I present a new model of how ex-ante skewness affects expected asset prices. The price that supports a given short position in a positively- (negatively-) skewed asset is further from (closer to) expected value than is the price that supports a long position of the same magnitude, even in a frictionless market. The average effect of this result, under market clearing of stochastic demand, produces the “skewness effect”—a documented negative relationship between ex-ante skewness and expected returns. The theory generates several new predictions about the cross section of expected stock returns, for which I provide empirical support.

KEYWORDS: ex-ante skewness; rational expectations; microstructure influences in asset pricing

JEL CLASSIFICATIONS: G12, D53, D82
1. INTRODUCTION

Why do (positively) skewed stocks trade at a premium? A long history of cross-sectional studies have demonstrated that stocks with higher (positive) skewness in their return profiles tend to have lower subsequent returns; i.e., higher prices. Standard models and their extensions to higher return moments (e.g., Co-Skewness) have not been able to account for this skewness effect—a negative relationship between ex-ante skewness of individual stocks and their expected subsequent returns. Proposed explanations have turned to ingredients such as market frictions (e.g., trading costs, illiquidity, or short-sales constraints), behavioral or heterogeneous investor preferences, heterogeneous investor beliefs, or characteristics related to financial distress (such as high leverage, credit deterioration, or probability of near-term failure). While these ingredients can certainly play a significant role in pricing, are they essential to the skewness effect? Or can this pricing effect be explained by a more fundamental mechanism of financial markets?

In this paper, I show that the skewness effect can arise intrinsically from a deeper mechanism—market clearing of stochastic demand for skewed assets—without reliance on the auxiliary features of existing explanations. In addition, this new explanation generates novel testable predictions about the relationship between skewness and expected returns, for which I provide empirical support. I start with a standard noisy rational expectations economy along the lines of Grossman and Stiglitz (1980), extended to accommodate multiple assets [e.g., Admati (1985)]. To this standard setup, I add small, independent, skewed securities. Note that there are no behavioral or heterogeneous investor preferences and risk aversion provides the only limit to arbitrage, since the model does not incorporate any market frictions. Also, my results do not rely on financial distress, since individual firms that might underly the tradable securities are not explicitly modeled. Instead, rational investors have identical preferences and trade without frictions on imperfect information about fundamentals, whose distributions are specified exogenously. I show in this framework that market clearing of stochastic demand is sufficient to generate the skewness effect: increases in skewness push average prices up and hence average returns down.\footnote{In Appendix B, I discuss conditions under which such a relationship between expected prices and expected returns holds.}

To understand how the skewness effect can emerge from market clearing of stochastic demand,\footnote{See Section 2 for background on these empirical results and on alternative explanations proposed in the literature.}
first consider a hypothetical rational agent facing a risky bet. Assume the agent prefers more to less and is strictly risk averse with non-increasing absolute risk aversion. What are the prices that would induce this agent to take different sides of this bet? The agent can buy the risk by paying the bid price upfront and collecting the risky outcome later; or, the agent can take the other side of the bet by collecting the offer price upfront and promising to pay the result of the risky outcome later. The general relationship among the buying price, the expected value (or fundamental value), and the offering price is widely-known and straightforward: the buying price is below the offering price, and the fundamental value is in between them.

But which of these two prices (bid price or offer price) is closer to the fundamental value? The answer depends on the skewness of the bet. If the bet is a simple lottery that pays 1 with probability $q$, and nothing otherwise, and $q < \frac{1}{2}$, then the bet is positively skewed. Since buying the bet entails more desirable upside risk (chance of a high outcome relative to expected value) whereas offering the bet involves more downside risk, the bid price can provide less risk compensation than the offer price. So, the offer price must be further above the expected value than the bid price is below it—an intuitive consequence of non-increasing absolute risk aversion, which entails that the agent prefers upside risk to downside risk. The case of negative skewness has a similar, but opposite, intuition. If $q > \frac{1}{2}$, then the bet is negatively skewed, and the opposite relationship holds: the offering price is closer to fundamental value than is the buying price.

Now, apply this intuition to understand the behavior of the average price the standard risk-averse agent faces in a simple hypothetical market. Suppose there are two equally-likely states of the world, and that in order to clear the market in the first state, the agent must be induced to buy, whereas in the second state, the agent must be induced to take the opposite position. For example, suppose there is a second agent—a liquidity trader—who, depending on the state of the world, needs to take a positive or negative position in the asset. Then, to clear the market, half of the time the first agent would face the bid price, and half of the time the first agent would face the offer price. For a positively-skewed asset, the average price in this market would exceed fundamental value because the bid price is closer to fundamental value than is the offer price. Similarly, for a negatively-skewed asset, the average price will be below the fundamental value. Average prices are inversely related to average returns. Hence, the directional effect of skewness on average returns arises from balancing

\footnote{Under mild regularity conditions on the outcome distribution this always holds.}
the prices associated with the opposite sides of a skewed bet. This basic analysis carries through to the more general theoretical setting of the paper, which includes many agents and deals with issues of learning from private information sources and informative prices.

In addition to generating the skewness effect through a new approach, my model generates several novel testable predictions. First, \((P1)\) higher ex-ante skewness generates lower expected returns in financial markets having classically ideal conditions of frictionless trade of risk-averse rational investors and liquidity traders. More specifically, the skewness effect can arise in financial markets with little or no influence of traders with behavioral preferences; with little or no short-selling restraints; with high trading activity; with highly liquid assets; or with assets backed by healthy fundamentals (e.g., stocks of financially healthy firms). Next, the skewness effect is deepened under \((P2)\) higher ex-ante skewness and/or \((P3)\) higher market-wide risk or risk aversion, all under ideal conditions corresponding to \(P1\). Moreover, \((P4)\) higher ex-ante skewness accelerates the deepening of the skewness effect under higher market-wide risk or risk aversion. Next, \((P5)\) at sufficient levels of skewness, expected returns (in excess of the riskfree rate) can be negative, again under ideal conditions. Furthermore, \(P5\) lends insight into why standard factor models and their extensions (e.g., Co-Skewness) will systematically fail to predict a skewed asset’s expected return by overshooting the expected return of such an asset (see Appendix C). Finally, \((P6)\) investor heterogeneity can weaken the skewness effect, in contrast to explanations which require heterogeneity as the ingredient to generate the skewness effect.

The paper is organized as follows. In Section 2, I review related literature. In Section 3, I discuss a simple background example that shows how the skewness effect arises from basic principles. In Section 4, I analyze the main model. I outline the model’s predictions in Section 5 and provide empirical support in Section 6. I conclude in Section 7. Appendix A contains all proofs for results in the main text. The Online Appendix contains extensions and robustness tests of the main results, including how the implications of the main model can help explain shortcomings of the CAPM and Co-Skewness models in accounting for the skewness effect.

2. RELATED LITERATURE

Early research on the impact of skewness on investors’ decisions recognized that since investors prefer upside risk to downside risk, they will choose portfolios that are more positively skewed,
all else being equal (Arditti (1967) and Scott and Horvath (1980)). In related studies, Rubenstein (1973), Kraus and Litzenberger (1976, 1983), and Harvey and Siddique (2000a) incorporated these observations into co-skewness models that focused on higher moment contributions to stochastic discount factors.\textsuperscript{4} These models implied no role for idiosyncratic skewness characteristics in pricing. However, a recent and growing literature has documented a negative relationship between ex-ante skewness of individual stocks and their subsequent expected returns, which is unexplained by standard risk-based asset pricing models such as the capital asset pricing model (CAPM), the Fama and French (1993) three-factor model, or co-skewness factors.

Ex-ante skewness has been difficult to measure [Harvey and Siddique (2000a)], but the skewness effect has been demonstrated recently using several different and effective measures: Boyer, Mitton, and Vorkink (2010) use firm characteristics to construct a monthly measure of predicted skewness for individual stocks; Amaya, Christoffersen, Jacobs, and Vasquez (2013) construct a measure of realized skewness (as an analog to realized volatility) using high-frequency returns data; Conrad, Dittmar, and Ghysels (2013) construct an options-implied ex-ante skewness measure; and Boyer and Vorkink (2014) demonstrate the skewness effect for returns on equity options using a measure based on moneyness and return volatility.

Explanations consistent with this recent evidence have invoked behavioral preferences such as cumulative prospect theory [Barberis and Huang (2008)], non-Bayesian beliefs [Brunnermeier and Parker (2005), and Brunnermeier, Gollier, and Parker (2007)], or heterogeneous preferences [Mitton and Vorkink (2007)]. In addition, connections have been proposed between the skewness effect and other pricing effects that pertain to individual security characteristics such as the financial distress and idiosyncratic volatility effects. In particular, Campbell, Hilscher, and Szilagyi (2008) suggest a connection to the financial distress effect and Boyer, Mitton, and Vorkink (2010) empirically explore a potential link with the idiosyncratic volatility effect of Ang, Hodrick, Xing, and Zhang (2006).

These explanations may indeed apply to price formation in real markets; however, I show how the skewness effect can arise without such ingredients. My approach partly parallels that of Barberis and Huang (2008) in the addition of small, independent, skewed assets to a standard multiple security framework, but I start from a very different framework. Barberis and Huang’s (2008) frame-

\textsuperscript{4}Note that co-skewness is typically measured by the covariation of excess returns with squared excess market returns following Kraus and Litzenberger (1976) and Harvey and Siddique (2000a).
work employs cumulative prospect theory preferences whereas investors in my model have constant absolute risk aversion (CARA) preferences. Moreover, my model is amenable to the consideration of private information. Goulding (2017a) establishes theoretical connections between the skewness effect as modeled in this paper and the impact of financial analysts’ forecast dispersion on asset prices, for which Goulding (2017b) provides strong supporting empirical evidence.

3. OPPOSITE SIDES OF A SKEWED BET

Consider a simple lottery risk $\tilde{\theta}$ that pays $1 with probability (w.p.) $q \in (0, 1)$ and $0$ otherwise:

$$
\tilde{\theta} = \begin{cases} 
1, & \text{w.p. } q, \\
0, & \text{w.p. } 1 - q,
\end{cases} \quad q \in (0, 1). \quad (1)
$$

Note that this distribution is positively skewed for $q < \frac{1}{2}$, negatively skewed for $q > \frac{1}{2}$, and has no skewness for $q = \frac{1}{2}$; specifically, its skewness is given by

$$
\text{Skew}(\tilde{\theta}) = \frac{\mathbb{E}[(\tilde{\theta} - q)^3]}{(\mathbb{E}[(\tilde{\theta} - q)^2])^{3/2}} = \frac{1 - 2q}{\sqrt{q(1-q)}} \begin{cases} 
> 0, & \text{for } q < \frac{1}{2}, \\
= 0, & \text{for } q = \frac{1}{2}, \\
< 0, & \text{for } q > \frac{1}{2}.
\end{cases} \quad (2)
$$

For comparison, Figure 1 shows how signed skewness manifests in a continuous distribution vs. a binary distribution.

In this section, I will focus on answering the following two questions regarding this simple lottery risk:

(Q1) At what prices would a rational agent be willing to take different sides of this bet (i.e., buying vs. offering/insuring the lottery)?

(Q2) How do prices for each side of the bet relate to the expected (or fundamental) value of the lottery (i.e., $q$)?

To answer these questions, consider a hypothetical rational agent with preferences in the hyperbolic absolute risk aversion (HARA) class, specifically the non-increasing absolute risk aversion
This figure depicts the probability distribution of a continuous random variable (top row) and for a binary random variable (bottom row) for negative skewness (left), zero skewness (middle), and positive skewness (right).

(NARA) subclass, represented by a utility function of the following form:

\[
U(w) = \frac{1-a}{a} \left( \frac{\gamma w}{1-a} + b \right)^a, \quad \gamma > 0, \quad \frac{\gamma w}{1-a} + b > 0, \quad 0 \neq a < 1,
\]  

(3)

where \( w \) is the agent’s wealth and \( a, b, \) and \( \gamma \) are constants. This class includes commonly used utility functions as special cases of affine transformations of \( U(w) \).\(^5\) All utility functions in this broad NARA class have a special property that higher-order derivatives alternate sign (Lemma 1). The first derivative is positive, entailing strictly increasing utility that captures the preference for more to less. The second derivative is negative, entailing strictly concave utility which captures risk aversion. The third derivative is positive, the fourth derivative is negative, and so on.

**Lemma 1.** The higher order derivatives of any utility function \( U(w) \) of form (3) alternate sign: \( U^{(n)}(w) > 0 \) for \( n \) odd, and \( U^{(n)}(w) < 0 \) for \( n \) even.

Given our hypothetical agent with utility function \( U(w) \) as in (3) and initial wealth \( w_0 > 1,^6\) following Pratt (1964) and LaValle (1968), consider the bid price \( B \) and ask (or offer) price \( S \) for lottery \( \tilde{\theta} \), such that:

---

\(^5\)For example, (1) \( \gamma = 1 \) and \( a \to 0 \) gives logarithmic utility, \( \ln(w+b) \); (2) \( \gamma = 1-a \) gives power utility, \( \frac{1}{1-\gamma}(w+b)^{1-\gamma} \); and (3) \( a \to -\infty \) and \( b = 1 \) gives negative exponential utility, \( -e^{-\gamma w} \).

\(^6\)\( w_0 > 1 \) ensures that final wealth is positive so that utility is well-defined for all outcomes of a single-unit position in \( \tilde{\theta} \).
**Definition 2.** $B$ denotes the largest price at which an agent with initial wealth $w_0$ is willing to obtain the lottery $\tilde{\theta}$ in (1): $B := \sup\{P \in \mathbb{R} : E[U(w_0 + (\tilde{\theta} - P))] = U(w_0)\}$. $S$ denotes the smallest price at which the agent is willing to offer or insure the lottery: $S := \inf\{P \in \mathbb{R} : E[U(w_0 - (\tilde{\theta} - P))] = U(w_0)\}$.

Direct application of Jensen’s inequality gives the widely-known result and answer to question (Q1) (Lemma 3): by a straightforward appeal to risk aversion, the bid price $B$ is below the fundamental value $q$ of the lottery, which is below the offer price $S$.

**Lemma 3.** $B < q < S$.

Now, turning to question (Q2), how do prices for each side of the bet relate to the fundamental value of the lottery, $q$? Is $B$ much further below $q$ than $S$ is above $q$; are they roughly the same distance on either side; or is $S$ much further above $q$ than $B$ is below $q$?

While the answer to the first question is straightforward and widely-known, an answer to the second question appears to be unexplored in the literature.

To answer the second question, consider the payoffs of obtaining or offering this lottery at prices $B$ and $S$, respectively. To obtain the lottery, the agent pays $B$ initially and later obtains $1$ with probability $q$. The overall payoff of obtaining the lottery at price $B$ can be expressed as a fixed part $q - B$ and an unbiased risk:

\[
-B + \begin{cases} 
1 \text{ w.p. } q \\
0 \text{ w.p. } 1-q
\end{cases} \equiv q - B + \begin{cases} 
1 - q \text{ w.p. } q \\
-q \text{ w.p. } 1-q
\end{cases} \equiv q - B + \tilde{\varepsilon}.
\]

(4)

To offer/insure the lottery, the agent receives $S$ initially and later pays $1$ with probability $q$. The overall payoff of offering/insuring the lottery at price $S$ can be expressed as a fixed part $S - q$ and the
opposite position in the same unbiased risk:

\[
S - \begin{cases}
  1 & \text{w.p. } q \\
  0 & \text{w.p. } 1-q \\
\end{cases} \equiv S - q - \begin{cases}
  1 - q & \text{w.p. } q \\
  -q & \text{w.p. } 1-q \\
\end{cases} \equiv S - q - \tilde{\varepsilon}. \tag{5}
\]

Specifically, the unbiased part of the risk is given by

\[
\tilde{\varepsilon} := \begin{cases}
  1 - q & \text{w.p. } q \\
  -q & \text{w.p. } 1-q \\
\end{cases}, \quad \mathbb{E}[\tilde{\varepsilon}] = 0, \tag{6}
\]

and has the property that the signs of its higher odd moments are the same and depend on the skewness of the underlying risk \( \tilde{\theta} \) (Lemma 4), which skewness is inversely one-to-one with \( q \).

**Lemma 4.** The higher odd moments \( (n > 1) \) of the unbiased part of the risk, \( \tilde{\varepsilon} \) (equation (6)), have the same sign, which depends on the skewness of the risk, which is one-to-one with \( q \):

\[
Even \ n : \quad \mathbb{E}[\tilde{\varepsilon}^n] = (1 - q)^n q + q^n (1 - q) > 0,
\]

\[
Odd \ n > 1 : \quad \mathbb{E}[\tilde{\varepsilon}^n] = (1 - q)^n q - q^n (1 - q)
\]

\[
\begin{cases}
  > 0 & \text{for } q < \frac{1}{2} \ (\Leftrightarrow \text{positive skewness}), \\
  = 0 & \text{for } q = \frac{1}{2}, \\
  < 0 & \text{for } q > \frac{1}{2} \ (\Leftrightarrow \text{negative skewness}).
\end{cases}
\]

Using the payoff decompositions into fixed components and unbiased risks ((4) and (5)), and the properties recorded in Lemma 1 and Lemma 4, I obtain the answer to question (Q2) (Theorem 5):

**Theorem 5.** The relative sizes of the gaps between \( B, q, \) and \( S \), depend on the skewness of \( \tilde{\theta} \):

\[
S - q \begin{cases}
  > q - B, & \text{for } q < \frac{1}{2} \ (\Leftrightarrow \text{positive skewness}), \\
  = q - B, & \text{for } q = \frac{1}{2}, \\
  < q - B, & \text{for } q > \frac{1}{2} \ (\Leftrightarrow \text{negative skewness}).
\end{cases}
\]

Note that the value of \( q \) impacts not only skewness but also mean and variance. However, for a given \( q \), the variance of the risk is the same from either the buying or selling perspective. So, intuition
for this result is linked to the difference between upside and downside risk. Intuitively, buying the lottery entails less expected risk compensation (q above the price B) when q < \( \frac{1}{2} \) because such a bet is positively skewed and dominated by upside risk. Similarly, the other side of such a bet is dominated by downside risk, so the offering price S must be relatively higher above q than B is below q.\(^7\)

Theorem 5 holds for a broader class of payoff distributions beyond the simple binary payoff (see Appendix D).

**Market Implications**

Now, consider the hypothetical agent in a simple market for asset \( \tilde{\theta} \). One can think of the market price for this asset as a lever that influences the agent’s willingness to take either a long or short position. If the market price \( P = B \), then, as described above, the agent is willing to buy, while if the price \( P = S \), then the agent is willing to take the other side of the risk. Suppose there is a second agent in the market, a liquidity trader, whose demand for the asset is random based on some exogenous reason unrelated to the price: \(^8\) half of the time the second agent buys the asset, and thus the price must be \( P = S \) in order for the first agent to take the other side of the risk to clear the market; likewise, half of the time the second agent shorts the asset, and thus the price must be \( P = B \) in order to clear the market. Therefore, an econometrician who observes the market clearing price over time in repeated instances of this simple setting (even if the details of the market participants are not observable) will document an average market price of \((B + S)/2\).

Corollary 6, which follows immediately from Theorem 5, summarizes how the average price of this system relates to the skewness of the asset payoff distribution.

**Corollary 6.**

\[
\text{average } P = \begin{cases} 
\frac{S+B}{2} > q, & \text{for } q < \frac{1}{2} \quad (\Leftrightarrow \text{positive skewness}), \\
\frac{S+B}{2} = q, & \text{for } q = \frac{1}{2}, \\
\frac{S+B}{2} < q, & \text{for } q > \frac{1}{2} \quad (\Leftrightarrow \text{negative skewness}).
\end{cases}
\]

So, the deviation of the average price from fundamental value varies directionally with the skewness.

---

\( ^7 \) The proof, which is in Appendix A, employs the Taylor’s series expansions of the agent’s expected utility around the fixed part of wealth aside from the unbiased risky part of the bet. The dependence of the sign of odd moments of the unbiased part of the risk on skewness (Lemma 4) and the alternating nature of higher-order derivatives of the utility function (Lemma 1) leads to dependence of the sign of the overall expansion on skewness.

\( ^8 \) I.e., the second agent’s demand is 1 with probability 50% and −1 with probability 50%.
of the asset risk. Accordingly, skewness has a negative relationship with expected returns.

Next, I will show how this insight pertains to the main analysis. So far I have only considered a simple market set-up with one strategic agent in which information did not play a role, including no information effects from the price itself. In the next section, I explore how the core insight from this limited setting plays out in a richer equilibrium set-up, which includes many agents and deals with issues of learning from private information sources and informative prices.

4. THE MODEL

The model has three dates, \( t \in \{0, 1, 2\} \), and its basic set-up is as follows:

**Assets:** There is a risk-free asset, with both its price and payoff normalized to one, and \( n + 1 \) risky assets: (1) \( n \) “baseline” risky assets, with price \( \tilde{p}(i) \) and payoff \( \tilde{\theta}(i) \) for \( i = 1, \ldots, n \); and (2) one additional small risky asset that is independent of all other assets, with price \( \tilde{p} \) and payoff \( \tilde{\theta} \).

**Investors:** A unit mass of investors with CARA preferences (with risk aversion parameter \( \gamma > 0 \)) consists of two possible types of investors: \( \lambda \in [0, 1] \) of the investors are “informed” investors who condition their trades on both price and private information, and the remaining \( 1 - \lambda \) are “uninformed” investors who condition their trades only on price. Investors trade the assets at date \( t = 1 \) in an adapted version of a noisy rational expectations financial market along the lines of Grossman and Stiglitz (1980) and Admati (1985) and the price of each risky asset is set to clear the market for that asset in every state of the economy.\(^9\)

**Stochastic demand:** Following the standard in Grossman and Stiglitz (1980), Admati (1985), and other noisy rational expectations models, there is stochastic demand for risky asset \( i \), \( \tilde{\eta}^{(i)} \), and for the additional risky asset, \( \tilde{\eta} \).\(^{10}\) Prices are set to clear the market of this stochastic demand for each risky asset in each state.

---

\(^9\) The main departure from the model of Grossman and Stiglitz (1980) and Admati (1985) will be the choice of the asset payoff distribution for the additional independent risky asset, to allow for skewness in its payoff distribution, and the structure of private information for this asset (Section 4.2). In Appendix E, I develop an auxiliary model which considers a more general class of payoff distributions and private information structures.

\(^{10}\) Black (1986) describes the importance of having some source of noise for strategic trading to occur and Grossman and Stiglitz (1980) discuss the role of noisy supply (or equivalently stochastic demand) to prevent fully revealing prices and to support profitable trade.
4.1. Investors’ Decisions and Market Clearing Prices

In this section, I consider the investment problem at date \( t = 1 \), after investors have observed any private signals about the risky assets, and I characterize the market clearing conditions that determine equilibrium prices. I show that the price of the small independent asset is not affected by, and does not affect, the prices of the other risky assets.\(^{11}\)

Each investor \( j \in [0, 1] \) has CARA utility over final wealth (i.e., \( -e^{-\gamma w} \)) with common risk aversion parameter \( \gamma > 0 \) and seeks a portfolio of the riskless and risky assets to maximize conditional expected utility given an information set:

\[
(x^{(1)}, \ldots, x^{(n)}, x) \in \arg \max_{(x^{(1)}, \ldots, x^{(n)}, x)} \mathbb{E}[ -e^{-\gamma(w_0 + \sum_{i=1}^n (\tilde{\theta}^{(i)} - \tilde{\beta}^{(i)})x^{(i)} + (\bar{\theta} - \bar{\beta})x)] | \mathcal{F}_j],
\]

where \( w_0 \) is initial wealth; \( x^{(i)} \) is the amount invested in baseline risky asset \( i \), for \( i = 1, \ldots, n \), and \( x \) is the amount invested in the small independent asset; and \( \mathcal{F}_j \) is investor \( j \)'s information set of private signals and prices,

\[
\mathcal{F}_j := \{ \tilde{s}_j = s^{(i)}, \tilde{p}^{(i)} = p^{(i)}, i = 1, \ldots, n, \tilde{s}_j = s, \tilde{p} = p \},
\]

where \( \tilde{s}_j \) is investor \( j \)'s private signal for baseline risky asset \( i \). In the case of an uninformed investor, the information set reduces to \( \mathcal{F}_j = \mathcal{F}_\emptyset := \{ \tilde{p}^{(i)} = p^{(i)}, i = 1, \ldots, n, \tilde{p} = p \} \). Prices are set to clear the market of stochastic demand for each asset in every state of the world:

\[
p^{(i)} : \int_j x^{(i)} d j = -\eta^{(i)}, \quad i = 1, \ldots, n,
\]

\[
p : \int_j x d j = -\eta.
\]

As indicated in Proposition 7, the pricing effects for the independent asset can be considered separately from those of the \( n \) baseline assets.\(^{12}\)

\(^{11}\)This property holds in my model despite no constraints on short-selling. For comparison, Barberis and Huang (2008), who provide a behavioral explanation for the skewness effect, state (pg. 2076): “In general, when a new security is introduced into an economy, the prices of existing securities are affected. An interesting feature of our proposed equilibrium ... is that, if the skewed security cannot be sold short, its introduction does not affect the prices of the \( J \) original risky assets: their prices ... are identical to what they were [when] there was no skewed security.”

\(^{12}\)Proposition 7 greatly simplifies the analysis by allowing consideration of a single stock rather than a portfolio. This useful result relies on the independence of the small asset and on CARA utility. Independence is a strong assumption;
Proposition 7 (Independence of the small asset’s price, $\tilde{p}$). The equilibrium price of the small independent asset, $\tilde{p}$, depends only on private signals $\tilde{s}_j, j \in [0,1]$ and stochastic demand $\tilde{\eta}$ that pertain directly to the independent asset. Moreover, $\tilde{p}$ is independent of the prices, private signals, and stochastic demands of the baseline risky assets. Furthermore,

$$x, \tilde{\eta} \in \arg\max_x E[-e^{-\gamma(\tilde{\theta} - \tilde{p})} | \tilde{s}_j = s, \tilde{p} = p]. \tag{11}$$

Accordingly, my analysis in the next subsections will focus on the specification details and equilibrium price of the small independent risky asset, which has price $\tilde{p}$, payoff $\tilde{\theta}$ and stochastic demand $\tilde{\eta}$. The other $n$ baseline risky assets need not be structured similarly and can assume more general or more special structures without changing any conclusions drawn for the small independent asset. For example, if the payoffs, private signals, and stochastic demands of the $n$ baseline risky assets are jointly normally distributed and no analysts follow those assets, then prices for those $n$ assets specialize to the closed-form solution given in Admati (1985). So, a simple way to view my model is that a small independent (and skewed) asset is introduced into the multiple asset noisy rational expectations economy of Admati (1985).

4.2. Distributions and Information for the Independent Asset

In this section, I detail the structure of uncertainty and information for the independent risky asset, whose payoff, stochastic demand, and investors’ signals are all independent of those of the other $n$ baseline risky assets.

4.2.1 Payoff Distribution

I continue with the binary distribution as introduced in the background example (Section 3, (1)) but generalize it to have the high and low payoff states as parameters.\textsuperscript{13} Specifically, let $\tilde{\theta} \in \{\theta_h, \theta_l\}$, however, few stocks are perfectly dependent, and their dependence virtually always lies somewhere between the two extremes. Likewise, CARA utility is a strong assumption since it exhibits no risk aversion sensitivity to wealth levels. However, extreme sensitivity of risk aversion to wealth levels would also be a strong assumption. Again, reality lies somewhere in the middle. Moreover, it is well-known that investors tend to hold a limited number of assets, and hence are not fully diversified. Analysis of a single stock can approximate features of a portfolio with a limited number of holdings. Therefore, the economic forces identified by my analysis have qualitative value and the quantitative impact of these simplifications is an empirical question. See Sections 5 and 6, respectively, for a discussion of the model’s useful predictions and supporting evidence.

\textsuperscript{13}In Appendix E, I develop an auxiliary model, which employs a large class of continuous payoff distributions, to show that the main results are robust to this specific choice for the payoff distribution.
where $\theta_h > \theta_l$ are the possible outcomes of the risky asset at date $t = 2$. Ex-ante (at date $t = 0$) the “high” outcome $\theta_h$ is realized with probability $q \in (0,1)$, and the “low” outcome $\theta_l$ is realized with probability $1 - q$:

$$
\tilde{\theta} = \begin{cases} 
\theta_h, & \text{w.p. } q, \\
\theta_l, & \text{w.p. } 1 - q,
\end{cases} \quad \theta_h > \theta_l, \quad q \in (0,1).
$$

(12)

As before, the skewness is one-to-one with $q$ for any choice of $(\theta_h, \theta_l)$: if $q < \frac{1}{2}$, then the payoff is positively skewed; if $q = \frac{1}{2}$, then the payoff is not skewed; if $q > \frac{1}{2}$, the payoff is negatively skewed. Skew $(\tilde{\theta}) = \frac{1 - 2q}{\sqrt{q(1-q)}}$ (cf. (2)).

Moreover, this three parameter binary distribution is one of the simplest distributions that allows the flexibility to specify independently its first three moments. Just as the binary payoff $\tilde{\theta}$ can be uniquely characterized by three parameters, $(\theta_h, \theta_l, q)$, equivalently, it can be uniquely characterized by its first three moments (Theorem 8).

**Theorem 8** (Three Moment Characterization of a Binary Payoff). Let $E$, $V(>0)$, and $S$ denote respectively the desired mean, variance, and skewness for the binary payoff $\tilde{\theta}$ of (12). These moments are achieved by the following unique parameterization:

$$
\theta_h = E + \sqrt{\frac{V(1-q)}{q}}, \quad \theta_l = E - \sqrt{\frac{Vq}{1-q}}, \quad q = \frac{1}{2} \left[ 1 - \frac{S}{\sqrt{4+S^2}} \right].
$$

This reparameterization opens the possibility to fix any two of the three moments and isolate the effects of changes in the third. Nevertheless, it will be more convenient to work with the $(\theta_h, \theta_l, q)$ parameterization for most of the analysis.

### 4.2.2 Stochastic Demand

The stochastic demand for the independent asset, $\tilde{\eta}$, is distributed with differentiable log-concave probability distribution function $g(\eta)$, which is symmetric around zero so that $\tilde{\eta}$ has zero mean.\(^{14}\)

\(^{14}\)Zero mean stochastic demand with zero supply is a common assumption in noisy rational expectations models with CARA utility since investors’ demand functions are independent of initial wealth; e.g., Verrecchia (1982). Barberis and Huang (2008) assume an infinitesimal deterministic supply for their independent skewed security such that it is small relative to their other (non-skewed) risky assets. In my model, zero supply operates similarly to an infinitesimal positive supply, with results essentially unaltered. In Appendix F, I discuss the impacts of a small positive increase to the supply for the independent asset.
4.2.3 Investors’ Information

Investors potentially have many sources of information that can lead to heterogeneous beliefs. In addition to information aggregated and conveyed by price, some investors may obtain information from exclusive sources such as private analyses or first-hand knowledge of factors affecting a firm’s stock price such as local demand for products, etc. For private signals about the independent asset, I use a simple trinary structure to model this notion of heterogeneity of information sources among privately-informed investors.

The goal is to allow for heterogeneous beliefs among investors at different levels of average optimism across agents. For example, consider the case of just two investors. If the first investor receives a more pessimistic private signal than the second investor, whose private signal is relatively neutral, then there will be heterogeneity in beliefs between these investors and the average level of optimism will be low. If, however, the first investor receives a more optimistic private signal than the second investor, whose private signal is again relatively neutral, then, again, there will be heterogeneity in beliefs between these investors, but the average level of optimism in this case will be high. So, heterogeneity can occur at different levels of overall optimism, which I model as follows.

Each informed investor receives one of three possible private signals, \( \tilde{s} \in \{L, M, H\} \) (“low”, “medium”, or “high”), according to one of two information scenarios, \( \tilde{\rho} \in \{LM, MH\} \), respectively either a “pessimistic” private information environment, \( \tilde{\rho} = LM \), (“low” and “medium” signals) or an “optimistic” private information environment, \( \tilde{\rho} = MH \), (“medium” and “high” signals). Under the pessimistic scenario, \( \tilde{\rho} = LM \), each informed investor receives either a low or medium signal, with equal probability, and no one receives a high signal. That is, \( P[\tilde{s} = L|\tilde{\rho} = LM] = P[\tilde{s} = M|\tilde{\rho} = LM] = \frac{1}{2} \) and \( P[\tilde{s} = H|\tilde{\rho} = LM] = 0 \). (See Figure 2 for an illustration of how signals are distributed.) Under the optimistic scenario, \( \tilde{\rho} = MH \), each informed investor receives either a medium or a high signal, with equal probability, and no one receives a low signal. That is, \( P[\tilde{s} = L|\tilde{\rho} = LM] = 0 \) and \( P[\tilde{s} = M|\tilde{\rho} = LM] = P[\tilde{s} = H|\tilde{\rho} = LM] = \frac{1}{2} \).

The occurrence of each information scenario depends on the (unknown) true asset payoff in a natural way. Specifically, when the true payoff is low, \( \tilde{\theta} = \theta_l \), a pessimistic scenario is more likely and occurs with probability

\[
\phi \in \left(\frac{1}{2}, 1\right); \tag{13}
\]
when the true payoff is high, $\tilde{\theta} = \theta_h$, an optimistic scenario is more likely and occurs with probability $\phi$. That is, $P[\tilde{\rho} = LM | \tilde{\theta} = \theta_l] =: \phi := P[\tilde{\rho} = MH | \tilde{\theta} = \theta_h]$. Figure 2 illustrates how the occurrence of the pessimistic or optimistic private information scenarios depends on the true payoff, and how the signals are distributed within each private information scenario.

This figure depicts the probability tree of an informed investor’s signal $\tilde{s}$ regarding the payoff $\tilde{\theta}$. Under the pessimistic scenario, $\tilde{\rho} = LM$, each informed investor receives either a low or medium signal, with equal probability, and no one receives a high signal. That is, $P[\tilde{s} = L | \tilde{\rho} = LM] = P[\tilde{s} = M | \tilde{\rho} = LM] = 1/2$ and $P[\tilde{s} = H | \tilde{\rho} = LM] = 0$. Under the optimistic scenario, $\tilde{\rho} = MH$, $P[\tilde{s} = L | \tilde{\rho} = LM] = 0$ and $P[\tilde{s} = M | \tilde{\rho} = LM] = P[\tilde{s} = H | \tilde{\rho} = LM] = 1/2$. When the true payoff is low, $\tilde{\theta} = \theta_l$, a pessimistic scenario is more likely and occurs with probability $\phi \in (1/2, 1)$; when the true payoff is high, $\tilde{\theta} = \theta_h$, an optimistic scenario is more likely and occurs with probability $\phi$. That is, $P[\tilde{\rho} = LM | \tilde{\theta} = \theta_l] =: \phi := P[\tilde{\rho} = MH | \tilde{\theta} = \theta_h]$.

The parameter $\phi$ (equation (13)) plays a dual role, governing two important properties of investors’ information environment. First, $\phi$ captures the precision of private signals. If $\phi$ is high, then low or high signals are more informative because they tend to correspond more strongly to the true payoff state. If $\phi$ is low, then there is an increased chance that low and high signals are inaccurate indicators of the (unknown) true payoff state.

Second, $\phi$ is tied to the degree of heterogeneity in investors’ beliefs, as indicated in Lemma 9. Specifically, let

$$\hat{\pi}_s := P[\tilde{\theta} = \theta_h | \tilde{s} = s], \quad s \in \{L, M, H\},$$

(14)

denote an investor’s belief regarding the high payoff state conditional on the private signal, $\tilde{s} = s$. 

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Lemma 9 (Investor Heterogeneity). Heterogeneity in investors’ beliefs increases with the precision of private signals, \( \phi \). Specifically, \( \frac{\partial (\bar{\pi}_M - \bar{\pi}_L)}{\partial \phi} > 0 \) and \( \frac{\partial (\bar{\pi}_H - \bar{\pi}_M)}{\partial \phi} > 0 \), where \( \bar{\pi}_L, \bar{\pi}_M, \) and \( \bar{\pi}_H \) are as in (14).

For two rational investors with a common prior to disagree strongly after updating their beliefs, such information must be sufficiently precise so that their posterior beliefs are swayed apart strongly from that prior and hence from each other’s beliefs. Thus, any parameter that governs the degree of heterogeneity in posterior beliefs also necessarily governs the informativeness of private signals.\(^\text{15}\) Investors would need heterogeneous priors in order to break this link between the two governing properties of \( \phi \).

4.3. Equilibrium Price of the Independent Asset

In this section, I apply the specific distributional properties from the previous section to determine the demand functions and market clearing price for the independent asset. Let Equation (11) be expressed as

\[
x_s := x \in \arg\max_x \left[ -e^{-\gamma(\bar{\theta} - \bar{p})x} | s = s, \tilde{p} = p \right], \quad s \in \{L, M, H, \emptyset\},
\]

where \( x_s \) denotes the optimal demand given \( s \in \{L, M, H, \emptyset\} \), \( s = \emptyset \) corresponds to an uninformed investor with no private information. Likewise, let \( \pi_{s,p} \) be the posterior probability that \( \tilde{\theta} = \theta_h \) given private signal \( s \) and price \( p \):

\[
\pi_{s,p} := P[\tilde{\theta} = \theta_h | s = s, \tilde{p} = p], \quad s \in \{L, M, H, \emptyset\}.
\]

The first-order condition is

\[
\gamma(\theta_h - p)e^{-\gamma(\theta_h - p)x_s} \pi_{s,p} + \gamma(\theta_l - p)e^{-\gamma(\theta_l - p)x_s} (1 - \pi_{s,p}) = 0,
\]

\(^\text{15}\)Note that this inherent link poses a potential complication for explanations of pricing effects that appeal to investor heterogeneity (or differences of opinion). In particular, investor heterogeneity can be coextensive with higher precision of private information and, consequently, linked to price informativeness (since equilibrium prices publicly transmit private information, with noise). That is, investor heterogeneity can go hand-in-hand with price informativeness, and would need to be disentangled. Accordingly, pricing effects that correlate with investor heterogeneity might be confounded or offset by pricing effects that are simply correlated with overall price informativeness.
which implies that

\[ x_s = \frac{1}{\gamma(\theta_h - \theta_l)} \ln \left( \frac{\theta_h - p}{p - \theta_l} \frac{\pi_{s,p}}{1 - \pi_{s,p}} \right). \]  

(16)

So, each investor’s demand for the independent risky asset has a simple form of a risk-scaled \((\gamma(\theta_h - \theta_l))\), log-transformed, payoff-adjusted version of an odds-ratio. The ratio consists of the probability of the high state (given all available public and private information) \(\pi_{s,p}\) times the net payoff of buying in the high state \(\theta_h - p\) relative to the probability of the low state \(1 - \pi_{s,p}\) times the net payoff of shorting in the low state \(p - \theta_l\).

### 4.3.1 Market Clearing and the Equilibrium Price Function

The equilibrium price function, \(P_\rho(\eta)\), depends on the realized private information scenario \(\rho\) (a binary variable representing either the pessimistic \((LM)\) or optimistic \((MH)\) signal scenario) and the realized stochastic demand \(\eta\). So, the price function can be determined as the solution to the following market-clearing conditions which specialize (10) (and which depend on price through each \(x_s\)):

\[ P_{LM}(\eta) : \lambda \left( \frac{1}{2} x_L + \frac{1}{2} x_M \right) + (1 - \lambda) x_\emptyset + \eta = 0, \quad \text{(pessimistic)} \]  

(17)

\[ P_{MH}(\eta) : \lambda \left( \frac{1}{2} x_M + \frac{1}{2} x_H \right) + (1 - \lambda) x_\emptyset + \eta = 0. \quad \text{(optimistic)} \]  

(18)

In the pessimistic scenario, half of the privately informed investors (fraction \(\lambda\) of the unit continuum) receive a low signal \((\tilde{s} = L)\) and demand \(x_L\) and the other half receive a medium signal \((\tilde{s} = M)\) and demand \(x_M\). In the optimistic scenario, half of the privately informed investors receive a medium signal \((\tilde{s} = M)\) and demand \(x_M\) and the other half receive a high signal \((\tilde{s} = H)\) and demand \(x_H\). In either scenario, the remaining fraction \(1 - \lambda\) of uninformed investors demand \(x_\emptyset\). The equilibrium price in each scenario must clear the market of informed demand, uninformed demand, and stochastic demand \(\eta\).

The interim price function that satisfies these market-clearing conditions can be expressed in closed form as summarized in Theorem 10:

**Theorem 10** (Interim Price Function). The interim stage equilibrium price function is given by

\[ P_{LM}(\eta) = (\theta_h - \theta_l) \frac{q}{q + (1 - q)e^{-f(\eta - \Delta)}} + \theta_l, \]  

(19a)
\[ P_{MH}(\eta) = (\theta_h - \theta_l) \frac{q}{q + (1-q)e^{-f(\eta+\Delta)}} + \theta_l, \]  

(19b)

where \( f \) is a closed-form strictly increasing non-linear function,

\[ f(x) := \gamma(\theta_h - \theta_l) \left[ x + \frac{1-\phi/2}{\gamma(\theta_h - \theta_l)} \ln \frac{(1-\phi)g(x+\Delta) + \phi g(x-\Delta)}{\phi g(x+\Delta) + (1-\phi)g(x-\Delta)} \right], \]  

(20)

and \( \Delta > 0 \) is a constant,

\[ \Delta = \frac{\lambda/2}{\gamma(\theta_h - \theta_l)} \ln \left[ \frac{\phi}{1-\phi} \right], \]  

(21)

and has the following properties:

1. \( P_{LM}(\eta) \) and \( P_{MH}(\eta) \) are strictly increasing in stochastic demand \( \eta \),
2. \( P_{LM}(\eta) < P_{MH}(\eta) = P_{LM}(\eta + 2\Delta) \), for all \( \eta \).

The price function in each of the pessimistic and optimistic scenarios is non-linear in \( \eta \) through an exponential-type adjustment in the denominator involving a strictly increasing non-linear function \( f \) of stochastic demand shifted by a constant. This adjustment accounts for the nature of the negative exponential risk aversion of the investors and for the impact of the inference problem of uninformed and medium-signal investors as to whether the price is associated with a pessimistic or optimistic private-information environment, of which such investors are uncertain.

Figure 3 illustrates the price function of Theorem 10 in a positively skewed case \( (q < \frac{1}{2}) \) and in which \( \eta \) is normally distributed. First, notice that as a function of \( \eta \), price is strictly increasing. This property is intuitive since higher stochastic demand will require a higher price in order to induce investors to decrease their demand in order to clear the market. Also, notice that the optimistic scenario price function strictly dominates the pessimistic scenario price function at any given level of \( \eta \). This property is also intuitive since investors in the optimistic scenario have higher valuations of the asset and thus price must be higher to induce them to have the same net demand required to clear the market at any given level of stochastic demand. The role of the adjustment function \( f \) as part of an inference problem is also illustrated in Figure 3. For example, consider an uninformed investor who observes market price \( p \) (e.g., \( p = 0.4 \)). Since this investor does not have any private information, this uninformed investor does not know with certainty whether the price level \( p \) is associated with the optimistic or pessimistic scenario price function. Instead, this uninformed investor must weight
This figure plots the equilibrium price of the risky asset $\tilde{\theta}$ as a function of stochastic demand $\eta$ (equations (19a) and (19b)) in the case of positive skewness ($q < \frac{1}{2}$), for normally distributed stochastic demand, $\tilde{\eta} \sim \mathcal{N}(0, \sigma^2)$, where $\sigma = 2$, $\theta_h = 1$, $\theta_l = 0$, $q = 0.25$, $\lambda = 0.8$, $\gamma = 1$, and $\phi = 0.7$.

these two possibilities by the likelihood of the demand shocks that would correspond to a price level $p$ in each scenario. This inference problem is reflected in the function $f$ by the appearance of a log ratio of different weighted averages of the stochastic demand density $g$.

### 4.3.2 Equilibrium Price under Uniform Stochastic Demand

The price function has very similar characteristics for different stochastic demand distributions, $g$, and so I will focus on the specific simplifying case of uniformly distributed stochastic demand, $\tilde{\eta} \sim \mathcal{U}[-k, k]$, going forward.\(^\text{16}\) The price function in the uniform demand shock case is recorded in Corollary 11 and illustrated in Figure 4.

**Corollary 11 (Uniform Demand Shock).** Let $\tilde{\eta}$ be uniformly distributed on $[-k, k]$ for some $k > 2\Delta$;

\(^\text{16}\)In Appendix G, I show by numerical illustration that the main results are robust to this specialization of noise to the uniform case.
i.e., \( g(\eta) = \frac{1}{2k} 1_{[\eta,-k,k]} \). Then, the interim stage equilibrium price function is given by

\[
P_{LM}(\eta) = \begin{cases} 
(\theta_h - \theta_l) \frac{1}{1 + \frac{e^{-\gamma(\theta_h - \theta_l)\eta}}{|\frac{k}{2}|}} + \theta_l, & \eta \in [-k,-k+2\Delta), \\
(\theta_h - \theta_l) \frac{1}{1 + \frac{e^{-\gamma(\theta_h - \theta_l)\eta}}{|\frac{k}{2}|}} + \theta_l, & \eta \in [-k+2\Delta,k], \\
(\theta_h - \theta_l) \frac{1}{1 + \frac{e^{-\gamma(\theta_h - \theta_l)\eta}}{|\frac{k}{2}|}} + \theta_l, & \eta \in [-k-2\Delta,k], \\
(\theta_h - \theta_l) \frac{1}{1 + \frac{e^{-\gamma(\theta_h - \theta_l)\eta}}{|\frac{k}{2}|}} + \theta_l, & \eta \in (k-2\Delta,k]. 
\end{cases}
\]

(22a)

\[
P_{MH}(\eta) = \begin{cases} 
(\theta_h - \theta_l) \frac{1}{1 + \frac{e^{-\gamma(\theta_h - \theta_l)\eta}}{|\frac{k}{2}|}} + \theta_l, & \eta \in [-k,-k+2\Delta), \\
(\theta_h - \theta_l) \frac{1}{1 + \frac{e^{-\gamma(\theta_h - \theta_l)\eta}}{|\frac{k}{2}|}} + \theta_l, & \eta \in [-k+2\Delta,k], \\
(\theta_h - \theta_l) \frac{1}{1 + \frac{e^{-\gamma(\theta_h - \theta_l)\eta}}{|\frac{k}{2}|}} + \theta_l, & \eta \in [-k-2\Delta,k], \\
(\theta_h - \theta_l) \frac{1}{1 + \frac{e^{-\gamma(\theta_h - \theta_l)\eta}}{|\frac{k}{2}|}} + \theta_l, & \eta \in (k-2\Delta,k]. 
\end{cases}
\]

(22b)

**FIGURE 4**

*Example of Price Function for Uniform Stochastic Demand*

This figure plots the equilibrium price of the risky asset \( \tilde{\theta} \) as a function of stochastic demand \( \eta \) (equations (19a) and (19b)) in the case of positive skewness \( (q < \frac{1}{2}) \), for uniformly distributed stochastic demand, \( \tilde{\eta} \sim \mathcal{U}[-k,k] \), where \( k = 3, \theta_h = 1, \theta_l = 0, q = 0.25, \lambda = 0.8, \gamma = 1, \) and \( \phi = 0.7 \).

In the uniform demand shock case in which shocks have finite support between two bounds, the inference adjustment function simplifies and we get two non-linear function segments for each of the optimistic and pessimistic scenarios, for a total of four segments to describe the price function. Intuitively, for example, the uninformed investor is typically weighing either the possibility (1) that investors are optimistic and the demand shock is low, or the possibility (2) that investors are pessimistic and demand shock is high. However, when the price is high enough, then the inference
problem for the uninformed becomes trivial: it must be an optimistic scenario because the required shock to support the pessimistic possibility would be above the shock upper bound. Similarly, when the price is low enough, then the inference problem for the uninformed becomes trivial: it must be a pessimistic scenario because the required shock to support the optimistic possibility would be below the shock lower bound. As a result, for shocks in an intermediate range the inference problem is non-trivial and there remain two possible scenarios from the point of view of the uninformed. However, for shocks near their extremes the inference problem is trivial, the scenario is revealed, and there is only one possible price function.

Both examples of Figure 3 and Figure 4 hint at the fact that the expected price, after averaging out demand shocks across their respective distributions, exceeds the fundamental value \( q \) for a positively skewed distribution. In both examples of Figure 3 and Figure 4, the payoff is positively skewed since \( q < \frac{1}{2} \). Also in both examples, the price function is non-linear and tends to take values increasingly in excess of \( q \) for positive demand shock values while taking values below, but closer to \( q \) for negative demand shock values. Hence, if one were to average these price functions according to a normal distribution weighting or uniform distribution weighting, respectively, one would expect the average interim price to exceed fundamental interim value \( q \) for the positively skewed case. This property of the expected price and its connection to the skewness effect is discussed formally in the next section.

4.4. The Skewness Effect

Up to this point I have characterized the equilibrium price of the small independent asset as a function of private information scenarios and the demand shock realization. In this section, I step back to the point in time prior to private information and trading decisions to characterize the expected price as a function only of the prior information \( q \). Hence the expectation will be taken with respect to the demand shock distribution and private information scenarios given \( q \). This interim expected price in the uniform demand shock case can be expressed in closed-form (Proposition 12).

**Proposition 12** (The Expected Price). The expected price in the uniform demand shock case, \( \hat{\eta} \sim \mathcal{U}[-k, k], k > 2\Delta \) (Corollary 11), as a function of prior success probability \( q \), is given by the following
closed-form expression:

\[
EP(q) := E[P_{\bar{q}}(\bar{\eta})] = \frac{1}{2k} \frac{1}{\gamma} \left[ (1 - h(q))u(q, X, Y) + h(q) u\left(q, \frac{1}{X}, \frac{1}{Y}\right) + u\left(q, Z, \frac{1}{Z}\right) \right] + \theta_l,
\]

where

\[
h(q) := q\phi + (1 - q)(1 - \phi),
\]

\[
u(q, x, y) := \ln \frac{1 - q(1 - x)}{1 - q(1 - y)},
\]

\[
X := \left(\frac{1 - \phi}{\phi}\right)^{1 - \lambda} e^{-\gamma(\theta_h - \theta_l)k},
\]

\[
Y := \frac{1 - \phi}{\phi} e^{-\gamma(\theta_h - \theta_l)k},
\]

\[
Z := \left(\frac{1 - \phi}{\phi}\right)^{1/2} e^{\gamma(\theta_h - \theta_l)k}.
\]

Moreover, \(EP(q)\) has the following properties:

1. \(EP(q)\) satisfies the following reverse-symmetric relationship for all \(q \in [0, 1]\):

\[
EP(q) - EP(0) = EP(1) - EP(1 - q).
\]

2. \(EP(0) = \theta_l\) and \(EP(1) = \theta_h\).

3. \(EP(q)\) is strictly increasing in \(q\), \(EP'(q) > 0\), for all \(q \in [0, 1]\).

4. \(EP(q)\) is strictly concave in \(q\), \(EP''(q) < 0\), for \(q \in \left[0, \frac{1}{2}\right]\).

5. \(EP(q)\) is strictly convex in \(q\), \(EP''(q) > 0\), for \(q \in \left(\frac{1}{2}, 1\right]\).

Working with the \((\theta_h, \theta_l, q)\) parameterization for the distribution of \(\bar{\theta}\) and the expected price as a function of \(q\) is most amenable to analytical analysis and provides the support needed to prove the main result, Theorem 13:

**Theorem 13 (The Skewness Effect).** The expected price, \(EP(q)\), of Proposition 12 (equation (23)) is higher than the expected payoff, \(q\theta_h + (1 - q)\theta_l\), when \(q < \frac{1}{2}\) (implying positive skewness) and lower
when \( q < \frac{1}{2} \) (implying negative skewness):

\[
EP(q) > q\theta_h + (1 - q)\theta_l, \quad \text{for} \ q \in (0, \frac{1}{2}), \quad (\Leftrightarrow \text{positive skewness}),
\]

\[
EP\left(\frac{1}{2}\right) = \frac{1}{2}\theta_h + \frac{1}{2}\theta_l, \quad (\Leftrightarrow \text{no skewness}),
\]

\[
EP(q) < q\theta_h + (1 - q)\theta_l, \quad \text{for} \ q \in (\frac{1}{2}, 1), \quad (\Leftrightarrow \text{negative skewness}).
\]

Theorem 13 shows that the same phenomenon illustrated in the background example of Section 3 carries through in this richer equilibrium framework. Conclusions based on comparing the expected price to the expected payoff, as above, carry over qualitatively to the expected return, if the coefficient of variation (CV) of the price—i.e., the ratio of the standard deviation of the price to its mean—is small, a typical property of stock prices.\(^{17}\) As with the expected price, the expected return can be expressed in closed form [Proposition 14]. However, due to the non-linear transformation of price in computing returns, some analytical tractability is lost including symmetries for positive and negative skewness that can be exploited by analyzing expected prices; therefore, I will focus the analysis on expected prices throughout and show using numerical illustrations of the expected return (via Proposition 14) how the implications carry over to expected returns.

**Proposition 14 (The Expected Return).** The expected return in the uniform demand shock case, \( \tilde{\eta} \sim \mathcal{U}[-k, k], k > 2\Delta, \) is given by the following closed-form expression:

\[
ERet := E \left[ \frac{\tilde{\theta}}{P_{\hat{\eta}}(\tilde{\eta})} - 1 \right]
= \frac{\theta_h q + \theta_l (1 - q)}{\theta_h} - \frac{H(1 - \phi)}{2k\gamma\theta_h\theta_l} \ln \frac{v(-1, k - 2\Delta)}{v(-\frac{1}{2}, k - 2\Delta)} - \frac{H(\phi)}{2k\gamma\theta_h\theta_l} \ln \frac{v(1, k - 2\Delta)}{v(\frac{1}{2}, 2\Delta - k)} - 1, \quad (30)
\]

where

\[
H(x) := \theta_h qx + \theta_l (1 - q)(1 - x), \quad (31)
\]

\[
v(r, s) := \theta_h \frac{q}{1 - q} \left[ \frac{\phi}{1 - \phi} \right]^r + \theta_l e^{s\gamma(\theta_h - \theta_l)}. \quad (32)
\]

As mentioned in Theorem 8, using the equivalent three moment characterization of the payoff

\(^{17}\)See Appendix B for more details.
distribution in terms of its mean, variance, and skewness, the effect of skewness can be isolated away from changes in the mean and variance. Numerical analysis indicates that the properties of expected price as a function of $q$, which is one-to-one with skewness, carry over to both the expected price and expected return as a function of skewness, $S$, holding the mean and variance of $\tilde{\theta}$ constant. The plots in Figure 5 show the skewness effect in action.\footnote{In Appendix G, I show by numerical illustration that the main results are robust to normally-distributed noise demand.} Hence, a negative relationship arises between skewness and expected returns—the skewness effect.

4.5. Discussion of the mechanism

The pricing mechanism in my model is fundamentally the same as in classic models [e.g., Grossman and Stiglitz (1980)]. So how does the simple departure from payoff symmetry to payoff asymmetry generate a pricing anomaly? When noise demand is positive and hence noise traders are buying on net, the price must increase relative to fundamental value in order to get investors to sell on net to clear the market of this noise demand. Similarly, when noise demand is negative and hence noise traders are selling on net, the price must decrease relative to fundamental value in order to get investors to buy on net to clear the market. That is, positive noise demand pushes prices above fundamental value and negative noise demand pushes prices below fundamental value. The question is whether these price deviations from fundamental value wash out on average. The answer is that it depends on the symmetry (or lack thereof) in the payoff distribution. When the payoff distribution is symmetric—as in classic models in which the payoff is normally distributed or as in my model when $q = \frac{1}{2}$—then payoff symmetry carries noise demand symmetrically into prices. For example, in classic jointly-normal linear models, normally distributed noise demand ends up linear and symmetric in prices. However, when the payoff distribution is asymmetric—$q \neq \frac{1}{2}$ in my model—then payoff asymmetry introduces noise demand asymmetrically into prices. In Appendix E, I develop and analyze an auxiliary model, which considers a class of continuous payoff distributions as nonlinear transformations of a normally-distributed payoff—of which the normal distribution is a special case—and I show that the introduction of such payoff nonlinearity transforms symmetric noise demand into an asymmetric component in prices, unlike the classic linear case.\footnote{I thank an anonymous referee for suggesting to explore this broader point.}

Skewness, a popular measure of asymmetry, gives rise to the skewness effect as market prices contend with clearing noise demand. Of course, skewness is not the only measure of asymmetry. Any
Plots (a) and (b) show the expected price and expected return, respectively, of the risky asset $\tilde{\theta}$ vs. skewness $S$ holding the payoff mean $E$ and variance $V$ fixed as in Theorem 8 (equation (23)), where $\gamma = 1$, $E = 8$, $V = 1$, $\phi = 0.8$, $\lambda = 0.8$, and $k = 1$.

payoff distribution asymmetry that differentiates upside and downside risk will translate unbiased noise demand into pricing anomalies.
4.6. Weakening of the Effects by Investor Heterogeneity

Investor heterogeneity is not a key driver of the skewness effect. Contrary to pre-existing explanations that appeal to investor heterogeneity, Theorems 13 shows that the skewness effect operates regardless of the level of investor heterogeneity, which is captured by $\phi$ (Lemma 9). Moreover, an increase in investor heterogeneity can actually weaken the skewness effect (Proposition 15).

**Proposition 15** (Skewness Effect Weakened by Information Heterogeneity). Let $\lambda = 1$. The skewness effect (Theorem 13) can be weakened by more information heterogeneity among investors (larger $\phi$, Lemma 9). Specifically, there exists $\tilde{\kappa} > 0$ such that for $\gamma(\theta_h - \theta_l)k \geq \tilde{\kappa}$,

\[
\frac{\partial E(P(q))}{\partial \phi} < 0, \quad \text{for } q \in (q_1, \frac{1}{2}), \quad (\iff \text{positive skewness}),
\]

\[
\frac{\partial E(P(q))}{\partial \phi} > 0, \quad \text{for } q \in (\frac{1}{2}, \overline{q}), \quad (\iff \text{negative skewness}),
\]

for some $q_1 < \frac{1}{2} < \overline{q}$.

Proposition 15 captures the fact that as the precision of private information $\phi$ increases—or equivalently as both heterogeneity and informativeness of the market increases—the skewness effect can decrease. Given sufficient risk or risk aversion, and sufficiently mild skewness, more informativeness leads to a reduction of the skewness effect. Proposition 15 is merely a sufficient condition, and numerical illustrations indicate that it can manifest more broadly than just for mild skewness and large risk or risk aversion. The phenomenon is illustrated in Figure 6, in which the expected price and expected return are each plotted as a function of skewness.

4.7. Deepening of the Effects by Risk/Aversion

Given that heterogeneity is not a key driver of the effects (Section 4.6), set aside investor heterogeneity for simplicity and consider the impact of market-wide risk or risk aversion on the skewness effect. Proposition 16 indicates that the skewness effect is amplified by an increase in market-wide risk (higher volatility of stochastic demand $k$) or risk aversion (higher $\gamma$).

**Proposition 16** (Effects Amplified by Risk/Aversion). Let $\lambda = 0$. The skewness effect (Theorem 13) is
FIGURE 6
Investor Heterogeneity Impact on the Skewness Effect

(a) Expected Price vs. Skewness

Plots (a) and (b) plot the expected price and expected return, respectively, of the risky asset \( \tilde{\theta} \) vs. skewness \( S \) holding the payoff mean \( E \) and variance \( V \) fixed as in Theorem 8 (equation (23)), where \( \gamma = 1, E = 8, V = 1, \lambda = 0.8, k = 1 \), and \( \phi = 0.6 \) (lower heterogeneity; solid lines) or \( \phi = 0.8 \) (higher heterogeneity; dashed lines).

amplified by higher risk \( (k) \) or risk aversion \( (\gamma) \):

\[
\frac{\partial EP(q)}{\partial \gamma} > 0, \quad \frac{\partial EP(q)}{\partial k} > 0, \quad \text{for } q \in (0, \frac{1}{2}), \quad (\Leftrightarrow \text{positive skewness}),
\]
\[
\frac{\partial EP(q)}{\partial \gamma} < 0, \quad \frac{\partial EP(q)}{\partial k} < 0, \quad \text{for } q \in \left(\frac{1}{2}, 1\right), \quad \text{(\(\Leftrightarrow\) negative skewness)}.
\]

This amplification effect is illustrated in Figure 7, which plots separately the expected price and expected return as a function of skewness under low and high risk/aversion scenarios.

The skewness effect already defies standard risk/reward explanations. Higher risk/aversion will not only leave the puzzle unresolved but, in fact, should make it worse. The intuition is as follows: the more risk averse investors are, the less aggressively they trade on perceived differences between their valuations and the price. Hence, larger price deviations are needed to induce the necessary market-clearing trades of such investors and price must deviate further from perceived value to contend with higher risk aversion, amplifying the skewness effect.

5. PREDICTIONS

The results of the main model generate several testable predictions, which contrast sharply with pre-existing explanations, extend the scope of pre-existing predictions, or have not previously been suggested by other theory or empirical evidence. In this section, I outline these predictions. I provide empirical support and further discussion in Section 6.

The first testable predictions of my model that extend beyond existing empirical studies are outlined below:

**P1**: Higher ex-ante skewness generates lower expected returns in financial markets having classically ideal conditions of frictionless trade of risk-averse rational investors and liquidity traders; or, more specifically, financial markets with:

- **P1(a)**: little or no influence of traders with behavioral preferences;
- **P1(b)**: little or no short-selling restraints;
- **P1(c)**: high trading activity;
- **P1(d)**: highly liquid assets; or,
- **P1(e)**: assets backed by healthy fundamentals (e.g., stocks of financially healthy firms).

Theorem 13 implies that the skewness effect arises in financial markets that have classically ideal conditions—frictionless trade of risk-averse investors and liquidity traders—as a fundamental consequence of market clearing of stochastic demand. Thus, the ramifications of Theorem 13 are exten-
Plots (a) and (b) plot the expected price and expected return, respectively, of the risky asset $\tilde{\theta}$ vs. skewness $S$ holding the payoff mean $E$ and variance $V$ fixed as in Theorem 8 (equation (23)), where $E = 8$, $V = 1$, $\phi = 0.8$, $\lambda = 0.8$, and $\gamma = k = 1$ (lower risk/aversion; solid lines) or one of $\gamma$ and $k$ is 1.25 (higher risk/aversion; dashed lines).

Moreover, for example, just consider the realm of existing explanations for the skewness effect. In direct contrast to those explanations, my theory predicts that the effect should exist in financial markets without investors with behavioral preferences (such as cumulative prospect theory preferences or...
optimal expectations); without market frictions such as short-selling constraints, transactions costs, or other barriers to trade; and for stocks of firms not in financial distress. More broadly, the scope of the effect is predicted to be much larger than indicated by existing theory or empirical evidence. The skewness effect should be first-order and arise in a market free of any other characteristic that departs from classical markets.

**P2** Higher ex-ante skewness deepens the skewness effect, even in financial markets with conditions corresponding to P1.

Numerical analyses of the main model—as illustrated in Figure 5—indicate that the impact of ex-ante skewness on expected returns deepens as the level of ex-ante skewness increases. The slopes of the expected price and expected return functions become steeper as skewness increases. These results comprise a new prediction, P2, which has not been suggested to exist by prior theory or empirical evidence. Intuitively, higher levels of ex-ante skewness inflate the differences between upside and downside risk on opposite sides of the bet. Each additional unit of downside risk requires an ever larger increase in price on the selling side of the bet due to the properties of risk aversion. So average prices diverge from fundamentals faster for each additional unit increase in ex-ante skewness.

**P3** Higher market-wide risk or risk aversion deepens the skewness effect, even in financial markets with conditions corresponding to P1.

Proposition 16 and numerical analyses illustrated in Figure 7 indicate that the impact of ex-ante skewness on expected returns deepens as the level of market-wide risk or risk aversion increases. The slopes of the expected price and expected return functions become steeper as $k$ or $\gamma$ increase, even holding the individual payoff volatility fixed. These results comprise new prediction P3.\(^\text{20}\)

**P4** Higher ex-ante skewness accelerates the deepening of the skewness effect under higher market-wide risk or risk aversion, even in financial markets with conditions corresponding to P1.

\(^{20}\)The possibility of higher market-wide risk or risk aversion deepening the skewness effect could be indirectly inferred from the limited context of the financial distress effect based on the observation that financially distressed stocks tend to be skewed stocks [Campbell, Hilscher, and Szilagyi (2008)]. However, this possibility has not been modeled nor tested empirically in the literature. Furthermore, P3 is a stronger prediction—the skewness effect deepens during episodes of higher risk/aversion, aside from characteristics associated with financial health.
The numerical analyses illustrated in Figure 7 indicate that predictions P2 and P3 extend further. The steepening of the slopes of the expected price and expected return functions as skewness increases remains in effect and is even stronger when risk/aversion is higher. These results comprise new prediction P4.

**P5** *Negative average excess returns:* At sufficient levels of skewness, expected returns (in excess of the riskfree rate) can be negative, even for assets in financial markets with conditions corresponding to P1.

Figure 5 associated with Theorem 13 shows that expected excess returns can be discernibly negative for sufficient levels of skewness. This prediction reinforces those from previous theories that negative expected returns can arise for sufficiently skewed assets [Harvey and Siddique (2000b); Barberis and Huang (2008)]. However, P5 extends this result to financial markets with classically ideal conditions. Moreover, P5 can help to explain shortcomings in accounting for the skewness effect of the CAPM and Co-Skewness models, which do not employ behavioral preferences [see Appendix C].

**P6** *Investor heterogeneity can weaken the skewness effect.*

Finally, Proposition 15 indicates that investor heterogeneity can weaken the skewness effect, yielding P6. This last prediction does not say that investor heterogeneity must weaken the effect, rather that its impact is not simplistic and not a sufficient explanation for the effect.

Note that the theory developed in the previous sections focused on the properties of the independent skewed asset, but the results extend easily to additional independent assets that might be added to the economy or that might already be among the baseline assets (which were left with unspecified structures), thereby having cross-sectional implications. That is, variation in skewness across a collection of such assets, with other cross-sectional characteristics being equal, should entail variations in expected returns. Hence, the above predictions can be tested cross-sectionally.

6. **EVIDENCE**

In this section, I test the new predictions summarized in the previous section on a sample of U.S. stocks. The theoretical insights can be approximated by a simple linear design for expected returns with a measure of ex-ante skewness as the main explanatory variable. Accordingly, I follow
the cross-sectional literature by implementing standard Fama and MacBeth (1973) regression tests on a linear approximation to the relationship between ex-ante skewness and expected returns.

6.1. Data

My sample is formed as the intersection of the Center for Research in Security Prices (CRSP) and COMPUSTAT universes. I obtain monthly stock returns (which I adjust for delisting bias\textsuperscript{21}), prices, volume, and shares outstanding from CRSP; and accounting data from COMPUSTAT. My sample covers all months in the 27-year period from January 1990 to December 2016.\textsuperscript{22}

My proxy for ex-ante skewness, SKEW, is the monthly expected idiosyncratic skewness measure for each stock as in Boyer, Mitton, and Vorkink (2010), provided by the authors through December 2016. Expected (or ex-ante) skewness is difficult to measure. As opposed to means, variances and covariances, skewness is not stable over time, and moreover, lagged skewness alone does not adequately forecast skewness [Harvey and Siddique (1999); Boyer, Mitton, and Vorkink (2010)]. Boyer, Mitton, and Vorkink (2010) (hereafter BMV) use firm-level variables to predict skewness [following the approach of Chen, Hong, and Stein (2001)]. Specifically, BMV develop measures of skewness each month that predict skewness of the return distribution over the next 60 months, based on firm characteristics in the prior 60 months, including lagged skewness, idiosyncratic volatility, momentum, turnover, size, exchange, and industry. BMV point out that although other variables, such as accounting variables, could be useful in predicting skewness, limiting variables to this collection allows the measure to be computed for every stock in CRSP with available history. As a result, using BMV’s proxies for skewness maintains a large sample. Moreover, BMV demonstrate that their measures of skewness exhibit a negative cross-sectional relationship with expected returns—the skewness effect.

I apply standard filters to include only common shares traded on NYSE, AMEX, or NASDAQ with price above $5. Firms must have positive market equity and book equity (for computing their book-to-market ratios). Accounting data is usually reported within a three-month delay after the fiscal end month, so firms must have a fiscal end date at least three months prior to the current month so that the information was plausibly available to market participants for the current month. Firms must have 12 months of returns (for computing momentum and short-term reversals) and a

\textsuperscript{21}This adjustment is suggested by Shumway (1997). My results are not sensitive to this adjustment.

\textsuperscript{22}The beginning of my sample is the first month that the VIX index is available in order to maintain a consistent sample for tests that I run later which use the VIX.
return for the subsequent month. Monthly variables derived from daily data such as idiosyncratic
volatility and illiquidity require a minimum of 10 days of daily returns in the month.\textsuperscript{23} Finally, my
measure of skewness implicitly requires at least 250 days of daily returns in the prior 60-month
period. Since most firms with 12 months of monthly returns also have 250 days of daily returns, this
implicit filter affects very few stocks. After applying these filters, there are 12,552 unique firms over
27 years (324 months) in the total sample with an average of 3,261 firms per month. My sample
consists of 1,053,301 firm-months.

Table 1 shows the means and percentiles of firm-month characteristics of my sample and the
distribution of the VIX index, which is used in later tests. Note that most individual stocks (over
95\%) have positive ex-ante skewness.

6.2. \textit{Fama and MacBeth (1973) regressions}

Following the literature, I run standard \textit{Fama and MacBeth (1973)} cross-sectional tests, using
individual stocks. I run the following cross-sectional regression for each month $t$ in my sample:

$$r_{t+1} = \gamma_{0,t} + \gamma_{1,t} \cdot SKEW_t + \phi' \cdot Z_t + \epsilon_t,$$

(33)

where $r_{t+1}$ is the return from end of month $t$ to $t + 1$, $SKEW_t$ is ex-ante skewness, and $Z_t$ is a vector
of controls, all observable at the end of month $t$.

Table 2 reports the time-series averages of the coefficient estimates ($\gamma$'s and $\phi$'s) with their \textit{Newey
and West (1987)} $t$-statistics$\textsuperscript{24}$ for seven models.$\textsuperscript{25}$ Model (1), which includes only $SKEW$ (and no other
right-hand-side variables), shows that the skewness effect operates in my sample unconditionally
since the coefficient on $SKEW$ is negative and significant.

In the remaining models reported in Table 2, I analyze the marginal effect of ex-ante skew-
ness conditional on two categories of controls. The first category corresponds to standard variables
commonly understood to explain cross-sectional variation in returns: size, book-to-market ratio, re-
turns momentum, idiosyncratic return volatility, and short-term return reversals.$\textsuperscript{26}$ I use the fol-

\textsuperscript{23}My results are robust to minimum day requirements among 5, 10, or 20 days.
\textsuperscript{24}Superscripts $^{***}$, $^{**}$, and $^*$ represent significance at the 1, 5, and 10 percent levels, respectively.
\textsuperscript{25}The intercept, which is omitted, is positive and significant in all tests.
\textsuperscript{26}In unreported tests, I also include factor loadings (betas) on the market return, size factor, book-to-market factor,
momentum factor, and coskewness, and obtain similar results.
TABLE 1
Distributions of Firm-Month Characteristics and of VIX

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>P01</th>
<th>P05</th>
<th>P10</th>
<th>P25</th>
<th>P50</th>
<th>P75</th>
<th>P90</th>
<th>P95</th>
<th>P99</th>
</tr>
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<tbody>
<tr>
<td><strong>Ex-ante skewness</strong>:</td>
<td></td>
<td></td>
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<td></td>
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<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>SKEW</td>
<td>0.95</td>
<td>−0.17</td>
<td>0.14</td>
<td>0.31</td>
<td>0.61</td>
<td>0.93</td>
<td>1.27</td>
<td>1.60</td>
<td>1.82</td>
<td>2.30</td>
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<td><strong>Valuation and prior returns</strong>:</td>
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<td></td>
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<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>ME ($ mil.)</td>
<td>3,266</td>
<td>11</td>
<td>24</td>
<td>39</td>
<td>101</td>
<td>359</td>
<td>1,436</td>
<td>5,243</td>
<td>12,071</td>
<td>55,960</td>
</tr>
<tr>
<td>B/M (%)</td>
<td>61.8</td>
<td>3.3</td>
<td>9.5</td>
<td>15.2</td>
<td>28.8</td>
<td>50.7</td>
<td>80.4</td>
<td>116.1</td>
<td>146.0</td>
<td>243.6</td>
</tr>
<tr>
<td>Price</td>
<td>47.5</td>
<td>4.6</td>
<td>5.9</td>
<td>7.0</td>
<td>10.7</td>
<td>18.8</td>
<td>32.1</td>
<td>50.1</td>
<td>65.7</td>
<td>117.1</td>
</tr>
<tr>
<td>MOM (%)</td>
<td>22.6</td>
<td>−65.4</td>
<td>−44.1</td>
<td>−31.5</td>
<td>−10.8</td>
<td>10.9</td>
<td>37.1</td>
<td>76.5</td>
<td>118.1</td>
<td>282.6</td>
</tr>
<tr>
<td>IVOL (%)</td>
<td>2.3</td>
<td>0.4</td>
<td>0.7</td>
<td>0.8</td>
<td>1.2</td>
<td>1.9</td>
<td>2.9</td>
<td>4.1</td>
<td>5.1</td>
<td>7.8</td>
</tr>
<tr>
<td>RET−1 (%)</td>
<td>1.0</td>
<td>−34.2</td>
<td>−19.0</td>
<td>−12.9</td>
<td>−5.4</td>
<td>0.6</td>
<td>6.8</td>
<td>14.9</td>
<td>21.7</td>
<td>41.2</td>
</tr>
<tr>
<td><strong>Frictions (investor type, trading restraints, financial health)</strong>:</td>
<td></td>
<td></td>
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<td></td>
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<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>IO (%)</td>
<td>40.4</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>8.6</td>
<td>39.0</td>
<td>67.9</td>
<td>84.5</td>
<td>92.1</td>
<td>100.0</td>
</tr>
<tr>
<td>TURN (%)</td>
<td>14.0</td>
<td>0.2</td>
<td>0.7</td>
<td>1.3</td>
<td>3.3</td>
<td>8.1</td>
<td>17.4</td>
<td>32.1</td>
<td>45.5</td>
<td>88.1</td>
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<tr>
<td>ILLIQ</td>
<td>114.5</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.2</td>
<td>1.6</td>
<td>19.3</td>
<td>154.7</td>
<td>382.1</td>
<td>1,681.4</td>
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<tr>
<td>ZSCORE</td>
<td>6.3</td>
<td>−0.7</td>
<td>0.6</td>
<td>0.7</td>
<td>1.3</td>
<td>3.0</td>
<td>5.6</td>
<td>10.8</td>
<td>18.2</td>
<td>54.7</td>
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<td><strong>Market risk/aversion</strong>:</td>
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<td></td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
<td></td>
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<tr>
<td>VIX</td>
<td>19.7</td>
<td>11.1</td>
<td>11.7</td>
<td>12.3</td>
<td>13.9</td>
<td>18.0</td>
<td>23.6</td>
<td>28.6</td>
<td>32.1</td>
<td>44.7</td>
</tr>
</tbody>
</table>

This table reports the mean and percentiles of firm-month characteristics for common shares traded on NYSE, AMEX, or NASDAQ with price above $5 and of the VIX index from January 1990 to December 2016. SKEW is the measure of expected idiosyncratic skewness as in Boyer, Mitton, and Vorkink (2010). ME is market equity. B/M is the ratio of book equity to market equity. MOM is the total return over the previous 12 months excluding the most recent month [Jegadeesh and Titman (1993)]. RET−1 is the return in the most recent month. IVOL is the standard deviation of residuals of excess daily returns regressed onto the Fama and French (1993) factors [Ang, Hodrick, Xing, and Zhang (2006)]. IO is percentage of shares outstanding held by institutions, where institutional share holdings are obtained from Thompson’s 13F filings data [D’Avolio (2002); Lewellen (2011)]. TURN is percentage trading volume per shares outstanding [D’Avolio (2002)]. ILLIQ is average daily absolute percent return per $1,000 volume [Amihud (2002)]. ZSCORE is Altman’s accounting-based measure of financial health [Altman (1968)]. VIX is implied volatility of the S&P 500 stock index.

Following standard proxies, respectively: log(ME), where ME is the market value of equity (in 1,000’s); log(B/M), where B/M is the log of the ratio of book equity to market equity using book equity from the most recent fiscal year statement at least three months old; MOM is the cumulative return over the prior twelve months excluding the most recent month [Jegadeesh and Titman (1993)]; IVOL is the standard deviation of residuals of excess daily returns regressed onto the factors of Fama and French (1993), with at least 10 days of returns within the month [Ang, Hodrick, Xing, and Zhang (2006)]; RET−1 is the return in the most recent month. Model (2) shows that the skewness effect remains negative and significant after controlling for these standard characteristics. Size is significant and negative in my sample, given all other controls, which is consistent with related studies. Coefficients

27My results are similar using total volatility in place of idiosyncratic volatility.
TABLE 2  
**FM Regressions: Skewness Effect Controlling for Frictions**

<table>
<thead>
<tr>
<th></th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
<th>(5)</th>
<th>(6)</th>
<th>(7)</th>
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<tbody>
<tr>
<td>Ex-ante skewness:</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SKEW</td>
<td>−0.440**</td>
<td>−0.467**</td>
<td>−0.418**</td>
<td>−0.453**</td>
<td>−0.404**</td>
<td>−0.466**</td>
<td>−0.456**</td>
</tr>
<tr>
<td></td>
<td>(−2.50)</td>
<td>(−2.13)</td>
<td>(−1.99)</td>
<td>(−2.17)</td>
<td>(−2.20)</td>
<td>(−2.10)</td>
<td>(−1.83)</td>
</tr>
<tr>
<td>Standard cross-sectional controls:</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>log(ME)</td>
<td>−0.102**</td>
<td>−0.122***</td>
<td>−0.098**</td>
<td>−0.085***</td>
<td>−0.108**</td>
<td>−0.122***</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(−2.54)</td>
<td>(−3.07)</td>
<td>(−2.46)</td>
<td>(−2.65)</td>
<td>(−2.49)</td>
<td>(−2.93)</td>
<td></td>
</tr>
<tr>
<td>log(B/M)</td>
<td>0.162*</td>
<td>0.159*</td>
<td>0.166*</td>
<td>0.168*</td>
<td>0.118</td>
<td>0.112</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(1.77)</td>
<td>(1.75)</td>
<td>(1.88)</td>
<td>(1.84)</td>
<td>(1.34)</td>
<td>(1.31)</td>
<td></td>
</tr>
<tr>
<td>MOM</td>
<td>0.002</td>
<td>0.002</td>
<td>0.002</td>
<td>0.002</td>
<td>0.002</td>
<td>0.002</td>
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<tr>
<td></td>
<td>(0.76)</td>
<td>(0.86)</td>
<td>(0.74)</td>
<td>(0.86)</td>
<td>(0.70)</td>
<td>(0.51)</td>
<td></td>
</tr>
<tr>
<td>IVOL</td>
<td>−0.151***</td>
<td>−0.153***</td>
<td>−0.124**</td>
<td>−0.156***</td>
<td>−0.149***</td>
<td>−0.141***</td>
<td></td>
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<td></td>
<td>(−2.67)</td>
<td>(−2.74)</td>
<td>(−2.49)</td>
<td>(−2.82)</td>
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<td>(−3.21)</td>
<td></td>
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<tr>
<td>RET−1</td>
<td>−0.019***</td>
<td>−0.019***</td>
<td>−0.019***</td>
<td>−0.018***</td>
<td>−0.019***</td>
<td>−0.017***</td>
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<tr>
<td></td>
<td>(−3.04)</td>
<td>(−3.02)</td>
<td>(−2.98)</td>
<td>(−3.15)</td>
<td>(−3.03)</td>
<td>(−3.34)</td>
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<tr>
<td>Frictions:</td>
<td></td>
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<td></td>
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</tr>
<tr>
<td>IO</td>
<td>−0.001</td>
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<td>0.051</td>
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</tr>
<tr>
<td></td>
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<td></td>
<td></td>
<td></td>
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<td>(0.32)</td>
</tr>
<tr>
<td>ILLIQ</td>
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<td>(1.00)</td>
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<td>(1.00)</td>
</tr>
<tr>
<td>ZSCORE</td>
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<td></td>
<td></td>
<td>−0.007**</td>
<td>−0.006**</td>
<td></td>
</tr>
<tr>
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<td></td>
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<td>(−2.21)</td>
<td>(−2.47)</td>
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<tr>
<td>Adjusted $R^2$</td>
<td>0.008</td>
<td>0.045</td>
<td>0.047</td>
<td>0.050</td>
<td>0.047</td>
<td>0.046</td>
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</table>

This table reports the results of Fama and MacBeth (1973) regressions of monthly returns on a proxy for ex-ante skewness. The regressions are estimated each month on lagged characteristics from January 1990 to December 2016 for common shares traded on NYSE, AMEX, or NASDAQ having price above $5 in the prior month. SKEW is the measure of expected idiosyncratic skewness as in Boyer, Mitton, and Vorkink (2010). log(ME) is the log of market equity and log(B/M) is the log of the ratio of book equity to market equity. MOM, IVOL, RET−1, IO, TURN, ILLIQ, and ZSCORE are as defined in Table 1. Average coefficients are reported along with Newey and West (1987) t-statistics (in parentheses). Significance at the 1%, 5%, and 10% level is indicated by ***, **, and *, respectively.

on book-to-market, momentum, idiosyncratic volatility, and short-term return reversals have signs and magnitudes similar to related studies.

The second category of controls corresponds to market frictions and other veins of pre-existing explanations for the skewness effect. These include variables that proxy for the influence of rational vs. behavioral investors; market frictions such as short-selling constraints, transaction costs, or
other barriers to trading; and the influence of financial health vs. financial distress. D’Avolio (2002) provides evidence that turnover and institutional ownership are good proxies for market frictions. Institutional ownership is also a proxy for the absence of behaviorally-motivated trades, which are important to explanations of the skewness effect by Barberis and Huang (2008), Brunnermeier and Parker (2005), and Brunnermeier, Gollier, and Parker (2007). Amihud (2002) provides a measure of illiquidity and evidence that it possesses explanatory power regarding market frictions’ effects on prices. Also, connections have been proposed between the skewness effect and other pricing effects that pertain to individual security characteristics such as the financial distress effect [Campbell, Hilscher, and Szilagyi (2008)]. So, I include a measure of financial health to control for this characteristic.

For brevity, I label all controls in this second category as “frictions.” Specifically, the proxies that I use for these frictions controls are as follows: Institutional ownership (IO) is the fraction of shares held by institutions per total shares outstanding, where institutional share holdings are obtained from Thompson’s 13F filings data.²⁸ TURN is percentage trading volume per shares outstanding [D’Avolio (2002)]. ILLIQ is average daily absolute percent return per $1,000 volume [Amihud (2002)]. ZSCORE is Altman’s accounting-based measure of financial health [Altman (1968)].²⁹

Models (3)–(7) of Table 2 show that the marginal effect of ex-ante skewness remains statistically negative after controlling for frictions. Moreover, only ZSCORE is significant but it does not absorb the explanatory power of ex-ante skewness.³⁰ These results provide support for the main predictions of my model, P1(a)–P1(e), since ex-ante skewness should not have such explanatory significance if behavioral preferences, short sales or other trading frictions, or financial distress are fundamental

²⁸ As documented by Lewellen (2011), institutional ownership has been increasing over the last several years and, due to how its calculated, can exceed 100% rarely for certain firms (less than %1 of the firm-months in my sample); as in that study, I censor IO above at 100%.

²⁹ The variables ZSCORE is defined as a specific weighted sum of five accounting factors (or ratios):

\[
ZSCORE = 3.3Z_1 + 0.999Z_2 + 0.6Z_3 + 1.2Z_4 + 1.4Z_5,
\]

where \( Z_1 := \frac{\text{earnings before interest and tax}}{\text{total assets}} \); \( Z_2 := \frac{\text{sales}}{\text{total assets}} \); \( Z_3 := \frac{\text{market equity}}{\text{book value of total liabilities}} \); \( Z_4 := \frac{\text{working capital}}{\text{total assets}} \); and \( Z_5 := \frac{\text{retained earnings}}{\text{total assets}} \). These accounting factors are not always available in COMPUSTAT for every firm, although most factors are available for most firms. Instead of throwing out observations in which some factors are missing, I instead replace any missing factor with a conservative representative value for all COMPUSTAT firms that have a non-missing value for that factor in the same year, in order to maximize the total number of observations included in my sample. Missing factors are replaced by the 70th percentile. My results are robust to dropping firms with missing observations for any of the five factors.

³⁰ In unreported tests, I consider Ohlson’s OSCORE [Ohlson (1980), model 1], an alternative measure for financial distress vs. financial health, and find similar results.
drivers of the effect.

6.3. *Fama and MacBeth (1973) regressions under “ideal conditions”*

In the previous section, my analyses focused on the effect of skewness holding various controls constant. In order to further examine prediction $P1$, I now examine whether the marginal effect of skewness is diminished when market conditions are “ideal”—closer to frictionless, rational markets. Translating “ideal” conditions to existing proxies, I test the effect of skewness when institutional ownership is high, turnover is high, illiquidity is low, and financial health is high by employing interactions between these proxies and skewness. For example, if behavioral trading is responsible for the skewness effect, then stocks with predominantly rational trading (e.g., high institutional ownership levels) should not exhibit the skewness effect. Likewise, if trading frictions or financial distress are responsible for the skewness effect, then stocks with low frictions or good financial health (high institutional ownership, high turnover, low illiquidity, and/or high ZSCORE) should not exhibit the skewness effect. Moreover, this should be true after holding the standard cross-sectional variables fixed.

My approach is to create a category variable that is 1 when the friction in question is most active (0 otherwise) and interact it with ex-ante skewness. By doing so, the coefficient on ex-ante skewness is properly interpreted as the marginal effect of skewness when the friction in question is turned “off” or least active. Specifically, I introduce the following category variables: $IO_{low}$ is equal to one if IO is below the median for the month; $TURN_{low}$ is equal to one if TURN is in the lower quartile for the month; $ILLIQ_{high}$ is equal to one if ILLIQ is in the upper quartile for the month; and $ZSCORE_{low}$ is equal to one if ZSCORE is in the lower quartile for the month.

Following the literature, I run standard *Fama and MacBeth (1973)* cross-sectional tests, using individual stocks. I run the following cross-sectional regression for each month $t$ in my sample:

$$ r_{t+1} = \gamma_{0,t} + \gamma_{1,t} \cdot SKEW_t + \gamma_{2,t} \cdot \text{Cat}_t + \gamma_{3,t} \cdot (SKEW_t \times \text{Cat}_t) + \phi_t \cdot Z_t + \varepsilon_t, $$

where $\text{Cat}_t$ is a vector of zero-one category variables and $Z_t$ is a vector of controls, all observable at the end of month $t$. Under this model, the marginal effect of ex-ante skewness depends on the category variables: $\frac{\partial r_{t+1}}{\partial SKEW} = \gamma_{1,t} + \gamma_{3,t} \cdot \text{Cat}_t$. Hence, the marginal effect of ex-ante skewness when
the category variables are “off” (i.e., zero) is given by the coefficient $\gamma_{1,t}$. Table 3 shows the results of these regressions for five models. The coefficient on SKEW in Models (8)–(11) is the marginal effect of ex-ante skewness when, respectively, IO is high, TURN is high, ILLIQ is low, or ZSCORE is high. Ex-ante skewness remains statistically negative at the 10% level in all models, including Model (12), which includes all category controls and their interactions with ex-ante skewness—providing additional support for predictions $P1(a)$–$P1(e)$.

These results are remarkable since the ex-ante skewness characteristic tends to be lower when each of these category variables are active. Table 4 shows the average of monthly average ex-ante skewness when the friction category variables assume either of their two values. Ex-ante skewness tends to be lower when IO is high (0.83) than when IO is low (1.12), lower when TURN is high (0.88) than when TURN is low (1.27), lower when ILLIQ is high (0.83) than when ILLIQ is low (1.42), and lower when ZSCORE is high (0.93) than when ZSCORE is low (1.11). Moreover, ex-ante skewness tends to be even lower when all of these market conditions are “ideal” (0.78) than when all of these frictions are their most active (1.43).

Thus, the characteristics of stocks in my sample are such that ex-ante skewness tends to be lower for stocks in more ideal trading conditions, likely diminishing the ability of these cross-sectional tests to detect a significant relationship and reject the null. This observation can also help to explain why the effect of ex-ante skewness could confused with other effects that tend to accompany markets that depart from ideal conditions. Furthermore, if the marginal effect of ex-ante skewness strengthens at higher levels of ex-ante skewness, as predicted by $P2$, then the tests of Models (8)–(12) are conservative, lending weight to the main prediction $P1$ that the skewness effect is first-order.

6.4. *Fama and MacBeth (1973) regressions under higher ex-ante skewness and/or market-wide risk or risk aversion*

In this section, I revisit the main Models (1)–(7) of Table 2 to examine whether the marginal effect of ex-ante skewness deepens at higher levels of ex-ante skewness ($P2$) and/or at higher levels of market-wide risk or risk aversion ($P3$). Table 5 shows the results of Models (1), (2), and (7) under three different sub-samples of my baseline sample.\(^{31}\)

$Higher \ ex-ante \ skewness: \ Models \ (1'), \ (2'), \ and \ (7') \ filter \ the \ baseline \ sample \ each \ month \ to$

---

\(^{31}\) Similar results hold for Models (3)–(6).
### TABLE 3
**FM Regressions: Marginal Effect of Skewness under Low Frictions**

<table>
<thead>
<tr>
<th></th>
<th>(8)</th>
<th>(9)</th>
<th>(10)</th>
<th>(11)</th>
<th>(12)</th>
</tr>
</thead>
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<tr>
<td><strong>Ex-ante skewness:</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SKEW</td>
<td>-0.581*</td>
<td>-0.365*</td>
<td>-0.394*</td>
<td>-0.299**</td>
<td>-0.419*</td>
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<tr>
<td></td>
<td>(-1.85)</td>
<td>(-1.67)</td>
<td>(-1.75)</td>
<td>(-1.98)</td>
<td>(-1.83)</td>
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<td><strong>Frictions categories and interactions:</strong></td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>IO\textsubscript{low}</td>
<td>-0.247*</td>
<td></td>
<td></td>
<td>-0.306**</td>
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</tr>
<tr>
<td></td>
<td>(-1.86)</td>
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<tr>
<td>IO\textsubscript{low} × SKEW</td>
<td>0.065</td>
<td></td>
<td></td>
<td>0.255**</td>
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</tr>
<tr>
<td></td>
<td>(0.46)</td>
<td></td>
<td></td>
<td>(1.98)</td>
<td></td>
</tr>
<tr>
<td>TURN\textsubscript{low}</td>
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<td></td>
<td>0.006</td>
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</tr>
<tr>
<td></td>
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<td>(-0.77)</td>
<td></td>
<td>(0.03)</td>
<td></td>
</tr>
<tr>
<td>TURN\textsubscript{low} × SKEW</td>
<td>-0.188</td>
<td></td>
<td>-0.250*</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(-1.46)</td>
<td></td>
<td>(-1.82)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>ILLIQ\textsubscript{high}</td>
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<td>-0.203</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>(-0.83)</td>
<td></td>
<td>(-1.06)</td>
<td></td>
</tr>
<tr>
<td>ILLIQ\textsubscript{high} × SKEW</td>
<td>-0.023</td>
<td></td>
<td>0.107</td>
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</tr>
<tr>
<td></td>
<td></td>
<td>(-0.15)</td>
<td></td>
<td>(0.71)</td>
<td></td>
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<tr>
<td>ZSCORE\textsubscript{low}</td>
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<td></td>
<td>0.060</td>
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<td>(0.29)</td>
<td>(0.58)</td>
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<tr>
<td>ZSCORE\textsubscript{low} × SKEW</td>
<td>-0.061</td>
<td></td>
<td>-0.058</td>
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<tr>
<td></td>
<td></td>
<td></td>
<td>(-0.34)</td>
<td>(-0.35)</td>
<td></td>
</tr>
</tbody>
</table>

**Standard controls:**
- log(ME), log(B/M), MOM, IVOL, RET\textsubscript{−1}
- Yes Yes Yes Yes Yes

Adjusted $R^2$
- 0.048 0.050 0.051 0.050 0.058

This table reports the results of Fama and MacBeth (1973) regressions of monthly returns on a proxy for ex-ante skewness. The regressions are estimated each month on lagged characteristics from January 1990 to December 2016 for common shares traded on NYSE, AMEX, or NASDAQ having price above $5 in the prior month. SKEW is the measure of expected idiosyncratic skewness as in Boyer, Mitton, and Vorkink (2010). IO\textsubscript{low} equals one if IO is below the median for the month. TURN\textsubscript{low} equals one if TURN is in the lower quartile for the month. ILLIQ\textsubscript{high} equals one if ILLIQ is in the upper quartile for the month. ZSCORE\textsubscript{low} equals one if ZSCORE is in the lower quartile for the month. Standard controls, log(ME), log(B/M), MOM, IVOL, and RET\textsubscript{−1}, and frictions controls used to define the dummy variables, IO, TURN, ILLIQ, and ZSCORE, are as defined in Table 1. Average coefficients are reported along with Newey and West (1987) $t$-statistics (in parentheses). Significance at the 1%, 5%, and 10% level is indicated by ***, **, and *, respectively.

Include only those firms whose ex-ante skewness is above the median for that month.\textsuperscript{32} Compared to the baseline model (1), model (1’) shows that the unconditional effect of skewness deepens

\textsuperscript{32}In unreported tests, I alternatively maintain the full sample but include a category variable for low SKEW and its interaction with SKEW so that the coefficient on SKEW represents the marginal effect of SKEW when SKEW is high. The results are similar.
This table reports the average of the monthly average ex-ante skewness for various splits of my sample of common shares traded on NYSE, AMEX, or NASDAQ from January 1990 to December 2016 having price above $5 in the prior month. Ex-ante skewness, SKEW, is the measure of expected idiosyncratic skewness as in Boyer, Mitton, and Vorkink (2010). Categories (High or Low) for each frictions variable is based on same splits that determine the dummy variables, IO_{low}, TURN_{low}, ILLIQ_{high}, and ZSCORE_{low}, which are defined as in Table 3.

from $-0.440^{**}(-2.50)$ to $-0.502^{***}(-2.62)$ when ex-ante skewness is high. Model (2') shows that, even after controlling for standard cross-sectional characteristics, the skewness effect deepens from $-0.467^{**}(-2.13)$ in model (2) to $-0.534^{***}(-3.48)$ when ex-ante skewness is high. Model (7') shows that, controlling for both standard characteristics and frictions, the skewness effect deepens, becoming more statistically negative with the $t$-statistic dropping from ($-1.83$) in model (7) to ($-2.89$) and estimates moving from the 10% to 1% significance level, when ex-ante skewness is high. These results provide strong support for $P_2$.

Higher market-wide risk or risk aversion: The VIX index, which tracks the implied volatility of the S&P 500 stock index, has been used in the literature as a proxy for market-wide risk or risk aversion since market volatility rises if (1) the level of uncertainty rises (holding risk aversion fixed) or (2) risk aversion rises (holding uncertainty fixed). Accordingly, to test $P_3$, models (1''), (2''), and (7'') filter the baseline sample to include only those months in which the VIX index is in the upper quartile over its monthly history. The results support that the skewness effect deepens under higher market-wide risk or risk aversion, since the estimated coefficients on ex-ante skewness are more negative and
This table reports the results of Fama and MacBeth (1973) regressions of returns on a proxy for ex-ante skewness under different restrictions of the baseline sample used in regressions of Table 2 based on thresholds for SKEW and/or VIX. SKEW\textsubscript{high} is the sub-sample in which SKEW above the median SKEW for the month; VIX\textsubscript{high} is the sub-sampled in which VIX is in upper quartile of VIX over the sample period. SKEW is the measure of expected idiosyncratic skewness as in Boyer, Mitton, and Vorkink (2010). Control variables for each restriction are defined as in Table 2. Average coefficients are reported along with Newey and West (1987) \( t \)-statistics (in parentheses). Significance at the 1%, 5%, and 10% level is indicated by \(*\ast\ast\ast\), \(*\ast\ast\), and \(*\ast\), respectively.

have higher statistical significance than in the baseline models, (1), (2), and (7), respectively. This deepening of the effect is robust to both standard and frictions controls, providing strong support for P3.

Higher ex-ante skewness and higher market-wide risk or risk aversion: Models (1\(\textsuperscript{''}\)), (2\(\textsuperscript{''}\)), and (7\(\textsuperscript{''}\)) filter the baseline sample to include only those months in which the VIX index is in the upper quartile over its monthly history and only those firms whose ex-ante skewness is above the median for that month. These models indicate that the skewness effect is deepened further by either higher

---

**TABLE 5**

*FM Regressions: Higher Ex-Ante Skewness and/or Market-Wide Risk/Aversion*

<table>
<thead>
<tr>
<th>SKEW\textsubscript{high}</th>
<th>(1′)</th>
<th>(2′)</th>
<th>(7′)</th>
</tr>
</thead>
<tbody>
<tr>
<td>SKEW</td>
<td>-0.502(\ast\ast\ast)</td>
<td>-0.534(\ast\ast\ast)</td>
<td>-0.438(\ast\ast\ast)</td>
</tr>
<tr>
<td></td>
<td>(−2.62)</td>
<td>(−3.48)</td>
<td>(−2.89)</td>
</tr>
<tr>
<td>Adjusted (R^2)</td>
<td>0.005</td>
<td>0.041</td>
<td>0.048</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>VIX\textsubscript{high}:</th>
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<th>(2″)</th>
<th>(7″)</th>
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<tbody>
<tr>
<td>SKEW</td>
<td>-1.086(\ast\ast)</td>
<td>-0.734(\ast\ast\ast)</td>
<td>-0.584(\ast\ast\ast)</td>
</tr>
<tr>
<td></td>
<td>(−2.45)</td>
<td>(−2.97)</td>
<td>(−2.87)</td>
</tr>
<tr>
<td>Adjusted (R^2)</td>
<td>0.007</td>
<td>0.066</td>
<td>0.079</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>SKEW\textsubscript{high} and VIX\textsubscript{high}:</th>
<th>(1″″)</th>
<th>(2″″)</th>
<th>(7″″)</th>
</tr>
</thead>
<tbody>
<tr>
<td>SKEW</td>
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<td>-1.095(\ast\ast\ast)</td>
<td>-0.974(\ast\ast\ast)</td>
</tr>
<tr>
<td></td>
<td>(−3.01)</td>
<td>(−4.00)</td>
<td>(−3.50)</td>
</tr>
<tr>
<td>Adjusted (R^2)</td>
<td>0.007</td>
<td>0.064</td>
<td>0.074</td>
</tr>
</tbody>
</table>

**Standard controls:**
- \text{log(\text{ME})}, \text{log(\text{B/M})}, \text{MOM}, \text{IVOL}, \text{RET}_{−1}

**Frictions controls:**
- \text{IO}, \text{TURN}, \text{ILLIQ}, \text{ZSCORE}

<table>
<thead>
<tr>
<th>Standard controls:</th>
<th>No</th>
<th>Yes</th>
<th>Yes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frictions controls:</td>
<td>No</td>
<td>No</td>
<td>Yes</td>
</tr>
</tbody>
</table>

42
ex-ante skewness or higher market-wide risk/aversion (given that the other one is high), and that this deepening of the effect is robust to both standard and frictions controls. At higher levels of ex-ante skewness and market-wide risk/aversion, a one-unit increase in ex-ante skewness predicts approximately a full one percentage point decrease in monthly returns. These results provide strong support for P4.

6.5. Negative expected excess returns

My results indicate that stocks with high enough levels of ex-ante skewness can have negative expected excess returns (P5). For example, the previous section shows how ex-ante skewness predicts a drop in monthly returns of one percent per additional unit of skewness, so that sufficiently high levels of skewness can translate into negative returns. On an excess return basis, the returns can be even more negative. Previous explanations of the skewness effect have predicted negative expected returns for sufficiently skewed assets [Harvey and Siddique (2000b); Barberis and Huang (2008)]. However, my prediction and empirical results apply to a broader scope of stocks, including those with low levels of frictions.

6.6. Investor heterogeneity

One could test whether investor heterogeneity can weaken the skewness effect (P6) if there were a reliable proxy for investor heterogeneity. For example, survey data on investors’ expectations about the stock market could potentially be used to test these predictions. The most commonly used proxy is dispersion in financial analysts’ forecasts. However, Goulding (2017b) demonstrates why this proxy is not necessarily appropriate, especially in the context of the skewness effect. Accordingly, I leave the testing of this prediction for future work.

7. CONCLUSION

In this paper, I have presented a new explanation for how skewness can stand in a negative relationship with expected returns—market clearing of stochastic demand for a skewed asset can move its price away from the asset’s expected value in the direction of the sign of skewness. My theory does not appeal to behavioral preferences, does not require trading frictions such as short-selling constraints or trading costs, and does not make an appeal to firm characteristics such as
financial distress. These alternative factors are significant ingredients in market pricing but, by not invoking assumptions beyond those common to a noisy rational expectations paradigm, my theory suggests a deeper mechanism for the skewness effect.

This mechanism can be understood from a simple examination of an investor’s trading decisions in different realizations of the economy, in which the investor may face either side of a skewed bet—as a buyer or seller. For a standard risk averse rational investor, the price that would induce a short position in a positively (negatively) skewed asset is further from (closer to) expected value than is the price that induces a long position of the same magnitude, even in a frictionless market. Consequently, the average of these two prices can deviate from the asset’s expected value in the direction of skewness, exceeding expected value for a positively skewed asset.

My results generate several new testable predictions for which I document strong empirical evidence on a large sample of U.S. stocks. First, the skewness effect is robust to a broad array of cross-sectional controls including controls for frictions. Next, the effect is deepened by higher levels of ex-ante skewness and/or higher levels of market-wide risk or risk aversion. Furthermore, stocks with high levels of ex-ante skewness can have negative expected excess returns, even for stocks with low levels of frictions.

Finally, given the broad scope of the mechanism, the ideas developed in this paper are a springboard for explaining additional pricing phenomena. For example, Goulding (2017a) explores the connections between the skewness effect as modeled in this paper and the impact of financial analysts’ forecast dispersion on asset prices, yielding a rich set of new predictions. Goulding (2017b) provides strong supporting empirical evidence for those predictions.

APPENDIX A. PROOFS OF STATEMENTS IN THE MAIN TEXT

Proof of Lemma 1. Recall that the utility function is given by

\[ U(w) = \frac{1-a}{a} \left( \frac{\gamma w}{1-a} + b \right)^a \] with \( a < 1 \) (equation (3)). By induction, the \( n \)th derivative of \( U(w) \) with respect to \( w \) satisfies:

\[ U^{(n)}(w) = (-1)^{n-1} \gamma^n \left( \prod_{i=1}^{n-1} \frac{i-a}{1-a} \right) \left( \frac{\gamma w}{1-a} + b \right)^{a-n}. \] (A1)

First, expression (A1) holds for \( n = 1 \), \( U'(w) = \gamma \left( \frac{\gamma w}{1-a} + b \right)^{a-1} \), where \( \prod_{i=1}^{0} = 1 \). Next, suppose (A1) holds for some \( n \geq 1 \). Then for \( n + 1 \),

\[ U^{(n+1)}(w) = \frac{d}{dw} U^{(n)}(w) = (-1)^{n} \gamma^{n+1} \]
\((\prod_{i=1}^{n} \frac{i-a}{1-a}) (\frac{n}{1-a} + b)^{-a} \), which completes the induction argument. Since \(\frac{n}{1-a} + b > 0\), \(\gamma^n > 0\), and \(\prod_{i=1}^{n-1} \frac{i-a}{1-a} > 0\) (equation (3) and \(a < 1 \leq i\)), the sign of \(U^{(n)}(w)\) is determined by the sign of \((-1)^{n-1}\), which alternates from positive to negative with each successive \(n\): \(\text{sign}(U^{(n)}(w)) = \text{sign}((-1)^{n-1}) = \begin{cases} +1, & \text{n odd,} \\ -1, & \text{n even.} \end{cases}\)

**Proof of Lemma 3.** Definition 2, Jensen's inequality under the strict concavity of \(U(w)\), and the fact that \(E[\tilde{\theta}] = q\) give: \(U(w_0) = E[U(w_0 + (\tilde{\theta} - B))] < U(E[w_0 + (\tilde{\theta} - B)]) = U(w_0 + q - B)\). Since \(U(w)\) is strictly increasing, this chain of inequalities implies that \(q - B > 0\), or equivalently \(B < q\). Similarly, \(U(w_0) = E[U(w_0 - (\tilde{\theta} - S))] < U(E[w_0 - (\tilde{\theta} - S)]) = U(w_0 - q + S)\), which implies \(q < S\). □

**Proof of Lemma 4.** Direct calculation of the \(n\)th moment of \(\tilde{\varepsilon}\) (equation (6)) gives:

\[ E[\tilde{\varepsilon}^n] = (1-q)^n q + (-q)^n (1-q) = \begin{cases} (1-q)^n q + q^n (1-q) > 0, & \text{n even,} \\ (1-q)^n q - q^n (1-q), & \text{n odd.} \end{cases} \]

For \(n = 1\), \(E[\tilde{\varepsilon}] = (1-q)q - q(1-q) = 0\). For odd \(n > 1\), \(E[\tilde{\varepsilon}^n] = (1-q)^n q - q^n (1-q) = (1-q)q ((1-q)^{n-1} - q^{n-1})\). Since \(q \in (0,1)\) and \(x^n\) is strictly increasing in \(x\) for \(n-1 > 0\), the sign is determined by the difference of \(1-q\) and \(q\): \(\text{sign}(E[\tilde{\varepsilon}^n]) = \text{sign}((1-q) - q) = \text{sign}(\frac{1}{2} - q)\). □

**Proof of Theorem 5.** Consider the Taylor's expansions of the expected utility of payoffs for each side of the bet ((4) and (5)), respectively,

\[
E[U(w_0 + q - B + \tilde{\varepsilon})] = U(w_0 + q - B) + E[\tilde{\varepsilon}]U'(w_0 + q - B) + \frac{1}{2} E[\tilde{\varepsilon}^2]U''(w_0 + q - B) + \frac{1}{3!} E[\tilde{\varepsilon}^3]U'''(w_0 + q - B) + \cdots, \quad (A2)
\]

\[
E[U(w_0 + S - q - \tilde{\varepsilon})] = U(w_0 + S - q) - E[\tilde{\varepsilon}]U'(w_0 + S - q) + \frac{1}{2} E[\tilde{\varepsilon}^2]U''(w_0 + S - q) - \frac{1}{3!} E[\tilde{\varepsilon}^3]U'''(w_0 + S - q) + \cdots, \quad (A3)
\]

which are each well-defined since \(\tilde{\varepsilon}\) is bounded by initial wealth, \(w_0 > 1 > \max(q, 1-q) \geq |\tilde{\varepsilon}|\) (equation (6)).
Let \( q < \frac{1}{2} \) and suppose (for contradiction) that \( q - B \geq S - q \). Subtract the two Taylor’s expansions ((A2) less (A3)), which is zero by Definition 2, and apply (6) and Lemmas 1 and 4, term by term:

\[
0 = \mathbb{E}[U(w + q - B + \varepsilon)] - \mathbb{E}[U(w + S - q - \varepsilon)] \\
= U(w + q - B) - U(w + S - q) + \mathbb{E}[\varepsilon(U'(w + q - B) + U'(w + S - q))] \\
+ \frac{1}{2!} \mathbb{E}[\varepsilon^2(U''(w + q - B) - U''(w + S - q))] \\
+ \frac{1}{3!} \mathbb{E}[\varepsilon^3(U'''(w + q - B) + U'''(w + S - q))] \\
+ \cdots
\]

\( > 0, \)

which obtains the contradiction. Thus, \( q - B < S - q \) for \( q < \frac{1}{2} \). The other cases are similar. \( \square \)

**Proof of Corollary 6.** Follows directly by simple rearrangement of the inequalities appearing in the statement of Theorem 5. \( \square \)

**Proof of Proposition 7.** The proof of Proposition 7 uses the conjecture and verify approach. Conjecture: The price \( p \) depends only on \( \mathcal{S} := \{s_j, j \in [0, 1]\} \) and \( \eta \); and each price \( \tilde{p}^{(i)} \) depends only on the collection of \( \mathcal{S}^{(i')} := \{s_j^{(i')}, j \in [0, 1]\} \) and \( \eta^{(i')} \), for \( i' = 1, \ldots, n \).

Then, \( p(\mathcal{S}, \eta) \) and \( p^{(i)}(\mathcal{S}^{(i')}, \eta^{(i')}); i' = 1, \ldots, n \) are independent for each \( i \), since their respective arguments are independent. Similarly, for each \( i = 1, \ldots, n \), note that \( \tilde{\eta}^{(i)} \) is independent of \( \mathcal{G}^1_j := \{s_j = s, \tilde{p} = p\} \), and \( \tilde{\eta} \) is independent of \( \mathcal{G}^1^{(1, \ldots, n)} := \{s_j = s^{(i)}, \tilde{p}^{(i)} = p^{(i)}, i = 1, \ldots, n\} \). Moreover, investor \( j \)'s information set \( \mathcal{F}_j \) (equation (8)) satisfies \( \mathcal{F}_j = \mathcal{G}_j \cup \mathcal{G}^1_j \). So, for each \( i = 1, \ldots, n \), \( \{\tilde{\eta}^{(i)}; \mathcal{F}_j \} = \{\tilde{\eta}^{(i)}; \mathcal{G}^1_j \} \) and \( \{\tilde{\eta}; \mathcal{F}_j \} = \{\tilde{\eta}; \mathcal{G}_j \} \). Since \( \mathcal{G}^1_j \) is independent of \( \mathcal{G}_j \) by the conjecture, we have \( \{\tilde{\eta}; \mathcal{F}_j \} \) is independent of \( \{\tilde{\eta}^{(i)}; \mathcal{F}_j \} \) for each \( i = 1, \ldots, n \). Thus, investor \( j \)'s maximization problem in (7) becomes

\[
\max_{(x^{(1)}, \ldots, x^{(n)}, \bar{x})} \mathbb{E}[e^{-\gamma L_0 + \sum_{i=1}^n \bar{\tilde{\eta}}^{(i)} - \tilde{p}^{(i)} x^{(i)} + \bar{\tilde{p}} \bar{x}} | \mathcal{F}_j] \\
= \max_{(x^{(1)}, \ldots, x^{(n)}, \bar{x})} -e^{-\gamma L_0} \cdot \mathbb{E}[e^{-\gamma L_0 + \sum_{i=1}^n \bar{\tilde{\eta}}^{(i)} - \tilde{p}^{(i)} x^{(i)} | \mathcal{F}_j}] \cdot \mathbb{E}[e^{-\gamma \bar{\tilde{p}} \bar{x}} | \mathcal{F}_j] \\
= -e^{-\gamma L_0} \min_{(x^{(1)}, \ldots, x^{(n)}, \bar{x})} \mathbb{E}[e^{-\gamma L_0 + \sum_{i=1}^n \bar{\tilde{\eta}}^{(i)} - \tilde{p}^{(i)} x^{(i)} | \mathcal{G}^1_j^{(1, \ldots, n)}}] \cdot \mathbb{E}[e^{-\gamma \bar{\tilde{p}} \bar{x}} | \mathcal{G}_j]
\]

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(by the separability property of sums in the exponential function), which implies that \( x_{q_j} = x_q \), which is independent of the prices of the \( n \) baseline risky assets. So, the market clearing condition for the small independent asset in (10) implies that price \( p \) depends only on \( s_j, j \in [0,1] \) and \( \eta \), thereby verifying the conjecture and completing the proof.

**Proof of Theorem 8.** Let \( q \equiv q_0 \). Given binary distribution parameterization \((\theta_h, \theta_l, q)\) with \( \theta_h = E + \sqrt{\frac{V(1-q)}{q}}, \theta_l = E - \sqrt{\frac{Vq}{1-q}} \), and \( q = \frac{1}{2} \left[ 1 - \frac{S}{\sqrt{4+S^2}} \right] \), the corresponding mean, variance, and skewness, respectively are: \( q \theta_h + (1-q) \theta_l = q \left[ E + \sqrt{\frac{V(1-q)}{q}} \right] + (1-q) \left[ E - \sqrt{\frac{Vq}{1-q}} \right] = E, q(\theta_h - E)^2 + (1-q)(\theta_l - E)^2 = q \frac{V(1-q)}{q} + (1-q) \frac{Vq}{1-q} = V \), and

\[
\frac{q(\theta_h - E)^3 + (1-q)(\theta_l - E)^3}{V^\frac{3}{2}} = \frac{q \left[ \frac{V(1-q)}{q} \right]^\frac{3}{2} - (1-q) \left[ \frac{Vq}{1-q} \right]^\frac{3}{2}}{V^\frac{3}{2}} = \frac{1 - 2q}{\sqrt{q(1-q)}} = \frac{\frac{S}{\sqrt{4+S^2}}}{\frac{1}{2} \sqrt{1 - \frac{S^2}{4+S^2}}} = S.
\]

Let \((\theta_h, \theta_l, q)\) and \((\theta'_h, \theta'_l, q')\) be two binary parameterizations having the same mean \( E \), variance \( V \), and skewness \( S \). Then, by the first two moment conditions for \( E \) and \( V \) each pair \((\theta_h, \theta_l)\) and \((\theta'_h, \theta'_l)\) are determined by \( E, V \), and respectively \( q \) and \( q' \): \( \theta_h = E + \sqrt{\frac{V(1-q)}{q}}, \theta_l = E - \sqrt{\frac{Vq}{1-q}}, \theta'_h = E + \sqrt{\frac{V(1-q')}{q'}}, \) and \( \theta'_l = E - \sqrt{\frac{Vq'}{1-q'}} \). Hence, the two parameterizations can only be different if \( q \) and \( q' \) are different. But, \( q \) and \( q' \) must satisfy the third moment condition, \( S = \frac{1-2q}{\sqrt{q(1-q)}} = \frac{1-2q}{\sqrt{q'(1-q')}} \), and since \( S(w) := \frac{1-2w}{\sqrt{w(1-w)}}\) is strictly decreasing, we have \( S(q) = S(q') \) if and only if \( q = q' \). Therefore, \((\theta_h, \theta_l, q)\) and \((\theta'_h, \theta'_l, q')\) characterize the same distribution.

**Proof of Lemma 9.** Informed investors’ beliefs about the probability of the high-state outcome (14) depend on their private signals and the public prior probability \( q \), as follows: \( \tilde{\pi}_s := P[\tilde{\theta} = \theta_h | \tilde{s} = s] \). In particular,

\[
\tilde{\pi}_L = \frac{q(1-\phi)}{q(1-\phi) + (1-q)\phi}, \quad (A4)
\]
\[
\tilde{\pi}_M = q, \quad (A5)
\]
\[
\tilde{\pi}_H = \frac{q\phi}{q\phi + (1-q)(1-\phi)}, \quad (A6)
\]

so that \( \frac{\partial \tilde{\pi}_M - \tilde{\pi}_L}{\partial \phi} \frac{q(1-q)}{[q(1-\phi) + (1-q)\phi]^2} > 0 \) and \( \frac{\partial \tilde{\pi}_H - \tilde{\pi}_M}{\partial \phi} \frac{q(1-q)}{[q(1-\phi) + (1-q)\phi]^2} > 0 \).
Proof of Theorem 10. For notational purposes: relabel the private investor signals $s_i$ for $i = 1, 2, 3 \equiv I$ by $s_1 \equiv L$, $s_2 \equiv M$, $s_3 \equiv H$; relabel information scenarios as $\rho_j$ for $j = 1, 2 \equiv J$ such that $\rho_1 \equiv LM$ and $\rho_2 \equiv MH$ and such that $\rho_1(s_1) = \rho_1(s_2) = \frac{1}{2}$, $\rho_1(s_3) = 0$, and $\rho_2(s_1) = 0$, $\rho_2(s_2) = \rho_2(s_3) = \frac{1}{2}$; let $\phi(\rho|\theta)$ denote the probability of sampling distribution $\rho$ given outcome $\theta$, such that $\phi(\rho_1|\theta_1) = \phi(\rho_2|\theta_h) = \phi$ and $\phi(\rho_1|\theta_h) = \phi(\rho_2|\theta_i) = 1 - \phi$; and let $P_j(\eta) \equiv P_{\rho_j}(\eta)$.

Plugging in the demand functions (16) and regrouping terms, the market clearing conditions equivalently become: $\lambda \sum_{i=1}^{I} \rho_j(s_i) \ln \frac{\pi_{s_i} P_{\rho_j}(\eta)}{1 - \pi_{s_i} P_{\rho_j}(\eta)} + (1 - \lambda) \ln \frac{\pi_{s_i} P_{\rho_j}(\eta)}{1 - \pi_{s_i} P_{\rho_j}(\eta)} + \ln \frac{\phi_{h-P_j(\eta)}}{\phi_{\theta_h - \theta_i}} + \gamma(\theta_h - \theta_i) \eta = 0$, for $j = 1, 2 \equiv J$, and for all $\eta$. Rearranging these implicit expressions for price so that price is partially isolated on the left-hand side yields:

$$P_j(\eta) = \frac{\theta_h \left[ \frac{\pi_{s_i} P_{\rho_j}(\eta)}{1 - \pi_{s_i} P_{\rho_j}(\eta)} \right]^{1-\lambda} \prod_{i=1}^{I} \frac{\pi_{s_i} P_{\rho_j}(\eta)}{1 - \pi_{s_i} P_{\rho_j}(\eta)} \lambda \rho_j(s_i) + \theta_i e^{-\gamma(\theta_h - \theta_i) \eta}}{\prod_{i=1}^{I} \frac{\pi_{s_i} P_{\rho_j}(\eta)}{1 - \pi_{s_i} P_{\rho_j}(\eta)} + e^{-\gamma(\theta_h - \theta_i) \eta}}.$$

Conjecture 1: For each $j = 1, 2$ the price function $P_j(\eta)$ is invertible in $\eta$ (which will be verified later in this proof). Then the posterior belief of the high-state outcome given private signal $s$ and price $p$ is:

$$\pi_{s,p} = \mathbb{P}[\tilde{\theta} = \theta_h | \tilde{s} = s, \tilde{\rho} = p] = \frac{\sum_j g(P_j^{-1}(p)) \mathbb{P}[\tilde{\theta} = \theta_h, \tilde{s} = s, \tilde{\rho} = \rho_j | \tilde{\rho} = p]}{\sum_j g(P_j^{-1}(p)) \mathbb{P}[\tilde{s} = s, \tilde{\rho} = p]}$$

$$= \frac{\sum_j g(P_j^{-1}(p)) \rho_j(s) \phi(\rho_j|\theta_h) q}{\sum_j g(P_j^{-1}(p)) \rho_j(s) \phi(\rho_j|\theta_h)}$$

where recall that $g(\cdot)$ is the probability density function of the supply shock $\tilde{\eta}$; the third equality follows by conditional independence of $\tilde{s}$ and $\tilde{\theta}$ given $\tilde{\rho}$. So, $\pi_{s,p} = \frac{q \sum_j g(P_j^{-1}(p)) \rho_j(s) \phi(\rho_j|\theta_h)}{1 - q \sum_j g(P_j^{-1}(p)) \rho_j(s) \phi(\rho_j|\theta_i)}$.

Conjecture 2: There exists some constants $\{\Delta_{jj'}\}$ such that $P_j^{-1}(P_j(\eta)) = \eta + \Delta_{jj'}$, for all $j, j'$. Equivalently, $P_j(\eta) = P_j(\eta + \Delta_{jj'})$, so to verify this conjecture it is required to show for $P_j(\eta) =$
\[
\theta_h q_j(\eta) + \theta_I [1 - q_j(\eta)], \text{ where } q_j(\eta) := \frac{1}{1 + e^{-\gamma(\theta_h - \theta_I) \eta}} \text{, that is,}
\]

\[
\theta_h q_j(\eta) + \theta_I [1 - q_j(\eta)] = \theta_h q_j(\eta + \Delta_j) + \theta_I [1 - q_j(\eta + \Delta_j)].
\]

After substitution and regrouping of terms, this verification condition becomes:

\[
(\theta_h - \theta_I) + \theta_I = \frac{1}{1 + e^{-\gamma(\theta_h - \theta_I) \eta} \left[ \frac{\sum_{j'=1}^{J} g(\eta + \Delta_j) \rho_j(\eta) \lambda_j}{\sum_{j'=1}^{J} g(\eta + \Delta_j) \rho_j(\eta) \lambda_j} \right]^{1-\lambda} \prod_{i=1}^{I} \left[ \frac{\sum_{j'=1}^{J} g(\eta + \Delta_j) \rho_j(\eta) \lambda_j}{\sum_{j'=1}^{J} g(\eta + \Delta_j) \rho_j(\eta) \lambda_j} \right]^{1-\lambda}} + \theta_I,
\]

or, equivalently,

\[
e^{-\gamma(\theta_h - \theta_I) \eta} \left[ \frac{\sum_{j'=1}^{J} g(\eta + \Delta_j) \rho_j(\eta) \lambda_j}{\sum_{j'=1}^{J} g(\eta + \Delta_j) \rho_j(\eta) \lambda_j} \right]^{1-\lambda} \prod_{i=1}^{I} \left[ \frac{\sum_{j'=1}^{J} g(\eta + \Delta_j) \rho_j(\eta) \lambda_j}{\sum_{j'=1}^{J} g(\eta + \Delta_j) \rho_j(\eta) \lambda_j} \right]^{1-\lambda} = e^{-\gamma(\theta_h - \theta_i)(\eta + \Delta_{j'})} \left[ \frac{\sum_{j'=1}^{J} g(\eta + \Delta_j) \rho_j(\eta) \lambda_j}{\sum_{j'=1}^{J} g(\eta + \Delta_j) \rho_j(\eta) \lambda_j} \right]^{1-\lambda} \prod_{i=1}^{I} \left[ \frac{\sum_{j'=1}^{J} g(\eta + \Delta_j) \rho_j(\eta) \lambda_j}{\sum_{j'=1}^{J} g(\eta + \Delta_j) \rho_j(\eta) \lambda_j} \right]^{1-\lambda}.
\]

Conjecture 3: \( \Delta_{j''j'} = \Delta_{j'j} + \Delta_{j''} \) for all \( j'', j, j' \). Under this third conjecture, the above condition for the second conjecture becomes:

\[
\prod_{i=1}^{I} \left[ \frac{\sum_{j'=1}^{J} g(\eta + \Delta_j) \rho_j(\eta) \lambda_j}{\sum_{j'=1}^{J} g(\eta + \Delta_j) \rho_j(\eta) \lambda_j} \right]^{1-\lambda} = e^{-\gamma(\theta_h - \theta_i)(\eta + \Delta_{j'})} \prod_{i=1}^{I} \left[ \frac{\sum_{j'=1}^{J} g(\eta + \Delta_j) \rho_j(\eta) \lambda_j}{\sum_{j'=1}^{J} g(\eta + \Delta_j) \rho_j(\eta) \lambda_j} \right]^{1-\lambda}.
\]

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where $f$ that function can be expressed as

$$\eta(x) = \frac{\lambda}{\gamma\theta - \theta_l} \sum_{i=1}^{I} (\rho_j(s_i) - \rho_j(s_l)) \ln \left[ \frac{\sum_{j=1}^{J} g(\eta + \Delta_jr_j) \rho_j(s_i) \phi(\rho_j|\theta_l)}{\sum_{j=1}^{J} g(\eta + \Delta_jr_j) \rho_j(s_l) \phi(\rho_j|\theta_l)} \right].$$

Thus, Conjecture 3 is verified since $\Delta_{11} = \Delta_{22} = 0$ and $\Delta_{12} = \frac{\lambda}{\gamma\theta - \theta_l} \ln \left[ \frac{\phi}{1-\phi} \right] = -\Delta_{21}$, which do not depend on $\eta$, and, in turn, Conjecture 2 is verified immediately by inspection. Therefore, the price function can be expressed as

$$P_j(\eta) = \frac{(\theta_h - \theta_l)}{1 + e^{-\gamma(\theta_h - \theta_l)}} + \theta_l,$$

which simplifies to

$$P_1(\eta) = (\theta_h - \theta_l) \frac{1}{1 + e^{-\gamma(\theta_h - \theta_l)}} + \theta_l \quad \text{and} \quad P_2(\eta) = P_1(\eta + \Delta_{12}),$$

and to

$$P_1(\eta) = (\theta_h - \theta_l) \frac{q}{q + (1-q)e^{-f(\eta-\Delta)}} + \theta_l$$

and

$$P_2(\eta) = (\theta_h - \theta_l) \frac{q}{q + (1-q)e^{-f(\eta+\Delta)}} + \theta_l,$$

where

$$\Delta := \frac{1}{2} \Delta_{12} = \frac{1/2}{\gamma(\theta_h - \theta_l)} \ln \frac{\phi}{1-\phi} \quad \text{and} \quad f(x) := \gamma(\theta_h - \theta_l) \left[ x + \frac{1/2}{\gamma(\theta_h - \theta_l)} \ln \frac{(1-\phi)g(x+\Delta) + \phi g(x-\Delta)}{\phi g(x+\Delta) + (1-\phi)g(x-\Delta)} \right].$$

Finally, I verify Conjecture 1 that the price function is invertible in $\eta$. A sufficient condition is that $f(x)$ is strictly increasing in $x$: $f'(x) = \gamma(\theta_h - \theta_l) + (1 - \frac{\lambda}{2}) \left[ \frac{g'(x+\Delta) + \phi g'(x-\Delta)}{g(x+\Delta) + \phi g(x-\Delta)} - \frac{\phi g'(x+\Delta) + g'(x-\Delta)}{\phi g(x+\Delta) + g(x-\Delta)} \right] > 0,$

where $\varphi := \frac{\phi}{1-\phi}$. The difference inside the square brackets must satisfy:

$$\frac{(1-\varphi^2)[g'(x + \Delta)g(x - \Delta) - g(x - \Delta)g'(x + \Delta)]}{\varphi(g(x + \Delta)^2 + g(x - \Delta)^2 + (1+\varphi^2)g(x - \Delta)g(x + \Delta)} > -\frac{\gamma(\theta_h - \theta_l)}{1 - \frac{\lambda}{2}},$$

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or equivalently,
\[
(1 - \varphi^2) \left[ \frac{g'(x + \Delta)}{g(x + \Delta)} - \frac{g'(x - \Delta)}{g(x - \Delta)} \right] > -\frac{\gamma(\theta_h - \theta_l)}{1 - \lambda/2} \left[ \varphi \left( \frac{g(x + \Delta)}{g(x - \Delta)} + \frac{g(x - \Delta)}{g(x + \Delta)} \right) + (1 + \varphi^2) \right].
\]

A sufficient condition for the above inequality to hold is that \(g(\eta)\) is differentiable and log-concave; i.e., \(g(\eta) = e^{v(\eta)}, v(\eta)\) concave. To see this, note that in terms of \(v(\cdot)\) this inequality becomes \((1 - \varphi^2) \left[ v'(x + \Delta) - v'(x - \Delta) \right] > -\frac{\gamma(\theta_h - \theta_l)}{1 - \lambda/2} \left[ \varphi \left( e^{v(x + \Delta) - v(x - \Delta)} + e^{v(x - \Delta) - v(x + \Delta)} \right) + (1 + \varphi^2) \right] \), and \(v(\eta)\) concave means that \(v'(\eta)\) is decreasing, so \(v'(x + \Delta) - v'(x - \Delta) \leq 0\). But \(\varphi > 1\) because \(\phi > \frac{1}{2}\), so \(1 - \varphi^2 < 0\). Hence, the left-hand side is nonnegative. Yet, the right-hand side is strictly negative. This result proves Conjecture 1 since \(g(\eta)\) was assumed to be log-concave and differentiable, and, in turn, proves this theorem. \(\Box\)

**Proof of Corollary 11.** Let \(M(x) := \frac{g(x + \Delta)(1 - \phi) + g(x - \Delta)\phi}{g(x + \Delta)\phi + g(x - \Delta)(1 - \phi)}\), such that (20) can be expressed as \(f(x) = \gamma(\theta_h - \theta_l) \left[ x + \frac{1 - \lambda/2}{\gamma(\theta_h - \theta_l)} \ln M(x) \right]\). Then, for \(\eta_0 \sim \mathbb{Z}[-k, k]\), we have \(g(\eta) = \frac{1}{2\pi} 1_{\{k \in [-k, k]\}}, M(\eta - \Delta) = \begin{cases} 1 - k \leq \eta < -k + 2\Delta, \\ \frac{1}{1 - \phi} k - 2\Delta < \eta \leq k, \end{cases}\) and \(M(\eta + \Delta) = \begin{cases} 1 - k \leq \eta < -k + 2\Delta, \\ \frac{1}{1 - \phi} k - 2\Delta < \eta \leq k, \end{cases}\) so that (20) exists for all \(\eta \in [-k, k]\) and satisfies:

\[
f(\eta - \Delta) = \begin{cases} \gamma(\theta_h - \theta_l)\eta + \ln \frac{1 - \phi}{\phi} & -k \leq \eta < -k + 2\Delta, \\ \gamma(\theta_h - \theta_l)\eta + \ln \left[ \frac{1 - \phi}{\phi} \right]^{1/2} & -k + 2\Delta \leq \eta \leq k, \end{cases}
\]

\[
f(\eta + \Delta) = \begin{cases} \gamma(\theta_h - \theta_l)\eta + \ln \left[ \frac{\phi}{1 - \phi} \right]^{1/2} & -k \leq \eta < -k + 2\Delta, \\ \gamma(\theta_h - \theta_l)\eta + \ln \frac{\phi}{1 - \phi} & k - 2\Delta < \eta \leq k. \end{cases}
\]

Use the identity \(\frac{\eta}{q + (1 - q)\eta} = \frac{1}{1 + \frac{1 - \phi}{\phi} \eta}\), \(q > 0, A > 0\), with \(A\) set to \(e^{-f(\eta - \Delta)}\) and \(e^{-f(\eta + \Delta)}\), respectively, to obtain the result. \(\Box\)

**Proof of Proposition 12.** Expected price function: Since \(\bar{\eta}\) is independent of \(\bar{\rho}\), the expected price function \(\text{EP}(q) := \mathbb{E}[P_{\bar{\rho}(\bar{\eta})}]\) satisfies

\[
\text{EP}(q) = \mathbb{E}[P_{LM}(\bar{\eta})](1 - h(q)) + \mathbb{E}[P_{MH}(\bar{\eta})]h(q), \tag{A7}
\]

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where \( h(q) := P[\bar{\rho} = MH] = q\phi + (1-q)(1-\phi) = 1-P[\bar{\rho} = LM] \). For \( \bar{\rho} = LM \), the expected price is

\[
E[P_{LM}(\tilde{\eta})] = \int_{-k}^{-k+2\Delta} \left[ (\theta_h - \theta_l) \frac{1}{1 + e^{\gamma(\theta_h-\theta_l)\eta}} + \theta_l \right] \frac{1}{2k} \, d\eta
\]

\[
+ \int_{-k}^{-k+2\Delta} \left[ (\theta_h - \theta_l) \frac{1}{1 + e^{\gamma(\theta_h-\theta_l)\eta}} + \theta_l \right] \frac{1}{2k} \, d\eta
\]

\[
= \frac{1}{2k} (\theta_h - \theta_l) \int_{-k}^{-k+2\Delta} \frac{1}{1 + e^{\gamma \theta_h \eta}} \, d\eta + \int_{-k}^{-k+2\Delta} \frac{1}{1 + e^{\gamma \theta_l \eta}} \, d\eta + \theta_l,
\]

where \( A := \frac{q}{1-q} \frac{1-\phi}{\phi} \), \( A_A := \frac{q}{1-q} \frac{1-\phi}{\phi} \frac{1}{2} \), and \( B := \gamma(\theta_h - \theta_l) \). Applying the fact \( \int_a^b \frac{1}{a + e^{bx}} \, dx = \frac{1}{b} \ln \left( \frac{1 + ce^{at}}{1 + ce^{bt}} \right) \), \( \forall a < b, \forall c \neq 0, \forall d \), gives \( E[P_{LM}(\tilde{\eta})] = \frac{1}{2k} \frac{1}{\gamma} \ln \left[ \frac{1-q(1-X_A)}{1-q(1-X_A)} \frac{1-q(1-Y_B)}{1-q(1-Y_B)} \right] + \theta_l \), where \( X_A := \frac{1-\phi}{\phi} e^{\gamma \theta_h \theta_l}(k-2\Delta), \)

\( Y_A := \frac{1-\phi}{\phi} e^{\gamma \theta_h \theta_l}(k-2\Delta), \)

\( X_B := (\frac{1-\phi}{\phi})^2 \frac{1}{\phi} e^{\gamma \theta_h \theta_l}(k-2\Delta), \)

\( Y_B := (\frac{1-\phi}{\phi})^2 \frac{1}{\phi} e^{\gamma \theta_h \theta_l}(k-2\Delta). \)

So the expected interim price in equation (A7) is

\[
EP(q) = \left[ (1-h(q)) \left( \frac{1}{2k} \frac{1}{\gamma} \ln \left[ \frac{1-q(1-X_A)}{1-q(1-Y_B)} \right] + \theta_l \right) \right]
\]

\[
= \frac{1}{2k} \frac{1}{\gamma} \left[ (1-h(q)) \ln \frac{1-q(1-X_A)}{1-q(1-Y_B)} + \theta_l \right]
\]

\[
= \frac{1}{2k} \frac{1}{\gamma} \left[ (1-h(q)) u(q, X, Y) + h(q) u(q, \frac{1}{\gamma}, \frac{1}{\gamma}) \right] + \theta_l,
\]

where

\[
u(q, x, y) := \ln \frac{1-q(1-x)}{1-q(1-y)}, \quad X := X_A, \quad Y := Y_A, \quad Z := X_B.
\]

Note that since \( \Delta = \frac{\lambda/\gamma(\theta_h - \theta_l)}{1-\phi} \ln \frac{\phi}{1-\phi} \), we have \( X = X_A = \frac{1-\phi}{\phi} \theta_h \theta_l(1-k+2\Delta) = \left( \frac{1-\phi}{\phi} \right)^2 e^{\gamma \theta_h \theta_l}(k-2\Delta), \)

\( Y = Y_A = \frac{1-\phi}{\phi} e^{\gamma \theta_h \theta_l}(k-2\Delta), \)

\( Z = X_B = \left( \frac{1-\phi}{\phi} \right)^2 e^{\gamma \theta_h \theta_l}(k), \)

coinciding with equations (26), (27), and (28), respectively.

I will prove the additional five statements of this proposition in the following order: (i) statement
2, (ii) statement 1, (iii) statement 4, (iv) statement 5, and then (v) statement 3.

(i) Statement 2: First note that \( u(0,x,y) = 0 \) for all \( x,y \). Thus,

\[
\text{EP}(0) = \frac{1}{2k_1} \left[ (1-h(0))u(0,X,Y) + h(0)u(0,\frac{1}{X},\frac{1}{X}) + u(0,Z,\frac{1}{Z}) \right] + \theta_l = \theta_l.
\]

Next, note that \( h(1) = \phi \) and \( u(1,x,y) = \ln \frac{x}{y} \) for all \( x,y \). Thus,

\[
\text{EP}(1) = \frac{1}{2k_1} \left[ (1-h(1))u(1,X,Y) + h(1)u(1,\frac{1}{X},\frac{1}{X}) + u(1,Z,\frac{1}{Z}) \right] + \theta_l
= \frac{1}{2k_1} \left[ (1-\phi)\ln \frac{x}{y} + \phi \ln \frac{1}{X} + \ln \frac{Z}{X} \right] + \theta_l
= \frac{1}{2k_1} \left[ \lambda \ln \left( \frac{\phi}{1-\phi} \right) + 2 \left( -\frac{1}{2} \ln \left( \frac{\phi}{1-\phi} \right) + \gamma (\theta_h - \theta_l)k \right) \right] + \theta_l = \theta_h.
\]

(ii) Statement 1 (reverse symmetry): First note that

\[
h(1-q) = (1-q)\phi + q(1-\phi) = 1-h(q)
\]

(A8)

and \( u(1-q,x,y) = \ln \frac{1-(1-q)(1-x)}{1-(1-q)(1-y)} = \ln \frac{x}{y} - \ln \frac{1-q(1-x)}{1-q(1-y)} = \ln \frac{x}{y} - u(q,\frac{1}{y},\frac{1}{x}) \). Then, by substitution,

\[
\text{EP}(1-q) - \theta_l = \frac{1}{2k_1} \left[ (1-h(1-q))u(1-q,X,Y) + h(1-q)u(1-q,\frac{1}{X},\frac{1}{X}) + u(1-q,Z,\frac{1}{Z}) \right]
= \frac{1}{2k_1} \left[ h(q) \left( \ln \frac{x}{y} - u(q,\frac{1}{X},\frac{1}{X}) \right) + (1-h(q)) \left( \ln \frac{x}{y} - u \left( q, \frac{1}{x}, \frac{1}{y} \right) \right) + (\ln Z^2 - u(q,Z,\frac{1}{Z}) \right]
= \frac{1}{2k_1} \left( \ln \frac{x}{y} + \ln Z^2 \right) - \frac{1}{2k_1} \left[ h(q) \left( \ln \frac{x}{y} - u(q,\frac{1}{X},\frac{1}{X}) \right) + (1-h(q))u(q,X,Y) + u(q,Z,\frac{1}{Z}) \right]
= \frac{1}{2k_1} \left[ \lambda \ln \left( \frac{\phi}{1-\phi} \right) + 2 \left( -\frac{1}{2} \ln \left( \frac{\phi}{1-\phi} \right) + \gamma (\theta_h - \theta_l)k \right) \right] - (\text{EP}(q) - \theta_l)
= \theta_h - \text{EP}(q),
\]

which, together with \( \theta_l = \text{EP}(0) \) and \( \theta_h = \text{EP}(1) \) from parts 1 and 3 of statement 2, respectively, yields the reverse symmetry property of \( \text{EP}(q) \).
(iii) Statement 4 (strict concavity of EP(q) for $q \in [0, \frac{1}{2}]$): First, note that

\begin{align*}
h(q) &= q\phi + (1-q)(1-\phi), \\
h'(q) &= 2\phi - 1 > 0, \\
h''(q) &= 0,
\end{align*}

where $h'(q) > 0$ because $\phi > \frac{1}{2}$, and

\begin{align*}
u(q, x, y) &= \ln \frac{1 - q(1-x)}{1 - q(1-y)}, \\
\frac{\partial}{\partial q} u(q, x, y) &= \frac{1 - y}{1 - q(1-y)} - \frac{1 - x}{1 - q(1-x)}, \\
\frac{\partial^2}{\partial q^2} u(q, x, y) &= \left( \frac{1 - y}{1 - q(1-y)} \right)^2 - \left( \frac{1 - x}{1 - q(1-x)} \right)^2, \\
\frac{\partial^3}{\partial q^3} u(q, x, y) &= 2 \left( \frac{1 - y}{1 - q(1-y)} \right)^3 - 2 \left( \frac{1 - x}{1 - q(1-x)} \right)^3.
\end{align*}

Next, drop the constant $\theta_l$ and positive scale $\frac{1}{\gamma_X}$ from EP(q), to define

\[
\overline{\text{EP}}(q) := (1 - h(q))a(q) + h(q)b(q) + c(q),
\]

where

\begin{align*}
a(q) &:= u(q, X, Y), \\
b(q) &:= u(q, \frac{1}{X}, \frac{1}{Y}), \\
c(q) &:= u(q, Z, \frac{1}{Z}).
\end{align*}

Then,

\[
\overline{\text{EP}}(q) = (1 - h(q))a'(q) - h'(q)a(q) + h(q)b'(q) + h'(q)b(q) + c'(q),
\]

and because $h''(q) = 0$,

\[
\overline{\text{EP}}''(q) = (1 - h(q))a''(q) - 2h'(q)a'(q) + h(q)b''(q) + 2h'(q)b'(q) + c''(q).
\]
Let
\[ L(q) := h(1 - q)a''(q) - 2h'(q)a'(q). \]  
(A22)

Applying (A13) and (A14) to (A17) and (A18) yields
\[ b'(q) = \frac{1 - \frac{1}{X}}{1 - q(1 - \frac{1}{X})} - \frac{1 - \frac{1}{Y}}{1 - q(1 - \frac{1}{Y})} = \frac{1 - Y}{1 - (1 - q)(1 - Y)} - \frac{1 - X}{1 - (1 - q)(1 - X)} \]
\[ = a'(1 - q), \]  
(A23)

and
\[ b''(q) = \left(\frac{1 - \frac{1}{X}}{1 - q(1 - \frac{1}{X})}\right)^2 - \left(\frac{1 - \frac{1}{Y}}{1 - q(1 - \frac{1}{Y})}\right)^2 = \left(\frac{1 - Y}{1 - (1 - q)(1 - Y)}\right)^2 - \left(\frac{1 - X}{1 - (1 - q)(1 - X)}\right)^2 \]
\[ = -a''(1 - q). \]  
(A24)

Thus,
\[ L(1 - q) = h(q)a''(1 - q) - 2h'(1 - q)a'(1 - q) \]
\[ = -h(q)b''(q) - 2h'(q)b'(q), \]  
(A25)

which follows by evaluating \( L(q) \) (equation (A22)) at \( 1 - q \) and by (A24) and (A23). Recall that \( h(1 - q) = 1 - h(q) \) (equation (A8)), and use (A22) and (A25) to express (A21) as
\[ \overline{EP''}(q) = L(q) - L(1 - q) + c''(q). \]  
(A26)

Next, I show that \( L(q) \) is increasing in \( q \) for all \( q \in [0, 1] \). Let \( \varphi := \frac{\theta_h - \theta_l}{\theta_h} > 1 \) (since \( \varphi > \frac{1}{2} \) by (13)) and \( T := \theta_h - \theta_l > 0 \) (since \( \theta_h > \theta_l \) by (12)). Then, \( \Delta = \frac{1}{T^2} \ln \varphi, \ X = \varphi^{1-1}e^{-\gamma kT}, \ Y = \varphi^{-1}e^{-\gamma kT}, \) and \( Z = \varphi^{-\frac{1}{2}}e^{\gamma kT} \) by (21), (26), (27), and (28), respectively. Note that
\[ 0 < Y \leq X < \frac{1}{\varphi} < 1 < Z, \]  
(A27)

where the inequality separating \( Y \) and \( X \) is strict for \( \lambda > 0 \), and both the third inequality \( (X < \frac{1}{\varphi}) \) and fifth inequality \( (1 < Z) \) follow from \( k > 2\Delta \) (Corollary 11), which implies that \( \varphi^{\lambda} < e^{\gamma kT}, \) so \( \varphi = \varphi^{1-\lambda}\varphi^{\lambda} < \varphi^{1-\lambda}e^{\gamma kT} = \frac{1}{X}, \) and \( 1 \leq \varphi^{\frac{1}{2}} = \varphi^{-\frac{1}{2}}\varphi^{\lambda} < \varphi^{-\frac{1}{2}}e^{\gamma kT} = Z. \)
Let
\begin{align*}
A(q) &:= \frac{1-Y}{1-q(1-Y)}, \\
B(q) &:= \frac{1-X}{1-q(1-X)}, \\
C(q) &:= (\varphi(1-q) + q)B(q).
\end{align*}
(A28)

Note that for all \( q \in [0,1] \),
\[ A(q) \geq B(q) > 0, \]  
(A31)
which follows from equation (A27), and
\[ B'(q) = (B(q))^2, \]
(A32)
\[ C'(q) = (\varphi(1-q) + q)B'(q) + (1-\varphi)B(q) \]
\[ = B(q)\left((\varphi(1-q) + q)B(q) + 1-\varphi\right). \]
(A33)

Then, \( \frac{1-X}{(B(q))^2} C'(q) = (\varphi(1-q) + q) + (1-\varphi)(1-q(1-X)) = 1-\varphi X > 0 \), where the final inequality follows from (A27). Since \( \frac{1-X}{(B(q))^2} > 0 \) (equation (A27)), \( C'(q) > 0 \), and thus
\[ \varphi(1-X) = C(0) = \min_{q \in [0,1]} C(q). \]  
(A34)

Next, applying (A14) and (A15) to (A17) and substitution of (A28) and (A29) yields:
\[ \frac{1}{2} \frac{a''''(q)}{a''(q)} = \frac{A(q)^3 - B(q)^3}{A(q)^2 - B(q)^2} \]
\[ = A(q) + B(q) - \frac{A(q)B(q)}{A(q) + B(q)}. \]
(A35)

Note that \( a''(q) > 0 \) by (A31), so for any \( q \in [0,1] \),
\[ 3a''(q)(\varphi - 1) \leq 3a''(q)\varphi(1-X) \leq 3a''(q)C(q) = 3a''(q)(\varphi(1-q) + q)B(q) \]
\[ \leq 3a''(q)(\varphi(1-q) + q)\left(\frac{1}{3}A(q) + \frac{2}{3}B(q)\right) \]
\[ = 2a''(q)(\varphi(1-q) + q)\left(A(q) + B(q) - \frac{A(q)B(q)}{2B(q)}\right) \]
\[ \leq 2a''(q)(\phi(1-q) + q) \left( A(q) + B(q) - \frac{A(q)B(q)}{A(q) + B(q)} \right) \]
\[ = (\phi(1-q) + q)a'''(q), \quad (A36) \]

where the first inequality follows from \( a''(q) > 0 \) and \((A27)\), the second inequality follows from \((A34)\), the third inequality follows from the definition of \( C(q) \) in \((A30)\), the fourth and sixth inequalities follow from \((A31)\), and the final inequality follows from \((A35)\).

Now,
\[ \frac{1}{1-\phi} L'(q) = \frac{1}{1-\phi} \left[ h(1-q)a'''(q) - h'(1-q)a''(q) - 2h'(q)a''(q) \right] = \frac{h(1-q)}{1-\phi} a'''(q) - 3a''(q) \frac{h'(q)}{1-\phi} = (\phi(1-q) + q)a'''(q) - 3a''(q)(\phi - 1) \]
\[ \geq 0, \quad (A37) \]

where the second inequality follows from \( h'(1-q) = h'(q) \) (equation \((A10)\)), the third inequality follows from the identities \( \frac{h(1-q)}{1-\phi} = \frac{(1-q)\phi + q(1-\phi)}{1-\phi} = (\phi(1-q) + q) \) (equation \((A9)\) and the definition \( \phi \equiv \frac{\phi}{1-\phi} \)) and \( \frac{h'(q)}{1-\phi} = \frac{2\phi - 1}{1-\phi} = \frac{\phi - (1-\phi)}{1-\phi} = \phi - 1 \) (equation \((A10)\) and the definition \( \phi \equiv \frac{\phi}{1-\phi} \)), the final inequality follows from \((A36)\). Because \( \phi \in \left( \frac{1}{2}, 1 \right) \), equation \((A37)\) implies that \( L(q) \) is increasing in \( q \). Furthermore, for \( q \in [0, \frac{1}{2}) \), \( q < 1 - q \) and hence \( L(q) \leq L(1-q) \). Thus,
\[ L(q) - L(1-q) \leq 0, \quad q \in [0, \frac{1}{2}). \quad (A38) \]

Next I show that second derivative of \( c(q) \) (equation \((A19)\)) is negative for \( q < \frac{1}{2} \). Note that
\[ c''(q) = \left( \frac{1 - \frac{1}{Z}}{1 - q(1 - \frac{1}{Z})} \right)^2 - \left( \frac{1 - Z}{1 - q(1 - Z)} \right)^2, \quad (A39) \]

and
\[ Z + \frac{1}{Z} > 2. \quad (A40) \]

To see why equation \((A40)\) holds, consider \( f(w) := w + \frac{1}{w} \), which is strictly convex on \( w > 0 \) \( (f'(w) = 1 - \frac{1}{w^2} \) and \( f''(w) = \frac{2}{w^3} > 0) \), and hence uniquely minimized at \( f'(w) = 0 \), or equivalently, at \( w = 1 \). Thus, \( 2 = f(1) < f(w) \) for all \( w > 1 \). Since \( Z > 1 \) (equation \((A27)\)), \( 2 < Z + \frac{1}{Z} \).
Multiply both sides of \((A40)\) by \(q - (1 - q)\), which is negative for \(q < \frac{1}{2}\), so \((q - (1 - q))(Z + \frac{1}{Z}) < 2(q - (1 - q))\). Expand the terms on each side of this inequality, rearrange, and regroup to obtain \((1 - \frac{1}{Z})(1 - q(1 - Z)) < (Z - 1)(1 - q (1 - \frac{1}{Z}))\). Divide both sides of this inequality by the positive product \((1 - q(1 - Z))(1 - q (1 - \frac{1}{Z}))\), to obtain \(\frac{1 - \frac{1}{Z}}{1 - q(1 - Z)} < \frac{Z - 1}{q(1 - Z)}\). Finally, since both sides of this inequality are positive, squaring each side yields \(\left(\frac{1 - \frac{1}{Z}}{1 - q(1 - Z)}\right)^2 < \left(\frac{Z - 1}{1 - q(1 - Z)}\right)^2\), which shows that \((A39)\) is negative for \(q < \frac{1}{2}\):

\[
c''(q) < 0, \quad q \in [0, \frac{1}{2}). \tag{A41}
\]

Equations \((A26), (A38),\) and \((A41)\) imply that \(EP''(q) < 0\) for \(q \in [0, \frac{1}{2})\). Since \(EP(q)\) is a positive scaling and translation of \(EP(q)\), \(\text{sign}(EP'(q)) = \text{sign}(EP'(q))\) and \(\text{sign}(EP''(q)) = \text{sign}(EP''(q))\). Thus, \(EP''(q) < 0\) for \(q \in [0, \frac{1}{2})\), which completes the proof of statement 4.

**(vi)** Statement 5 (strict convexity of \(EP(q)\) for \(q \in (\frac{1}{2}, 1]\)): By statement 2 (reverse symmetry), for all \(q \in [0, 1]\

\[
EP(q) = EP(1) + EP(0) - EP(1 - q),
\]

\[
EP'(q) = EP'(1 - q),
\]

\[
EP''(q) = -EP''(1 - q). \tag{A42}
\]

So, \(q \in (\frac{1}{2}, 1]\) implies \((1 - q) \in [0, \frac{1}{2})\) and \(EP''(1 - q) < 0\) by statement 4. Thus, by \((A42)\), \(EP''(q) > 0\) for \(q \in [0, \frac{1}{2})\), which proves statement 5.

**(v)** Statement 3 (\(EP(q)\) strictly increasing for \(q \in [0, 1]\)): First, note that \(EP'(\frac{1}{2}) > 0\):

\[
\begin{align*}
EP'(\frac{1}{2}) &= (1 - h(\frac{1}{2})) a'(\frac{1}{2}) - h'(\frac{1}{2}) a(\frac{1}{2}) + h(\frac{1}{2}) b'(\frac{1}{2}) + h'(\frac{1}{2}) b(\frac{1}{2}) + c'(\frac{1}{2}) \\
&= a'(\frac{1}{2}) + c'(\frac{1}{2}) + (2\phi - 1) \left( b(\frac{1}{2}) - a(\frac{1}{2}) \right) \\
&= 2 \left[ \frac{1 - Y}{1 + Y} - \frac{1 - X}{1 + X} \right] + 4 \frac{1 - \frac{1}{2}}{1 + \frac{1}{2}} + (2\phi - 1) \ln \frac{1 + \frac{1}{Y}}{1 + \frac{1}{X}} (1 + Y), \tag{A43}
\end{align*}
\]

where the first equality follows from \((A20)\); the second equality follows from \(h(\frac{1}{2}) = \frac{1}{2}\) (equation \((A9)\)), \(h'(\frac{1}{2}) = 2\phi - 1\) (equation \((A10)\)), and \(a'(\frac{1}{2}) = b'(\frac{1}{2})\) (equation \((A23)\)); and the last equality follows from \((A13)\) applied to \((A17), (A18),\) and \((A19)\) at \(q = \frac{1}{2}\). Consider \(f_1(w) := \frac{1 - w}{1 + w}\) and \(f_2(w) := w + \frac{1}{w}\), which, for \(0 < w < 1\), are strictly decreasing \((f_1'(w) = -\frac{2}{(1 + w)^2} < 0\) and \(f_2'(w) = 1 - \frac{1}{w^2} < 0\)) and positive \((f_1(w) > 0\)
and \( f_2(w) > 0 \). Then, since \( 0 < Y < X < 1 \) (equation (A27)), \( f_1(Y) \geq f_1(X) = \frac{1}{1+X} > 0 \) and \( Y + \frac{1}{Y} = f_2(Y) \geq f_2(X) = X + \frac{1}{X} \). So, the first and third terms of (A43) are non-negative: \( 2\left( \frac{1}{1+Y} - \frac{1}{1+X} \right) \geq 0; \) and \( \frac{(1+\frac{1}{2})(1+Y)}{(1+\frac{1}{2})(1+X)} = \frac{2Y+\frac{1}{2}}{2X+\frac{1}{2}} \geq 1 \) and \( \phi > \frac{1}{2} \). Also, since \( 0 < \frac{1}{2} < 1 \) (equation (A27)), \( \frac{1}{1+Z} = f_1(Z) > 0 \), which implies that the second term of (A43) is positive: \( 4\frac{1}{1+Z} > 0 \). Thus, \( EP^l(\frac{1}{2}) > 0 \). Since \( \text{sign}(EP^l(q)) = \text{sign}(EP^r(q)) \), \( EP^r(\frac{1}{2}) > 0 \).

Now, by statement 4, \( EP^r(q) \) is strictly concave for \( q \in [0, \frac{1}{2}] \). So, \( EP^r(q) \) is strictly decreasing for \( q \in [0, \frac{1}{2}] \), and hence \( EP^r(q) > EP^r(\frac{1}{2}) > 0 \) for \( q \in [0, \frac{1}{2}] \). Similarly, by statement 5, \( EP^r(q) \) is strictly convex for \( q \in \left( \frac{1}{2}, 1 \right] \). So, \( EP^r(q) \) is strictly increasing for \( q \in \left( \frac{1}{2}, 1 \right] \), and hence \( 0 < EP^r(\frac{1}{2}) < EP^r(q) \) for \( q \in \left( \frac{1}{2}, 1 \right] \). Together, these properties imply that \( EP(q) \) is strictly increasing for all \( q \in [0, 1] \).

**Proof of Theorem 13.** For \( q \in (0, \frac{1}{2}) \), \( EP(q) > (1-q)EP(0) + qEP(1) = (1-q)q + q\theta_h \), where the first inequality follows because \( EP(q) \) is strictly concave for \( q \in [0, \frac{1}{2}] \) (Proposition 12 and \( q = (1-q)0 + q1 \)), and the final equality follows from statement 2 of Proposition 12. Similarly, for \( q \in (\frac{1}{2}, 1) \), \( EP(q) < (1-q)EP(0) + qEP(1) = (1-q)q + q\theta_h \), where the first inequality follows because \( EP(q) \) is strictly convex for \( q \in (\frac{1}{2}, 1) \) (Proposition 12), and the final equality follows from statement 2 of Proposition 12.

Finally, apply statement 1 (reverse symmetry) and statement 2 of Proposition 12 at \( q = \frac{1}{2} \) to obtain \( EP(\frac{1}{2}) - \theta_l = \theta_h - EP(\frac{1}{2}) \), which implies \( EP(\frac{1}{2}) = \frac{1}{2} \theta_h + \frac{1}{2} \theta_l \).

**Proof of Proposition 14.**

\[
\mathbb{E}\left[ \frac{\tilde{\theta}}{P^l(\tilde{\eta})} \right] = \int_{-k}^{k} \left( P[\tilde{\rho} = LM] \mathbb{E}[\tilde{\theta} | \tilde{\rho} = LM] \right) \frac{1}{2k} d\eta
\]

\[
= \frac{\theta_h q(1-\phi) + \theta_l (1-q)\phi}{2k} \int_{-k}^{k} \frac{d\eta}{P_{LM}(\eta)} + \frac{\theta_h q \phi + \theta_l (1-q)(1-\phi)}{2k} \int_{-k}^{k} \frac{d\eta}{P_{MH}(\eta)}.
\]

Using the formula \( \int_{a}^{b} \left( -\frac{\theta_h - \theta_l}{\theta_h - \theta_l} + \theta_l \right)^{-1} dx = \frac{b-a}{\theta_h} - \frac{1}{\gamma \theta_h \theta_l} \ln \frac{\theta_h B + \theta_l e^{-\gamma(\theta_h - \theta_l)}}{\theta_h B + \theta_l e^{-\gamma(\theta_h - \theta_l)}} \) to evaluate the integrals above, the previous equality continues as:

\[
\mathbb{E}\left[ \frac{\tilde{\theta}}{P^l(\tilde{\eta})} \right] = \frac{\theta_h q(1-\phi) + \theta_l (1-q)\phi}{2k} \left( \frac{2k}{\theta_h} - \frac{1}{\gamma \theta_h \theta_l} \ln \frac{\theta_h q 1-\phi + \theta_l e^{-(k+2\gamma)(\theta_h - \theta_l)}}{\theta_h q 1-\phi + \theta_l e^{-(k+2\gamma)(\theta_h - \theta_l)}} \right)
\]
Now consider 

\[
- \frac{1}{\gamma \theta_h \theta_t} \ln \frac{\theta_h q}{1-q} \left( \frac{1-q}{\phi} \right)^{\frac{1}{2}} + \theta_t e^{-\left( k \gamma \left( (\theta_h - \theta_t) \right) \right)} \\
+ \frac{\theta_h q \phi + \theta_t (1-q)(1-\phi)}{2k} \left( \frac{2k}{\gamma \theta_h \theta_t} \ln \frac{\theta_h q}{1-q} \left( \frac{1-q}{\phi} \right)^{\frac{1}{2}} + \theta_t e^{-\left( k -\Delta \right) \gamma \left( (\theta_h - \theta_t) \right)} \right)
\]

Let \( \nu(r,s) := \theta_h \frac{q}{1-q} \left( \frac{\phi}{1-\phi} \right)^r + \theta_t e^{s \gamma \left( (\theta_h - \theta_t) \right)} \) and \( \nu(x) := \theta_h q x + \theta_t (1-q)(1-x) \). Then,

\[
\mathbb{E} \left[ \frac{\tilde{\theta}}{P_{\tilde{\theta}}(\tilde{\eta})} \right] = \theta_h q (1-\phi) + \theta_t (1-q) q \phi \left[ \frac{2k}{\gamma \theta_h \theta_t} \ln \frac{\nu(-1,k-2\Delta)}{\nu(-1,k)} \frac{\nu(-\frac{1}{2},k-2\Delta)}{\nu(-\frac{1}{2},k)} \right] \\
+ \theta_h q \phi + \theta_t (1-q)(1-\phi) \left[ \frac{2k}{\gamma \theta_h \theta_t} \ln \frac{\nu(1,k-2\Delta)}{\nu(1,k)} \frac{\nu(\frac{1}{2},k-2\Delta)}{\nu(\frac{1}{2},k)} \right]
\]

which further simplifies to

\[
\mathbb{E} \left[ \frac{\tilde{\theta}}{P_{\tilde{\theta}}(\tilde{\eta})} \right] = \theta_h q (1-\phi) + \theta_t (1-q) q \phi \left[ \frac{2k}{\gamma \theta_h \theta_t} \ln \frac{\nu(-1,k-2\Delta)}{\nu(-1,k)} \frac{\nu(-\frac{1}{2},k-2\Delta)}{\nu(-\frac{1}{2},k)} \right] \\
+ \theta_h q \phi + \theta_t (1-q)(1-\phi) \left[ \frac{2k}{\gamma \theta_h \theta_t} \ln \frac{\nu(1,k-2\Delta)}{\nu(1,k)} \frac{\nu(\frac{1}{2},k-2\Delta)}{\nu(\frac{1}{2},k)} \right]
\]

and finally:

\[
\mathbb{E} \left[ \frac{\tilde{\theta}}{P_{\tilde{\theta}}(\tilde{\eta})} \right] = \frac{\theta_h q + \theta_t (1-q)}{\theta_h} \frac{H(1-\phi)}{2k \gamma \theta_h \theta_t} \ln \frac{\nu(-1,k-2\Delta)}{\nu(-1,k)} \frac{\nu(-\frac{1}{2},k-2\Delta)}{\nu(-\frac{1}{2},k)} - \frac{H(\phi)}{2k \gamma \theta_h \theta_t} \ln \frac{\nu(1,k-2\Delta)}{\nu(1,k)} \frac{\nu(\frac{1}{2},k)}{\nu(\frac{1}{2},k)} - 1.
\]

Proof of Proposition 15. Note that \( \mathbb{E} \nu(q) \) (equation (23)) is invariant to \( \phi \) at \( q = \frac{1}{2} \): \( \mathbb{E} \nu(\frac{1}{2}) = \frac{1}{2} \theta_h + \frac{1}{2} \theta_t \) (Theorem 13). However, the slope of \( \mathbb{E} \nu(q) \) at \( q = \frac{1}{2} \), \( \mathbb{E} \nu(\frac{1}{2}) \), depends on \( \phi \) for \( \lambda = 1 \). It is enough to show that this slope increases in \( \phi \) for \( \gamma (\theta_h - \theta_t) k \) sufficiently large: since \( \mathbb{E} \nu(q) \) is continuous with continuous derivative, if its slope at \( q = \frac{1}{2} \) increases in \( \phi \), then there exists a neighborhood of \( q = \frac{1}{2} \), say, \( (q, q) \) with \( q < \frac{1}{2} < \overline{q} \), such that \( \mathbb{E} \nu(q) \) decreases for \( q < q < \frac{1}{2} \) and \( \mathbb{E} \nu(q) \) increases for \( \frac{1}{2} < q < \overline{q} \). Moreover, it is enough to show it for the positively-scaled version of the slope, \( \mathbb{E} \nu(\frac{1}{2}) \) (equation (A43))
in the proof of Proposition 12). Let \( \varphi := \frac{\phi}{1 - \varphi} > 1 \) (equation (13)), \( T := \theta_h - \theta_l > 0 \) (equation (12)). Note that for \( \lambda = 1 \), \( X = e^{-\gamma kT} \), \( Y = \varphi^{-1}X \), and \( \frac{1}{Z} = \varphi^{\frac{1}{2}}X \) by (26), (27), and (28), respectively. In particular, \( X \) is invariant to \( \varphi \). So,

\[
\frac{\partial}{\partial \varphi} \mathbb{E}_P \left( \frac{1}{Z} \right) = \frac{\partial}{\partial \varphi} \left( 2 \left( 1 - \frac{Y}{1 + Y} \right) - \frac{1}{Z} \right) + 4 \left( 1 - \frac{1}{Z} \right) \ln \left( \frac{1 + \frac{1}{Z}}{1 + \frac{1}{Y}} \right) \\
= -4 \frac{\partial}{\partial \varphi} Y \frac{1}{(1 + Y)^2} - 8 \frac{\partial}{\partial \varphi} \left( \frac{1}{1 + \frac{1}{Z}} \right)^2 + (2\varphi - 1) \frac{\partial}{\partial \varphi} \left( \frac{\partial}{\partial \varphi} \frac{1}{Y} + \frac{\partial}{\partial \varphi} \frac{Y}{1 + Y} \right) \\
= \left( 4Y \frac{1}{(1 + Y)^2} - \frac{4}{Z^2} \right) \left( 1 - \frac{1}{Z} \right) \frac{1}{1 + \frac{1}{Z}} \frac{\partial}{\partial \varphi} \frac{1}{Y} + 2\varphi - 1 \frac{\partial}{\partial \varphi} \frac{1}{Y} + 2\varphi - 1 \frac{\partial}{\partial \varphi} \frac{Y}{1 + Y}. \quad (A44)
\]

The last term in (A44), \( 2\ln \left( \frac{1 + \frac{1}{Z}}{1 + \frac{1}{Y}} \right) \), was shown to be non-negative in the proof of statement 3 of Proposition 12; \( \frac{1}{\varphi} > 0 \); and \( \frac{\partial \varphi}{\partial \varphi} = \left( 1 - \frac{1}{\varphi^2} \right) > 0 \). So, (A44) is positive if \( 4Y \frac{1}{(1 + Y)^2} - \frac{4}{Z^2} \left( 1 - \frac{1}{Z} \right) \frac{1}{1 + \frac{1}{Y}} > 0 \), or equivalently, \( 4Y + (2\varphi - 1)Y^2 > \frac{4}{Z^2} \left( 1 + Y \right)^2 \). So, a sufficient condition is that \( 4\varphi^{-1}X + (2\varphi - 1)(1 - (\varphi^{-1}X)^2) > \frac{4}{Z^2} \left( 1 + \varphi^{-1}X \right)^2 \).

Since

\[
\lim_{X \to 0} \left[ 4\varphi^{-1}X + (2\varphi - 1)(1 - (\varphi^{-1}X)^2) \right] = 2\varphi - 1 > 0 = \lim_{X \to 0} \left[ 4 \frac{\varphi^{\frac{1}{2}}X}{(1 + \varphi^{\frac{1}{2}}X)^2} \left( 1 + \varphi^{-1}X \right)^2 \right],
\]

there exists some \( \hat{X} > 0 \) such that (A44) is positive for all \( X < \hat{X} \). Equivalently, since \( X = e^{-\gamma kT} \), there is some \( \hat{k} > 0 \) such that (A44) is positive for all \( \gamma kT > \hat{k} \), thereby completing the proof. \( \square \)

**Proof of Proposition 16.** Let \( \theta_h \equiv 1 \) and \( \theta_l \equiv 0 \) for simplicity. (The proof operates similarly with arbitrary \( \theta_h > \theta_l \).) With \( \lambda = 0 \), the expected price is

\[
\mathbb{E}_P(q) = \int_{\eta = 0}^{\eta = k} \frac{q}{q + (1 - q)e^{-\gamma \eta}} \frac{1}{2k} \, d\eta \\
= q \int_{\eta = 0}^{\eta = k} \left( \frac{1}{q + (1 - q)e^{-\gamma \eta}} + \frac{1}{q + (1 - q)e^{\gamma \eta}} \right) \frac{1}{2k} \, d\eta, \quad (A45)
\]

by symmetry around zero. Let \( f_1(\gamma) := e^{-\gamma \eta} + e^{\gamma \eta} \), \( f_2(w) := \frac{w - 2}{q^2 + q(1 - q)w + (1 - q^2)} \), and use the identity \( \frac{1}{q + (1 - q)e^{-\gamma \eta}} + \frac{1}{q + (1 - q)e^{\gamma \eta}} = 2 + (1 - 2q)(1 - q) \frac{e^{\gamma \eta} + e^{-\gamma \eta} - 2}{q^2 + q(1 - q)e^{\gamma \eta} + (1 - q^2)} \) to express (A45) as

\[
\mathbb{E}_P(q) = q \int_{\eta = 0}^{\eta = k} \left( 2 + (1 - q)(2q - 1)f_2(f_1(\gamma)) \right) \frac{1}{2k} \, d\eta. \quad (A46)
\]

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Since $f_1'(\gamma) = \eta(e^{\gamma\eta} - e^{-\gamma\eta}) > 0$ for $\eta > 0$, $f_2'(w) = \frac{1}{[q^2 + q(1-q)w + (1-q)\gamma^2w^2]} > 0$, and the integrand in (A46) is continuous with continuous partial derivative,

$$\frac{\partial}{\partial \gamma} \text{EP}(q) = (2q - 1)q(1-q) \int_{\eta=0}^{\eta=+\infty} f_2'(f_1(\gamma))f_1'(\gamma) \frac{1}{2k} d\eta \begin{cases} > 0 & \text{for } q \in (0, \frac{1}{2}), \\ < 0 & \text{for } q \in (\frac{1}{2}, 1). \end{cases}$$

For $\lambda = 0$, Proposition 12 implies $\text{EP}(q) = \frac{1}{2k} \frac{1}{k} \ln \frac{1-q(1-e^{\gamma\theta_h - \eta\theta_l})}{1-q(1-e^{-\gamma\theta_h - \eta\theta_l})}$. Since $k$ appears on the right-hand side everywhere $\gamma$ appears, the partial derivative of the right-hand side with respect to $\gamma$ or $k$ are the same.

REFERENCES


APPENDIX B. EXPECTED PRICES VS. EXPECTED RETURNS

In general, for two arbitrary random variables \( \tilde{x} \) and \( \tilde{y} \), the expected value of their ratio is not equal to the ratio of their expected values, \( E \left[ \frac{\tilde{x}}{\tilde{y}} \right] \neq \frac{E[\tilde{x}]}{E[\tilde{y}]} \). However, the latter is a reasonable approximation to the former under certain conditions. The Taylor’s expansion of the ratio \( \frac{x}{y} \) as a function of \( y \) around \( \mu_y \equiv E[\tilde{y}] \) is given by:

\[
\frac{x}{y} = \frac{x}{\mu_y} - \frac{x(y - \mu_y)}{\mu_y^2} + \frac{x(y - \mu_y)^2}{\mu_y^3} - \frac{x(y - \mu_y)^3}{\mu_y^4} + \cdots.
\]

Applied to expectations of random variables, the expected ratio,

\[
E \left[ \frac{\tilde{x}}{\tilde{y}} \right] = \frac{E[\tilde{x}]}{E[\tilde{y}]} + \left[ - \frac{E[\tilde{x}(\tilde{y} - \mu_y)]}{\mu_y^2} + \frac{E[\tilde{x}(\tilde{y} - \mu_y)^2]}{\mu_y^3} - \frac{E[\tilde{x}(\tilde{y} - \mu_y)^3]}{\mu_y^4} + \cdots \right],
\]

is reasonably approximated by the ratio of expectations if \( |\mu_y| \) is large and \( |\tilde{y} - \mu_y| \) is small relative to \( |\mu_y| \) with sufficiently high probability—in other words, if the CV (coefficient of variation) of \( \tilde{y} \) is small. Under this condition, the sum in the brackets is well-behaved with alternating terms partially offsetting each other and higher-order terms vanishing quickly.

This analysis is particularly relevant when \( \tilde{x} \) represents fundamental stock values (i.e., \( \tilde{\theta} \)) and \( \tilde{y} \) represents stock prices (i.e., \( \tilde{p} \)), which typically have low CVs since stock price changes tend to be small relative to the price level itself. For example, the monthly CV of the price of the S&P 500 market index (using a two-year rolling estimation window of monthly prices) is approximately one-tenth and the monthly CV of prices for IBM Corp.’s stock is about one-eighth.\(^{34}\) For small price CVs such as these, when the expected price is larger (smaller) than the expected payoff, \( E[\tilde{p}] \gtrless E[\tilde{\theta}] \), so that \( \frac{E[\tilde{\theta}]}{E[\tilde{p}]} - 1 \lesssim 0 \), then the expected return is negative (positive) within a small approximation error: \( E \left[ \frac{\tilde{\theta}}{\tilde{p}} - 1 \right] \lesssim 0 + \epsilon \). Larger gaps of \( E[\tilde{p}] \gtrsim E[\tilde{\theta}] \) correspond to larger gaps of \( E \left[ \frac{\tilde{\theta}}{\tilde{p}} - 1 \right] \gtrsim 0 \), so the indications for expected prices vs. fundamentals carry over qualitatively to expected returns.

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\(^{33}\)I assume \( \mu_y \neq 0 \).

\(^{34}\)I use a payoff CV of one-eighth in my numerical illustrations in the main text.
APPENDIX C. THE SKEWNESS EFFECT UNDER CAPM AND CO-SKEWNESS

As discussed in the main text, there is a negative relationship between the ex-ante skewness of individual securities and their subsequent average returns—the skewness effect. Moreover, this relationship is not explained by standard risk factor models such as the Capital Asset Pricing Model (CAPM) or the co-skewness model. In this appendix, I adapt the pricing implications of skewness obtained in the main text to a simple returns-based model with independent assets and show how the CAPM and co-skewness models can over-predict the average returns of a small, independent, positively skewed asset.

As in Section 4.4 and Theorem 13, the small positively skewed independent asset \( \tilde{\theta} \) (i.e., with \( q_1 \in (0, \frac{1}{2}) \)) has an expected price that exceeds its expected payoff, thereby yielding a negative expected return. In this appendix, I show that the expected return for an asset of this nature can be overshot by both the standard CAPM estimates and coskewness-based estimates (following the coskewness methodology of Kraus and Litzenberger (1976) and Harvey and Siddique (2000a)), thereby consistent with empirical evidence that has documented a negative relationship between total (or idiosyncratic) skewness and expected returns even after controlling for covariance and coskewness with the market.

C.I. Model

Specifically, let \( \tilde{r}^{(i)} \) be the return (in excess of the riskfree rate) on risky asset \( i \) and \( \delta^{(i)} \in (0, 1) \) its weight in the market portfolio, for \( i = 1, \ldots, n \). Moreover, let the combined weighted excess return on these \( n \) assets, \( \sum_{i=1}^{n} \delta^{(i)} \tilde{r}^{(i)} \) have finite positive mean and variance and, for simplicity, zero skewness. Let \( \tilde{r} \) be the (excess) return on a separate, positively skewed, asset, which is independent of all other assets with small weight \( \delta > 0 \) in the market portfolio (so that its contribution to the market expected return is negligible).

The return on this small positively skewed asset has variance \( v := \text{Var}[\tilde{r}] > 0 \) and skewness \( s := \text{Skew}[\tilde{r}] > 0 \) such that its mean is negative: \( e := \mathbb{E}[\tilde{r}] < 0 \). Accordingly, the (excess) market return on all assets is \( \tilde{r}_m := \delta \tilde{r} + \sum_{i=1}^{n} \delta^{(i)} \tilde{r}^{(i)} \), where \( \delta + \sum_{i=1}^{n} \delta^{(i)} = 1 \) and \( \delta > 0 \) is sufficiently small so that \( \mathbb{E}[\tilde{r}_m] = \delta e + \mathbb{E}(\sum_{i=1}^{n} \delta^{(i)} \tilde{r}^{(i)}) > 0 \).

\( ^{35} \)This assumption is similar to Barberis and Huang (2008) who also compare the pricing implications of their model to predictions of the CAPM (under expected utility) by considering an independent skewed asset with infinitesimal supply so that its contribution to the market return is insignificant.
C.II. CAPM

Suppose that the expected excess return on $\tilde{r}$ is related linearly to the first moment of the excess market return as in the standard CAPM:

$$E[\tilde{r}] = \beta_r E[\tilde{r}_m], \tag{C1}$$

where $\beta_r$ is to be estimated from returns data. Consider a linear regression of $\tilde{r}$ on $\tilde{r}_m$, after all variables have been de-meaned, where $\beta_r$ is the population value (or probability limit) of the estimated coefficient (beta) on the excess market return, $\tilde{r}_m$. Then, a standard result is

$$\beta_r = \frac{\text{Cov}(\tilde{r}, \tilde{r}_m)}{\text{Var}(\tilde{r}_m)}. \tag{C2}$$

Lemma 17. $\text{Cov}(\tilde{r}, \tilde{r}_m) = \delta > 0$.

Corollary 18. The CAPM model overshoots the expected return: $\beta_r E[\tilde{r}_m] > e$.

As indicated in the corollary, this simple model demonstrates that the CAPM model can overshoot the expected return of a skewed asset. Consistent with the model of the main text, the key property here is that enough positive ex-ante skewness can lead to negative returns, whereas the covariation between an independent asset’s return and the market return is necessarily non-negative.

C.III. Co-Skewness

Now, suppose that the expected excess return of $\tilde{r}$ is related linearly to both the first and second moments of the excess market return [as in Kraus and Litzenberger (1976) and Harvey and Siddique (2000a)]:

$$E[\tilde{r}] = A E[\tilde{r}_m] + B E[\tilde{r}_m^2]. \tag{C3}$$

Consider a linear regression of $\tilde{r}$ on $\tilde{r}_m$ and $\tilde{r}_m^2$, after all variables have been de-meaned, where $A$ and $B$ are the population values of the estimated coefficients (betas) on the excess market return, $\tilde{r}_m$, and the excess market return squared, $\tilde{r}_m^2$, respectively. Then,

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} \text{Var}(\tilde{r}_m) & \text{Cov}(\tilde{r}_m, \tilde{r}_m^2) \\ \text{Cov}(\tilde{r}_m, \tilde{r}_m^2) & \text{Var}(\tilde{r}_m^2) \end{bmatrix}^{-1} \begin{bmatrix} \text{Cov}(\tilde{r}, \tilde{r}_m) \\ \text{Cov}(\tilde{r}, \tilde{r}_m^2) \end{bmatrix}$$
\[
\begin{bmatrix}
\text{Var}(\tilde{r}_m)\text{Cov}(\tilde{r},\tilde{r}_m) - \text{Cov}(\tilde{r}_m,\tilde{r}^2_m)\text{Cov}(\tilde{r},\tilde{r}_m^2)
\end{bmatrix}
\begin{bmatrix}
\text{Var}(\tilde{r}_m)\text{Var}(\tilde{r}_m^2) - (\text{Cov}(\tilde{r}_m,\tilde{r}^2_m))^2
\end{bmatrix}^{-1}.
\]

(C4)

Lemma 19. 1. \(\text{Cov}(\tilde{r},\tilde{r}_m^2) = \delta v (\delta s \sqrt{\nu} + 2 \mathbb{E}[\tilde{r}_m]) > 0;\)
2. \(\text{Cov}(\tilde{r}_m,\tilde{r}_m^2) = \delta^3 v s \sqrt{\nu} + 2 \mathbb{E}[\tilde{r}_m] \text{Var}[\tilde{r}_m] > 0;\)
3. \(\lim_{\delta \searrow 0} A = \lim_{\delta \searrow 0} B = 0.\)

Corollary 20. If \(\delta > 0\) is sufficiently small, then the coskewness model overshoots the expected excess return of the small positively skewed asset: \(A \mathbb{E}[\tilde{r}_m] + B \mathbb{E}[\tilde{r}_m^2] > e.\)

This result shows that coskewness can fail to explain the expected return on such a skewed asset, and that it does so by overshooting the expected return when total skewness is positive, which is consistent with the evidence in Boyer, Mitton, and Vorkink (2010) and Conrad, Dittmar, and Ghysels (2013) that document the negative relationship between total (and/or idiosyncratic) skewness with expected returns even after controlling for coskewness along the lines of Kraus and Litzenberger (1976) and Harvey and Siddique (2000a).

C.IV. Proofs

Proof of Lemma 17. \(\text{Cov}(\tilde{r},\tilde{r}_m) = \text{Cov}(\tilde{r}, \delta \tilde{r} + \sum_{i=1}^n \delta^{(i)} \tilde{r}^{(i)}) = \delta \text{Var}(\tilde{r}) = \delta v > 0,\) where the second equality follows from independence of \(\tilde{r}\) and \(\sum_{i=1}^n \delta^{(i)} \tilde{r}^{(i)}.\)

Proof of Corollary 18. By Lemma 17, \(\text{Cov}(\tilde{r},\tilde{r}_m) > 0,\) so that \(\beta_r > 0\) (equation (C2)). Since \(\mathbb{E}[\tilde{r}_m] > 0,\) the CAPM model expected return of (C1) is positive, \(\beta_r \mathbb{E}[\tilde{r}_m] > 0,\) and hence greater than expected return, \(e,\) which is negative for the positively skewed independent asset.

Proof of Lemma 19. Statement 1: The third central moment of the independent return \(\tilde{r}\) is

\[
\mathbb{E}[(\tilde{r} - e)^3] = \mathbb{E}[v^3] - 3 \mathbb{E}[\tilde{r}^2] e + 3 \mathbb{E}[\tilde{r}] e^2 - e^3
= \mathbb{E}[\tilde{r}^3] - 3 e \mathbb{E}[\tilde{r}^2] + 2 e^3
= \mathbb{E}[\tilde{r}^3] - e \mathbb{E}[\tilde{r}^2] - 2 e (\mathbb{E}[\tilde{r}^2] - e^2)
= \mathbb{E}[\tilde{r}^3] - e \mathbb{E}[\tilde{r}^2] - 2 e v.
\]

(C5)
The covariance with the square of the market return is

\[
\text{Cov}(\bar{r}, \bar{r}_m^2) = \text{Cov}(\bar{r}, (\delta \bar{r} + (\bar{r}_m - \delta \bar{r}))^2)
\]

\[
= \text{Cov}(\bar{r}, \delta^2 \bar{r}^2) + 2 \text{Cov}(\delta \bar{r}, (\bar{r}_m - \delta \bar{r}))
\]

\[
= \delta^2(\mathbb{E}[\bar{r}^3] - \mathbb{E}[\bar{r}]\mathbb{E}[\bar{r}^2]) + 2 \delta(\mathbb{E}[\bar{r}^2(\bar{r}_m - \delta \bar{r})] - \mathbb{E}[\bar{r}]\mathbb{E}[\bar{r}(\bar{r}_m - \delta \bar{r})])
\]

\[
= \delta^2(\mathbb{E}[\bar{r}^3] - e\mathbb{E}[\bar{r}^2]) + 2 \delta \mathbb{E}[\bar{r}]\mathbb{E}[\bar{r}_m - \delta e]
\]

\[
= \delta^2(\mathbb{E}[\bar{r}^3] - e\mathbb{E}[\bar{r}^2]) + 2 \delta \mathbb{E}[\bar{r}_m - \delta e] + 2 \delta \mathbb{E}[\bar{r}_m - \delta e]
\]

\[
= \delta \mathbb{E}[\bar{r}^3] - e\mathbb{E}[\bar{r}^2] + 2 \delta \mathbb{E}[\bar{r}_m] + 2 \mathbb{E}[\bar{r}_m - \delta e]
\]

where the second and fourth inequalities follow from independence of \(\bar{r}\) and \(\bar{r}_m - \delta \bar{r}\); the sixth inequality follows from (C5); the seventh inequality follows from the definition of skewness: \(s = \frac{\mathbb{E}[(\bar{r} - e)^3]}{\sqrt[3]{\mathbb{E}[(\bar{r} - e)^2]}}\); and the final inequality follows because the market excess return is positive, \(\mathbb{E}[\bar{r}_m] > 0\), and the excess return, \(\bar{r}\), is positively skewed, \(s > 0\).

Statement 2: The third central moment of the excess market return can be expressed in two equivalent ways:

\[
\mathbb{E}[(\bar{r}_m - \mathbb{E}[\bar{r}_m])^3] = \mathbb{E}[\bar{r}_m^3] - 3 \mathbb{E}[\bar{r}_m^2] \mathbb{E}[\bar{r}_m] + 3 \mathbb{E}[\bar{r}_m] (\mathbb{E}[\bar{r}_m])^2 - (\mathbb{E}[\bar{r}_m])^3
\]

\[
= \mathbb{E}[\bar{r}_m^3] - \mathbb{E}[\bar{r}_m^2] \mathbb{E}[\bar{r}_m] - 2 \mathbb{E}[\bar{r}_m] \text{Var}(\bar{r}_m)
\]

\[
= \text{Cov}(\bar{r}_m, \bar{r}_m^2) - 2 \mathbb{E}[\bar{r}_m] \text{Var}(\bar{r}_m),
\]

(C6)

and

\[
\mathbb{E}[(\bar{r}_m - \mathbb{E}[\bar{r}_m])^3] = \mathbb{E}[(\delta \bar{r} - \delta e) + (\bar{r}_m - \delta \bar{r} - \mathbb{E}[\bar{r}_m - \delta \bar{r}])^3]
\]

\[
= \delta^3 \mathbb{E}[(\bar{r} - e)^3] + (\mathbb{E}[(\bar{r}_m - \delta \bar{r} - \mathbb{E}[\bar{r}_m - \delta \bar{r}])])^3
\]

\[
= \delta^3 \mathbb{E}[\bar{r}^3] - \delta \mathbb{E}[\bar{r}^2] - \delta \mathbb{E}[\bar{r}_m] + \mathbb{E}[\bar{r}_m] - \delta \mathbb{E}[\bar{r}_m - \delta \bar{r}]
\]

\[
= \delta^3 \mathbb{E}[\bar{r}^3] - \delta \mathbb{E}[\bar{r}^2] + \mathbb{E}[\bar{r}_m] - \delta \mathbb{E}[\bar{r}_m - \delta \bar{r}]
\]

\[
= \delta^3 \mathbb{E}[\bar{r}^3] - \delta \mathbb{E}[\bar{r}^2] + \mathbb{E}[\bar{r}_m] - \delta \mathbb{E}[\bar{r}_m - \delta \bar{r}]
\]

(C7)

where the second equality in the second expression follows from the independence of \(\bar{r}\) and \(\bar{r}_m - \delta \bar{r}\).
Together, (C6) and (C7) yield: \( \text{Cov}(\bar{r}_m, \bar{r}_m^2) = \delta^3 v \sqrt{v} s + 2E[\bar{r}_m] \text{Var}(\bar{r}_m) > 0. \)

Statement 3: The denominators of \( A \) and \( B \) are positive and finite for all \( \delta \geq 0 \) since the covariance matrix for \( \bar{r}_m \) and \( \bar{r}_m^2 \) is positive definite so that its determinant, in both denominators in (C4), is positive. Note that the expected value of the market return, its variance, and the variance of its square are finite and positive for all \( \delta \geq 0: E[\bar{r}_m] > 0, \text{Var}(\bar{r}_m) > 0 \) and \( \text{Var}(\bar{r}_m^2) > 0 \). Thus, taking limits of the covariance expressions in Lemma 17 and in statements 1 and 2, respectively, yields:

\[
\lim_{\delta \searrow 0} \text{Cov}(\bar{r}, \bar{r}_m) = \lim_{\delta \searrow 0} \delta v = 0,
\]

\[
\lim_{\delta \searrow 0} \text{Cov}(\bar{r}, \bar{r}_m^2) = \lim_{\delta \searrow 0} \delta v (\delta s \sqrt{v} + 2E[\bar{r}_m]) = 0,
\]

\[
\lim_{\delta \searrow 0} \text{Cov}(\bar{r}_m, \bar{r}_m^2) = \lim_{\delta \searrow 0} (\delta^3 v \sqrt{v} s + 2E[\bar{r}_m] \text{Var}(\bar{r}_m)) = \lim_{\delta \searrow 0} 2E[\bar{r}_m] \text{Var}(\bar{r}_m) > 0,
\]

where the last limit is finite. Thus,

\[
\lim A = \lim_{\delta \searrow 0} \frac{\text{Var}(\bar{r}_m) \text{Cov}(\bar{r}, \bar{r}_m) - \text{Cov}(\bar{r}_m, \bar{r}_m^2) \text{Cov}(\bar{r}, \bar{r}_m^2)}{\text{Var}(\bar{r}_m) \text{Var}(\bar{r}_m^2) - (\text{Cov}(\bar{r}_m, \bar{r}_m^2))^2} = 0,
\]

\[
\lim B = \lim_{\delta \searrow 0} \frac{\text{Var}(\bar{r}_m) \text{Cov}(\bar{r}, \bar{r}_m^2) - \text{Cov}(\bar{r}_m, \bar{r}_m^2) \text{Cov}(\bar{r}, \bar{r}_m)}{\text{Var}(\bar{r}_m) \text{Var}(\bar{r}_m^2) - (\text{Cov}(\bar{r}_m, \bar{r}_m^2))^2} = 0.
\]

\( \square \)

**Proof of Corollary 20.** Since \( E[\bar{r}_m] \) and \( E[\bar{r}_m^2] \) are positive and finite for all \( \delta \geq 0 \), by Lemma 19, \( \lim_{\delta \searrow 0} A E[\bar{r}_m] + B E[\bar{r}_m^2] = 0. \) But the expected return on the small independent skewed asset is negative: \( e < 0 \). Thus, there exists a \( \delta > 0 \) small enough such that \( A E[\bar{r}_m] + B E[\bar{r}_m^2] \) is close enough to zero to exceed \( e \).

\( \square \)

**APPENDIX D. OPPOSITE SIDES OF A SKEWED BET: GENERALIZATION TO BOUNDED UNIMODAL DISTRIBUTIONS**

Theorem 5 applies to a larger class of payoff distributions for \( \bar{\theta} \) than just the binary case. The key condition in the proof of Theorem 5 was that higher odd central moments of the payoff distribution of \( \bar{\theta} \) have the same sign. If this condition is met, and if the Taylor expansions converge, then the proof still carries through. Specifically, if the distribution of \( \bar{\theta} \) has bounded support with finite mean \( \mu \), i.e., \( |\bar{\theta} - \mu| \leq a \) for some \( a < \infty \), if all of its higher odd central moments have the same sign, and
if the initial wealth of the agent $w_0$ is sufficiently large, $w_0 \geq a$, so that the utility over final wealth $(w_0 - B + \tilde{\theta}$ or $w_0 + S - \tilde{\theta})$ is well-defined for all outcomes, then the result generalizes: the sign of $(S - \mu) - (\mu - B)$ is determined by the sign of the skewness of $\tilde{\theta}$.

Bélisle (1991) provided a simple intuitive sufficient condition for the payoff distribution to have the key property that its higher odd central moments all have the same sign. Let the random variable $\tilde{\theta}$ have a unimodal probability density $f$ with finite mean $\mu$. If $f$ is symmetric, then all odd central moments are zero. Suppose $f$ is not symmetric. Start at the modal value $x$, which corresponds to the maximum density level, denoted by $y_{\max} := \sup_{x \in \mathbb{R}} f(x)$. Track the midpoint $x(y)$ of the horizontal cross-section of the density function as the height $y$ of the density decreases from its maximum $y_{\max}$ (i.e., $x(y) := \frac{1}{2} \left[ \inf_{x \in \mathbb{R}} f(x) < y \right] + \sup_{x \in \mathbb{R}} f(x) > y$]). As long as this cross-sectional midpoint $x(y)$ moves to the right as the density value $y$ decreases, then all of the higher central moments will be positive. Specifically, if $f$ is unimodal, non-symmetric, and $\inf \{ y \in (0, y_{\max}) : h(y) < \mu \} \geq \sup \{ y \in (0, y_{\max}) : h(y) > \mu \}$, then $E[(\tilde{\theta} - \mu)^n] > 0$ for all odd $n > 1$.

APPENDIX E. AUXILIARY MODEL

In this Appendix, I develop and analyze an auxiliary model as a robustness check that the key ingredients of the main model of Section 4—market clearing of stochastic demand for an asset with a skewed payoff distribution—are indeed responsible for generating the skewness effect and that it is not the main model’s use of a binary payoff structure, discrete private information signals, or stochastic demand distribution driving the result. The central differences between this auxiliary model and the main model are (1) the auxiliary model’s risky asset has a flexible continuous payoff distribution structure whereas the main model has a three-parameter binary payoff distribution; (2) the auxiliary model’s private signals follow a continuous jointly normal structure, as is more common in the literature for noisy rational expectation models, whereas the main model has a discrete signal structure; and (3) the auxiliary model employs position limits with risk-neutral investors as its limit to arbitrage whereas the main model employs risk aversion as its limit to arbitrage.

Proofs for results in this appendix are given in Section E.III.

E.I. Auxiliary model and analysis

There are three dates, $t \in \{0, 1, 2\}$. 
**Assets:** There is a risk-free asset, with both its price and payoff normalized to one and a risky asset with price $\tilde{p}$ and payoff $\tilde{\theta}$. The payoff $\tilde{\theta}$ is distributed as a payoff function, $R(\tilde{z})$, of an underlying unobservable normally-distributed random variable, $\tilde{z}$, which has mean $\mu$ and precision $\tau$ (i.e., variance $\frac{1}{\tau}$):

$$\tilde{\theta} = R(\tilde{z}), \quad \tilde{z} \sim \mathcal{N}\left(\mu, \frac{1}{\tau}\right),$$

where

$$R(z) = \begin{cases} a + \frac{b}{c} (e^{cz} - 1), & c \neq 0, \\ a + bz, & c = 0, \end{cases}$$

(E1)

for some parameters $a, b > 0$, and $c$. Note that $R(z)$ is continuous in $c$ since $\lim_{c \to 0} a + \frac{b}{c} (e^{cz} - 1) = a + bz$. Note that the payoff distribution spans the collection of scaled and shifted lognormal random variables, and includes the normal distribution as a special case.

**Investors:** There is a unit mass of informed investors with risk-neutral preferences and position limits of $\pm 1$ share in the risky asset. At date $t = 1$, each investor $j \in [0, 1]$ receives a noisy private signal, $\tilde{s}_j = \tilde{z} + \tilde{\epsilon}_j$, about the uncertainty underlying the payoff, where $\tilde{\epsilon}_j \sim \mathcal{N}\left(0, \frac{1}{\tau}\right)$ is independent of $\tilde{z}$ and independent and identically distributed across investors. Investors condition their trades on their private signal and the market price and investor $j$ chooses demand $x_j \in [-1, 1]$ to maximize conditional expected net payoff: $x_j \in \arg\max_{x \in [-1, 1]} E[x(\tilde{\theta} - \tilde{p})|\tilde{s}_j, \tilde{p}]$.

**Stochastic demand:** Stochastic demand, $\tilde{\eta}$, is independent of $\tilde{z}$ and of all private signals, and is distributed according to a transformation of a mean-zero normally distributed random variable with variance $k > 0$ from the real line into the interval $[-1, 1]$. This transformation, which I discuss in more detail after Proposition 21, gives $\tilde{\eta}$ a cumulative distribution function $G(\eta) := \Phi\left(\frac{1}{\sqrt{k}} \Phi^{-1}\left(\frac{1+\eta}{2}\right)\right)$ on the interval $[-1, 1]$, where $\Phi(\cdot)$ is the cumulative distribution function of a standard normal random variable. Figure 8 illustrates the shape of the probability density of $\tilde{\eta}$, $g(\eta)$, for various values of $k$. When $k = 1$, then $\tilde{\eta}$ simplifies to a uniform random variable on $[-1, 1]$. Smaller values for $k$ shift more weight into the center of the interval, whereas larger values for $k$ shift more weight into the tails of the interval.

The market price is set to clear the market of this stochastic demand: $X := \int_{-1}^{1} x_j d\tilde{\eta} = -\tilde{\eta}$.  

---

For expositional convenience, I set position limits at $\pm 1$ share, and take the asset to be in zero net supply. My results can be readily extended to general position limits $[\underline{x}, \overline{x}]$ as long as the positions are not unlimited ($-\infty < \underline{x} < \overline{x} < \infty$).
This figure illustrates the shape of the distribution of stochastic demand, $g(\eta) \equiv \phi \left( \frac{1}{\sqrt{k}} \Phi^{-1} \left( \frac{1+\eta}{2} \right) \right) / \left[ 2\sqrt{k} \phi \left( \Phi^{-1} \left( \frac{1+\eta}{2} \right) \right) \right]$, $k > 0, \eta \in [-1, 1]$, for various levels of $k$.

**Proposition 21.** The equilibrium price of the asset $\tilde{\theta} \equiv R(\tilde{z})$ at date $t = 1$ is given by

$$\tilde{p} = a + \frac{b}{c} \left( \exp \left[ c \frac{\tau \mu + (1 + \frac{1}{k}) \tau \epsilon (\tilde{z} + \tilde{\delta})}{\tau + (1 + \frac{1}{k}) \tau \epsilon} + \frac{1}{2} \frac{c^2}{\tau + (1 + \frac{1}{k}) \tau \epsilon} - 1 \right],$$

where

$$\tilde{\delta} := \frac{1}{\sqrt{\tau \epsilon}} \Phi^{-1} \left( \frac{1 + \tilde{\eta}}{2} \right)$$

(E2)

and $\Phi(\cdot)$ is the standard normal cumulative distribution function. Moreover, $\tilde{\delta} \sim \mathcal{N} \left( 0, \frac{k}{\tau \epsilon} \right)$.

Rearranging the definition of $\tilde{\delta}$ in (E2) gives $\tilde{\eta} = 2\Phi \left( \sqrt{\tau \epsilon} \tilde{\delta} \right) - 1$, which is a transformation of a mean-zero, variance-$k$, normally-distributed random variable, $\sqrt{\tau \epsilon} \tilde{\delta}$, into the interval $[-1, 1]$, since $\tilde{\delta} \sim \mathcal{N} \left( 0, \frac{k}{\tau \epsilon} \right)$ and $2\Phi(\cdot) - 1 \in [-1, 1]$.

The expected price at date $t = 0$ follows directly from Proposition 21 by properties of the lognormal distribution:

**Proposition 22.** The expected price at date $t = 0$ is

$$EP := \mathbb{E}[\tilde{p}] = a + \frac{b}{c} \left( \exp \left[ c \mu + \frac{1}{2} \frac{c^2}{\tau + (1 + \frac{1}{k}) \tau \epsilon} \left( 1 + \frac{(1 + \frac{1}{k})^2 \tau \epsilon}{\tau + (1 + \frac{1}{k}) \tau \epsilon} \right) - 1 \right].$$

(E3)
The expected return at date \( t = 0 \) can be calculated by numerical integration of the following expression with respect to \( \tilde{z} \) and \( \tilde{\delta} \), which are independent normal random variables, since \( \tilde{y} = \tilde{z} + \tilde{\delta} \):

\[
ERet := E \left[ \frac{\tilde{\theta}}{\tilde{p}} - 1 \right] = E \left[ \frac{a + \frac{b}{c} \left( e^{\tilde{z}} - 1 \right)}{\tilde{p}} - 1 \right],
\]

(E4)

where

\[
\tilde{p} = a + \frac{b}{c} \left( \exp \left[ c \frac{\tau \mu + \left( 1 + \frac{1}{k} \right) \tau \epsilon (\tilde{z} + \tilde{\delta})}{\tau + \left( 1 + \frac{1}{k} \right) \tau \epsilon} + \frac{1}{2} c^2 \frac{1}{\tau + \left( 1 + \frac{1}{k} \right) \tau \epsilon} - 1 \right) \right),
\]

as in Proposition 21.

In the following, I show that the structure of \( R(z) \) permits independent specification of the mean, variance, and skewness of the payoff distribution. As in the main model, this property permits analysis of the expected price and expected return as skewness varies while holding the mean and variance fixed.

**Proposition 23.** Given values \( E, V, \) and \( S \) for the mean, variance, and skewness of the payoff, respectively, the parameters \( a, b, \) and \( c \) are uniquely determined as follows:

\[
a = \begin{cases} 
E - \frac{b}{c} \left( e^{c \mu} e^{\frac{2}{\pi} c^2} - 1 \right), & c \neq 0, \\
E - b \mu, & c = 0;
\end{cases}
\]

(E5)

\[
b = \begin{cases} 
\frac{|c|}{e^{c \mu} e^{\frac{2}{\pi} c^2}} \sqrt{\frac{V}{e^{c^2}} - 1}, & c \neq 0, \\
\sqrt{\tau V}, & c = 0;
\end{cases}
\]

(E6)

\[
c = \text{Sign}(S) \sqrt{\tau \ln \left( \frac{A + 1}{A} - 1 \right)}, \quad \text{where} \quad A := \left( 1 + \frac{S^2 + |S| \sqrt{4 + S^2}}{2} \right)^{\frac{1}{2}}.
\]

(E7)

Therefore, the payoff function (E1), which extends continuously to \( c = 0 \),

\[
R(z) := \begin{cases} 
\frac{a + \frac{b}{c} (e^{c z} - 1)}{\tilde{p}}, & c \neq 0, \\
\frac{a + b z}{\tilde{p}}, & c = 0,
\end{cases}
\]

(E1)

generates a payoff \( \tilde{\theta} = R(\tilde{z}) \) with mean \( E \), variance \( V \), and skewness \( S \) when \( a, b, \) and \( c \) are considered as functions of \( E, V, \) and \( S \) as given in (E5), (E6), and (E7), respectively.
E.II. Discussion

In Figure 9, applying Proposition 23 to (E3) and (E4), I plot the expected price and expected return, respectively, versus skewness while holding the mean and variance fixed. Figure 9 shows that the same phenomenon arising from the main model also arises here. When there is no skewness, the expected price of the risky asset precisely equals the expected payoff of the risky asset. As skewness increases above zero, the expected price increasingly exceeds the expected payoff. Likewise, as skewness decreases below zero, the expected price increasingly falls short of the expected payoff. These effects carry over to expected returns such that, holding all else equal, the expected return decreases in the skewness of the payoff—the skewness effect.

Limits to arbitrage constrain any individual investor’s trades. While the main model limits the risk any one individual bears, this auxiliary model uses position limits as a limit to arbitrage. Price deviations in the auxiliary model push many traders to the same position limit (some to +1, the rest to −1) in order to generate enough overall trading differences between the two groups to offset noise demand. When the payoff distribution is asymmetric, then the collection of investors’ posterior expectations are also asymmetric across the investor population. Positive skewness entails that the mass of investors with optimistic outcome beliefs are more thinly spread, requiring large positive price deviations to generate enough selling to offset noise buying; i.e., a large group trading −1 compared to the group trading +1. In either the main model or here, price moves asymmetrically to generate enough selling (buying) mass to offset positive (negative) noise demand. So, average prices deviate from fundamental value in the direction of payoff asymmetry.

As with the main model, heterogeneity of investor beliefs is parameterized. A decrease in the precision of private signals (i.e., a decrease in $\tau_\varepsilon$) increases the variance of such signals, so that investors beliefs are more widely spread out. Hence, there is additional heterogeneity in their beliefs. Figure 10 illustrates that the skewness effect can drop in magnitude with such an increase in investor heterogeneity. These results are consistent with those of the main model. In sum, it is possible, even in the absence of market frictions such as short-selling constraints, for the skewness effect to weaken when there is more disagreement among investors.

Although the auxiliary model employs a different mechanism to limit arbitrage than in the main model, so that there is no risk aversion per se, it is still possible to analyze the impact of uncertainty
on average pricing. Figure 11 shows that holding all else equal, an asset with high payoff variance will show a strengthening of the skewness effect. However, this result cannot be tied to either risk or risk aversion, unlike in the main model. The main model is able to provide additional insight that an increase to the limits of arbitrage, in the form of more uncertainty or more aversion to that uncertainty, strengthens the skewness effect. Moreover, the key insights of the paper are more directly exhibited by the main model and its motivating example. Therefore, this auxiliary model best serves as a robustness test for the main model.

Finally, the flexible structure of the payoff distribution in this auxiliary model includes the normal distribution, a mainstay of classic models. The implication of these results is that the assumption of a normally distributed payoff, which lacks skewness, masks the average pricing effects of clearing the market of stochastic demand. Under the normal distribution, the pricing impacts of stochastic demand perfectly offset due to the symmetry of that distribution. However, even the smallest departure from normality (e.g., moving the parameter $c$ away from zero) upsets that balance, resulting in the skewness effect.

E.III. Proofs

Proof of Proposition 21. I focus on the decision of a single investor $j$ and drop the subscript $j$. I use the conjecture and verify method in parts of this proof. First, suppose the natural monotonicity condition that the conditional expected asset payoff valuation, $\hat{V}(s, p) := E[\hat{\theta}|s, \hat{p} = p]$ is increasing in the private signal realization $s$ for all realized equilibrium prices $p$. This conjecture says that at any equilibrium price $p$, if an informed investor’s private signal $s$ increases, the investor’s prediction of the asset’s payoff would increase.

Each investor compares the prediction of the asset’s payoffs, $\hat{V}(s, p)$, to the price, $p$, to form a trading decision, and under the monotonicity described above, this comparison is equivalently characterized by a comparison between the private signal, $s$, and some private signal threshold level, $s^*(p)$, which depends on the price. The intuition for this is as follows. Note that because of risk neutrality, an informed investor optimally either invests up to the upper position limit, $x = 1$, or sells to the lower position limit, $x = -1$, or is indifferent; hence, an investor with private signal $s$ chooses demand at price $p$ as follows\(^{37}\): $x(s, p) = -1$ if $E[R(\theta)|s, p] > p$; $x(s, p) = 0$ if $E[R(\theta)|s, p] = p$; $x(s, p) = 1$ if $E[R(\theta)|s, p] < p$.

\(^{37}\)I assume the investor does not trade when indifferent.
FIGURE 9
Skewness Effect Holding Mean and Variance Fixed

(a) Expected Price vs. Skewness

(b) Expected Return vs. Skewness

Plots (a) and (b) show the expected price and expected return, respectively, of the risky asset \( \tilde{\theta} \) vs. skewness \( S \) holding the payoff mean \( E \) and variance \( V \) fixed, where \( E = 20, V = 3, \mu = 1, \tau = 1, \tau_\varepsilon = 1, \) and \( k = 1. \)

and \( x(s, p) = 1 \) if \( E[R(\theta)|s, p] < p. \)

Note that since private signals are distributed across the entire real line, for any given equilibrium price \( p, \) there is always some level of an individual’s private signal that is high enough such
Plots (a) and (b) show the expected price and expected return, respectively, of the risky asset $\tilde{\theta}$ vs. skewness $S$ holding the payoff mean $E$ and variance $V$ fixed, where $E = 20$, $V = 3$, $\mu = 1$, $\tau = 2$, $k = 1$, and $\tau_\epsilon = 1$ (solid lines) or $\tau_\epsilon = \frac{1}{4}$ (dashed lines).

that the individual values the asset above the price. Similarly, there is always some level of the private signal low enough such that an individual values the asset below the equilibrium price. By continuity, it holds that there is some level of the private signal, $s^*(p)$, such that an individual would value the asset precisely at the market price. Clearly this private signal threshold, $s^*(p)$, depends on
FIGURE 11

Uncertainty Impact on the Skewness Effect

(a) Expected Price vs. Skewness

(b) Expected Return vs. Skewness

Plots (a) and (b) show the expected price and expected return, respectively, of the risky asset \( \hat{\theta} \) vs. skewness \( S \) holding the payoff mean \( E \) and variance \( V \) fixed, where \( E = 20, \ V = 1 \) (solid lines) or \( V = 2 \) (dashed lines), \( \mu = 1, \ \tau = 2, \ \tau_\epsilon = 1, \) and \( k = 1. \)

Therefore, I obtain the following equivalent characterization of an investor’s equilibrium demand
\[ x(s, p) = \begin{cases} 
1, & s > s^*(p), \\
0, & s = s^*(p), \\
-1, & s < s^*(p). 
\end{cases} \]

I will focus on equilibria of this form and later verify that the monotonicity conjecture indeed holds.

Next, I show that the market-clearing condition generates a market price signal of the form
\[ \tilde{y} = \tilde{z} + \tilde{\delta}, \]
where \( \tilde{\delta} \) is a transformation of the stochastic demand: \( \tilde{\delta} := \frac{\tau}{\sqrt{2\pi}} \Phi^{-1}\left(\frac{1+\tilde{\eta}}{2}\right) \), where \( \Phi(\cdot) \) is the standard normal cumulative distribution function. For each realization of underlying state, \( z \), and equilibrium price, \( p \), the mass of informed investors that sell is equal to the measure of individuals whose signals are below the signal threshold level \( s^*(p) \). Because the continuum of informed investors is measure one, the fraction who sell is captured by the following conditional probability:\(^{38}\)

\[ \mathbb{P}[\tilde{s} < s^*(\tilde{p})|\tilde{z} = z, \tilde{p} = p] = \Phi\left(\sqrt{\frac{1+\tilde{\eta}}{2}}s^*(p) - z\right). \] (E8)

The first equality follows from the structure of the private signal: \( \tilde{s} = \tilde{z} + \tilde{\epsilon} \). The second equality follows from the fact that \( s^*(\tilde{p}) \) and \( \tilde{z} \) are measurable with respect to the conditioning arguments and hence treated as constants, while the the idiosyncratic noise, \( \epsilon \), is independent of the conditioning arguments, and is mean-zero normally distributed with standard deviation \( \frac{1}{\sqrt{2\pi}} \).

The complement of the fraction given in (E8) is the measure of individuals who buy: \( 1 - \Phi(\sqrt{\frac{1+\tilde{\eta}}{2}}s^*(p) - z) \)). So, adding one share for the fraction of individuals who buy and subtracting one share for the fraction who sell gives the net informed demand:

\[
X(z, p) = (+1)[1 - \Phi(\sqrt{\frac{1+\tilde{\eta}}{2}}s^*(p) - z)] + (-1)\Phi(\sqrt{\frac{1+\tilde{\eta}}{2}}s^*(p) - z)]
= 1 - 2\Phi(\sqrt{\frac{1+\tilde{\eta}}{2}}s^*(p) - z).
\] (E9)

I use (E9) together with the market-clearing condition, \( X(z, p) = -\eta \), to solve for the private signal threshold \( s^*(p) \), which gives: \( s^*(p) = z + \frac{1}{\sqrt{2\pi}}\Phi\left(\frac{1+\tilde{\eta}}{2}\right) \). So I have the following characterization of the

---

\(^{38}\)It can be shown by standard arguments that since the continuum of investors is Lebesgue measure one the Lebesgue measure of investors who sell, \( \mathcal{L}\{j \in [0,1]: x(s, p) = -1\} \), is given by the displayed probability.
equilibrium price correspondence:

\[ p(z, \eta) \in \left\{ p \in \mathbb{R} : s^*(p) = z + \frac{1}{\sqrt{\tau_z}} \Phi^{-1}\left(\frac{1+\eta}{2}\right) \right\}. \]

Let \( \tilde{\delta} := \frac{1}{\sqrt{\tau_z}} \Phi^{-1}\left(\frac{1+\eta}{2}\right) \), which is independent of \( \tilde{z} \) since \( \tilde{\eta} \) is independent of \( \tilde{z} \). Thus, liquidity demand \( \tilde{\eta} \) is translated into additive noise in the market price signal, \( \tilde{y} := \tilde{z} + \tilde{\delta} \). Note that \( \tilde{\delta} \sim \mathcal{N}\left(0, \frac{k}{\tau_z}\right) \) since \( \tilde{\delta} = \sqrt{k} \frac{1}{\sqrt{k}} \Phi^{-1}\left(\frac{1+\eta}{2}\right) \) and \( \frac{1}{\sqrt{k}} \Phi^{-1}\left(\frac{1+\eta}{2}\right) \) is standard normally distributed: \( P\left[\frac{1}{\sqrt{k}} \Phi^{-1}\left(\frac{1+\eta}{2}\right) \leq w\right] = P\left[\eta \leq 2\Phi\left(\sqrt{k}w\right) - 1\right] = G\left(2\Phi\left(\sqrt{k}w\right) - 1\right) = \Phi(w) \). Note also that \( \tilde{y} = \tilde{z} + \tilde{\delta} \) is normally distributed: \( \tilde{y} \sim \mathcal{N}\left(\mu, \frac{1}{\tau_z} + \frac{1}{\tau_z}\right) \).

Next, I make another basic conjecture that the equilibrium price \( p \) is equal to a function \( P(\cdot)(y) \) of the observable market price signal \( y \); i.e., \( p = P(y) \). In such an equilibrium, \( s^*(P(y)) = y \) for every market signal level \( y \). This conjecture implies that \( P \) is invertible\(^{39}\) with \( s^* = P^{-1} \), and hence the market price signal, \( y \), is a sufficient statistic for the equilibrium price; i.e., observing the equilibrium price, \( p \), is equivalent to observing the market signal, \( y \). I will verify this second conjecture later.

An investor who observes private signal \( s = z + \varepsilon \) and the market signal \( y = z + \delta \) values the asset at:

\[ V(s, y) := E[\tilde{\theta} | \tilde{s} = s, \tilde{y} = y], \]

where the first conditioning argument represents the value of the insider’s private signal realization, \( z + \varepsilon = s \), the second conditioning argument represents the value of the market signal realization, \( z + \delta = y \), and recall that \( \tilde{\theta} = R(\tilde{z}) \). Thus, each informed investor sums the payoff, \( \theta = R(z) \), over the possible states of the world, \( z \), using the posterior distribution over these states, denoted by \( z|s, y \).

By straightforward Bayesian updating, this posterior distribution is normally distributed as follows:

\[ \tilde{z}|s, y \sim \mathcal{N}\left(\frac{\tau \mu + \tau \varepsilon + \frac{1}{k} \tau \varepsilon y}{\tau + (1 + \frac{1}{k}) \tau \varepsilon}, \frac{1}{\tau + (1 + \frac{1}{k}) \tau \varepsilon}\right). \]

so that

\[ V(s, y) = a + \frac{b}{c} \left( e^{d \frac{\tau \mu + \tau \varepsilon + \frac{1}{k} \tau \varepsilon y}{\tau + (1 + \frac{1}{k}) \tau \varepsilon}} - 1 \right). \]

\(^{39}\)To see this note that \( P(y) \neq P(y') \) implies that \( s^*(P(y)) = y \neq y' = s^*(P(y')) \) because \( P(y) \) is a function and hence cannot assume two different values for the same argument.
Since $V(s,y)$ is strictly increasing in $s$ for any price $p$ which uniquely conveys information $y$, this result verifies the first conjecture.

Some investors receive high private signals (relative to $s^*(p)$) and take a long position in the risky asset. Some investors receive low private signals (relative to $s^*(p)$) and take the opposite position in the risky asset. There is one marginal investor who receives a private signal just at the threshold: $s = s^*(p)$. This investor values the asset at precisely the equilibrium price and is indifferent between buying, selling, or not trading. If an investor receives the cutoff signal level $s = s^*(p)$ then $s = s^*(p) = s^*(P(y)) = y$ since $p = P(y)$ and $s^*(p) = P^{-1}(p)$. This marginal investor values the asset at $V(y,y)$. So, we have the following characterization of the level of the equilibrium price function for any realization of the economy:

$$P(y) = V(y,y).$$

Thus, the equilibrium price equal a sum of the payoff, $\theta = R(z)$, over the possible states of the world, $z$, using the posterior distribution over these states, denoted by $z|\{y,y\}$. By straightforward Bayesian updating, this posterior distribution is normally distributed as follows:

$$\tilde{z}|\{y,y\} \sim \mathcal{N}\left(\frac{\tau \mu + \left(1 + \frac{1}{\kappa}\right) \tau \epsilon y}{\tau + \left(1 + \frac{1}{\kappa}\right) \tau \epsilon}, \frac{1}{\tau + \left(1 + \frac{1}{\kappa}\right) \tau \epsilon}\right).$$

Let $\tilde{z}'$ denote the conditional random variable $\tilde{z}|(\tilde{y},\tilde{y})$. Then, the equilibrium market price for the risky asset with payoff $\tilde{\theta} = R(\tilde{z})$ is distributed as:

$$\tilde{p} = R(\tilde{z}'), \quad \tilde{z}' \sim \mathcal{N}\left(\frac{\tau \mu + \left(1 + \frac{1}{\kappa}\right) \tau \epsilon \tilde{y}}{\tau + \left(1 + \frac{1}{\kappa}\right) \tau \epsilon}, \frac{1}{\tau + \left(1 + \frac{1}{\kappa}\right) \tau \epsilon}\right), \quad \tilde{y} = \tilde{z} + \tilde{\delta}.$$

Hence,

$$\tilde{p} = a + b \left(\exp\left[ c \frac{\tau \mu + \left(1 + \frac{1}{\kappa}\right) \tau \epsilon \tilde{y}}{\tau + \left(1 + \frac{1}{\kappa}\right) \tau \epsilon} + \frac{1}{2} c^2 \frac{1}{\tau + \left(1 + \frac{1}{\kappa}\right) \tau \epsilon}\right] - 1\right),$$

which is a function of $\tilde{y}$. By inspection, this result verifies the second conjecture, and completes the proof. \[\square\]
Proof of Proposition 23. The mean of $\tilde{\theta}$ is

$$E = a + \frac{b}{c} \left( e^{c \mu + \frac{c^2}{2}} - 1 \right), \quad c \neq 0,$$

(E10)

with $\lim_{c \to 0} E = a + b\mu$. The variance of $\tilde{\theta}$ is

$$V = \left( \frac{b}{c} \right)^2 \left( e^{\frac{c^2}{2}} - 1 \right) e^{2c\mu + \frac{c^2}{2}}, \quad c \neq 0,$$

(E11)

with $\lim_{c \to 0} V = \frac{b^2}{\tau}$. The skewness of $\tilde{\theta}$ is

$$S = \text{Sign}(c) \cdot \left( e^{\frac{c^2}{2}} + 2 \right) \sqrt{e^{\frac{c^2}{2}} - 1}.$$

(E12)

Given values $E$, $V$, and $S$ for the mean, variance, and skewness of the payoff, respectively, the parameters $a$, $b$, and $c$ are uniquely determined. First, it is straightforward that $S$ is one-to-one increasing with $c$ since $\left( e^{\frac{c^2}{2}} + 2 \right) \sqrt{e^{\frac{c^2}{2}} - 1}$ in (E12) is strictly increasing in $c$ for $c > 0$, $S = 0$ at $c = 0$, and $\text{Sign}(c)$ flips the sign of $S$ for $c < 0$. In fact, $S$ is a strictly increasing odd function of $c$: $S(-c) = -S(c)$ for all real $c$. Hence, $S(c)$ is invertible, with inverse given by:

$$c = \text{Sign}(S) \cdot \sqrt{\tau \ln \left( A + \frac{1}{A} - 1 \right)}, \quad \text{where} \quad A := \left( 1 + \frac{S^2 + |S| \sqrt{4 + S^2}}{2} \right)^{\frac{1}{2}}.$$

Second, given $c \neq 0$, it is straightforward that $V$ is one-to-one increasing with $b$ for $b > 0$ since $\left( e^{\frac{c^2}{2}} - 1 \right) e^{2c\mu + \frac{c^2}{2}}$ in (E11) is non-negative. In the limiting case as $c \to 0$, $b = \sqrt{\tau V}$. Because $c$ is uniquely determined by $S$, then $b$ is uniquely determined by $V$ and $S$ according to:

$$b = \begin{cases} \frac{|c|}{e^{c \mu + \frac{c^2}{2}}} \sqrt{\frac{V}{e^{\frac{c^2}{2}} - 1}}, & c \neq 0, \\ \sqrt{\tau V}, & c = 0, \end{cases}$$

where $c$ is the function of $S$ in (E7).

Finally, given $E$, $V$, and $S$, since $b$ is uniquely determined by $V$ and $S$ and $c$ is uniquely deter-
mined by \( c \), it is straightforward from (E10) that \( a \) is also uniquely determined:

\[
a = \begin{cases} 
E - \frac{b}{c} \left( e^{c \mu + \frac{c^2}{2 \tau}} - 1 \right), & c \neq 0, \\
E - b \mu, & c = 0.
\end{cases}
\]

\[ \Box \]

**APPENDIX F. POSITIVE AVERAGE ASSET SUPPLY, \( \delta > 0 \)**

Suppose the supply of the positively skewed asset, \( \tilde{\theta} \), is increased by a small constant \( \delta > 0 \) so that the market clearing conditions in (17) and (18) become, respectively,

\[
P_{q_1,LM}(\eta) : \lambda \left( \frac{1}{2} x_L + \frac{1}{2} x_M \right) + (1 - \lambda)x_\varnothing + \eta = \delta, \quad \text{(pessimistic)}
\]

(F1)

\[
P_{q_1, MH}(\eta) : \lambda \left( \frac{1}{2} x_M + \frac{1}{2} x_H \right) + (1 - \lambda)x_\varnothing + \eta = \delta. \quad \text{(optimistic)}
\]

(F2)

Then, it is straightforward to show that Theorem 10 still holds but with the additive adjustment \(-\gamma(\theta_h - \theta_l)\delta \) to equation (20). That is, replace the function \( f \) in (19a), (19b), and (20) by

\[
f_\delta(x) := \gamma(\theta_h - \theta_l) \left[ x - \delta + \frac{1 - \lambda/2}{\gamma(\theta_h - \theta_l)} \ln \left( \frac{(1 - \phi)g(x + \Delta) + \phi g(x - \Delta)}{\phi g(x + \Delta) + (1 - \phi) g(x - \Delta)} \right) \right],
\]

(F3)

such that (19a) and (19b) are expressed as

\[
P^\delta_{q_1, LM}(\eta) = (\theta_h - \theta_l) \frac{q_1}{q_1 + (1 - q_1)e^{-f_\delta(\eta - \Delta)}} + \theta_l,
\]

(F4a)

\[
P^\delta_{q_1, MH}(\eta) = (\theta_h - \theta_l) \frac{q_1}{q_1 + (1 - q_1)e^{-f_\delta(\eta + \Delta)}} + \theta_l,
\]

(F4b)

and \( \Delta := \frac{\lambda/2}{\gamma(\theta_h - \theta_l)} \ln \left[ \frac{\phi}{1 - \phi} \right] \) of (21) remains unaltered.

Similarly, it is straightforward to show that the expected price function of Proposition 12 still holds but with the constants \( X, Y, \) and \( Z \) of equations (26), (27), and (28), respectively, each scaled by \( e^{-\gamma(\theta_h - \theta_l)\delta} \):

\[
EP_\delta(q_1) = \frac{1}{2k} \gamma \left[ (1 - h(q_1))u(q_1, X_\delta, Y_\delta) \right.
\]

\[
+ h(q_1) u \left( q_1, \frac{1}{Y_\delta}, \frac{1}{X_\delta} \right) + u \left( q_1, Z_\delta, \frac{1}{Z_\delta} \right) \left] + \theta_l, \quad \text{(F5)}
\]

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where

\[ h(q) := q\phi + (1 - q)(1 - \phi), \]  
\( (F6) \)

\[ u(q, x, y) := \ln \frac{1 - q(1 - x)}{1 - q(1 - y)}, \]  
\( (F7) \)

\[ X_\delta := \left( \frac{1 - \phi}{\phi} \right) 1 - \lambda e^{-\gamma(\theta_h - \theta_l)(k + \delta)}, \]  
\( (F8) \)

\[ Y_\delta := \frac{1 - \phi}{\phi} e^{-\gamma(\theta_h - \theta_l)(k + \delta)}, \]  
\( (F9) \)

\[ Z_\delta := \left( \frac{1 - \phi}{\phi} \right)^{3/2} e^{\gamma(\theta_h - \theta_l)(k - \delta)} . \]  
\( (F10) \)

Moreover, it’s clear that the expected price decreases in \( \delta \) since the price function is uniformly shifted downward for \( \delta > 0 \). The effect on expected price of a small upward shift in average supply \( \delta \) is illustrated in Figure 12. For infinitesimal \( \delta \), or sufficiently small, positive average supply of the asset relative to the supply of all other assets, the expected price function is essentially unaltered and the results of the previous sections hold qualitatively.

**FIGURE 12**

*Positive Supply Impact on Skewness Effect*

This figure illustrates the impact of increased supply on the skewness effect as discussed in Section F, where the solid line is positive supply (\( \delta = 0.1 \)) and the dotted line is zero supply (\( \delta = 0 \)), and where \( \gamma = 2, \theta_h = 1, \theta_l = 0, k = 2, \lambda = 0.4, \) and \( \phi = 0.55 \).

For more significant values of \( \delta > 0 \), the expected price still exhibits similar properties to the baseline case of \( \delta = 0 \) (such as increasing concavity below a threshold and increasing convexity above
it) except that the pivot value of $q$ such that expected price exceeds expected payoff occurs at a value below $\frac{1}{2}$. In this case, a minimum level of positive skewness is required before the expected payoff becomes negative. This captures an “unconditional” risk premium that appears from increasing the endowment of the asset of each investor, thereby requiring a downward shift in price to induce them to hold on to that extra endowment in equilibrium. The inflection point at which the expected price function changes from being concave to convex remains at zero conditional skewness.

Empirical evidence on skewness shows that the typical stock has significantly positive skewness, so it is still reasonable to expect to see negative average returns for some stocks even after accounting for this unconditional risk premium. If this unconditional risk premium is what is being captured by market factor models such as the CAPM or Fama and French (1993) factors, then the residual (or idiosyncratic) expected return above and beyond this component would then essentially re-assume the characteristics analyzed in earlier sections of this paper. That is, if factor models are doing a good job of picking-up the unconditional risk premium induced by average positive supply or volume effects, then the results of this section suggest why those same factors could then fail to pick up the skewness effect which arises in a separate way.

APPENDIX G. SKEWNESS EFFECT UNDER NORMALLY-DISTRIBUTED STOCHASTIC DEMAND

In this appendix, I provide numerical illustrations of results from the main text when noise demand is normally distributed. Figure 13 parallels the results of Figure 5 in the main text, using the same parameters, but with $\mathcal{N}(0,1)$ noise demand instead of $\mathcal{U}[0,1]$. Relative to the uniform noise demand case of the main model, the qualitative results are unaltered.
Plots (a) and (b) show the expected price and expected return, respectively, of the risky asset \( \bar{\theta} \) vs. skewness \( S \) holding the payoff mean \( E \) and variance \( V \) fixed as in Theorem 8 (equation (23)), where \( \gamma = 1, E = 8, V = 1, \phi = 0.8, \lambda = 0.8, \) and \( \tilde{\eta} \sim \mathcal{N}(0, 1). \)