shot game $(q_m, q_h, \pi, 0)$, which equals to $(1 - \pi)k(q_m) + \pi(k(q_m + q_h) - k(q_h))$. By Proposition 8, the market maker receives the same expected payoff in the sequential pricing one-shot game.

For $\pi(1 - F(x^*))x^*a(x)(\pi)k(q_h) < C$, in the simultaneous pricing game, the market maker receives payoff $(1 - F(x^*))x^*a(x)(\eta\pi)k(q_m)$ by the indifference condition where

$$a(x)(\eta\pi) = \frac{C}{\pi(1 - F(x^*))x^*k(q_h)}.$$ 

Thus, the market maker’s expected payoff is $\frac{C}{\pi k(q_h)}k(q_m)$, which equals to the expected payoff in a one-shot sequential pricing game by Proposition 8.

Next consider the case when $k(q_m) < \pi k(q_h)$. In a one-shot simultaneous pricing game,

$$a(x)(\pi) = \frac{k(q_m + q_h) - k(q_m)}{k(q_h)}.$$ 

If $\pi(1 - F(x^*))x^*a(x)(\pi)k(q_h) = \pi(1 - F(x^*))x^*(k(q_m + q_h) - k(q_m)) \geq C$, in a simultaneous pricing game, the market maker’s expected payoff is $(1 - F(x^*))x^*a(x)k(q_m)$. This is the same as the expected payoff in a sequential pricing game. If $\pi(1 - F(x^*))x^*a(x)(\pi)k(q_h) < C$, in a simultaneous pricing game, the market maker receives payoff $(1 - F(x^*))x^*a(x)(\eta\pi)k(q_m)$ where

$$a(x)(\eta\pi) = \frac{C}{\pi(1 - F(x^*))x^*k(q_h)}.$$ 

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Thus, the market maker’s expected payoff is \( C \pi_k(q)k(q_m) \), which equals to the expected payoff in the sequential pricing game. ■

C Multiple Markets Extension

In this extension, consider the case where \( n \) markets coexist. Each market has a market maker with net worth \( w_0 \) at the beginning of period 0. In each period, \( n \) market makers determine shareholdings and spreads simultaneously. A short-run HFT then arrives with net worth \( q_h \). She observes all market maker’s shareholdings and spreads and chooses to enter a market. We assume in this extension that \( \pi = 1 \) and \( C = 0 \). To simplify the model, I further assume that the buyer in each market is homogeneous and \( G \) follows an exponential distribution with mean \( \frac{1}{\lambda} \). We also assume that a market maker never observes the entry of the HFT or other market makers’ shareholdings and spreads.34

**Theorem 7** Suppose \( B = \frac{\delta}{1-\delta}(1 - F(x^*))x^* > 1 \). Let \( \bar{q}_h \) satisfy \( Be^{-\lambda q_h} + \lambda \bar{q}_h = \frac{B+12+\sqrt{B^2+8B}}{16} + \ln(B + \sqrt{B^2 + 8B}) - 2ln2 \). If \( q_h < \bar{q}_h \), there exists a symmetric steady state equilibrium where all market makers hold \( q_m = \frac{\ln B}{\lambda} - q_h \). They use the same mixed strategy to post spreads in \([x, x^*]\). \( x \) satisfies \( (1 - F(x))x = e^{-\lambda q_h}(1 - F(x^*))x^* \).

No spread is posted with positive probability. The HFT undercuts the market maker with the highest spread on the equilibrium path.

C.1 Proof of Theorem 7

C.1.1 One Market with \( \pi = 1 \) and Exponential Demand

In the following calculation, suppose a buyer’s demand follows an exponential distribution with mean \( \frac{1}{\lambda} \). To simplify notation, define \( B = \frac{\delta}{1-\delta}(1 - F(x^*))x^* \). We assume \( B > 1 \). If not, the market maker will liquidate in period 0 even if no HFT exists. To see this, notice that when no HFT exists, the market maker’s steady state capital commitment satisfies \( B(1 - G(\bar{q})) = 1 \). This implies \( \bar{q} = \frac{\ln B}{\lambda} \).

First determine the market maker’s tight spread when his shareholding is \( q_m \). The

34 These assumptions aim at avoiding lengthy off equilibrium path discussions. If market makers can observe all realizations and have passive beliefs, the result still holds.
indifference condition for the HFT is:

\begin{equation}
(1 - F(x)) x k(q_h) = (1 - F(x^*)) x^* [k(q_m + q_h) - k(q_m)] .
\end{equation}

(39)

Thus,

\begin{equation}
a(x(q_m)) = \frac{k(q_m + q_h) - k(q_m)}{k(q_h)} = e^{-\lambda q_m} .
\end{equation}

(40)

This shows a very special property of the game when \( G \) is an exponential distribution: The market maker’s defensive spread only depends on the market maker’s shareholding. If the market maker holds \( q_m \) shares, his defensive spread can be uniquely pinned down by the equation \( (1 - F(x_m)) x_m = e^{-\lambda q_m} (1 - F(x^*)) x^* \).

Now consider a sequential pricing game with \( \pi = 1 \). In this game, if the market maker adopts the wide spread strategy in the equilibrium, his capital commitment satisfies

\begin{equation}
B(1 - G(q_m + q_h)) = 1 .
\end{equation}

(41)

Thus, \( q_m = \frac{\ln B}{\lambda} - q_h \) and the equilibrium payoff for the market maker given he posts the wide spread is

\begin{equation}
V_a = B(k(q_m + q_h) - k(q_m)) + (w_0 - q_m)
= \frac{1}{\lambda} B(G(q_m + q_h) - G(q_m)) + (w_0 - q_m)
= \frac{1}{\lambda} B(e^{-\lambda q_h} - \frac{1}{B}) + (w_0 - q_m)
= \frac{1}{\lambda} (Be^{-\lambda q_h} - 1) + w_0 - \left( \frac{\ln B}{\lambda} - q_h \right) .
\end{equation}

(42)

Since \( q_m \geq 0 \) implies \( e^{-\lambda q_h} \geq \frac{1}{B} \), this payoff is decreasing in \( q_h \).

If the market maker is adopting the defensive strategy in the equilibrium, he chooses \( q_m \) to maximize

\begin{equation}
V_d(q_m) = Be^{-\lambda q_m} \cdot k(q_m) + (w_0 - q_m) .
\end{equation}

(43)

Take derivative to get

\begin{equation}
V'_d(q_m) = B(2e^{-2\lambda q_m} - e^{-\lambda q_m}) - 1
\end{equation}

(44)

Let \( y = e^{-\lambda q_m} \in (0, 1] \). Since \( B > 1, 2y^2 - y - \frac{1}{B} \) cross zero only once for \( y \in (0, 1] \).
Moreover, $2y^2 - y - \frac{1}{B} > 0$ when $y = 1$. Thus, $V_d(q_m)$ has a unique maximizer when $e^{-\lambda q_m} = y = \frac{1 + \sqrt{1 + \frac{B}{4}}}{2}$. We get

$$q_m = \frac{1}{\lambda}(ln(\sqrt{B^2 + 8B} - B) - ln2)$$

and

$$V_d = \frac{1}{\lambda} \cdot \frac{B - 4 + \sqrt{B^2 + 8B}}{16} + w_0 - \frac{1}{\lambda}(ln(\sqrt{B^2 + 8B} - B) - ln2).$$

Since $V_d$ is independent of $q_h$ and $V_a$ is decreasing in $q_h$, there exists a $\bar{q}_h$ such that for $q_h > \bar{q}_h$, the market maker adopts the defensive strategy and for $q_h < \bar{q}_h$, the market maker posts a wide spread. Specifically, $\bar{q}_h$ satisfies

$$Be^{-\lambda \bar{q}_h} + \lambda \bar{q}_h = \frac{B + 12 + \sqrt{B^2 + 8B}}{16} + ln(B + \sqrt{B^2 + 8B}) - 2ln2. \quad (45)$$

C.1.2 n-market game

Notice that $x$ satisfies

$$(1 - F(x))xk(q_m) = (1 - F(x^*))x^*[k(q_m + q_h) - k(q_h)]. \quad (46)$$

That is, the market maker is indifferent between selling at spread $x$ with no HFT and selling at spread $x^*$ with the presence of the HFT of probability 1. Let $H$ be a cdf with support $[e^{-\lambda q_h}, 1]$. $H(a(x))$ is the probability that a market maker posts a spread smaller or equal to $x$ such that $a(x) = \frac{1-F(x)}{(1-F(x^*))x^*}$. By lemma A.1, there is a bijection between $a$ and $x$. Market makers’ pricing strategy is characterized by the indifference condition:

$$a(x)B(H(a(x)))^{n-1}[k(q_m + q_h) - k(q_h)] + a(x)B[1-(H(a(x)))^{n-1}]k(q_m) = B[k(q_m + q_h) - k(q_h)]. \quad (47)$$

Easy to see that at any level of shareholding, no market maker has incentive to post a spread smaller than $x$. Thus, I only need to check that a market maker has no incentive to deviate by choosing a different $q_m$ and then post a spread (or spreads) between $x$ and $x^*$. Since the HFT acts after market makers and market makers cannot observe deviation, a market maker has no incentive to play a mixed strategy in deviation. First, a market maker has no incentive to decrease $q_m$ and post a spread in $[x, x^*]$ such that the HFT never choose to undercut him. This is because a market
maker’s equilibrium payoff is equal to the equilibrium payoff in a one market game with \( \pi = 1 \) in the wide spread region. Since \( q_h < \bar{q}_h \), this dominates all possible payoffs a market maker can obtain from the defensive strategy.

We also need to consider a potential type of deviation that a market maker deviates by fixing a spread in \([x, x^*] \) and chooses a different level of capital commitment. This possibility can be ruled out by showing that the marginal benefit of capital commitment is 1 under any spread in equilibrium. Since \( q_m = \frac{lnB}{\lambda} - q_h \), \( B(1 - G(q_m + q_h)) = 1 \). This means the marginal benefit of capital commitment at spread \( x^* \) is 1 in the equilibrium. \( x \in [x, x^*] \), the marginal benefit of capital commitment is

\[
\begin{align*}
  a(x)BQ(x)[1 - G(q_m + q_h)] + a(x)B[1 - Q(x)][1 - G(q_m)] ,
\end{align*}
\]

where \( Q(x) = H(a(x))^{n-1} \). To show this also equals to 1, notice that from the indifference condition on spread,

\[
\begin{align*}
  a(x)Q(x) + a(x)[1 - Q(x)]e^{\lambda q_h} = 1 .
\end{align*}
\]

Then,

\[
\begin{align*}
  \frac{a(x)BQ(x)[1 - G(q_m + q_h)] + a(x)B[1 - Q(x)][1 - G(q_m)]}{B(1 - G(q_m + q_h))} \\
= aQ(x) + \frac{a(x)[1 - Q(x)][1 - G(q_m)]}{1 - G(q_m + q_h)} \\
= a(x)Q(x) + a(x)[1 - Q(x)]e^{\lambda q_h} \\
= 1 .
\end{align*}
\]

This shows that the marginal benefit of capital commitment over any spread within \([x, x^*] \) is one and a market maker has no incentive to deviate.

**D Capital Commitment when G has Non-decreasing Hazard Rate**

This section provides a more detailed analysis of the market maker’s capital commitment strategy when the buyer’s demand \( G \) follows a distribution with increasing hazard rate. Particularly, under any fixed HFT shareholding \( q_h \), the market maker has a unique optimal steady state tight spread strategy.

69
Proposition 15 If $G$ has non-decreasing hazard rate, $\max_y B \frac{k(y+q_h) - k(y)}{k(q_h)} k(y) + (w_0 - y)$ has a unique solution $q_m \in [0, \bar{q}]$. $B = \frac{\delta}{1-\delta} (1 - F(x^*)) x^*.$

Notice that $a(x) = \frac{k(q_m + q_h) - k(q_m)}{k(q_h)}$. By posting spread $x_m$ satisfying $(1 - F(x_m)) x_m = (1 - F(x^*)) x^* a(x^*)$, a short-run HFT with $q_h$ shares has no incentive to undercut the market maker.

Proof. The first order condition is

$$W' (y) = \frac{B}{k(q_h)} [(1 - G(y)) (k(y + q_h) - k(y)) - (G(y + q_h) - G(y)) k(y)] - 1 = 0 \quad (51)$$

When $y = 0$, $W'(0) = B - 1 > 0$. When $y \geq \bar{q}$, $W'(y) < B(1 - G(q)) \frac{k(y + q_h) - k(y)}{k(q_h)} - 1$. Since $B(1 - G(q)) = 1$, $W'(y) < 0$. By continuity, $W'(y)$ cross zero at least once for $y \in [0, \bar{q}]$. If I can show that $W'$ only cross zero once, then a unique maximizer exists.

Consider any $q_m$ such that $W'(q_m) = 0$. We have

$$(1 - G(q_m)) [k(q_m + q_h) - k(q_m)] - (G(q_m + q_h) - G(q_m)) k(q_m) = k(q_h) (1 - G(q)) > 0 \quad (52)$$

Thus,

$$\frac{k(q_m + q_h) - k(q_m)}{k(q_m)} > \frac{G(q_m + q_h) - G(q_m)}{1 - G(q_m)} \quad (53)$$

Next I show that $W''(q_m) < 0$. $W''(q_m) < 0$ is equivalent to

$$g(q_m) [k(q_m + q_h) - k(q_m)] + k(q_m) [g(q_m + q_h) - g(q_m)] + 2(1 - G(q_m)) [G(q_m + q_h) - G(q_m)] > 0 \quad (54)$$

Since $G(q_m + q_h) - G(q_m) > 0$, a sufficient condition for inequality (54) is

$$g(q_m) [k(q_m + q_h) - k(q_m)] > k(q_m) [g(q_m) - g(q_m + q_h)] \quad (55)$$

Since $G$ has non-decreasing hazard rate, $\frac{g(q_m + q_h)}{1 - G(q_m + q_h)} \geq \frac{g(q_m + q_h)}{1 - G(q_m)}$. Thus, $g(q_m) - g(q_m + q_h) \leq \frac{G(q_m + q_h) - G(q_m)}{1 - G(q_m)} g(q_m)$. This implies inequality (53) is sufficient for inequality (55).

In sum, there exists a $q_m \in [0, \bar{q}]$ such that $W'(q_m) = 0$. Moreover, for any $q_m$ such that $W'(q_m) = 0$, $W''(q_m) < 0$. This implies that $W(y)$ has a unique maximum.

Proposition 16 Suppose $G$ has non-decreasing hazard rate. Consider two simulta-
neous pricing games where the market maker has discount rate $\delta_1$ in the first game and discount rate $\delta_2$ in the second game. Suppose $\delta_1 > \delta_2$ and all other parameters are the same. Let $q_m^1$ ($q_m^2$) be the market maker’s steady state capital commitment at the first game (the second game). Then $q_m^1 > q_m^2$.

**Proof.** Let $B_1 = \frac{\delta_1}{1-\delta_1} (1 - F(x^*)) x^*$; $B_2 = \frac{\delta_2}{1-\delta_2} (1 - F(x^*)) x^*$. If the market maker is using the wide spread strategy in both games, then $q_m^1 > q_m^2$ directly follows from the first order condition. If the market maker is using the tight spread strategy in both games, then by the first order condition,

$$\frac{B_1}{k(q_h)} [(1 - G(q_m^1))(k(q_m^1 + q_h) - k(q_m^1)) - (G(q_m^1 + q_h) - G(q_m^1))k(q_m^1)] - 1 = 0.$$ 

Since $B_1 > B_2$, we have

$$\frac{B_2}{k(q_h)} [(1 - G(q_m^1))(k(q_m^1 + q_h) - k(q_m^1)) - (G(q_m^1 + q_h) - G(q_m^1))k(q_m^1)] - 1 < 0.$$ 

Then by Proposition 15, there exists a unique $q_m^2 < q_m^1$ such that

$$\frac{B_2}{k(q_h)} [(1 - G(q_m^2))(k(q_m^2 + q_h) - k(q_m^2)) - (G(q_m^2 + q_h) - G(q_m^2))k(q_m^2)] - 1 = 0,$$

and $q_m^2$ maximize the market maker’s expected payoff given he is using a tight spread strategy in the steady state. If the market maker is using the wide spread strategy in the first game and the tight spread strategy in the second game, combine the result about with Theorem 3 yield the result that $q_m^1 > q_m^2$. This covers all situations when the market maker is using the wide spread strategy in the first game.

Now consider the situation where the market maker is using the tight spread strategy in the first game. Let $q_t^1$ and $q_w^1$ ($q_t^2$ and $q_w^2$) be the market maker’s shareholding under the optimal tight and wide spread strategy in the first (second) game. Since the market maker is using the tight spread strategy in the first game, $q_m^1 = q_t^1$. By the discussion above, $q_t^1 > q_t^2$; $q_w^1 > q_w^2$. If $q_t^1 > q_w^2$, the claim is true. Thus, we only consider the case when $q_t^1 \leq q_w^2$.

From the optimality condition,

$$\frac{\delta_1}{1-\delta_1} M(q_t^1) + (w_0 - q_t^1) \geq \frac{\delta_1}{1-\delta_1} M(q_w^1) + (w_0 - q_w^1) > \frac{\delta_1}{1-\delta_1} M(q_w^2) + (w_0 - q_w^2), \quad (56)$$
where $M(\cdot)$ is the expected profit of the market maker in a one-shot game. If $M(q^1_t) > M(q^2_w)$, since $q^1_t \leq q^2_w$, we have

$$\frac{\delta_2}{1 - \delta_2} M(q^2_t) + (w_0 - q^2_t) > \frac{\delta_2}{1 - \delta_1} M(q^1_t) + (w_0 - q^2_t) > \frac{\delta_2}{1 - \delta_2} M(q^2_w) + (w_0 - q^2_w).$$

Thus, the market maker would use the tight spread strategy in the second game and $q^1_m > q^2_m = q^2_t$.

If $M(q^1_t) \leq M(q^2_w)$, from equation 56 and $\frac{\delta_1}{1 - \delta_1} > \frac{\delta_2}{1 - \delta_2}$, we also have

$$\frac{\delta_2}{1 - \delta_2} M(q^2_t) + (w_0 - q^2_t) > \frac{\delta_2}{1 - \delta_2} M(q^2_w) + (w_0 - q^2_w).$$

This implies $q^1_m > q^2_m = q^2_t$ and concludes the proof.

This result is important for an extension of the simultaneous pricing game. Notice that the equilibrium I construct in the simultaneous pricing game might not be sub-game perfect. In the sub-game where the market maker commits less capital than the steady state level, it might not be optimal for the market maker to stick to the strategy specified in the equilibrium since cumulating capital can provide him additional benefit. However, this is not a problem since I can embed the result into game where the HFT is uncertainty about the market maker’s discount rate and try to infer it from the market maker’s capital commitment. This result guarantees a separating equilibrium where the market maker’s discount rate can be uniquely determined by the HFT through observing the market maker’s capital commitment.\footnote{When the market maker’s capital commitment cannot be mapped to any $\delta$, I specify that the HFT assumes that the market maker sticks to maximizing the short term profit.}

In this sense, the equilibrium I propose is a perfect Bayesian equilibrium.

**Proposition 17** Suppose $G$ has non-decreasing hazard rate. If $q_h \geq \frac{q}{2}$, $\text{argmax}_y W(y) \in [0, \frac{q}{2}]$.

**Proof.** By Proposition 15, if $W'(\frac{q}{2}) \leq 0$, then $\text{argmax}_y W(y) \in [0, \frac{q}{2}]$. Thus, it is sufficient to show that for all $q_h \geq \frac{q}{2}$, $W'(\frac{q}{2}) \leq 0$.

This is equivalent to

$$k(q_h)(1 - G(\bar{q})) + (G(\frac{\bar{q}}{2} + q_h) - G(\frac{\bar{q}}{2}))k(\frac{\bar{q}}{2}) - [1 - G(\frac{\bar{q}}{2})][k(\frac{\bar{q}}{2} + q_h) - k(\frac{\bar{q}}{2})] \geq 0. \tag{57}$$
When \( q_h = \bar{q}_2 \), the LHS of inequality \((57)\) becomes
\[
(1 - G(\frac{\bar{q}}{2})) [2k(\frac{\bar{q}}{2}) - k(\bar{q})] .
\] (58)
This quantity is greater than zero since \( 2k(\frac{\bar{q}}{2}) > k(\bar{q}) \). Denote the LHS of inequality \((57)\) by \( J(q_h) \). If \( J(q_h) \) is increasing in \( q_h \), the lemma is proved.
\[
J'(q_h) = (1 - G(q_h))(1 - G(\bar{q})) + g(\frac{\bar{q}}{2})k(\frac{\bar{q}}{2}) - [1 - G(\frac{\bar{q}}{2})](1 - G(\frac{\bar{q}}{2} + q_h)) .
\] (59)
A sufficient condition of \( J'(q_h) \geq 0 \) is \( \frac{1 - G(\bar{q})}{1 - G(\frac{\bar{q}}{2})} \geq \frac{1 - G(\frac{\bar{q}}{2} + q_h)}{1 - G(q_h)} \). Since \( q_h \geq \frac{\bar{q}}{2} \), it is sufficient to have \( \frac{1 - G(\bar{q} + z)}{1 - G(\frac{\bar{q}}{2} + z)} \) decreasing in \( z \). Take derivative to get
\[
-g(\bar{q} + z)(1 - G(\frac{\bar{q}}{2} + z)) + g(\frac{\bar{q}}{2} + z)(1 - G(\bar{q} + z)) \leq 0 .
\] (60)
This condition is satisfied due to the increasing hazard rate of \( G \).

\[\text{E Extension: Selling with a Supply Schedule}\]

In this section, I consider an extension where the market maker can sell shares at different spreads. I analyze additional trade-offs brought by the market maker’s flexibility and discuss how it affects market quality in the steady state. To keep this problem tractable, I maintain the assumption that the HFT can only supply her shares at one spread.

The market maker’s pricing strategy can be uniquely represented by his capital commitment \( q_m \) and supply schedule \( \Psi \). Formally, fix the market maker’s capital commitment \( q_m \), a supply schedule can be described by a distribution with CDF \( \Psi(x), x \in [0, \hat{x}] \). \( q_m \Psi(x) \) is the quantity of shares supplied by the market maker with spreads less or equal to \( x \). Since in the steady state, the market maker posts the supply schedule to maximize his expected profit, it is suffice to first solve the one period problem with arbitrary \( q_m \) and then consider the marginal value of capital commitment to determine the steady state equilibrium. I first characterize the steady state when the market maker is a monopolist. Then I consider the situation where the HFT might enter the market.
E.1 No HFT

Consider a one-shot problem where the market maker with \( q_m \) shares maximizes expected profit in a single period.\(^{36} \) Under this circumstance, the market maker’s best pricing strategy is to sell all shares at the monopolistic spread \( x^* \). Formally we have the following proposition:

**Proposition 18** In a one-shot problem, the market maker would optimally set the supply profile to be \( \Psi(x) = I_{\{x \geq x^*\}} \), which coincides the optimal spread when the market maker has to supply all shares at one price.

**Proof.** Obviously, it is not optimal for the market maker to sell any share at a spread higher than \( x^* \). Then without loss of generality, I only consider the situation where the market maker set spreads lower than \( x^* \). The proof consists of two steps. I first show that if the market maker can supply shares with \( n \) spreads \( x_1, ..., x_n \) with \( \sum_{i=1}^{n} q_i = q_m \), then he should optimally set \( x_1 = ... = x_n = x^* \). Then I show that the market maker’s payoff under any supply schedule \( \Psi(x) \) can be approximated with arbitrary precision by a \( n \)-spreads supply plan with a large enough \( n \).

Consider the situation when \( n = N \). Without loss of generality, suppose \( x_1 \leq x_2 \leq ... \leq x_N \leq x^* \). Define \( q_0 = 0 \). The market maker’s expected payoff is

\[
\sum_{i=1}^{N} (1 - F(x_i))x_i[k(\sum_{j=0}^{i} q_j) - k(\sum_{r=0}^{i-1} q_r)] .
\]

Note that the market maker can increase his expected payoff by setting \( x_1 = x_2 \) since \((1 - F(x))x\) is increasing in \( x \in [0, x^*] \). This reduce the problem to \( n = N - 1 \) situation. By induction, for arbitrary \( n \), \( x_1 = ... = x_n = x^* \) is the optimal supply schedule.

Next consider the approximation procedure under arbitrarily fixed \( q_m \). For arbitrary \( \Psi(x) \), divide its support into \( n \) intervals \( \{I_1, ..., I_n\} \). The \( I_i \) interval is from \( \frac{i-1}{n} \)th quantile to \( \frac{i}{n} \)th quantile. Consider a new supply schedule that supply shares at \( n \) spreads. Specifically, in the new schedule, the market maker supplies \( q_i \) shares at spread \( x_i \) for \( i = 1, ..., n \). Let \( x_i = E_q(x|x \in I_i) \); \( q_i = \frac{q_m}{n} \) for all \( i \). Under any buyer’s demand and valuation, realized profits of this new schedule and schedule \( \Psi \)

\(^{36}\)As in the baseline model, the market maker can sell all shares at price 1 back to the inter-dealer market at the end of the period.
differ by at most a factor of $\frac{q_m}{n}$, which goes to 0 as $n \to \infty$. Thus, expected profit from any supply schedule $\Psi$ can be approximated to an arbitrarily close level by a schedule with $n$ spreads when $n$ is large enough. This establish the fact that the optimal supply schedule is to sell all shares at the spread $x^*$. ■

This Proposition implies that the infinite horizon steady state equilibrium is the same as the equilibrium under the original setting when no HFT exists. Moreover, when the market maker can post spreads flexibly, I can further show that the market maker never pay dividend when his net worth is smaller than the steady state capital commitment.

**Corollary 7** When no HFT exists, the steady state equilibrium is the same as the benchmark case with no HFT. Moreover, the market maker does not pay dividend when his net worth is smaller than the steady state capital commitment $\bar{w}$.

**Proof.** The first statement is a straightforward result from Proposition 18. For the second statement, if the dividend payout is non-zero, the market maker can always achieve a higher payoff by refraining from paying dividend and supply the extra amount of shares at the spread $x^*$ and payout the total return from the extra shares in the next period. ■

### E.2 With HFT

When the HFT may enter the market, the market maker with pricing flexibility would not use a single spread pricing strategy. The following proposition shows that the market maker never sticks to the wide spread strategy in the steady state.\(^{37}\)

**Proposition 19** For any $q_h$ and $\pi > 0$, selling all shares at the spread $x^*$ ($\Psi(x) = I_{\{x \geq x^*\}}$) is not the optimal strategy for the market maker in the steady state equilibrium.

**Proof.** Suppose for some $\pi$ and $q_h$ there exists a steady state equilibrium with capital commitment $q_m$ and supply schedule $\Psi(x) = I_{\{x \geq x^*\}}$. Then upon entering the market, the HFT would set spread $x_h = x^*$. The market maker’s expected dividend payout

\(^{37}\)It can be easily shown that the market maker would not use a single spread pricing strategy at other spreads, either. The proof strategy is similar.
each period would be
\[ \pi(1 - F(x^*))x^*[k(q_m + q_h) - k(q_h)] + (1 - \pi)(1 - F(x^*))x^*k(q_m) . \]

Consider a deviation of the market maker by selling \(\epsilon\) shares at the spread \(x_\epsilon\) satisfying
\[ (1 - F(x^*))x^*(k(\epsilon + q_h) - k(\epsilon)) = (1 - F(x_\epsilon))x_\epsilon k(q_h) \]
and \(q_m - \epsilon\) shares at the spread \(x^*\). Then the HFT would still set spread \(x_h = x^*\) and the market maker’s expected dividend payout would be
\[ \text{div}(\epsilon) = (1 - F(x_\epsilon))x_\epsilon k(\epsilon) + \pi(1 - F(x^*))x^*[k(q_m + q_h) - k(q_h + \epsilon)] + (1 - \pi)(1 - F(x^*))x^*[k(q_m) - k(\epsilon)] . \]

Easy to check
\[ \text{div}'(0) = (1 - F(x^*))x^*\pi G(q_h) > 0 . \]

Thus, the market maker can deviate in pricing to achieve a higher expected payoff. Contradiction. ■

Moreover, with the ability to flexibly sell shares, an immediate lower bound exists on the market maker’s capital commitment (depending on \(\pi\) and \(q_h\)). Intuitively, if the capital commitment is smaller than the lower bound, the market maker can always improve his expected payoff by committing more shares and sell additional shares at the spread \(x^*\).

**Proposition 20** The market maker would commit at least \(w\) unit of capital in market making, where \(w\) is either the solution of
\[ \frac{\delta}{1 - \delta}(1 - F(x^*))x^*\pi(1 - G(w + q_h)) + (1 - \pi)(1 - G(w))] = 1 \]
or 0.

**Proof.** If not, the market maker can commit additional \(w - q_m\) unit of capital and sell them at the spread \(x^*\) to achieve a higher payoff. ■

Although the market maker’s equilibrium pricing strategy \(\Psi(x)\) cannot be expressed explicitly, it can be characterized by the following result:

**Proposition 21** The market maker’s equilibrium pricing strategy \(\Psi(x)\) satisfies three conditions:
1. \( \Psi(x^*) = 1 \).

2. \( \Psi(\cdot) \) has no mass point for \( x < x^* \).

3. The HFT achieves the same expected payoff by setting any \( x_h \in [\underline{x}, x^*] \) where \( \Psi(\underline{x}) = 0 \).

**Lemma 5** In any steady state equilibrium, the HFT set \( x_h = x^* \).

**Proof.** Suppose not, then \( \lim_{x \to x^-} \Psi(x) < 1 \). The market maker would achieve a higher expected payoff by sell \( q_m(\lim_{x \to x^-} \Psi(x) - \lim_{x \to x^-} \Psi(x)) \) shares at the spread \( x^- \). ■

**Proof. Proposition 21** First note that if \( \Psi(x^*) < 1 \), the market maker can become better off by selling all shares with spreads higher than \( x^* \) at spread \( x^- \).

Suppose \( \Psi(x) \) has a mass point at \( x < x^* \). If the HFT is strictly prefers posting \( x_h = x^* \), then there exists an \( \epsilon \) such that the market maker can sell these shares at the spread \( x + \epsilon \) to achieve higher payoff. If the HFT is indifferent, then there must exist an \( \epsilon \) such that the HFT is strictly prefer setting \( x_h = x^* \) than setting \( x_h = x + \epsilon \). If the HFT is indifferent between posting \( x \) and \( x^* \), since \( x \) is a mass point, there exists a \( \epsilon > 0 \) such that the HFT strictly prefers setting \( x_h = x^* \) to setting \( x_h \in (x, x + \epsilon) \).

The market maker can then improve his pricing by selling all shares within the spread range \( (x, x + \epsilon) \) and some shares at the spread \( x \) to the spread \( x + \epsilon \).

The next step is to show that for any \( \Psi \) violating the indifference condition of the HFT, the market maker can always find a better pricing plan. Specifically, I consider this problem holding \( q_m \Psi(x^-) \) and \( q_m \) constant. First notice that \( x \) can be uniquely pinned down by

\[
(1 - F(x^*)x^*)k(q_h + q_m \Psi(x^-)) - k(q_m \Psi(x^-)) = (1 - F(x)\underline{x})k(q_h) .
\]

Denote the pricing distribution satisfying the HFT’s indifference condition by \( \Psi(\cdot) \). Then for all \( x \in [\underline{x}, x^*] \), \( \Psi(x) \geq \Psi(\underline{x}) \). Otherwise the HFT will not set \( x_h = x^* \) and the pricing distribution cannot be optimal at the steady state. Suppose \( \Psi \neq \Psi \), let \( \hat{\epsilon} = \inf_{x} \{ \Psi(x) > \Psi(\underline{x}) \} \). Since \( \Psi(x) \) does not have mass point, there exists \( \xi > 0 \) such that \( \Psi(x) > \Psi(\underline{x}) \) for \( x \in (\hat{\epsilon}, \hat{\epsilon} + \xi) \) and \( \Psi(\hat{\epsilon} + \xi) > \Psi(\hat{\epsilon}) \). Then by the same

\footnote{Notice that the second result is implied by the third result. If there is a mass point for \( x < x^* \), the indifference condition cannot hold everywhere.}
approximation and moving mass argument, the market maker is better off selling shares in the spread interval \((\hat{x}, \hat{x} + \xi)\) at the spread \(\hat{x} + \xi\).

With these results, the market maker’s equilibrium capital commitment and pricing strategy can be numerically computed using the following procedure: (i) Fix \(q_{ml}\), the amount of share sold by the market maker with spreads smaller than \(x^\ast\). (ii) Let \(q_m = q\) if \(q_{ml} \leq q\); \(q_m = q_{ml}\) otherwise. (iii) \(\Psi(x)\) can be uniquely pinned down by equation

\[
(1 - F(x))x[k(\Psi(x)q_m + q_h) - k(\Psi(x)q_m)] = (1 - F(x^\ast))x^\ast[k(q_{ml} + q_h) - k(q_{ml})]
\]

for \(x \in [\hat{x}, x^\ast)\) and \(\Psi(x^\ast) = 1\). (iv) As in the baseline case, let \(M(q_m)\) denote the expected per-period payoff of the market maker. For \(q_m = q\), define \(M(q)\) to be the maximum payoff for \(q_{ml} \in [0, q]\). (v) The market maker’s capital commitment \(q_m\) is maximizing

\[
\frac{\delta}{1 - \delta}M(q_m) + (w_0 - q_m)
\]

for \(q_m \in [q, \hat{q}]\). The pricing strategy is pinned down from the calculation above.

When the buyer’s demand follows an exponential distribution, the market maker’s supply schedule can be explicitly characterized. Specifically, let \(\psi(x) = \Psi'(x)\). Then for \(x \in [\hat{x}, x^\ast)\), \(\psi(x) \propto \frac{1}{x} - \frac{f(x)}{1 - F(x)}\).

### E.3 Discussion

This extension suggests that competition is a potential reason for us to observe shares supplied with different spreads in the market. Specifically, when no HFT exists, it is optimal for the monopolistic market maker to supply all shares at the monopolistic spread. On the contrary, even with a small probability of HFT entry, it is optimal for the market maker to supply shares at different spreads. Generally, higher HFT entry probability leads to lower average spread of the market maker. Yet the liquidity effect can be ambiguous since the market maker would also commit less capital in market making.