Optimal Capital Structure and Bankruptcy Choice: Dynamic Bargaining vs Liquidation

Samuel Antill† Steven R. Grenadier‡

January 30, 2018

Abstract

We model a firm’s optimal capital structure decision in a framework in which it may later choose to enter either Chapter 11 reorganization or Chapter 7 liquidation. Creditors anticipate equityholders’ ex-post reorganization incentives and price them into the ex-ante credit spreads. Using a realistic dynamic bargaining model of reorganization, the implied capital structure results in both higher credit spreads and dramatically lower leverage than existing models. If reorganization is less efficient than liquidation, the added option of reorganization can actually make equityholders worse off ex-ante, even when they liquidate on the equilibrium path.

†Graduate School of Business, Stanford University, 655 Knight Way, Stanford, CA 94305 (e-mail: santill@stanford.edu)
‡Graduate School of Business, Stanford University, 655 Knight Way, Stanford, CA 94305 (e-mail: sgren@stanford.edu)
1 Introduction

In 2016, the U.S. bankruptcy court system received nearly 38,000 commercial bankruptcy filings. For publicly traded firms, 80% of bankruptcies are handled under Chapter 11, while only 20% are Chapter 7 liquidations.\(^1\) Given the significance of Chapter 11 filings among reasonably-sized firms, the possible contingency of a future reorganization must be priced into the debt issued by such firms. We propose a model in which equityholders can choose both their timing of default and the chapter of bankruptcy,\(^2\) then examine how this flexibility alters capital structure decisions. In addition to providing potential explanations for debt conservatism and the “credit spread puzzle,” this model allows us to answer other important questions: how do the characteristics of the Chapter 11 process impact the capital structure of firms? Conversely, given that firms adjust their capital structures infrequently,\(^3\) can the capital structure of firms impact their choice of bankruptcy chapter?

In this work, we develop and solve a realistic continuous-time dynamic bargaining model of Chapter 11. For tractability, we must make some simplifying assumptions, but we make every effort to ensure our model accords with the U.S. Bankruptcy Code and its implementation. We include many features of the Chapter 11 process, such as automatic stay, suspension of dividends, the exclusivity period, post exclusivity proposals by creditors, forced conversion to Chapter 7, absolute priority rule in liquidation, and the unanimity rule (by creditor class) in approval of a reorganization plan. The reorganized firm may issue new debt and continue operating. Chapter 11 entails inefficiencies which are distinct from Chapter 7, such as professional fees and a decline in the cashflows that accumulate during reorganization. Moreover, both debtors and creditors face uncertainty over future asset values as they debate reorganization plans. In our model, creditors and equityholders are fully strategic in proposing and accepting plans, and we solve for a unique Markov perfect equilibrium outcome in closed form. This outcome turns out to be Pareto optimal, despite potential delays in agreement.

Using this equilibrium bargaining outcome, we extend the classic Leland (1994) model of endogenous default\(^4\) by allowing firms to choose between Chapter 7 or Chapter 11 when they default. As is standard in these models, equityholders receive nothing in liquidation. It follows that at the moment of default, equityholders choose Chapter 11 if and only if the expected bargaining

---

\(^1\)In a Chapter 7 liquidation, the firm’s assets are sold and the firm typically stops operating. Chapter 11 is a reorganization in which equityholders and creditors reach an agreement for the firm to stay in business. Aggregate statistics are from the American Bankruptcy Institute, data from Epiq Systems: http://www.abi.org/newsroom/bankruptcy-statistics. The 80% figure is based on Compustat and the UCLA-LoPucki Bankruptcy Research Database (see, for example, Corbae and D’Erasmo (2017)).

\(^2\)While we focus on debtor-chosen bankruptcy, in some cases, creditors can file for an involuntary bankruptcy in their chosen chapter under Section 303 of the U.S. Bankruptcy Code. However, courts only enforce a controverted filing if “the debtor is generally not paying such debtors debts as such debts become due,” and in this case the debtor still “may file an answer to a petition under this section” to choose the chapter (Section 303(d,h), http://law.abi.org/title11/303). Iverson (2017) reports that fewer than 2% of Chapter 11 filings are involuntary. For further reasons why involuntary bankruptcy petitions are a “lesser-used creditor tool,” see http://bankruptcy.cooley.com/2012/05/articles/business-bankruptcy-issues/forced-into-bankruptcy-the-involuntary-bankruptcy-process/.

\(^3\)See, for example, Fama and French (2002), Leary and Roberts (2005), or Korteweg, Schwert, and Streibulaev (2014).

\(^4\)See Streibulaev and Whited (2012) or Sundaresan (2013) for a survey of this model and the ensuing literature.
outcome exceeds the fixed cost they must pay to enter Chapter 11.\textsuperscript{5} Intuitively, in the subgame following debt issuance (ex-post), Chapter 11 is optimal for equityholders when the firm is sufficiently profitable at the moment of default. Taking into account the ex-post, strategic behavior of equityholders, the firm issues rationally priced debt at time zero (ex-ante) to exploit tax benefits.

The time zero capital structure decision depends on the relative efficiencies of Chapter 7 and Chapter 11 (traded off against the tax benefits of leverage). Specifically, depending on model parameters, there are three possible scenarios. The first two cases are rather stark and straightforward. In the first case, Chapter 11 is significantly more efficient than Chapter 7, so equityholders naturally find the former more attractive ex-post upon default. Thus, at time zero, debtholders demand a higher credit spread to compensate them for the rents equityholders extract in the event of a future reorganization. Notably, equityholders are willing to pay this higher spread since it is the rational expectation of their contingent Chapter 11 proceeds. The net effect is an increase in ex-ante firm value from the added option of a Chapter 11, due to the lower default costs of Chapter 11. Alternately, in the second case, Chapter 11 is extremely wasteful relative to Chapter 7. It follows that equityholders are not willing to incur the fixed cost of entering reorganization. Thus at time zero, equityholders optimally issue the same coupon as in the Leland (1994) model (which neglects Chapter 11), and ex-post liquidate at the same stopping time as well. In this case, the added option of Chapter 11 has no effect on ex-ante firm value.

The third case, in which Chapter 11 is slightly less efficient than Chapter 7, is the most interesting. In this case, if equityholders issue the optimal coupon from Leland (1994), they will ex-post find it optimal to enter Chapter 11. This is because large coupons imply the firm defaults in profitable states of the world, where the prospects of Chapter 11 for equityholders justify the fixed cost of entry. Debtholders thus demand a higher credit spread at time zero for such a coupon. Equityholders are hesitant to pay this spread since reorganization destroys more value than Chapter 7. For such parameters, equityholders have two choices. They can issue a large coupon to reap tax benefits, and accept that they will pay for the ex-post Chapter 11 inefficiencies with a higher credit spread at time zero. We call this the “optimal inefficient Chapter 11” strategy. Alternately, equityholders can issue a smaller coupon consistent with ex-post optimal Chapter 7. In this case they sacrifice the tax benefits of a larger coupon, but they enjoy a lower cost of debt due to the rational expectation of a future, more efficient liquidation.\textsuperscript{6} We call this the “constrained debt Chapter 7” strategy. Counterintuitively, under either of these strategies,\textsuperscript{7} the added option of Chapter 11 actually reduces ex-ante firm value.

Graham (2000) points out that “paradoxically, large, liquid, profitable firms with low expected

\textsuperscript{5}As we discuss in section 3.1, firms typically incur nontrivial professional fees prior to filing. These fees cannot be reimbursed in bankruptcy, and thus might reasonably be incurred by equityholders. There is empirical evidence for a significant fixed cost component to professional fees (LoPucki and Doherty (2011)).

\textsuperscript{6}As in all other models like Leland (1994), equityholders only issue debt once prior to default. In reality, debt covenants are often written with tight interest coverage ratios, such that further issuance is restricted even at the inception of the loan (Chava and Roberts (2006)). In Section 6, we explore how our results might change in an alternate model with multiple debt issuances in the style of Goldstein, Leland, and Ju (2001).

\textsuperscript{7}Equityholders endogenously choose one of these two strategies ex-ante. Which one is optimal depends on the parameters.
distress costs use debt conservatively” and “the typical firm could double tax benefits by issuing debt until the marginal tax benefit begins to decline.” When the “constrained debt Chapter 7” strategy is optimal, our model suggests a potential novel mechanism for explaining this puzzle. If the relative inefficiencies of Chapter 11 are large compared to the tax benefits of debt, equityholders optimally issue a modest coupon. For such a coupon, they will find Chapter 7 optimal ex-post, which lowers the cost of debt ex-ante. Since this entails forgoing tax benefits, equityholders issue the largest coupon consistent with future Chapter 7. In this case, for reasonable parameters, our model predicts a leverage ratio of 40%. For the same parameters, the Leland (1994) model predicts a 70% leverage ratio. To an econometrician, our model looks identical to the Leland model for such parameters: a firm issues debt then eventually liquidates. However, the off equilibrium considerations introduced by our bargaining model lead the firm to issue a much smaller coupon than in the standard Leland model. In this case, our model predicts lower leverage than the Leland model, even for the 65% of (public and private) firms that liquidate in Chapter 7 (Bernstein, Colonnelli, and Iverson (2017), henceforth BCI).

Endogenous default models of capital structure tend to underestimate credit spreads (Huang and Huang (2012)). Under the “optimal inefficient Chapter 11” strategy, our model suggests that high credit spreads could be due to the rational expectation of future rents extracted by equityholders in Chapter 11. For these parameter ranges, firms are unwilling to sacrifice tax benefits to get a lower cost of debt from issuing a low coupon consistent with ex-post Chapter 7. Instead, they accept the higher cost of debt and issue a large coupon for the tax shield. Since the higher default costs are internalized by equityholders when they issue debt, the overall coupon is still lower than in the Leland model. For reasonable parameter values, credit spreads can be 20 basis points higher than in the Leland model, even with an optimal leverage ratio that is 7 percentage points lower.

Finally, our model generates many other testable implications about the relationship between Chapter 11 and capital structure. Consider the following list. Creditor rights might be interpreted as the relative bargaining power of creditors in bankruptcy. Under this interpretation, stronger creditor rights lead to higher optimal leverage and shorter bankruptcies, consistent with empirical evidence. Firms with higher growth rates should be more likely to choose Chapter 11, so comparing Chapter 11 and Chapter 7 by the value of assets at the end of bankruptcy might overstate the efficiency of Chapter 11. Firms with more volatile cashflows or lower growth rates should have longer bankruptcies. Firms that choose Chapter 11 should have both more valuable assets and higher leverage ratios at the time of default than firms which choose Chapter 7. When the “constrained debt chapter 7” strategy is optimal, anything that makes Chapter 11 less appealing (for example, higher legal costs) will actually improve firm value. Changes in parameter values can have surprising comparative statics when they cause the firm to shift from Chapter 11 to Chapter 7 or vice versa.

bankruptcy or Chapter 7 liquidation. Papers such as Fan and Sundaresan (2000)\textsuperscript{9} have extended the Leland (1994) framework to allow for costless renegotiation in private workouts.\textsuperscript{10} Articles such as Hackbarth, Hennessy, and Leland (2007) and Hackbarth and Mauer (2012) study the optimal mixture of bank and public debt, where bank debt may be renegotiated in a private workout. These works document important links between private workouts and optimal capital structure decisions, but the details of the Chapter 11 procedure are not modeled.

François and Morellec (2004) and Broadie, Chernov, and Sundaresan (2007) are more similar to our work. François and Morellec (2004) extend the model of Fan and Sundaresan (2000) to study the Chapter 11 bankruptcy procedure.\textsuperscript{11} In their model, equityholders choose a threshold at which to enter Chapter 11. While the asset value is below this threshold, equityholders and debtholders split the cashflow according to Nash Bargaining. If asset values do not rise back above the same threshold before an exogenous window of time expires, then the firm liquidates.\textsuperscript{12} Broadie, Chernov, and Sundaresan (2007) numerically extend this by keeping track of accumulated earnings and accumulated arrears during the bankruptcy. In their framework, the firm emerges from bankruptcy when accumulated earnings are sufficient to pay off the accumulated arrears, where an exogenous fraction of the debt is forgiven. They study equity and debt values when creditors pick the bankruptcy threshold compared to the same values when equityholders choose the thresholds. Both François and Morellec (2004) and Broadie, Chernov, and Sundaresan (2007) also consider a time zero capital structure decision. These papers capture the impact of bankruptcy procedure on time zero capital structure, but only allow for liquidation after the firm has already entered Chapter 11. In reality, the majority of firms go straight to Chapter 7 without ever entering Chapter 11 (BCI (2017)). By allowing equityholders to choose either Chapter 7 or Chapter 11, our model produces implications for the choice of bankruptcy procedure. This also has important implications for the time zero capital structure decision which are impossible to produce in either of the previously mentioned models. To our knowledge, the only model which allows firms to enter Chapter 7 or Chapter 11 is Corbae and D’Erasmo (2017), who set up and estimate a structural discrete time general equilibrium model. Their model is extremely different from ours (for example, bankruptcies always last one period and all debt matures in one period), so our analysis complements theirs while providing novel insights.

A novel methodological contribution of our paper relative to all previous work is our bargaining model of Chapter 11. We use a new\textsuperscript{13} continuous time formulation of the stochastic bargaining

\textsuperscript{9}See also Anderson and Sundaresan (1996), Mella-Barral and Perraudin (1997).
\textsuperscript{10}More recent papers which jointly consider renegotiation and investment include Sundaresan and Wang (2007), Shibata and Nishihara (2015). Christensen et al (2014) have a similar model with dynamic capital structure.
\textsuperscript{11}Earlier theoretical papers studying Chapter 11 include Franks and Torous (1989) and Longstaff (1990) who model Chapter 11 as a right to extend the maturity date of debt. Lambrecht and Perraudin (1996) study a creditor race in bankruptcy where multiple creditors might preempt one another in seizing assets (Bruche (2011) examines a similar setup). These models have interesting implications regarding bankruptcy procedure, but do not consider any decision to enter bankruptcy, or any capital structure decision.
\textsuperscript{12}For related approaches where the time spent in Chapter 11 or the lowest asset value hit during Chapter 11 matter, see Moraux (2002) and Galai et al (2007).
\textsuperscript{13}In a companion theoretical paper, we solve a more general version and explore features of the solution in greater
model from Merlo and Wilson (1995). This captures two important features of the bankruptcy process. First, all impaired classes of creditors (including equity) must unanimously\textsuperscript{14} agree to a reorganization plan to exit bankruptcy. The models mentioned previously assume that equity or debt unilaterally decide the timing of the exit. Second, Chapter 11 bankruptcies can take as long as 10 years, and all parties face significant uncertainty over how the value of the firm’s assets will change in this time. The previously mentioned models, such as Corbae and D’Erasmo (2017), assume that the split between equity and debt is either determined exogenously or by Nash Bargaining at the moment of entering Chapter 11. Our stochastic bargaining framework allows parties to change their bargaining strategies as they observe the resolution of uncertainty.\textsuperscript{15}

The article proceeds as follows. Section 2 describes the model and reviews the Leland (1994) setup. Section 3 provides institutional details then describes and solves for the equilibrium of the Chapter 11 reorganization timing game. Section 4 derives the optimal decision to enter Chapter 11 or Chapter 7. Section 5 describes the time zero capital structure choice, and provides results on capital structure. Section 6 gives additional empirical implications, and discusses possible generalizations of the model. Section 7 concludes.

2 Model Setup

In this section we begin by describing the setup of our model of the leveraged firm. In particular, we focus on the timing of the decision to enter Chapter 11 reorganization or Chapter 7 liquidation, the continuous-time bargaining game that occurs during reorganization, and the endogenous emergence from reorganization. This model is solved in Sections 3 and 4, working backwards through time. As a useful benchmark, and for our own model prior to and following reorganization, in Section 2.2 we provide a brief derivation of the standard solution where Chapter 11 reorganization is not considered.

2.1 Outline and Assumptions of the Baseline Model

We consider a continuous time infinite horizon model of a firm. At all times at which the firm is operating, its assets in place produce earnings before interest and taxes (EBIT) $\delta_t$. We assume the existence of a risk neutral measure with risk free rate $r$ under which $\delta_t$ follows a geometric Brownian motion

$$d\delta_t = \delta_t \mu dt + \delta_t \sigma dB_t$$  \hspace{1cm} (1)
where $B_t$ is a Brownian motion under the risk neutral measure. The cashflow $\delta_t$ is subject to effective corporate tax rate $\tau$, where we follow the literature in assuming full loss offset provision.

The model is separated into four distinct times, $T = 0, 1, 2, 3$. Figure 1 presents a graphical timeline of the model. Time 0 represents the initial determination of the capital structure of the firm. Specifically, at time 0, the firm can issue consol debt with total perpetual coupon $C_0$. Debt entails a tax shield, so the firm subsequently pays taxes $\tau(\delta_t - C_0)dt$ per unit time. The firm chooses $C_0$ to maximize firm value, the determination of which will depend on expectations of future strategic decisions.

Once the firm has issued debt with coupon $C_0$, they progress to time 1, which is the period of time in which the firm operates prior to the realization of bankruptcy or liquidation. In time 1, equityholders receive after tax cashflow $(1 - \tau)(\delta_t - C_0)dt$ per unit time. There are two potential stopping times that the owners must contemplate. One is the standard liquidation decision that is common in the literature. At any stopping time $T_L$, the equityholders may choose to liquidate, at which point equityholders receive 0, and debtholders receive the liquidation value

$$\zeta \delta_{T_L} = \frac{(1 - \tau)(1 - \alpha)\delta_{T_L}}{\tau - \mu}$$

The liquidation value $\zeta \delta_{T_L}$ is the expected discounted value of receiving $(1 - \tau)\delta_t$ in perpetuity given the current value of $\delta_{T_L}$, multiplied by a constant $(1 - \alpha)$. As standard in the literature, we assume a fraction $\alpha \in (0, 1)$ of the firm value is lost in liquidation. If the firm chooses to liquidate, the game ends.

Our novel contribution is the additional option for the firm to enter a Chapter 11 reorganization. At any time $T_B$, the firm may declare bankruptcy and enter into a Chapter 11 reorganization. In this case, they pay a fixed cost $B > 0$ and enter time 2.

Time 2 represents the time spent in Chapter 11 reorganization. In this period, debtholders and equityholders play a continuous-time bargaining game to determine when to emerge from bankruptcy. During the Chapter 11 process, the automatic stay provision prevents creditors from demanding payments. At the same time, dividend suspension prevents debtors (equityholders) from paying themselves dividends. We assume that the firm continues to receive cashflows $(1 - \tau)h\delta dt$ per unit time, where $h \in [0, 1]$ is a haircut representing the inefficiency of operations during bankruptcy, and these cashflows accumulate. At any stopping time $T_R$, the debtholders and equityholders may agree to a reorganization plan. At this time, they receive the firm value $V(\delta_{T_R})$ minus a fixed reorganization cost $R_0 > 0$. They also receive the accumulated earnings, for a total payment of

$$P_{T_R} = V(\delta_{T_R}) - R_0 + (1 - \tau)h \int_{T_B}^{T_R} \delta_s ds$$

\footnote{We describe the process in more detail and provide justifications for our assumptions in section 3.1.}

\footnote{We do not distinguish between equityholders and firm management in this paper. In all cases, the “debtors” are the equityholders.}
In Broadie, Chernov, and Sundaresan (2007), the authors provide a model of Chapter 11 that assumes equityholders receive the residual firm value after paying arrears at a time chosen by equityholders. We depart from this by modeling the reorganization as a bargaining process. Consistent with the laws of Chapter 11, equityholders and debtholders alternate filing plans for how to split the total $P_t$, and the process ends at the first time $T_R$ when one party makes a proposal the other party accepts.

Time 2 ends by either the debtholders and equityholders agreeing to a reorganizing plan at stopping time $T_R$, or by a judge-mandated liquidation. With probability $\nu dt$ per unit time, the judge converts the Chapter 11 reorganization into a Chapter 7 liquidation. In the event of liquidation, debtholders receive the liquidation value $\zeta$ plus the accumulated earnings net of fees, equityholders receive nothing consistent with Absolute Priority Rule (APR), and the game ends. While this occurs exogenously, agents anticipate the possibility of liquidation and may endogenously increase the likelihood of liquidation by stalling. If equityholders and debtholders agree to a reorganization prior to liquidation, the game proceeds to period 3.

Time 3 represents the operation of the reorganized firm. Just as at time 0, the new equityholders of the reorganized firm issue new debt $C_1$ to maximize firm value. For the remainder of time 3, equityholders receive a payment $(1 - \tau)(\delta_t - C_1)dt$ per unit time. For simplicity, in the basic model we assume that the option to reorganize no longer exists and the firm may only exit through liquidation.\footnote{It is unlikely this assumption substantially affects our results, as discussed in Section 6.} Thus, at any stopping time $T_{L,1}$, equityholders may liquidate the firm. As described previously, at the time of liquidation equityholders receive 0, the new debtholders receive $\zeta \delta_{T_{L,1}}$, and the game ends.

### 2.2 Benchmark Model: The Leland Model with Only Chapter 7 Liquidation

In the standard Leland model, a levered firm with coupon $C$ chooses a liquidation time $T_L$ to maximize equity value:

$$E^L(\delta) = \sup_{T_L \in F^\delta} \mathbb{E}^\delta_0 \left[ \int_0^{T_L} e^{-rt}(1 - \tau)(\delta_t - C)dt \right] \quad (4)$$

where throughout the paper, $\mathbb{E}^\delta$ represents expectation with respect to the probability law of the process $\delta_t$ starting at $\delta_0 = \delta$.\footnote{We require $T_L$ is a stopping time with respect to the filtration $F^\delta$ generated by $\delta_t$.} It is worth noting that there could be a time $t < T_L$ such that the cashflow to equity is negative. Consistent with the prior literature, we assume in this case that equityholders issue new shares and dilute their equity in order to pay the coupon to debtholders. For the optimal $T_L$, the value of equity will always be positive for $t < T_L$, consistent with limited liability, so such dilution is possible.

In the region where liquidation is not optimal, standard dynamic programming arguments show the value of equity $E^L(\delta)$ satisfies the following ordinary differential equation (ODE):
\( rE^L(\delta) = DE^L(\delta) + (1 - \tau)(\delta - C) \)  

(5)

where \( D \) is the differential operator from Ito’s lemma for a smooth function of \( \delta \):\(^{20}\)

\[ Df(\delta) = f'(\delta)\mu \delta + f''(\delta)\frac{\sigma^2}{2} \delta^2 \]  

(6)

As \( \delta \to \infty \), the value of the option to liquidate should become worthless. This implies the value of equity \( E^L(\delta) \) should approach the value of receiving the aftertax cashflows less the debt payments in perpetuity, which is \((1 - \tau)[\frac{\delta}{r-\mu} - \frac{C}{r}]\). Imposing this condition, the general solution of (5) is

\[ E^L(\delta) = A_1 \delta^\psi + (1 - \tau)[\frac{\delta}{r-\mu} - \frac{C}{r}] \]

where \( A_1 \) is an arbitrary positive constant and \( \psi \) is the negative root of the characteristic polynomial

\[ r - \mu z - \frac{\sigma^2}{2} z(z - 1) = 0 \]

It can be verified that the optimal liquidation time \( T_L \) is a hitting time \( T_L = \inf\{t : \delta_t \leq \delta_L\} \) for some barrier \( \delta_L \). The constant \( A_1 \) and the liquidation threshold \( \delta_L \) are determined by value matching and smooth pasting at \( \delta_L \). Since equity receives nothing in liquidation, the value matching and smooth pasting conditions are

\[ A_1 \delta^\psi + (1 - \tau)[\frac{\delta}{r-\mu} - \frac{C}{r}] = 0 \]  

(7)

\[ A_1 \psi \delta_{\psi-1} + \frac{(1 - \tau)}{r - \mu} = 0 \]  

(8)

This system of equations has the usual unique solution

\[ \delta_L = \frac{\psi}{\psi - 1} \frac{r - \mu}{r} C \]  

(9)

\[ A_1 = \delta_L^{-\psi}(1 - \tau)[\frac{C}{r} - \frac{\delta_L}{r-\mu}] \]  

(10)

Taking the liquidation threshold \( \delta_L \) as given, the value of the debt \( D^L(\delta) \) satisfies an ODE similar to (5) prior to liquidation:

\[ rD^L(\delta) = DD^L(\delta) + C \]  

(11)

\(^{20}\)Throughout the paper, “smooth” means continuously differentiable, and twice continuously differentiable almost everywhere.
and similar logic shows this has general solution

\[ D^L(\delta) = \frac{C}{r} + A_2 \delta \psi \quad (12) \]

for an arbitrary constant \( A_2 \). As discussed in the previous section, at the time of liquidation \( T_L \) a fraction of firm value \( \alpha \) is lost, leaving value \( \zeta \delta_L \) for the debtholders. Imposing that \( D^L(\delta_L) = \zeta \delta_L \) uniquely determines the constant \( A_2 \), which gives the rational expectations value of consol debt with coupon \( C \):

\[ D^L(\delta) = \frac{C}{r} + \delta \psi \delta^{-\psi}_L \left[ -\frac{C}{r} + (1 - \alpha)(1 - \tau) \frac{\delta_L}{r - \mu} \right] \quad (13) \]

The standard Leland model features a time zero capital structure decision. Specifically, at time 0 equityholders choose the coupon \( C \) for their consol debt to maximize the total firm value \( E^L(\delta_0) + D^L(\delta_0) \).\(^{21}\) As in the standard tradeoff theory, the firm weighs the tax benefits of debt with the loss of firm value in liquidation. For any arbitrary \( \delta_0 \), we can plug in (9) for \( \delta_L \) and the resulting expression for \( E^L(\delta_0) + D^L(\delta_0) \) is concave in \( C \). Solving the first order condition gives the unique optimal \( C^* \):

\[ C^* = \delta_0 \frac{r}{r - \mu} \psi \left[ \frac{1}{\psi (1-\tau) \alpha + (\psi - 1) \tau} \right]^{\frac{1}{\psi}} \quad (14) \]

Note that the optimal coupon \( C^* \) is linear in the starting cashflow \( \delta_0 \). Since the liquidation barrier \( \delta_L \) is linear in \( C \), it can be seen from equations (9, 10, 13, 14) that at the optimal coupon,

\[ E^L(\delta_0) + D^L(\delta_0) = \theta \delta_0 \quad (15) \]

for a constant \( \theta \) that is a known function of the model primitives given in Appendix A.

### 3 Chapter 11 as a Stochastic Bargaining Game

We begin with a brief description of what will appear in this section. We first discuss some features of the Chapter 11 process that are important for our model. We then look at the firm under reorganization solving the bargaining/exercise game. Notably the payoff is simplified by our Leland Model benchmark from above. Then, we determine the optimal timing of reorganization by using a Pareto optimality result. Then, conditional on this optimal timing, we solve for the optimal bargaining split.

Recall that our model is divided in four distinct times. Since we rule out a second reorganization,\(^{22}\) in time 3 the equityholders solve the standard liquidation problem described in section 2.2. Likewise, at the beginning of time 3 they issue an optimal level of debt as described above. In this section, we describe and solve time 2, the Chapter 11 process.

---

\(^{21}\) They maximize this sum since they receive the market value of the debt \( D^L(\delta) \) at issuance.

\(^{22}\) In practice, very few firms have gone through multiple Chapter 11 reorganizations. For exceptions that prove the rule: [http://www.therichest.com/business/companies-business/5-big-corporations-that-have-survived-multiple-bankruptcies/](http://www.therichest.com/business/companies-business/5-big-corporations-that-have-survived-multiple-bankruptcies/).
3.1 Relevant Features of Chapter 11

In order to be tractable, our model makes some simplifying assumptions regarding the Chapter 11 process. However, we make every attempt to ensure that our model is broadly consistent with some of the most salient features of the actual Bankruptcy Code. In this section we summarize some of the most important aspects of the Chapter 11 process that inform much of our modeling assumptions. A comprehensive description of Chapter 11 is far beyond the scope of this paper.\textsuperscript{23}

When we reference “Sections,” we are quoting or paraphrasing the referenced section of Title 11 of the U.S. Bankruptcy Code.\textsuperscript{24}

First, the automatic stay provision of Chapter 11 (Section 362) prohibits all entities from “any act to obtain possession of property of the estate.” In particular, debtholders stop receiving coupons and equityholders stop receiving dividends.

Second, in order to confirm a reorganization plan and exit Chapter 11, every impaired class of creditors must accept the plan by Section 1129(a).\textsuperscript{25} This gives equityholders, who constitute a class of claims, some power to hold up the reorganization process and potentially extract rents. The absolute priority rule (APR) refers to the idea, in Chapter 7 or 11, that each creditor should only be compensated once all senior creditors are paid in full. However, equityholders are often able to use their bargaining power to violate this in Chapter 11. Bris, Welch, and Zhu (BWZ, 2006) find in their sample that APR is always followed in Chapter 7, while it is violated in 12% of Chapter 11 cases. Weiss (1990) finds violations in 29 of the 37 Chapter 11 cases he studies.

Third, at the start of the Chapter 11 process, equityholders enjoy an “exclusivity period.” Specifically, the debtor-in-possession (DIP) enjoys the exclusive right to propose reorganization plans for 120 days under Section 1121(a). Small businesses have 180 days. The debtors then have another 60 days to get the plan approved by creditors. After this window, any party in interest may file a plan. Under Section 1121(d), the court may reduce or increase this window. Since this is at the judge’s discretion, both equityholders and creditors face uncertainty as to the length of the exclusivity period, although it cannot exceed 18 months (20 months for small businesses).

Fourth, it is common for bankruptcy cases which begin as Chapter 11 reorganizations to be converted to Chapter 7 liquidations. In the sample analyzed in BWZ (2006), 14% of the cases which began in Chapter 11 were converted,\textsuperscript{26} while as many as 40% of cases were converted in the sample of BCI (2017). While debtors may in principle choose to convert to Chapter 7, and creditors may petition for such a conversion, the ultimate decision lies with the judge. This suggests that modelling the conversion as exogenous and random is a reasonable approximation of reality. Indeed, BCI (2017) use the random assignment of judges to bankruptcy cases as exogenous variation in the probability of conversion:

\textsuperscript{23}For a readable introduction, see http://www.uscourts.gov/services-forms/bankruptcy/bankruptcy-basics/chapter-11-bankruptcy-basics
\textsuperscript{24}This can be found at http://law.abi.org/title11
\textsuperscript{25}In some cases, a plan may still be crammed down (Section 1129(b)) at the discretion of the judge if one class refuses. Section 1126 states that a class accepts a plan if it is accepted by creditors that “hold at least two-thirds in amount and more than one-half in number of the allowed claims” within the class.
\textsuperscript{26}Specifically, 42 of the 117 chapter 7 cases began as chapter 11. The 14% is calculated based on their sample of 257 Chapter 11 cases that were not converted.
U.S. bankruptcy courts use a blind rotation system to assign cases to judges, effectively randomizing filers to judges within each court division. While there are uniform criteria by which a judge may convert a case from Chapter 11 to Chapter 7, there is significant variation in the interpretation of these criteria across judges.

Fifth, debtors and creditors hire professionals (i.e., accountants, lawyers, investment bankers, financial advisors) who charge nontrivial fees. Fees incurred during bankruptcy are typically reimbursed from the estate (the firm’s assets):

“The large bulk of bankruptcy professional fees and expenses are awarded under Bankruptcy Code Section 330(a). Section 330(a) awards are to professionals employed by the DIP... or employed by an official committee... the DIP pays the awards from the estate (LoPucki and Doherty (2011)).”

Creditors have further opportunities for fee reimbursement through Sections 503(b) and 506(b). Weiss (1990) estimates that such fees average 3.1% of firm value, but LoPucki and Doherty (2011) give many reasons why this is an underestimate. In extreme cases like the bankruptcy of Allied Holdings, fees can reach 22% of firm assets (LoPucki and Doherty (2011) Appendix A). While these fees are certainly higher for larger firms and longer bankruptcy, there is empirical evidence for a fixed cost component. For example, Table 1 of LoPucki and Doherty (2004) presents results of a regression of log fees on log assets, log number of days in bankruptcy, the number of professional firms, and a constant. They find significant coefficients of 0.414 and 0.535 on assets and length of bankruptcy, respectively. The constant term, however, is large and positive with a t-statistic nearly three times as large as either assets or bankruptcy length. This suggests that larger and longer bankruptcies entail higher fees, but there is a significant fixed cost faced by all firms.27

While these fees are typically awarded and thus subtracted from the total estate to be split between creditors, prepetition fees are an important exception. Firms hire professionals prior to filing for Chapter 11 (prepetition), and the court “does not award prepetition fees” (LoPucki and Doherty (2011)). Indeed, while firms are supposed to report prepetition fees and expenses in connection with a future Chapter 11 under Section 329(a), they frequently fail to report. Within the dataset used for LoPucki and Doherty (2011), prepetition fees averaged 43% of the subsequent total 330(a) awards in cases where the firm reported both.28 These fees must be paid by the firm (i.e., equityholders) since they are not awardable.

Finally, there are indirect costs to Chapter 11. These might include loss of reputation, reduced customer loyalty, and impaired employee retention. Altman (1984) attempts to measure these using analyst earnings estimates, and finds the average ratio of indirect costs to firm value to be 20%. While such costs likely are larger for larger firms, it is reasonable that a firm’s reputation suffers the moment they declare Chapter 11. Other indirect costs, such as rehiring managers and employees, issuing new debt, and ramping up firm operations, are likely incurred at the end of the reorganization.

27For example, Farm Fresh Inc., with less than $200 million in assets upon declaring bankruptcy, spent three million in fees on a 44 day bankruptcy (LoPucki and Doherty (2011) Appendix A).
28The data is generously available from the authors at http://lopucki.law.ucla.edu/ProfessionalFees.htm.
3.2 The Dynamic Reorganization Game

At the beginning of time 2, the firm enters Chapter 11, and the period ends with a reorganization or a forced liquidation. Based on the analysis of Section 2.2, the total firm value available to be split among debtholders and equityholders if a reorganization occurs at time \( T_R \) is \( \theta \delta_{T_R} \). This expression takes into account the value of the debt the new equityholders will issue.

As discussed previously, there are no payments during the Chapter 11 process. We assume that after-tax earnings accumulate into an account, and that the accumulated earnings \( \int_{T_B}^{T_R} (1 - \tau) h \delta_t dt \) are split among equity and debtholders. We allow for the possibility that the firm operates less efficiently during bankruptcy by including a haircut \( h \in [0, 1] \). This also can include a flow of professional fees.

We assume that with probability \( \iota dt \) per unit time, exogenous to the decisions of any agent, the bankruptcy case is converted and the firm is liquidated.\(^{29}\) While debtors may in principle choose to convert to Chapter 7, and creditors may petition for such a conversion, the ultimate decision lies with the judge. If a conversion occurs at time \( T_c \), we assume APR is upheld so equityholders receive 0 and debtholders receive the liquidation value \( \zeta \delta_{T_c} \) plus the accumulated earnings net of fees described below. While this occurs exogenously, agents anticipate the possibility of liquidation and may endogenously increase the likelihood of liquidation by stalling. However the firm emerges from bankruptcy, they must pay a fixed cost \( R_0 > 0 \). This represents the direct and indirect costs discussed in section 3.1.

In summary, if the reorganization occurs at a time \( T_R < T_c \), then equityholders and debtholders split \( P_{T_R} \), where \( P_t \) is as defined in (3):

\[
P_t = \theta \delta_t - R_0 + (1 - \tau) \int_{T_B}^{t} h \delta_s ds
\]

The accumulated earnings complicate the problem, since now the current value \( \delta_t \) is not sufficient to determine the potential payoff. To handle this, we introduce a second state variable \( R_t \):

\[
R_t = R_0 - (1 - \tau) \int_{T_B}^{t} h \delta_s ds \tag{16}
\]

This allows us to write the reorganization payoff as \( P_t = \theta \delta_t - R_t \) and the liquidation payoff as \( \zeta \delta_t - R_t \).

One of our primary contributions relative to the literature is modelling Chapter 11 reorganization as a bargaining process. As discussed in section 3.1., both the debtors and creditors have opportunities to propose reorganization plans, and approval must be unanimous. Further, the bargaining process is inherently dynamic. The average Chapter 11 case lasts two and a half years (BWZ (2006)), so it is inevitable that the value of underlying assets fluctuates stochastically during this period. In light of this, we model the Chapter 11 process as a dynamic, stochastic bargaining game between debtholders and equityholders.

\(^{29}\)This is purely for realism, there are no issues when \( \iota = 0 \).
The bargaining procedure is the continuous time equivalent of the bargaining game in Merlo and Wilson (1995, 1998). The two players bargain over a time $T_R$ to emerge from bankruptcy, which must be agreed upon unanimously, and a split of the firm value $\theta \delta R - R_T$. If a forced conversion occurs, the game ends and debtholders receive the entire liquidation payoff $\zeta \delta R - R_T$. At any moment in the game, exactly one player (equity or debt) is the proposer. The proposer may make offers to the other player in any second, and the receiving player instantaneously decides to accept or reject their proposed share of the payoff. The game ends when a proposed split is accepted. The proposer in any instant is given exogenously by a time homogeneous Markov chain $s_t$ taking values in two states which we label \{e, d\}.\footnote{This labeling is merely for clarity, one can think of e, d as arbitrary real numbers.} When $s_t = e$, equityholders get to propose splits, and when $s_t = d$, debtholders get to propose splits. For simplicity, we assume the Markov chain has constant transition intensities, so the probability of transitioning from state $i$ to state $j$ per unit time is $\lambda_i dt$, $i = e, d$.

The stochastic proposer bargaining protocol is standard in the literature (see Merlo and Wilson (1995, 1998); Baron and Ferejohn (1989); Yildiz (2003); Hart and Mas-Colell (1996); Rubinstein and Wolinsky (1985); Binmore and Dasgupta (1987)). The rates of transitions are a tractable representation of bargaining power. In this setting, equityholders have a strong bargaining position if $\lambda_e$, the rate of transition away from state $e$, is low, and if the rate of transition $\lambda_d$ into state $e$ is high. Likewise, equityholders have a weak bargaining position if $s_t$ leaves state $e$ quickly and transitions into state $e$ infrequently. Virtually all bargaining models (including continuous time models like Perry and Reny (1993) and Admati and Perry (1987)) assume there is some discrete length of time during which one player cannot make offers. In our model, that length is stochastic, but for any fixed $dt$ there exist transition probabilities such that all players have the chance to make offers within the interval $[t, t + dt]$ with arbitrarily high probability. Merlo (1997) uses a structural estimation to show the stochastic proposer model fits empirical data on government negotiations well.

The main advantage of the stochastic proposer model is that it facilitates the analysis of time homogeneous strategies and equilibria. However, giving equityholders a window of exogenously stochastic length during which they have the exclusive right to propose splits is actually a highly realistic model of the exclusivity period. After the exclusivity period, creditors may file a competing plan, and equityholders may file additional plans. If the reader would prefer a model in which equityholders and debtholders may both make offers in any instant, letting $\lambda_e, \lambda_d$ approach infinity accomplishes this.

Given this bargaining protocol and the model primitives, equityholders (player e) and debtholders (player d) form strategies. We will focus on equilibria in stationary strategies that only depend on the current state ($\delta, R, s$). A stationary strategy for player $i$ consists of

1. A region $O_i \subset \mathbb{R}^2$ of ($\delta, R$) values for which they make an offer when they are the proposer.
2. An offer function $\omega_i : O_i \to \mathbb{R}$ such that they offer $\omega_i(\delta, R)$ to player $j$ when $(\delta_t, R_t) \in O_i$.
3. A correspondence $A_i : \mathbb{R}^2 \to \mathbb{R}$ mapping current ($\delta, R$) values to the set of offers that they
Stationary strategies allow for a great deal of flexibility. Each player chooses a triple of infinite dimensional objects. However, restricting attention to stationary strategies does rule out some possibilities. For example, players may not condition their actions on previous offers. They also may not make decisions as explicit functions of the elapsed time since the start of the bargaining.

The benefit of focusing on stationary strategies is that they clearly induce outcomes. If we fix a stationary strategy \((O_i, A_i, \omega_i)\) for each player, we can define

\[ T_i = \inf \{ t : s_t = i, (\delta_t, R_t) \in O_i, \omega_i(\delta_t, R_t) \in A_j(\delta_t, R_t) \} \]

as the first time that player \(i\) is proposer and the value of \((\delta_t, R_t)\) is such that player \(i\) makes a proposal which is accepted by player \(j\). It follows that \(T = T_e \wedge T_d\) is the time at which the game ends (unless liquidation occurs first), according to the fixed strategies. When the game ends in reorganization, the payoff to player \(i\) depends on whose proposal is accepted. It will be convenient to define the terminal payoff for player \(i\), given fixed strategies, as

\[ J_i(\delta, R, s) = 1(s = i)[\theta \delta - R - \omega_i(\delta, R)] + 1(s = j)\omega_j(\delta, R) \]

Intuitively, \(J_i(\delta, R, s)\) equals the offer which player \(j\) makes to player \(i\) if \(s = j\), while if the game ends with a proposal from player \(i\) then it equals the stochastic payoff minus the offer made by player \(i\). Finally, given these definitions of \(T, J_i\), we can define the outcome induced by the fixed strategies. The expected payoff to equityholders, conditional on a starting state \((\delta, R, s)\) and following the fixed stationary strategies, can be written as

\[ E(\delta, R, s) = \mathbb{E}(\delta, R, s)[1(T < T_e)e^{-rT}J_e(\delta_T, R_T, s_T)] \]  \(17\)

while the expected payoff to creditors is

\[ D(\delta, R, s) = \mathbb{E}(\delta, R, s)[1(T < T_e)e^{-rT}J_d(\delta_T, R_T, s_T) + 1(T > T_e)e^{-rT_e}(\zeta \delta_{T_e} - R_{T_e})] \]  \(18\)

The expected payoffs take into account the possibility of a forced conversion, in which case equityholders receive 0 and debtholders receive \(\zeta \delta_{T_e} - R_{T_e}\). Given the expected payoffs \(E(\delta, R, s), D(\delta, R, s)\) induced by stationary strategies, we can define our equilibrium concept.

**Definition:** A Markov Perfect Equilibrium (MPE) consists of a stationary strategy \((O_i, A_i, \omega_i)\) for each player such that

1. Taking the opponents’ strategies as given, for every \((\delta, R, s)\), player \(e\)’s strategy maximizes \(E(\delta, R, s)\) and player \(d\)’s strategy maximizes \(D(\delta, R, s)\).

2. Player \(e\) finds it optimal to set an acceptance policy \(A_e(\delta, R) = [E(\delta, R, d), \infty)\) and player \(d\) finds it optimal to set an acceptance policy \(A_d(\delta, R) = [D(\delta, R, e), \infty)\).
Our definition of an MPE is highly intuitive. Condition 1 ensures that the equilibrium strategies correspond to a Nash equilibrium in stationary strategies for any starting values. Condition 2 is our notion of subgame perfection in continuous time: players must accept offers if and only if the offer exceeds their continuation value in the equilibrium.\footnote{The standard concept of subgame perfection in discrete time requires optimality starting in any history. Given discounting, the one shot deviation principle implies it is sufficient that players behave optimally in any history, conditional on both players returning to equilibrium strategies the next period. If we fix a stationary Nash Equilibrium (i.e., strategies satisfying Condition 1) and impose such a condition on strategies in off equilibrium histories, this could never rule out any proposing strategies \( \omega_i, O_i \), since the proposer takes no action immediately after proposing. Clearly, it restricts accepting strategies by enforcing players accept offers greater than their equilibrium outcome.}

The value functions \( E(\delta, R, s), D(\delta, R, s) \) corresponding to a MPE solve a fixed point problem. Given the strategies, the expected equilibrium payoffs are \( E(\delta, R, s), D(\delta, R, s) \), and given the opponent’s strategy, each player finds it optimal to set an acceptance policy equal to their value function. Nonetheless, the fixed point problem simplifies the calculation of such equilibria, since now we only need to search for value functions, offer regions \( O_i \), and offer functions \( \omega_i \). The following lemma simplifies analysis further:

**Lemma 3.1** In any MPE, \( \omega_e(\delta, R) \leq D(\delta, R, e) \) and \( \omega_d(\delta, R) \leq E(\delta, R, d) \) for all \( \delta, R \). For any MPE, there exists another MPE with identical value functions in which all equilibrium offers are accepted and the above inequalities hold with equality for all \( \delta, R \).

The lemma is sufficiently obvious that we do not provide a proof.\footnote{Intuitively, there is no reason for player \( i \) to ever offer more to player \( j \) than necessary to make player \( j \) accept, and doing so leads to a strictly profitable deviation. Likewise, any offers which are accepted with probability 0 cannot affect the equilibrium outcome and can be ignored.} As a consequence of this lemma and the definition of MPE, it is without loss of generality to characterize a MPE by a collection of value functions \( E(\delta, R, s), D(\delta, R, s) \) and offer regions \( O_i \), with the interpretation that the game ends the first time \( (\delta, R, s) \in O_i \times \{i\} \) for any \( i \). The outcome is player \( i \) proposing an offer equal to player \( j \)'s value function, and player \( j \) accepting. Given this lemma, we can prove the bargaining outcome must be Pareto optimal:

**Proposition 1** In any MPE, \( E(\delta, R, s) + D(\delta, R, s) = V(\delta, R) \), where \( V(\delta, R) \) is the value function of a social planner who picks the efficient reorganization time:

\[
V(\delta, R) = \sup_{T_R \in F^{\delta, R}} E^{(\delta, R)} \left[ 1(T_R < T_c) e^{-r T_R (\theta \delta T_R - R_{T_R})} + 1(T_c < T_R) e^{-r T_c (\zeta \delta T_c - R_{T_c})} \right] \tag{19}
\]

The proof is given in appendix A, but it follows from three simple observations. First, the sum of the value functions cannot exceed \( V \). Second, letting \( T_R \) denote the optimal reorganization time solving (19), any player can deviate to force the game to end at the maximum of \( T_R \) and the equilibrium time \( T \). For this to not be profitable, each player must weakly prefer to receive their
terminal payoff at $T$ rather than $T \lor T_R$. The final observation is that in any cases where $T > T_R$, it must be that waiting until $T$ is just as good as waiting until $T_R$ or else the proposer at time $T_R$ has a profitable deviation.

### 3.3 The optimal timing of reorganization

In light of proposition 1, the first step in calculating the equilibrium is to find the social planner’s value function defined by the optimal stopping problem in (19). By standard dynamic programming arguments, in the region where continuation is optimal, the continuation value $V(\delta, R)$ solves a partial differential equation (PDE):

$$rV(\delta, R) = L V(\delta, R) + \iota[\xi \delta - R - V(\delta, R)]$$

where $L$ is a differential operator defined on smooth functions of $\delta, R$ by

$$Lf = \delta \mu f_\delta + \frac{\sigma^2}{2} \delta^2 f_{\delta\delta} - (1 - \tau) h \delta f_R$$

The first two terms are familiar from Ito’s lemma, and represent the sensitivity of the value function to changes in EBIT. The third term represents the fluctuations in the continuation value due to the accumulation of earnings. The final term in (20) captures the compensation for the risk of a forced conversion to Chapter 7 liquidation.

Following Bartolini and Dixit (1991), we solve the PDE by making a change of variables. In appendix B, we solve for the general solutions of equation (20). We impose the intuitive boundary condition that for fixed $R > 0$, the value function stays bounded by the liquidation value as $\delta \to 0$, since reorganization could never be optimal if $\delta = 0$ and $R > 0$. The unique solution of equation (20) satisfying this is $V(\delta, R) = \delta v(\frac{R}{\delta})$, where the function $v : \mathbb{R} \to \mathbb{R}$ is defined by

$$v(x) = A_3 x^\gamma M(-\gamma, -2(\gamma - 1) + \frac{2\mu}{\sigma^2}, -2h(1 - \tau) + \frac{\iota \xi + \frac{h(1-\tau)\mu}{r+\tau}}{r+\tau} - \frac{\iota x}{r+\tau})$$

In this definition, $A_3$ is an arbitrary constant, $\gamma$ is the negative root of the polynomial

$$0 = -(r + \iota - \mu) - \mu z + \frac{\sigma^2}{2} z(z - 1)$$

and $M(a, b, z)$ is the confluent hypergeometric function

$$M(a, b, z) = 1 + \frac{a}{b} z + \frac{1}{2!} \frac{a(a+1)}{b(b+1)} z^2 + \frac{1}{3!} \frac{a(a+1)(a+2)}{b(b+1)(b+2)} z^3 + ...$$

The confluent hypergeometric function can be thought of as a generalization of the exponential function.\(^{34}\)

\(^{33}\)Subscripts denote partial derivatives.

\(^{34}\)Since $\gamma < 0$, we show in Appendix B that the confluent hypergeometric portion of equation (22) is strictly positive and increases monotonically to 1 as $x \to \infty$. Of course, the $x^\gamma$ term decreases monotonically to 0 as $x \to \infty$, so this solution satisfies the previously mentioned boundary condition since $x = \frac{R}{\delta} \to \infty$ as $\delta \to 0$. 17
In the region where the social planner finds it optimal to immediately reorganize, we have $V(\delta, R) = \theta\delta - R$. Given the form of the value function in the continuation region, we conjecture there exists a threshold $\bar{x}$ such that immediate reorganization is optimal if and only if $x = \frac{R}{\delta} \leq \bar{x}$. In this case, $\delta v\left(\frac{R}{\delta}\right)$ should value match and smooth paste with $\theta\delta - R = \delta(\theta - \frac{R}{\delta})$ on the curve $\frac{R}{\delta} = \bar{x}$. This is equivalent to the following system:

$$A_3\bar{x}^\gamma M(-\gamma, -2(\gamma - 1) + \frac{2\mu}{\sigma^2}, \frac{-2h(1 - \tau)}{\sigma^2 \bar{x}}) + \frac{\nu\delta + \frac{h(1-\tau)\mu}{r + \tau - \mu}}{r + \tau} - \frac{\nu\bar{x}}{r + \tau} = \theta - \bar{x} \quad (23)$$

$$\frac{d}{dx}A_3\bar{x}^\gamma M(-\gamma, -2(\gamma - 1) + \frac{2\mu}{\sigma^2}, \frac{-2h(1 - \tau)}{\sigma^2 \bar{x}}) - \frac{\nu\bar{x}}{r + \tau} = -1 \quad (24)$$

This system of algebraic equations is simple to solve numerically. However, we still must verify that the optimal policy is in fact a barrier policy as conjectured. We prove the following proposition in Appendix B:

**Proposition 2** Suppose $A_3, \bar{x}$ solve (23, 24), and the following two conditions are met:

$$\bar{x} \leq -\frac{h(1 - \tau) + \mu\theta + \nu(\zeta - \theta) - r\theta}{r} \quad (25)$$

$$v(x) \geq \theta - x \quad (26)$$

where $v(x)$ is the function given in (22). Then the stopping time $T_R = \inf\{t : R_t < \bar{x}\delta_t\}$ solves (19) with associated value function

$$V(\delta, R) = \delta v\left(\frac{R}{\delta}\right) \quad R \geq \bar{x}\delta \quad (27)$$

$$V(\delta, R) = \theta\delta - R \quad R \leq \bar{x}\delta \quad (28)$$

The conditions of proposition 2 are intuitive: the first one ensures that reorganization does not happen too early according to the barrier strategy, while the second one ensures it does not occur too late. They are also easy to check numerically for a candidate $A_3, \bar{x}$.\footnote{We have yet to find a case where these conditions are not satisfied. It is likely possible to prove they must hold generically, but for our purposes such a result is irrelevant.} In summary, a social planner would watch the movement of the EBIT and the accumulation of the earnings and emerge from bankruptcy when the current EBIT is large or when the accumulated earnings have offset enough of the fixed cost of emerging from bankruptcy. To be clear, the fixed cost of exiting bankruptcy makes this analogous to a real option. For some firms, this option is “in the money” at default so the reorganization is instantaneous, while for other firms, the option value of reorganizing in the future leads to efficient delay and lengthy reorganizations.
3.4 Calculating the Split

From proposition 2, the social planner chooses to emerge from bankruptcy when $(\delta, R) \in O^* = \{(\delta, R) : R \leq \bar{x}\delta\}$. Proposition 1 then implies that the game cannot end when $(\delta, R) \notin O^*$. Intuitively, in the region where a single agent would optimally choose to wait, in equilibrium the proposer chooses to wait. Then the value function of each player in this region must equal the discounted expectation of receiving their value function a second later. If we conjecture that both value functions are smooth, then by a standard dynamic programming argument, this implies the following system of linked PDEs:

\[ rE(\delta, R, e) = LE(\delta, R, e) + \lambda_e[E(\delta, R, d) - E(\delta, R, e)] + \iota[0 - E(\delta, R, e)] \quad (29) \]
\[ rE(\delta, R, d) = LE(\delta, R, d) + \lambda_d[E(\delta, R, e) - E(\delta, R, d)] + \iota[0 - E(\delta, R, d)] \quad (30) \]
\[ rD(\delta, R, e) = LD(\delta, R, e) + \lambda_e[D(\delta, R, d) - D(\delta, R, e)] + \iota[\zeta\delta - R - D(\delta, R, e)] \quad (31) \]
\[ rD(\delta, R, d) = LD(\delta, R, d) + \lambda_d[D(\delta, R, e) - D(\delta, R, d)] + \iota[\zeta\delta - R - D(\delta, R, d)] \quad (32) \]

which must hold for all $(\delta, R) \notin O^*$. Next, recall that in the definition of a MPE, each player $i$ must find it optimal in every instant where $s_t \neq i$ to accept an offer equal to their value function. Player $i$’s outside option should they reject would be to wait a second and receive their value function. This suggests that for the players receiving offers, their value functions should always equal the discounted expectation of receiving their value function a moment later, even in the region where offers are made. If player $i$’s value function in state $s \neq i$ were ever strictly less than the expected discounted value of waiting a second, it is suboptimal for player $i$ to follow their equilibrium strategy of accepting offers equal to their value function. Likewise, if player $i$’s value function in state $s \neq i$ were ever strictly greater than the expected discounted value of waiting a second, then player $i$ should be willing to accept an offer just below their value function. This suggests that in an MPE with smooth value functions, the value functions should satisfy (29)-(32) for all $(\delta, R) \notin O^*$, and the receiving value functions $E(\delta, R, d), D(\delta, R, e)$ should satisfy (30, 31) everywhere. The following proposition, which is proved in appendix C, shows this constitutes an MPE.

**Proposition 3** Assume the conditions of proposition 2 hold. Let $E(\delta, R, s), D(\delta, R, s)$ be smooth functions such that $E(\delta, R, s) + D(\delta, R, s) = V(\delta, R)$. Assume (29)-(32) are satisfied for all $(\delta, R) \notin O^*$, and (30, 31) hold everywhere. Then the following strategy for each player $i$ constitutes a MPE, with value functions $E(\delta, R, s), D(\delta, R, s)$:

1. Offer player $j$ their value function if and only if $(\delta, R) \in O^*$
2. Accept an offer equal to or greater than player $i$’s value function at any time, for any $(\delta, R)$.

Proposition 3 allows us to calculate the MPE for our bargaining game. In appendix C we prove proposition 3 and calculate the unique smooth MPE value functions $E(\delta, R, s), D(\delta, R, s)$ in closed
form. The solution, which requires solving a linked system of PDEs, combines the methods of proposition 2 with Markov chain techniques appearing in Guo, Miao, Morellec (2005), among other papers.

4 Analysis of the decision to reorganize or liquidate and capital structure

4.1 The levered firm with the option to reorganize

First, we consider the decision of the equityholders in period 1 to enter Chapter 11. Ultimately, equityholders will have the option to reorganize or liquidate, but the first step to solve this problem is to ignore the option to liquidate. We assume the bargaining game starts with equityholders making proposals in the exclusivity period (i.e., in state $s_0 = e$). Thus if equityholders choose to enter Chapter 11, they receive $E(\delta, R, 0, e) - B$, where $E(\delta, R, s)$ is the unique smooth MPE value function for equityholders and $B$ is a fixed cost of entering bankruptcy. $B$ might represent, for example, the prepetition fees of section 3.1. The function $E(\delta, R, e)$ is calculated in closed form in appendix C, for simplicity of notation we define $E(\delta) = E(\delta, R, 0, e)$.

Prior to bankruptcy, equityholders receive cashflow $(1 - \tau)(\delta - C_0)dt$ per unit time, where $C_0$ is the optimally chosen coupon at time 0. In this section, we assume the firm only has the option to enter Chapter 11. In this case, equityholders choose a stopping time $T_B$ at which point the firm enters Chapter 11 to solve

$$E^B(\delta) = \sup_{T_B \in F^\delta} E^\delta[\int_0^{T_B} e^{-rt}(1 - \tau)(\delta_t - C_0)dt + e^{-rT_B}(E(\delta_{T_B}) - B)]$$

(33)

To solve for the optimal time to enter Chapter 11, we conjecture a lower barrier $\delta_B$ such that equityholders declare bankruptcy the first time $\delta_t \leq \delta_B$. Following the logic of Section 2.2, the value of equity prior to entering bankruptcy must be

$$E^B(\delta) = A_4 \delta^\psi + (1 - \tau)[\frac{\delta}{r - \mu} - \frac{C_0}{r}]$$

$A_4$ is an arbitrary constant and $\psi$ is again the negative root of

$$0 = -r + \mu z + \frac{\sigma^2}{2} z(z - 1)$$

The constant $A_4$ is determined by value matching and smooth pasting on the bargaining value at the point of bankruptcy. Using the closed form for $E(\delta)$, we solve the nonlinear system...

20
$A_4 \delta_B^\psi + (1 - \tau)\left[\frac{\delta_B}{r - \mu} - \frac{C_0}{r}\right] = \mathcal{E}(\delta_B) - B \quad (34)$

$A_4 \psi \delta_B^{\psi - 1} + (1 - \tau)\left[\frac{1}{r - \mu}\right] = \mathcal{E}'(\delta_B) \quad (35)$

Proposition 4 provides conditions analogous to proposition 2 under which the barrier strategy is optimal:

**Proposition 4** Suppose $A_4, \delta_B$ solve (34, 35), and the following two conditions are met:

$$-r(\mathcal{E}(\delta) - B) + \mu \delta \mathcal{E}'(\delta) + \frac{\sigma^2 \delta^2}{2} \mathcal{E}''(\delta) \leq -(1 - \tau)(\delta - C_0) \quad \delta \leq \delta_B \quad (36)$$

$$A_4 \delta^\psi + (1 - \tau)\left[\frac{\delta}{r - \mu} - \frac{C_0}{r}\right] \geq \mathcal{E}(\delta) - B \quad \delta \geq \delta_B \quad (37)$$

Then the stopping time $T_B = \inf\{t : \delta_t < \delta_B\}$ solves (33) with associated value function

$$E^B(\delta) = A_4 \delta^\psi + (1 - \tau)\left[\frac{\delta}{r - \mu} - \frac{C_0}{r}\right] \quad \delta \geq \delta_B \quad (38)$$

$$E^B(\delta) = \mathcal{E}(\delta) - B \quad \delta \leq \delta_B \quad (39)$$

The proof appears in appendix D. Finally, once we have solved for $\delta_B$, the calculation for the value of debt is straightforward. Debt has value

$$D^B(\delta) = A_5 \delta^\psi + \frac{C_0}{r} \quad (40)$$

and $A_5$ is calculated by value matching at $\delta_B$:

$$A_5 \delta_B^\psi + \frac{C_0}{r} = D(\delta_B, R_0, e) \quad (41)$$

Once we plug in the closed form solutions for $D(\delta, R_0, e), \mathcal{E}(\delta), \mathcal{E}'(\delta)$, equations (34-41) represent a system of algebraic equations which are easily solved numerically. Likewise, the second derivative in (36) is available in closed form, allowing us to numerically check the conditions for the verification.

### 4.2 The levered firm with the option to reorganize or liquidate

In this section, we consider the decision of the equityholders in time 1 to enter Chapter 11 or Chapter 7. In the previous subsection we solved for the equity value $E^B$ when equityholders may only choose Chapter 11, and showed the corresponding optimal stopping time $T_B$ is a barrier strategy with threshold $\delta_B$. In section 2.2, we derived the equity value $E^L$ when equityholders could only liquidate, with corresponding optimal liquidation time $T_L$ and associated threshold $\delta_L$. 

21
In this section, we study the decision of how to optimally choose a time of liquidation $T_L$ and time of bankruptcy $T_B$ to maximize

$$E_0(\delta) = \sup_{T_L, T_B \in F^\delta} \mathbb{E}^\delta \left[ \int_0^{T_B \wedge T_L} e^{-rt}(1 - \tau)(\delta_t - C_0) dt + 1(T_B < T_L)e^{-rT_B}[\mathcal{E}(\delta_{T_B}) - B] \right]$$ (42)

This decision is equivalent to picking a time of default $T_D = T_L \wedge T_B$ and whether to enter Chapter 7 or Chapter 11 at that time. Using our results from Section 3, the latter decision is trivial: either the bargaining value net of fixed costs $\mathcal{E}(\delta_{T_D}) - B$ is larger than zero, so Chapter 11 is optimal, or it is less than 0, so liquidation is optimal. Define

$$g(\delta) = \max(\mathcal{E}(\delta) - B, 0)$$ (43)

Then the decision of when to enter Chapter 7 or Chapter 11 is equivalent to

$$E_0(\delta) = \sup_{T_D \in F^\delta} \mathbb{E}^\delta \left[ \int_0^{T_D} e^{-rt}(1 - \tau)(\delta_t - C_0) dt + e^{-rT_D}g(\delta_{T_D}) \right]$$ (44)

Since $g$ is continuous and nonnegative, standard results (Øksendal (2003) Chapter 10) show that $E_0(\delta)$ exists, with associated exercise region $S = \{\delta : E_0(\delta) = g(\delta)\}$.

In reality, firms default in bad states of the world. However, if creditors have no rights in Chapter 11, then equityholders might use Chapter 11 in good states of the world as an opportunity to default on their existing debt, issuing more debt afterward to take advantage of the tax shield. Since Chapter 11 is an opportunity to reduce, not increase a firm’s debt, so we rule out this unrealistic case with the following assumption:

**Assumption 1:** The bargaining power of debtholders is high enough that

$$\lim_{\delta \to \infty} \frac{(1 - \tau)\delta}{r - \mu} = -\infty$$

This intuitive assumption says that as firms become infinitely profitable, the unlevered firm value exceeds the value to equity of defaulting and entering Chapter 11. We give a specific condition on underlying parameters that is sufficient for this in Appendix D. When this assumption holds, we can obtain a clean characterization for the equityholders’ optimal policy in (42).

**Proposition 5:** Suppose Assumption 1 holds. For any fixed $C$, let $S(C) = \{\delta : E_0(\delta) = g(\delta)\}$ denote the set of $\delta$ values where the firm defaults immediately, and let $\delta_L, \delta_B$ be the optimal liquidation and reorganization thresholds from Sections 2.2 and 4.1. Then $\tilde{\delta}(C) = \sup S(C)$ is finite. Further, $\tilde{\delta}(C)$ equals the liquidation trigger $\delta_L$ if and only if $\mathcal{E}(\tilde{\delta}(C)) \leq B$ and it equals the bankruptcy threshold $\delta_B$ if and only if $\mathcal{E}(\tilde{\delta}(C)) \geq B$.

---

36 And of course, for any fixed $C$, the levered firm value
This proposition says that for any fixed $C$, at a large enough $\delta$ the firm knows with certainty which of Chapter 11 or Chapter 7 they will eventually enter, and it will occur at a lower threshold. We next show that which of these occurs will depend on $C$:

**Proposition 6** Suppose Assumption 1 holds. The default threshold $\delta(C)$ is a weakly increasing and continuous function of $C$, and $\lim_{C \to \infty} \delta(C) = \infty$. There exists $\tilde{C}$ such that $E(\tilde{\delta}(C)) = B$ and $C > \tilde{C}$ implies $E(\tilde{\delta}(C)) > B$.

Proposition 6 delivers the central intuition of the choice between Chapter 7 and Chapter 11 in our model. When the firm has a larger coupon, they default at higher $\delta$ values. We see from Figure 2 that equity’s value in Chapter 11 $E(\delta)$ is strictly increasing in $\delta$, so equity’s prospects in Chapter 11 are more likely to justify the fixed cost $B$ of entering Chapter 11 when $\delta$ is high. Proposition 6 shows the existence of a $\tilde{C}$ such that when the firm has issued more debt than $\tilde{C}$, they will default at a sufficiently profitable $\delta$ that Chapter 11 is preferable to Chapter 7 at that $\delta$. Given the strict monotonicity of $E$ and $\tilde{\delta}(C)$ we observe numerically, equityholders will strictly prefer liquidation for $C < \tilde{C}$, by the same logic.

### 4.3 Analysis of the capital structure with Chapter 11 Reorganization

In this section, we consider the decision of equityholders at time 0 of how much debt to issue. For simplicity of notation, let $F_0(\delta_0, C_0) = E_0(\delta_0, C_0) + D_0(\delta_0, C_0)$ denote the firm value for a given coupon $C_0$. Likewise, let $F_j(\delta_0, C_0) = E_j(\delta_0, C_0) + D_j(\delta_0, C_0)$, $j = L, B$ denote firm value assuming a future Chapter 7 ($j = L$) or Chapter 11 ($j = B$) bankruptcy. Since equityholders receive the proceeds of the initial debt issue, at time 0 equity chooses the optimal coupon $C_0$ to maximize the sum of the values of equity and debt $F_0(\delta_0, C_0)$, subject to the constraint that equity will subsequently decide between Chapter 7 and Chapter 11 to maximize equity value. Under Assumption 1, as long as $\delta_0$ is large relative to $C_0$, equity will know a second after they issue debt whether they will eventually enter Chapter 7 or Chapter 11 (proposition 5), so $E_0(\delta_0, C_0) = \max\{E^B(\delta_0, C_0), E^L(\delta_0, C_0)\}$ and

\[
\begin{align*}
F_0(\delta_0, C_0) &= F^B(\delta_0, C_0) & E^B(\delta_0, C_0) > E^L(\delta_0, C_0) \tag{45} \\
F_0(\delta_0, C_0) &= F^L(\delta_0, C_0) & E^B(\delta_0, C_0) < E^L(\delta_0, C_0) \tag{46} \\
F_0(\delta_0, C_0) &= \max(F^L(\delta_0, C_0), F^B(\delta_0, C_0)) & E^B(\delta_0, C_0) = E^L(\delta_0, C_0) \tag{47}
\end{align*}
\]

In words, the value of the firm for a given coupon is either the value of the firm conditional on eventual liquidation, or the value of the firm conditional on eventual Chapter 11. Which of these cases occurs is determined by which is better ex-post for equityholders. As is standard in dynamic models of capital structure, equityholders lack commitment power. For a given coupon $C_0$, equityholders might be able to get a better price on debt (and higher overall time 0 value) if

---

37 We give numerically verifiable conditions for this in appendix D.
they could commit to a future Chapter 7. However, if that $C_0$ implies equity will prefer Chapter 11, the debt will be priced at time 0 under the rational expectation of future Chapter 11. From Proposition 6, and the observation that in all numerical examples $E(\delta)$ is strictly increasing, we can obtain a cleaner characterization of how debt will be priced at a given coupon: there will always exist $\bar{C}$ such that for large $\delta_0$,

$$F_0(\delta_0, C_0) = \max\{F_L(\delta_0, C_0), F_B(\delta_0, C_0)\}$$

$$C_0 = \bar{C}$$

It follows that at time zero, the firm has two options. They may choose the coupon $C_{B,\text{constr}} \in [\bar{C}, \infty)$ that maximizes the firm value $F_B(\delta_0, \cdot)$ under the rational expectation of a future Chapter 11 reorganization, or they may choose the coupon $C_{L,\text{constr}} \in [0, \bar{C}]$ that maximizes the firm value $F_L(\delta_0, \cdot)$ under the rational expectation of a future Chapter 7 liquidation.$^{38}$ We distinguish these constrained solutions from the unconstrained solutions $C_B, C_L$ which maximize $F_B(\delta_0, \cdot)$ and $F_L(\delta_0, \cdot)$ over all possible coupons. In the next section, we examine cases where the constrained solutions and unconstrained solutions coincide, and cases where the constraints imposed by lack of commitment bind.

Using the solutions derived in the last two sections (and Section 2.2), we numerically maximize both functions $F_B(\delta_0, \cdot)$, $F_L(\delta_0, \cdot)$ on a grid of possible coupons, and calculate the optimal coupon $C^*$ as the one of $C_{B,\text{constr}}, C_{L,\text{constr}}$ which leads to a higher firm value. The procedure is summarized in the following equations.

$$\bar{C} = \max\{C_0 : E_B(\delta_0, C_0) \leq E_L(\delta_0, C_0)\}$$

$$C_{L,\text{constr}} = \arg\max_{C_0 \in [0, \bar{C}]} F_L(\delta_0, C_0)$$

$$C_{B,\text{constr}} = \arg\max_{C_0 \in [\bar{C}, \infty)} F_B(\delta_0, C_0)$$

$$F_0(\delta_0, C^*) = \max\{F_L(\delta_0, C_{L,\text{constr}}), F_B(\delta_0, C_{B,\text{constr}})\}$$

Finally, in presenting the results it will be useful to have notation for the optimal coupons equity would choose if they only had the option of Chapter 7 or only Chapter 11:

$$C_L = \arg\max_{C_0 \in [0, \infty)} F_L(\delta_0, C_0)$$

$$C_B = \arg\max_{C_0 \in [\bar{C}, \infty)} F_B(\delta_0, C_0)$$

$^{38}$Numerically, we evaluate $F_L(\delta_0, \cdot), F_B(\delta_0, \cdot)$ on a grid and always find a unique optimizer.
5 Capital Structure and Empirical Predictions

5.1 Capital Structure Decisions and the Relative Efficiency of Chapter 11

In this section, we analyze the optimal capital structure of the firm with the option to reorganize or liquidate. As is standard in capital structure models, the equityholders internalize the inefficiency of their ex-post optimal bankruptcy procedure when they issue debt. Put differently, the price equityholders can charge for their debt will exactly reflect the inefficiency of their future preferred bankruptcy procedure. When one form of bankruptcy (Chapter 11 or Chapter 7) is so inefficient relative to the other that equityholders would never find it optimal ex-post, equityholders can credibly ignore that option. In these cases, equityholders are unconstrained by their lack of commitment when choosing the coupon to maximize the tax benefits given their preferred future bankruptcy form. Debtholders will correctly infer the future strategy of equityholders when they price the debt.

However, when Chapter 11 is slightly less efficient than Chapter 7, our model predicts a more nuanced capital structure decision. In this region of the parameter space, debtholders prefer Chapter 7 liquidation, since Chapter 11 reorganization only allows them to capture a fraction of a slightly smaller pie. Equityholders would like to issue a large coupon to take advantage of tax benefits, and commit to future liquidation to obtain a low cost of debt. However, for these parameters, the result of Proposition 6 implies that these two goals conflict with each other: large coupons imply equityholders will ex-post find Chapter 11 optimal. Debtholders recognize this and pay less for debt with such a coupon at time 0. Since equityholders cannot formally commit to Chapter 7, they have two choices. They can issue a large coupon to maximize tax benefits, and accept that debtholders will charge extra for the future Chapter 11 inefficiencies. Alternately, equityholders can issue the largest coupon $\bar{C}$ consistent with Chapter 7 being optimal for equity ex-post. This allows equityholders to get a better price for the debt they issue, but they forgo tax benefits since for these parameters $\bar{C}$ is smaller than the coupon they would otherwise issue.

To an econometrician, in this latter case our model looks identical to the Leland model: a firm issues debt then eventually liquidates. However, the off-equilibrium considerations introduced by our bargaining model lead the firm to issue a much smaller coupon than in the standard Leland model. In this case, our model predicts lower leverage than the Leland model, even for the 65% of firms that liquidate in Chapter 7 (BCI (2017)).

To illustrate the capital structure decision in more detail, we now present examples of each case. First, we briefly motivate our baseline parameters. Our model shares parameters $\mu, \sigma, \tau, r, \alpha$ with the standard Leland model. For these parameters, we follow the literature (see Strebulaev and Whited (2012) Table 3). For bargaining power parameters, we choose $\lambda_e$ to correspond to the exclusivity period in Chapter 11. Specifically, since equityholders begin with a 120 day window (or longer) to exclusively make offers, we choose $\lambda_e = 3$ so that in expectation the equityholders first offer window lasts 120 days.\footnote{All parameters are annualized.} To start the analysis, we set $\lambda_d = \lambda_e$. For the rate of forced conversion to Chapter 7, we set a baseline value of $\iota = 0.06$, which corresponds to the 14% of
Chapter 11 cases converted to liquidations in the sample of (BWZ, 2006), given the average length of 2.5 years for a Chapter 11 case. For ease of interpretation, we use $\delta_0 = 1$ for all our numerical analysis. Firm values may then be interpreted as years of earnings. These baseline parameters are listed in table 1, and unless otherwise stated all analysis uses these values.

The parameters $h, R, B$ representing the inefficiencies of Chapter 11 are difficult to quantify with empirical analysis, so we vary these to illustrate the different cases mentioned previously. To succinctly describe regions of the $(h, R, B)$ parameter space, we introduce two measures of efficiency. The first measure is $\text{RelEff}$, which is the ratio of total firm value upon entering Chapter 11 to the total firm value at the moment of liquidation:

$$\text{RelEff} = \frac{V(\delta_{\text{default}}, R_0) - B}{\zeta \delta_{\text{default}}}$$

Recall the numerator is the total firm value in Chapter 11 reorganization, which incorporates a partial loss of earnings during Chapter 11 and a fixed cost of exiting Chapter 11, minus the fixed cost $B$ of entering Chapter 11. The denominator is the liquidation value debtholders receive by selling the assets for their perpetuity value minus proportional liquidation costs. Since Chapter 11 entails fixed costs in our model while Chapter 7 does not, the value of this ratio is sensitive to the $\delta$ at which it is evaluated. Intuitively, spending several years in court over a firm worth one dollar would be extraordinarily wasteful relative to liquidating such a firm, regardless of the overall efficiency of each procedure. We thus exogenously set $\delta_{\text{default}} = 0.3674$, which is motivated by Corbae and D’Erasmo (2017).\(^{40}\)

$\text{RelEff}$ has no direct significance in the solution of our model, but is helpful for concisely summarizing the inefficiencies of Chapter 11 and Chapter 7 without delving into the optimal strategy of equityholders.\(^{41}\) The extent to which these inefficiencies impact firm value is of course endogenous. It will be helpful to define a second measure which measures how much firm value is changed by the added option of Chapter 11:

$$\text{Choicevalue} = \frac{F_0(\delta_0, C^*)}{F_L(\delta_0, C_L)}$$

The numerator is the time zero value of the firm with the option to liquidate or enter Chapter 11, evaluated at the optimal coupon. The denominator is the time zero firm value in the Leland model with only Chapter 7, evaluated at the corresponding optimal coupon. This measure provides clearer intuition on how the optimal strategy changes with the Chapter 11 parameters. We will use $\text{Choicevalue}$ to partition the space of $(h, R, B)$ values into cases corresponding to distinct optimal strategies, then reference $\text{RelEff}$ to describe the exogenous inefficiencies which induce each case.

**Case 1:** $\text{Choiceval} > 1$. Suppose that Chapter 7 liquidation is less efficient than Chapter 11 reorganization. Since the tax benefits of debt are large empirically, equityholders like to issue

---

\(^{40}\)This number is the ratio of average firm assets for firms entering Chapter 11 to average firm assets of healthy firms in table 1 of Corbae and D’Erasmo (2017).

\(^{41}\)In particular, there is no theorem stating that $\text{Choicevalue} > 1$ if and only if $\text{RelEff} > 1$. 


large coupons. Such coupons imply equityholders will default in profitable states of the world (Proposition 6), and these are the states of the world where equity’s prospects in Chapter 11 justify the fixed costs of entering Chapter 11 (Proposition 5). Debtholders might dislike sharing the firm with equityholders in Chapter 11, but since Chapter 11 is much more efficient, the overall pie is bigger. Equityholders are thus willing to take a higher cost of debt associated with Chapter 11 being ex-post optimal, since they are compensated by the rents they eventually extract in Chapter 11, and they issue a large coupon to take full advantage of the tax shield.

Figure 3 plots firm value as a function of \( C_0 \), the time zero perpetual coupon on the consol debt, for a parameter set where Chapter 11 is 15% more efficient than Chapter 7 by our metric RelEff. The blue dotted line plots \( F^L(\delta_0, C_0) \), the firm value under the assumption of future liquidation, as a function of the coupon \( C_0 \). As usual, the tradeoff between the tax shield of debt and the efficiency loss in liquidation leads to an inverted U shape for firm value as a function of the coupon. The dotted line descending from the peak of the blue curve marks \( C_L \), the optimal unconstrained coupon under future liquidation, on the x-axis. The black dashed line plots \( F^B(\delta_0, C_0) \), the firm value under the assumption of future Chapter 11. Again there is a tradeoff between the tax shield of debt and the inefficiency of Chapter 11. The dotted line descending from the peak of the black dashed curve marks \( C_B \), the unconstrained optimal coupon under future Chapter 11 reorganization, on the x-axis. Since we have assumed here that the bankruptcy costs in Chapter 11 are less extreme than those in Chapter 7, the optimal coupon \( C_B \) is larger than \( C_L \) as expected.

The red line plots the actual firm value \( F_0(\delta_0, C_0) \) as a function of the coupon \( C_0 \). Since equityholders lack commitment power, \( F_0 \) is either equal to \( F^B \) or \( F^L \), depending upon whether equityholders will subsequently find it optimal to enter Chapter 11 or liquidate. The first vertical dotted line marks \( \bar{C} \), the threshold coupon for Chapter 11 vs Chapter 7, on the x-axis. For \( C < \bar{C} \), the red curve \( F_0 \) follows the liquidation value \( F^L \) since the firm will subsequently find it optimal to liquidate. For \( C \geq \bar{C} \), the red curve follows \( F^B \), since equityholders will subsequently enter Chapter 11. In particular, if equityholders want to sell debt for the value under liquidation \( D^L \), the largest coupon they may issue is \( \bar{C} \). Since Chapter 11 is good for firm value in this instance, equityholders find it optimal to issue \( C_B \) and credibly signal a future Chapter 11 reorganization, since this is the maximal point on the red curve. Equityholders are thus unconstrained by their lack of commitment when they decide to maximize the firm value under future Chapter 11.

**Case 2: \( \text{Choiceval} = 1 \).** Another possible case is that Chapter 7 liquidation is much more efficient than Chapter 11 reorganization. In this case, if equity had commitment power, they would get the greatest time zero value by committing to a future Chapter 7, and issuing debt with the coupon \( C_L \) that maximizes the firm value conditional on future Chapter 7. However, if Chapter 11 reorganization is very inefficient, there will be no commitment problem. Specifically, in an inefficient Chapter 11 process, equityholders would have to default in a very profitable state of the world in order for their bargaining prospects to justify the fixed costs of Chapter 11. They will only default in such a state of the world if they have issued debt with a coupon \( \bar{C} \) much larger than \( C_L \). So even without formal commitment power, equity can issue their favorite coupon \( C_L \) and credibly...
promise a future Chapter 7 liquidation. This is depicted graphically in Figure 4, with a parameter set that corresponds to a Chapter 11 process that is 50% less efficient than Chapter 7.

The interpretation of Figure 4 is exactly the same as Figure 3. The point on the x-axis where the red curve $F^0$ drops down from the blue curve $F^L$ to the black curve $F^B$ corresponds to the threshold coupon $\bar{C}$. In this figure, we see that the red curve $F^0$ is maximized at the point where $F^L$ is maximized, with corresponding coupon $C_L$. Since $C_L < \bar{C}$, equityholders are able to issue $C_L$, credibly promise a future liquidation, and receive $F^L$. In this sense, equityholders are unconstrained in their decision to maximize the firm value under future liquidation.

**Case 3: **$\text{Choiceval} < 1$. Perhaps the most interesting case is when the Chapter 11 procedure is less efficient than Chapter 7, but efficient enough that equityholders still find it attractive ex-post for reasonable levels of debt. The inefficiency of Chapter 11, combined with equity’s lack of commitment power, will actually reduce firm value in this region, relative to the value if Chapter 11 were not an option. To see this intuitively, suppose that Chapter 7 is slightly more efficient than Chapter 11. In such a situation, equityholders might be able to get a much lower cost of debt by committing to a future Chapter 7 liquidation. If equityholders had commitment power, in this case they would promise a future Chapter 7 liquidation and issue the coupon $C_L$ that optimally trades off tax benefits with the liquidation costs. However, such a coupon $C_L$ would imply that equityholders default in a profitable state of the world, where the reasonably efficient Chapter 11 is appealing to equityholders. Since equityholders lack commitment power, debtholders recognize that the coupon $C_L$ will correspond to a future Chapter 11, and charge a higher cost of debt.

This leaves the equityholders with two choices. If the higher cost of debt associated with Chapter 7 is small in magnitude relative to the tax benefits of debt, equityholders will optimally issue the coupon $C_B$ which maximizes tax benefits relative to Chapter 11 inefficiencies. In this case, equityholders are optimally choosing an inefficient Chapter 11 process and a high cost of debt, because it allows them to capture tax benefits. An example of this case is shown in figure 5, which plots firm value as a function of the coupon $C_0$, for parameters with RelEff= 0.85.

Figure 5 has the same interpretation as Figures 3 and 4. The highest firm value is associated with coupon $C_L$ on the blue curve depicting firm value under future liquidation. However, Chapter 11 is sufficiently attractive that $\bar{C} < C_L$, so equity’s lack of commitment power prevents them from obtaining this firm value. The highest attainable value on the red curve corresponds to $C_B$, the optimal coupon given a future Chapter 11. Thus, even though Chapter 11 is less efficient than Chapter 7, the tax benefits of a large coupon outweigh the increased cost of debt so equity optimally chooses Chapter 11. We call this the “optimal inefficient Chapter 11” strategy.

The other choice that equity can make in Case 3 is to issue $\bar{C}$. If the higher cost of debt associated with Chapter 11 is large relative to the tax benefits of a larger coupon, equityholders will sacrifice some tax benefits to issue a coupon that credibly commits them to a future Chapter 7. Specifically, equityholders will optimally issue $\bar{C}$, the largest coupon such that they will subsequently find Chapter 7 optimal.\footnote{We model the coupon choice as a grid search over finitely many values so such a point exists. If one views this}
Figure 6 plots firm value as a function of the initial coupon $C_0$, with a parameter set corresponding to RelEff = 75%. Once again, the highest point on the graph occurs on the blue curve, at $F^L(\delta_0, C_L)$, which corresponds to the firm value if equity could issue $C_L$ and commit to Chapter 7. However, this value is unattainable for equityholders. If they issue debt larger than $\bar{C}$, which is the point on the x axis where the red line drops from the blue curve to the black curve, then the value they receive is $F^B$. As a result, the best equityholders can do is to issue $\bar{C}$ and receive $F^L(\delta_0, \bar{C})$. We refer to this as the “constrained debt Chapter 7” strategy.

We reiterate that the equityholders optimally issue a lower coupon than in the Leland (1994) model, even though they subsequently face the exact same liquidation costs. This is because we have found a novel agency cost of debt: it encourages equityholders to destroy firm value in Chapter 11 bankruptcy. The coupon which optimally trades off tax benefits with liquidation costs and this novel agency cost is lower than the one predicted by the Leland model.

5.2 Results on Capital Structure

In Graham (2000), he finds that “paradoxically, large, liquid, profitable firms with low expected distress costs use debt conservatively” and “the typical firm could double tax benefits by issuing debt until the marginal tax benefit begins to decline.” From the lens of the Leland model, this is a puzzle. For example, when Strebulaev and Whited (2012) describe the endogenous default model, they write

“The leverage ratio in the endogenous default case varies between 70% (liquidation) and 80% ... far in excess of the observed leverage ratios. For example, the equally-weighted quasi-market leverage ratio in the COMPUSTAT universe consistently appears within the 20-25% range.”

In Table 2, we present outcomes from our model and the Leland model for a variety of parameter values. We start with the baseline parameters from Table 1, and parameters $h = 0.9, R = 4, B = 0.3$ for which the “constrained debt Chapter 7” strategy is optimal. As can be seen from the first row, this corresponds to a RelEff of 75% and an ex-post optimal liquidation (column “Liq?”). For these parameters, Chapter 11 is efficient enough that equityholders find it attractive ex-post, but inefficient enough that their best strategy is to issue the lower coupon $\bar{C}$ to prevent a future Chapter 11. The first three columns give the optimal coupon $C_L$ and leverage from the Leland model with only Chapter 7, and the value of the levered firm $F^L$ relative to the unlevered firm value $U = \frac{(1-r)\delta_0}{r-\mu}$. The remaining columns give the corresponding optimal coupon, leverage, firm value, credit spread, and bankruptcy procedure for the model with Chapter 7 and Chapter 11. For the “constrained debt Chapter 7” strategy considered in the first row, equityholders must issue a lower coupon $\bar{C}$ in order to credibly commit to a future liquidation, relative to the optimal

---

43We follow the literature in evaluating leverage as $\frac{D_0(\delta_0, C^*)}{E_0(\delta_0, C^*)}$, the value of debt at issuance divided by the sum of equity and debt values at issuance, all evaluated at the optimal coupon.
coupon $C_L$ they would choose in the absence of a Chapter 11 option. Specifically, in the first row the optimal coupon in the Leland model is $C_L = 1.36$ with a corresponding leverage ratio of 70%, while in our model equity optimally issues $\bar{C} = 0.64$ with a leverage ratio of just 40%. As noted in Graham (2000), this makes it look like firms are not optimally taking advantage of tax benefits. The columns titled $F^L$, $F^U$ compare the time zero levered firm value (with just liquidation and both Chapter 7 and liquidation, respectively) to the unlevered firm value. Without the Chapter 11 option, the tax benefits of debt (net the liquidation inefficiencies) would add 11% to the unlevered firm value, while in order to credibly commit to Chapter 7 firms can only add 8% to their unlevered firm value.

**Chapter 11 Efficiency and Capital Structure:** In this region of the parameter space, anything which makes Chapter 11 less appealing will increase $\bar{C}$. Intuitively, when Chapter 11 gets worse for equityholders, it is easier for them to promise not to file for Chapter 11, which lets them issue a higher coupon while receiving the lower cost of debt corresponding to Chapter 7. When we increase $R$ from 3.5 to 4, the optimal coupon (which in this case is $C^* = \bar{C}$) increases from 0.61 to 0.64. Leverage increases from 38% to 40%. Since the marginal benefit of debt is positive, this increases firm value from 107% of unlevered firm value $U$ to 108% of $U$. Counterintuitively, a less efficient Chapter 11 process is actually increasing firm value. This general effect can be observed in many parameters which affect the relative attractiveness of Chapter 11 for equityholders. Increasing the rate of conversion $\iota$ to Chapter 7, exacerbating the haircut to cashflows $h$, and increasing the cost to equity $B$ of entering Chapter 11 all lead to higher leverage and firm value. Of course, all of these results depend upon the firm finding the “constrained debt Chapter 7” strategy optimal. In Table 3, we present the same outcomes with $h = 1$, $R = 3.35$, $B = 0.1$ (RelEff=85%), for which the “optimal inefficient Chapter 11” strategy is optimal. Here the firm is optimally choosing Chapter 11, so for small declines in the efficiency of Chapter 11 (driven by $h, R, B$) the firm optimally reduces debt. This is the same logic that drives firms to lower leverage in the Leland model when the liquidation inefficiency $\alpha$ rises. This decline in efficiency is accompanied by a decline in firm value. However, an increase in $\iota$, the rate of conversion to Chapter 7, can still increase leverage. While such an increase makes Chapter 11 slightly less efficient, it also makes Chapter 11 much better for debtholders since it endogenously increases their bargaining power.44 As a result, equityholders take advantage of the lower cost of debt by issuing more debt, increasing firm value.

**Creditor Rights and Capital Structure:** There is a growing empirical literature examining the real effects of creditor rights. Li, Whited, and Wu (2016) study the enactment of antirecharacterization laws in seven states in the late 1990s and early 2000s. These laws protected the rights of creditors who used special purpose vehicles to conduct secured borrowing, and several papers argue these represent an exogenous increase in creditor rights. Li et al (2016) find this led to an increase in leverage. Mann (2015) studies the same laws and also finds an increase in long term debt over assets.

---

44A forced conversion benefits debtholders more than equityholders. Increasing the rate of forced conversions thus increases the outside option of debtholders and consequently their bargaining power.
In our model, creditor rights might reasonably be interpreted as the relative bargaining power of debtholders. Laws like the anticharacterization laws certainly reduce the relative bargaining power of equityholders since they may no longer hold up creditors with the threat of recharacterizing assets held in special purpose vehicles. As noted in Section 3, the timing in the dynamic bargaining game is not affected by the relative bargaining power of creditors and debtors. However, the bargaining power of creditors affects the fraction of firm value that can be captured by equityholders in Chapter 11, which factors into the capital structure decision of equityholders. Recall that lower $\lambda_d$ values correspond to longer offer windows for debtholders and stronger creditor rights. In the “constrained debt Chapter 7” strategy considered in Table 2, an increase in creditor rights from $\lambda_d = 10$ to $\lambda_d = 3$ increases leverage from 33% to 40%. This is the same mechanism discussed above: when creditor rights improve, Chapter 11 becomes less attractive to equityholders, so they may issue a larger coupon $\bar{C}$ while still credibly committing to a Chapter 7. In Table 3 where Chapter 11 is optimal, better creditor rights still lead to an increase in leverage, because it makes Chapter 11 more appealing to debtholders and lowers the cost of debt. Thus our model generally predicts that higher creditor rights lead to higher leverage, consistent with the empirical evidence. The only exception to this is when an increase in creditor rights pushes equityholders from Chapter 11 to Chapter 7. For example, in Table 3, when creditor rights increase from $\lambda_d = 0.5$ to $\lambda_d = 0.25$, equityholders prefer the lower cost of debt associated with Chapter 7 to the rents they can extract in Chapter 11, so they drastically reduce their coupon from $C_B$ to $\bar{C}$ and choose liquidation instead of Chapter 11.

In most cases, when creditor rights improve, the resulting increase in leverage leads to an improvement in firm value. Under the “constrained debt Chapter 7” strategy, this is because the marginal benefits of debt for the firm are positive at the optimum and equity’s constraint becomes looser with higher creditor rights. Thus the increase in debt has a net positive effect on firm value. Under the “optimal inefficient Chapter 11” strategy (Table 3), when creditor rights improve equity waits longer to default for any given coupon. Thus equity can issue a larger coupon, increasing the tax benefits of debt, while ex-post they will still default at a similar $\delta_B$ so the discounted inefficiencies of Chapter 11 stay constant. As a result, our model predicts that higher creditor rights should improve firm value. There is empirical evidence for this comparative static. Ponticelli and Alencar (2016) find an increase in value after an increase in the enforceability of creditor rights, and Ersahin (2017) finds greater productivity after the antirecharacterization laws discussed previously. However, these empirical results have different mechanisms than the tax benefits of debt which drives the result in our model.

Other Model Primitives: In general, the effects of $r, \mu, \sigma, \tau, \alpha$ on capital structure are similar in our model to in the Leland (1994) model. However, the commitment problem of equityholders leads to some new implications. For example, in Table 2, an increase in $\mu$ from 0.01 to 0.02 increases the relative efficiency of Chapter 11, making it more attractive to equityholders. As a result, equityholders must issue a lower coupon with lower leverage to credibly commit to Chapter 7, which they do at the optimum. This is in contrast to the standard mechanism that higher $\mu$
means higher expected cashflows and thus higher expected tax benefits, implying higher optimal leverage. Nonetheless, in most cases this standard mechanism prevails in our model as well (for example, Table 3). In the Leland model, higher volatility can increase or decrease the coupon, lowers leverage, and lowers the overall levered firm value. Most of these hold true in our model, but the last effect means that the reorganized firm value is lower when volatility is higher. This tends to lower the relative efficiency of Chapter 11, so for the “constrained debt Chapter 7” strategy this can lead to higher leverage in our model.

Higher taxes typically increase the tax benefits of debt and lead to higher leverage. However, Chapter 11 includes an embedded option to relever upon reorganizing, making it more appealing to equityholders. Table 2 presents an example where, when taxes increase from \( \tau = 0.15 \) to \( \tau = 0.2 \), the commitment effect outweighs the time zero increase in tax benefits and the firm optimally lowers its coupon to commit to Chapter 7. There are also cases in our model where higher taxes increase leverage (Table 3). Finally, even when liquidation inefficiencies are tiny (\( \alpha = 0.005 \)), the commitment problem in our model can lead to leverage as low as 44%, compared to 78% in the Leland model.

### 6 Additional Results and Extensions

#### 6.1 The Decision to Enter Chapter 7 or Chapter 11

There is a lot of interest in empirical research about the causal effect of bankruptcy procedure on future firm asset performance. The main challenge in such work is overcoming the selection bias, that firms choosing Chapter 11 are inherently different from those choosing Chapter 7. Any statements our model might generate about the causal effect of Chapter 11 vs Chapter 7 would be dependent on the parameter values we assume. However, our model generates much more general predictions about what types of firms choose Chapter 11 or Chapter 7.

**Profitability, Asset Value, and Choice of Bankruptcy Procedure:** In our model, when equityholders default, they choose Chapter 11 if and only if their value function in the subsequent bargaining justifies their fixed cost of entering Chapter 11 (proposition 5). Since the value function is increasing in the current EBIT (\( \delta \)), this implies that firms which are more profitable at default will choose Chapter 11. It is standard in the Leland model to define the firm asset value as the unlevered firm value \( U \), which is linear in \( \delta \), so this also implies firms with more valuable assets at default should choose Chapter 11. These predictions of our model are supported by BWZ (2006), BCI (2017), and Corbae and D’Erasmo (2017). Specifically, in table 1 of BWZ (2006), they find that the average asset value of firms entering Chapter 11 is nearly four times as large as the average asset value of firms entering Chapter 7. Their table 2 shows in a Probit model that conditional on being a reasonable size, firms with more valuable assets are more likely to choose Chapter 11. Table 1 of Corbae and D’Erasmo (2017) similarly shows that firms entering Chapter 11 are roughly four times as large as those entering Chapter 7, and table 1 of BCI (2017) shows firms in Chapter 11 have four times as many plants as firms entering Chapter 7. While none of these tables show
statistics on unnormalized EBIT, firms entering Chapter 11 have a higher EBITDA normalized by assets than firms entering Chapter 7 (table 1 of Corbae and D’Erasmo (2017)). Also, multiplying the median firm’s EBITDA over assets by the median firm’s assets in the same table suggests a higher unnormalized EBITDA for firms entering Chapter 11.

**Debt and Chapter 11:** In proposition 6, we show that firms with higher coupons tend to choose Chapter 11 (specifically, those with a coupon above some threshold \( \bar{C} \)). This implies the prediction that defaulting firms should be more likely to choose Chapter 11 when they have a lot of debt. Table 1 of BWZ (2006) shows that firms entering Chapter 11 have a higher Debt/Assets ratio, and combining this with the higher denominator for firms in Chapter 11 mentioned previously, firms entering Chapter 11 have more debt. This is also a significant predictor of Chapter 11 in their Probit regressions. Table 1 of Corbae and D’Erasmo (2017) confirms these findings, consistent with our model. In summary, propositions 5 and 6 characterize the decision between Chapter 7 and Chapter 11 in our model, and both of the mechanisms enjoy empirical support.

**Creditor Rights and Chapter 11:** As mentioned previously, higher creditor rights (lower \( \lambda_d \)) make Chapter 11 less appealing to equityholders, and thus tend to encourage Chapter 7. This can be seen, for example, in the decrease from \( \lambda_d = 0.5 \) to \( \lambda_d = 0.25 \) in Table 3. One empirical proxy for creditor rights within a particular firm is the number of secured creditors. As discussed in BWZ (2006), a large number of secured creditors will have a harder time coordinating in Chapter 11 - while our model could be extended to accommodate multiple creditors, we consider this in reduced form as a decrease in creditor bargaining power. BWZ (2006) find that firms with more secured creditors are much more likely to file Chapter 11 than Chapter 7, consistent with our model’s prediction.

**Other Comparative Statics:** Higher growth firms value the tax shield of debt more, which makes them want to issue a large coupon. Higher \( \mu \) also increases the relative efficiency of Chapter 11 as discussed previously. This means that ceteris paribus, firms with higher \( \mu \) tend to prefer Chapter 11, since larger coupons encourage Chapter 11 ex-post. To our knowledge, this is a novel empirical prediction. It also suggests that estimates of the benefits of Chapter 11 might be overstated, since the firms which chose to enter Chapter 11 might have had higher growth rates on average. Higher volatility has the opposite effect as it decreases the relative efficiency of Chapter 11 and, in most cases in our model, lowers the optimal coupon. Intuitively, when the rate of conversion to Chapter 7 increases, or the fixed cost \( B \) that equity must pay to enter Chapter 7 increases, the equityholders are more likely to choose Chapter 7.

### 6.2 Length of Chapter 11

Given parameters for which the firm optimally chooses Chapter 11, our model produces predictions on the length of the Chapter 11 process. The length of Chapter 11 is stochastic, since it depends upon the path of the EBIT \( \delta_t \) during bankruptcy. Rather than simulate the average length, we use the metric \( \frac{\delta_B}{\delta_B} \). The denominator \( \delta_B \) is the endogenous threshold at which equity defaults and enters Chapter 11. Recall from Section 3 that the firm emerges from Chapter 11 the first time
\( \delta_t \geq \frac{R_t}{\bar{x}} \), where \( \bar{x} \) is endogenous. When \( h = 0 \), we have \( R_t = R_0 \) for all \( t \), so \( \frac{R_0}{\delta_t} \) reflects the factor by which cashflows must improve to exit Chapter 11.

From Table 3, we see that as creditor rights improve (\( \lambda_d \) falls), this ratio falls, suggesting shorter Chapter 11 procedures in expectation. This is supported by BWZ (2006): in their table 6, they find that the number of secured creditors is the only significant determinant of the length of the Chapter 11 procedure. Since a larger number of secured creditors likely corresponds to coordination problems and lower bargaining power for creditors, this matches our prediction that lower creditor rights lead to longer bankruptcies.

Other parameters of our model produce variation in the length of Chapter 11. Based on table 3, we find that lower growth firms, higher volatility firms, and firms with higher liquidation values all have longer bankruptcy procedures. We are unaware of any empirical papers which test these findings.

### 6.3 Credit Spreads

As in the standard Leland setup, our model produces credit spreads by the formula

\[
CS = \frac{C^*}{D_0(\delta_0, C^*)} - r
\]

where \( D_0(\delta_0, C^*) \) is the value of debt at the optimal coupon at issuance. Both tables 2 and 3 present credit spreads for the Leland model (i.e., when the firm can only enter Chapter 7) and our model. Tables 2 and 3 show that, unless a parameter change leads to a change in bankruptcy chapter, credit spreads demonstrate similar comparative statics with respect to growth, volatility, and discount rate as in the Leland model. The Leland model predicts that higher taxes lead to higher debt and leverage and thus riskier debt with higher credit spreads. Table 3 shows the same pattern in the case of Chapter 11. As discussed in Section 5, in the “constrained debt Chapter 7” strategy the firm will sometimes lower its coupon in the presence of higher taxes, which correspondingly lowers credit spreads in contrast to the Leland model.

Tables 2 and 3 suggest that changes in Chapter 11 costs and the bargaining parameters have the same effect on credit spreads as they do on leverage and the optimal coupon. This is intuitive since higher coupons always lead to earlier default and thus riskier debt. What is perhaps most interesting in our model is not the comparative statics of credit spreads but the levels. In cases where equityholders finds Chapter 11 to be ex-post optimal, debtholders demand a higher cost of debt to compensate them for the rents that equityholders will extract in Chapter 11. This can be seen from the higher credit spreads (relative to the Leland model) in cases where Chapter 11 occurs on the equilibrium path. In cases where Chapter 7 occurs on the equilibrium path, equityholders issue less debt than in the Leland model, so the debt is safer and has a lower credit spread.

The credit spread puzzle suggests that models like Leland (1994) tend to underestimate credit spreads on risky debt. By adding the option of Chapter 11, we can produce credit spreads much higher than those in the Leland model. In the “optimal inefficient Chapter 11” strategy, the higher
default costs lead equityholders to issue less debt than if they only were able to liquidate. This is because they internalize the default costs when they issue debt at time zero. However, the debtholders still demand compensation for the future Chapter 11 reorganization. As a result, in this case the model can simultaneously generate a credit spread 20 basis points higher than the Leland model while producing an optimal leverage ratio that is 7 percentage points lower (Table 3).

6.4 Extensions

In this section, we informally consider how two assumptions in our model might impact our results. First, we assume that firms can only enter Chapter 11 once, so after reorganizing equityholders must liquidate if they subsequently default. If we were to extend the model to allow for two Chapter 11 opportunities, we believe this would not significantly change the results. Suppose we allow two Chapter 11 opportunities, and consider a history in which the firm chose to default, entered Chapter 11, and has just emerged. When this reorganized firm is choosing its new capital structure, it looks exactly like a firm at time 0 in our model, and makes the exact capital structure decision we described in the previous section. If Choiceval > 1, then this relevered firm value is more attractive than the corresponding reorganized firm value with only the choice of future Chapter 7. Then, considering a history where the firm has not yet defaulted, Chapter 11 looks more appealing, since the reorganized firm value is even higher. Since Choiceval > 1, the firm was likely already going to choose Chapter 11, and now with two Chapter 11 opportunities this is even more appealing, so the firm chooses Chapter 11 at the first default. Thus, in this case the choice of Chapter 7 vs Chapter 11 is unchanged, and the firm likely issues slightly more debt at time 0 since the first Chapter 11 is now less costly. It is unlikely that this increase in debt is significant, since the reduction in the cost of Chapter 11 only occurs in an unlikely state of the world (where the firm defaults and reemerges before a forced conversion) that is heavily discounted.

If Choiceval = 1, then when the firm emerges from the first Chapter 11, they ignore the option to enter a second Chapter 11 and issue a coupon consistent with future Chapter 7. But then in a history where the firm has not yet defaulted the first time, the first Chapter 11 looks exactly as attractive as it would if there were no second Chapter 11 opportunity. In this case, the firm still chooses Chapter 7 at the first default, and the time 0 coupon is unchanged.

If Choiceval < 1, it is possible the second Chapter 11 might change behavior. In particular, the firm value upon emerging from the first Chapter 11 will now be lower than in the current model, which makes the first Chapter 11 less appealing. This could slightly increase debt in the “constrained debt Chapter 7” strategy, since it is easier for equity to promise ex-post not to enter Chapter 11. However, it could also shift the strategy from “optimal inefficient Chapter 11” to the “constrained debt Chapter 7” strategy, since now the inefficiency of the first Chapter 11 could outweigh the tax benefits of the larger coupon $C_B > \bar{C}$.

In summary, adding a second Chapter 11 might lead to slightly higher debt in some cases, while it could also drastically reduce debt if it causes a firm to shift from the “optimal inefficient Chapter
11” strategy to the “constrained debt Chapter 7” strategy. There is no change in the intuition if we were to add a third, fourth, or general nth Chapter 11 opportunity.

Second, in our model we assume the firm can only issue debt once (as in Leland (1994)). One potential means of relaxing this assumption would be to let firms issue callable debt with an indenture restricting further issuance prior to calling existing debt. This is the assumption made in the dynamic capital structure literature (Leland (1998), Goldstein, Leland and Ju (2001), Strebulaev (2007)). The nonlinearity of equity’s Chapter 11 bargaining value function would violate the scaling property needed to solve such models. However, we imagine this would not change the implications of our model too drastically. In general, the tax benefits of any particular debt issue are lower in these models since the firm may always refinance with a larger coupon if profitability improves. Thus at time zero firms would generally issue less debt, and in our Case 3 it is more likely that firms would find the “constrained debt Chapter 7” strategy optimal than the “optimal inefficient Chapter 11” strategy since large coupons are less valuable.

Importantly, the ability to refinance and issue more debt in such a setting would not change the ability of equityholders to credibly commit to Chapter 7. Should equityholders issue a coupon \( \tilde{C} \) consistent with future Chapter 7, the creditors purchasing this debt would recognize that if equityholders want to issue more debt, the firm would first need to call the outstanding debt at par, defending the existing creditors from devaluation by the shift to an ex-post optimal Chapter 11. This informal logic suggests our “constrained debt Chapter 7” strategy would be robust to allowing multiple debt issuances.

7 Conclusion

This paper studies the choice of bankruptcy chapter and its relationship to capital structure decisions. We provide a model of an equity value-maximizing firm that decides how much debt to issue, then subsequently chooses when and under which bankruptcy chapter to default. We model Chapter 11 reorganization with a novel continuous-time stochastic bargaining model in the style of Merlo and Wilson (1995). Specifically, equityholders and debtholders observe the firm’s assets and accumulated cashflows evolve stochastically, and they must unanimously agree when to emerge from Chapter 11 and how to split the firm. The reorganized firm can then issue new debt and continue operating.

There may often be a conflict between the desires of equityholders and creditors in terms of their relative treatment in the two chapters of bankruptcy. Equityholders with larger debt obligations endogenously default in more profitable states, in which they prefer the prospect of reorganization. Creditors typically enjoy higher recovery rates in Chapter 7 liquidation due to APR. Thus, when the firm issues debt, creditors take these incentives into account and demand higher credit spreads for large coupons that imply a subsequent Chapter 11. In some cases, the model predicts equityholders will optimally issue a coupon that implies a future inefficient Chapter 11, leading to lower leverage and higher credit spreads than the Leland (1994) model. In other cases, equityholders optimally
issue a small coupon, such that they will find Chapter 7 optimal ex-post, to obtain a lower credit spread. For a reasonable parameterization of this case, our model predicts an optimal leverage ratio of 40% while the Leland (1994) model predicts 70%, even though the firm liquidates on the equilibrium path in both models. Stated another way, while in our model the observed bankruptcy behavior may be identical to that of the Leland model in which only liquidation may be undertaken, the off-the-equilibrium threat of reorganization delivers a much lower optimal leverage ratio. The added option of Chapter 11 actually reduces ex-ante firm value in these cases, since equityholders cannot commit to a future Chapter 7 liquidation.

Several extensions of our model may prove illuminating. The model could be generalized to incorporate asymmetric information between equityholders and debtholders, given the rich tradition in the corporate finance literature exploiting the implications of hidden information. In addition, the model could be extended to a multiple-firm industry equilibrium (similar to Lambrecht (2001); Grenadier (2002); Miao (2005)). The latter extension could produce interesting interactions wherein firms might push rivals toward Chapter 7 rather than Chapter 11 in order to reduce competition. Finally, the model’s framework might prove useful for empirical work aimed at estimating the relative inefficiencies of Chapter 7 and Chapter 11.
References


This figure shows $E(\delta) - B$, the payoff to equity if they default at a value $\delta$ and choose Chapter 11, as a function of $\delta$. 
This figure shows firm value (Equity + Debt) as a function of the coupon $C_0$. The blue dotted line plots $F^L(\delta_0, C_0)$, the firm value under the assumption of future liquidation. The dotted line descending from the peak of the blue curve marks $C_L$, the optimal unconstrained coupon under future liquidation, on the x-axis. The black dashed line plots $F^B(\delta_0, C_0)$, the firm value under the assumption of future Chapter 11. The dotted line descending from the peak of the black dashed curve marks $C_B$, the unconstrained optimal coupon under future Chapter 11 reorganization, on the x-axis. The red line plots the actual firm value $F_0(\delta_0, C_0)$ as a function of the coupon $C_0$. Finally, the line ending in $C_{\text{bar}}$ on the x-axis marks the largest coupon for which equity finds liquidation optimal ex-post. The parameters corresponding to this figure are the baseline parameters from Table 1, with $R = 0.6, h = 1, B = 0.1$. 
This figure shows firm value (Equity + Debt) as a function of the coupon $C_0$. The blue dotted line plots $F^L(\delta_0, C_0)$, the firm value under the assumption of future liquidation. The dotted line descending from the peak of the blue curve marks $C_L$, the optimal unconstrained coupon under future liquidation, on the x-axis. The black dashed line plots $F^B(\delta_0, C_0)$, the firm value under the assumption of future Chapter 11. The dotted line descending from the peak of the black dashed curve marks $C_B$, the unconstrained optimal coupon under future Chapter 11 reorganization, on the x-axis. The red line plots the actual firm value $F_0(\delta_0, C_0)$ as a function of the coupon $C_0$. Finally, the line ending in $C_{bar}$ on the x-axis marks the largest coupon for which equity finds liquidation optimal ex-post. The parameters corresponding to this figure are the baseline parameters from Table 1, with $R = 4.4, h = 0.4, B = 2$. 
This figure shows firm value (Equity + Debt) as a function of the coupon $C_0$. The blue dotted line plots $F^L(\delta_0, C_0)$, the firm value under the assumption of future liquidation. The dotted line descending from the peak of the blue curve marks $C_L$, the optimal unconstrained coupon under future liquidation, on the $x$-axis. The black dashed line plots $F^B(\delta_0, C_0)$, the firm value under the assumption of future Chapter 11. The dotted line descending from the peak of the black dashed curve marks $C_B$, the unconstrained optimal coupon under future Chapter 11 reorganization, on the $x$-axis. The red line plots the actual firm value $F_0(\delta_0, C_0)$ as a function of the coupon $C_0$. Finally, the line ending in $C_{\text{bar}}$ on the $x$-axis marks the largest coupon for which equity finds liquidation optimal ex-post. The parameters corresponding to this figure are the baseline parameters from Table 1, with $R = 3.35$, $h = 1$, $B = 0.1$. 

Figure 5: Capital Structure Choice: RelEff = 85%
This figure shows firm value (Equity + Debt) as a function of the coupon $C_0$. The blue dotted line plots $F^L(\delta_0, C_0)$, the firm value under the assumption of future liquidation. The dotted line descending from the peak of the blue curve marks $C_L$, the optimal unconstrained coupon under future liquidation, on the x-axis. The black dashed line plots $F^B(\delta_0, C_0)$, the firm value under the assumption of future Chapter 11. The dotted line descending from the peak of the black dashed curve marks $C_B$, the unconstrained optimal coupon under future Chapter 11 reorganization, on the x-axis. The red line plots the actual firm value $F_0(\delta_0, C_0)$ as a function of the coupon $C_0$. Finally, the line ending in $C_{bar}$ on the x-axis marks the largest coupon for which equity finds liquidation optimal ex-post. The parameters corresponding to this figure are the baseline parameters from Table 1, with $R = 4$, $h = 0.9$, $B = 0.3$. 
Table 1: Baseline Parameter Values

<table>
<thead>
<tr>
<th></th>
<th>( \mu )</th>
<th>( \sigma )</th>
<th>( r )</th>
<th>( \tau )</th>
<th>( \alpha )</th>
<th>( \iota )</th>
<th>( \lambda_d )</th>
<th>( \lambda_e )</th>
<th>( \delta_0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.02</td>
<td>0.25</td>
<td>0.05</td>
<td>0.2</td>
<td>0.1</td>
<td>0.06</td>
<td>3</td>
<td>3</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 2: Chapter 7 with Constrained Debt, RelEff= 75%

<table>
<thead>
<tr>
<th>Only Chapter 7</th>
<th>Chapter 7 or Chapter 11</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( C_L )</td>
</tr>
<tr>
<td>Benchmark</td>
<td>1.36</td>
</tr>
<tr>
<td>( \mu = )</td>
<td>0.01</td>
</tr>
<tr>
<td>( \mu = )</td>
<td>0.03</td>
</tr>
<tr>
<td>( \sigma = )</td>
<td>0.2</td>
</tr>
<tr>
<td>( \sigma = )</td>
<td>0.3</td>
</tr>
<tr>
<td>( r = )</td>
<td>0.03</td>
</tr>
<tr>
<td>( r = )</td>
<td>0.07</td>
</tr>
<tr>
<td>( \tau = )</td>
<td>0.15</td>
</tr>
<tr>
<td>( \tau = )</td>
<td>0.3</td>
</tr>
<tr>
<td>( \alpha = )</td>
<td>0.005</td>
</tr>
<tr>
<td>( \alpha = )</td>
<td>0.3</td>
</tr>
<tr>
<td>( \iota = )</td>
<td>0.03</td>
</tr>
<tr>
<td>( \iota = )</td>
<td>0.09</td>
</tr>
<tr>
<td>( R = )</td>
<td>3.5</td>
</tr>
<tr>
<td>( R = )</td>
<td>4.5</td>
</tr>
<tr>
<td>( h = )</td>
<td>0.8</td>
</tr>
<tr>
<td>( h = )</td>
<td>0.5</td>
</tr>
<tr>
<td>( \lambda_e =)</td>
<td>1</td>
</tr>
<tr>
<td>( \lambda_e =)</td>
<td>10</td>
</tr>
<tr>
<td>( \lambda_d =)</td>
<td>1</td>
</tr>
<tr>
<td>( \lambda_d =)</td>
<td>10</td>
</tr>
<tr>
<td>( \lambda_d, \lambda_e =)</td>
<td>20</td>
</tr>
<tr>
<td>( B = )</td>
<td>0.1</td>
</tr>
<tr>
<td>( B = )</td>
<td>0.5</td>
</tr>
</tbody>
</table>

The parameters in row 1 are the baseline parameters from Table 1, with \( R = 4, h = 0.9, B = 0.3 \). Subsequent rows change one parameter value at a time. Credit Spreads (CS) are in percentage points.
Table 3: Optimal Inefficient Chapter 11, RelEff= 85%

<table>
<thead>
<tr>
<th></th>
<th>Only Chapter 7</th>
<th>Chapter 7 or Chapter 11</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$C_L$</td>
<td>Lev</td>
</tr>
<tr>
<td>Benchmark</td>
<td>1.36</td>
<td>0.70</td>
</tr>
<tr>
<td>$\mu = 0.01$</td>
<td></td>
<td>1.05</td>
</tr>
<tr>
<td>$\mu = 0.03$</td>
<td></td>
<td>2.00</td>
</tr>
<tr>
<td>$\sigma = 0.2$</td>
<td></td>
<td>1.30</td>
</tr>
<tr>
<td>$\sigma = 0.3$</td>
<td></td>
<td>1.46</td>
</tr>
<tr>
<td>$r = 0.03$</td>
<td></td>
<td>2.62</td>
</tr>
<tr>
<td>$r = 0.07$</td>
<td></td>
<td>1.11</td>
</tr>
<tr>
<td>$\tau = 0.15$</td>
<td></td>
<td>1.28</td>
</tr>
<tr>
<td>$\tau = 0.3$</td>
<td></td>
<td>1.46</td>
</tr>
<tr>
<td>$\alpha = 0.005$</td>
<td></td>
<td>1.61</td>
</tr>
<tr>
<td>$\alpha = 0.3$</td>
<td></td>
<td>1.04</td>
</tr>
<tr>
<td>$\iota = 0.03$</td>
<td></td>
<td>1.36</td>
</tr>
<tr>
<td>$\iota = 0.09$</td>
<td></td>
<td>1.36</td>
</tr>
<tr>
<td>$R = 3.15$</td>
<td></td>
<td>1.36</td>
</tr>
<tr>
<td>$R = 3.6$</td>
<td></td>
<td>1.36</td>
</tr>
<tr>
<td>$h = 0.8$</td>
<td></td>
<td>1.36</td>
</tr>
<tr>
<td>$h = 0.7$</td>
<td></td>
<td>1.36</td>
</tr>
<tr>
<td>$\lambda_e = 1$</td>
<td></td>
<td>1.36</td>
</tr>
<tr>
<td>$\lambda_e = 10$</td>
<td></td>
<td>1.36</td>
</tr>
<tr>
<td>$\lambda_d = 0.25$</td>
<td></td>
<td>1.36</td>
</tr>
<tr>
<td>$\lambda_d = 0.5$</td>
<td></td>
<td>1.36</td>
</tr>
<tr>
<td>$\lambda_d = 1$</td>
<td></td>
<td>1.36</td>
</tr>
<tr>
<td>$\lambda_d = 10$</td>
<td></td>
<td>1.36</td>
</tr>
<tr>
<td>$\lambda_d, \lambda_e = 20$</td>
<td></td>
<td>1.36</td>
</tr>
<tr>
<td>$B = 0.05$</td>
<td></td>
<td>1.36</td>
</tr>
<tr>
<td>$B = 0.15$</td>
<td></td>
<td>1.36</td>
</tr>
</tbody>
</table>

The parameters in row 1 are the baseline parameters from Table 1, with $R = 3.35, h = 1, B = 0.1$. Subsequent rows change one parameter value at a time. Credit spreads (CS) are in percentage points.
A Solving Chapter 11 Efficiently

First, we provide an expression for the constant \( \theta \). In the notation of Section 2, let

\[
p^1 = \frac{r}{r - \mu} \frac{\psi - 1}{\psi} \left[ \psi - \frac{1}{\psi} \left( 1 - \tau \right) \alpha + \left( \psi - 1 \right) \left( \psi - 1 \right) \right] \frac{1}{\psi}
\]

\[
p^2 = \frac{\psi}{\psi - 1} \frac{r - \mu}{r}
\]

so \( C^* = p^1 \delta \) and \( \delta_L = p^2 C^* = p^1 p^2 \delta \). Summing the values of equity and debt,

\[
E^L(\delta) + D^L(\delta) = \delta^p \delta_L^\psi \left( 1 - \frac{C^*}{r - \mu} - \frac{\delta_L}{r - \mu} \right) \left( 1 - \frac{(1 - \tau) C^*}{r - \mu} + \frac{C^*}{r - \mu} + \delta_L \psi \left( 1 - \frac{(1 - \tau) C^*}{r - \mu} \right) \right)
\]

\[
= 1 - \frac{\tau}{r - \mu} \delta + \frac{\tau C^*}{r} - \delta^\psi \delta_L^\psi \left( 1 - \frac{C^*}{r} + \alpha (1 - \tau) \frac{\delta_L}{r - \mu} \right)
\]

Evaluating at \( \delta_0 \) and plugging in the above formulas, this is

\[
= 1 - \frac{\tau}{r - \mu} \delta_0 + \frac{\tau p^1 \delta_0}{r} - \delta_0^\psi \left( p^1 p^2 \delta_0 \right)^\psi \left( \frac{p^1 \delta_0}{r} - \frac{(1 - \tau) p^1 p^2 \delta_0}{r - \mu} \right)
\]

\[
= \frac{1 - \tau}{r - \mu} + \frac{\tau p^1}{r} - \left( p^1 p^2 \right)^\psi \left( \frac{p^1}{r} - \frac{(1 - \tau) p^1 p^2}{r - \mu} \right) \delta_0 = \theta \delta_0
\]

Next, in many of the proofs in this appendix, we will need to apply dominated convergence. This next lemma allows us to do so under the assumption of \( r > \mu \), whenever the function in question can be bounded by an affine function of \( \delta, R \).

**Lemma A.1** For any fixed constants \( \delta_0, R_0, B_1, B_2 > 0 \),

\[
E^{\delta_0, R_0} \left[ \sup_t e^{-rt} \left( B_1 \delta_t - B_2 R_t \right) \right] < \infty
\]

**Proof of Lemma:** Since \( r > \mu \), for an arithmetic Brownian motion \( Z_t = (-r + \mu - \frac{\sigma^2}{2}) t + \sigma B_t \), the supremum of \( Z_t \) over all \( t \) has an exponential distribution with parameter \( \hat{\lambda} = \frac{2(1 - r + \mu - \frac{\sigma^2}{2})}{\sigma^2} > 1 \) (see, for example, Graversen and Peskir (1998)). It follows that

\[
E^{\delta_0} \left[ \sup_t e^{-rt} \right] = \delta_0 E \left[ e^{2t} \right] = \delta_0 \frac{\hat{\lambda}}{\hat{\lambda} - 1} < \infty
\]

and, decomposing \( r = r_1 + r_2 \) with \( r_1 > \mu, r_2 > 0 \),

\[
E^{\delta_0} \left[ \sup_t e^{-rt} \right] \int_0^t \delta_s ds \leq E^{\delta_0} \left[ \sup_t e^{-rs} \delta_s ds \right] \leq E \left[ \int_0^\infty e^{-r_s} \delta_s ds \right]
\]

\[
= E^{\delta_0} \left[ \int_0^\infty e^{-r_1 s} e^{-r_2 s} \delta_s ds \right] \leq E^{\delta_0} \left[ \int_0^\infty e^{-r_1 s} \left( \sup_s e^{-r_2 s} \delta_s \right) ds \right]
\]

50
\[= \mathbb{E}^{\delta_0}[(\sup_s e^{-r_1 s} \delta_s) \int_0^\infty e^{-r_2 s} ds] = \frac{1}{r_2} \mathbb{E}^{\delta_0}[(\sup_s e^{-r_1 s} \delta_s)]\]

which is similarly finite. Then putting everything together, we have that

\[\mathbb{E}^{\delta_0, R_\theta}[\sup_t e^{-r t} (B_1 \delta_t - B_2 R_t)] = \mathbb{E}^{\delta_0}[\sup_t e^{-r t} (B_1 \delta_t + B_2 h(1 - \tau) \int_0^t \delta_s ds - B_2 R_0)]\]

\[\leq \mathbb{E}^{\delta_0}[\sup_t e^{-r t} (B_1 \delta_t + B_2 h(1 - \tau) \int_0^t \delta_s ds)] \leq B_1 \mathbb{E}^{\delta_0}[\sup_t e^{-r t} \delta_t] + B_2 h(1 - \tau) \mathbb{E}^{\delta_0}[\sup_t e^{-r t} \int_0^t \delta_s ds] < \infty\]

completing the proof. As an immediate corollary, for any fixed \(\delta, R\),

\[V(\delta, R) = \sup_{T \in E^{\delta, R}} \mathbb{E}^{(\delta, R)}[1(T < T_e) e^{-r T_e} (\theta \delta_t - R_t) + 1(T_e < T) e^{-r T_e} (\zeta \delta_t - R_e)] < \infty\]

(61)

since, letting \(T = T_R \wedge T_c\), there exist \(B_1, B_2\) such that the expression in the expectation is less than

\[e^{-r T} (B_1 \delta_T - B_2 R_T + B_2 R_0)\]

with probability 1.

**Proof of Proposition 1**: First, some simplifying notation. Let \(O^* = \{ (\delta, R) : V(\delta, R) = \theta \delta - R \}\) be the set of values where the social planner’s value function \(V(\delta, R)\) equals the payoff \(\theta \delta - R\). Fix a MPE with value functions \(E, D\) and equilibrium stopping time \(T\). It will be convenient to define the set \(\mathcal{E} = \cup_i O_i \times \{ i \}\) so the game ends when \((\delta, R, s) \in \mathcal{E}\). From this point on, \(T\) is always defined as the first hitting time of \(\mathcal{E}\). Let \(V^e(\delta, R, s) = E(\delta, R, s)\) and \(V^d(\delta, R, s) = D(\delta, R, s)\), and let \(y = \delta \delta - R\) and \(z = \zeta \delta - R\).

First, from the definition of \(J_i(\delta, R, s)\), we see that \(\sum_i J_i(\delta, R, s) = y\). The sum of all offers and the payoff minus offers equals the payoff. Given this, we have

\[V^e(\delta, R, s) + V^d(\delta, R, s) = \mathbb{E}^{(\delta, R, s)}[1(T < T_e) e^{-r T} y_T + 1(T_e < T) e^{-r T_e} z_T] \leq V(\delta, R)\]

(62)

by the definition of \(V(\delta, R)\). Second, we claim that if \((\delta, R) \in O^*\), then \(V^e(\delta, R, s) + V^d(\delta, R, s) = V(\delta, R)\). From the first observation, the leftside cannot be strictly greater. If it were strictly less, then letting \(s' \neq s\),

\[y - V^{s'}(\delta, R, s) = V(\delta, R) - V^{s'}(\delta, R, s) > V^s(\delta, R, s)\]

where the first equality is the definition of \(O^*\). It follows that player \(s\) would have a strictly profitable deviation to offer the other player their value function. This implies that if \((\delta, R) \in O^*\), then
\[ \mathbb{E}^{(\delta, R, s)}[1(\mathcal{T} < T_c)e^{-rT}y_\mathcal{T} + 1(T_c < \mathcal{T})e^{-rT_c}z_{T_c}] = V^e(\delta, R, s) + V^d(\delta, R, s) \]  
\[ = V(\delta, R) = \mathbb{E}^{(\delta, R)}[1(\tau_s < T_c)e^{-r\tau_s}y_{\tau_s} + 1(T_c < \tau_s)e^{-rT_c}z_{T_c}] \]

where \( \tau^* = \inf\{t : V^*(\delta_t, R_t, s_t) = y_t \} \) solves the optimal stopping problem. Third, any player can demand the game end at the maximum \( \mathcal{T} \lor \tau^* \) with payoffs \( J_i \). Specifically, any player \( i \) can deviate to making offers when \( s_t = i \) and \( (\delta, R) \in O_i \cap O^* \) and accepting offers from player \( j \) when \( (\delta, R) \in O_j \cap O^* \). For this to be an MPE, this cannot be a profitable deviation for each player. Summing across \( i \), we have

\[ \sum_i V^i(\delta, R, s) \geq \mathbb{E}^{(\delta, R, s)}[1(\mathcal{T} \lor \tau_s < T_c)e^{-rT}y_{\mathcal{T} \lor \tau_s} + 1(T_c < \mathcal{T} \lor \tau_s)e^{-rT_c}z_{T_c}] \]

\[ = \mathbb{E}^{(\delta, R, s)}[1(\mathcal{T} > \tau_s)(1(\mathcal{T} < T_c)e^{-rT}y_{\mathcal{T}} + 1(T_c < \mathcal{T})e^{-rT_c}z_{T_c})] \]

\[ + \mathbb{E}^{(\delta, R, s)}[1(\mathcal{T} < \tau_s)(1(\tau_s < T_c)e^{-r\tau_s}y_{\tau_s} + 1(T_c < \tau_s)e^{-rT_c}z_{T_c})] \]

Now, fix \((\delta_0, R_0, s_0)\). Let \( F_t \) be the filtration generated by \((\delta, R, s)\), which are jointly Markov. We have \( F_0 \subset F_{\tau_s} \) where \( \tau_s, \delta_{\tau_s}, R_{\tau_s}, s_{\tau_s}, 1(\mathcal{T} > \tau_s) \) are all \( F_{\tau_s} \) measurable. Then

\[ \mathbb{E}^{(\delta_0, R_0, s_0)}[1(\mathcal{T} > \tau_s)(1(\mathcal{T} < T_c)e^{-rT}y_{\mathcal{T}} + 1(T_c < \mathcal{T})e^{-rT_c}z_{T_c})] \]

\[ = \mathbb{E}[1(\mathcal{T} > \tau_s) \mathbb{E}^{(\delta_{\tau_s}, R_{\tau_s}, s_{\tau_s})}(1(\mathcal{T} < T_c)e^{-rT}y_{\mathcal{T}} + 1(T_c < \mathcal{T})e^{-rT_c}z_{T_c}) | F_0] \]

Applying the Markov property,

\[ = \mathbb{E}[1(\mathcal{T} > \tau_s) \mathbb{E}^{(\delta_{\tau_s}, R_{\tau_s})}(1(\tau_s < T_c)e^{-r\tau_s}y_{\tau_s} + 1(T_c < \tau_s)e^{-rT_c}z_{T_c}) | F_0] \]

and since \((\delta_{\tau_s}, R_{\tau_s}) \in O^*\) by definition, applying (63),

\[ = \mathbb{E}[1(\mathcal{T} > \tau_s) \mathbb{E}^{(\delta_{\tau_s}, R_{\tau_s})}(1(\tau_s < T_c)e^{-r\tau_s}y_{\tau_s} + 1(T_c < \tau_s)e^{-rT_c}z_{T_c}) | F_0] \]

\[ = \mathbb{E}^{(\delta_0, R_0, s_0)}[1(\mathcal{T} > \tau_s)(1(\tau_s < T_c)e^{-r\tau_s}y_{\tau_s} + 1(T_c < \tau_s)e^{-rT_c}z_{T_c})] \]

Plugging this in to (64), we have that

\[ \sum_i V^i(\delta, R, s) \geq \mathbb{E}^{(\delta, R, s)}[1(\mathcal{T} > \tau_s)(1(\mathcal{T} < T_c)e^{-rT}y_{\mathcal{T}} + 1(T_c < \mathcal{T})e^{-rT_c}z_{T_c})] \]

\[ + \mathbb{E}^{(\delta, R, s)}[1(\mathcal{T} < \tau_s)(1(\tau_s < T_c)e^{-r\tau_s}y_{\tau_s} + 1(T_c < \tau_s)e^{-rT_c}z_{T_c})] \]

\[ = \mathbb{E}^{(\delta, R, s)}[1(\tau_s < T_c)e^{-r\tau_s}y_{\tau_s} + 1(T_c < \tau_s)e^{-rT_c}z_{T_c}] = V(\delta, R) \]

completing the proof.
B Efficient Chapter 11

Recall the HJB is

\[ rV = -h(1 - \tau)\delta V_R + \delta \mu V_\delta + \frac{\sigma^2}{2} \delta^2 V_\delta + \iota [\zeta \delta - R - V] \]

where \( \zeta = (1 - \alpha)(1 - \tau) \frac{1}{r - \mu} \) such that \( \zeta \delta - R \) is the liquidation value of the firm, and \( \iota dt \) is the probability of liquidation per unit time.

Define \( v = \frac{V}{\delta} \) and \( x = \frac{R}{\delta} \). Note this means we expect exercise at low values of \( x \). Straightforward calculus shows \( v' = V R - x v' = V \delta - v, v'' x^2 = \delta V_\delta \). Then dividing by \( \delta \) and substituting, we get

\[ (r + \iota - \mu)v = -(\mu x + h(1 - \tau))v' + \frac{\sigma^2}{2} x^2 v'' + \iota (\zeta - x) \]

B.1 General Solution of the Homogeneous Equation

To start, consider the homogeneous equation

\[ (r + \iota - \mu)v = -(\mu x + h(1 - \tau))v' + \frac{\sigma^2}{2} x^2 v'' \]

Conjecture a solution \( v = x^\beta w(x) \) for some function \( w \) and constant \( \beta \). This implies derivatives

\[ v' = \beta x^{\beta - 1} w + x^\beta w' \]
\[ v'' = \beta(\beta - 1)x^{\beta - 2} w + 2\beta x^{\beta - 1} w' + x^\beta w'' \]

Plugging this conjecture in, we get

\[ (r + \iota - \mu)v = -(\mu x + h(1 - \tau))[\beta x^{\beta - 1} w + x^\beta w'] + \frac{\sigma^2}{2} x^2 [\beta(\beta - 1)x^{\beta - 2} w + 2\beta x^{\beta - 1} w' + x^\beta w''] \]

First, gather \( v \) terms:

\[ 0 = v[-(r + \iota - \mu) - \mu \beta + \frac{\sigma^2}{2} \beta(\beta - 1)] \]

We define \( \beta \) such that this equals 0. That is, \( \beta \) is a positive or negative root of

\[ 0 = -(r + \iota - \mu) - \mu \beta + \frac{\sigma^2}{2} \beta(\beta - 1) \]

Let \( \kappa \) be the positive root and \( \gamma \) be the negative root, and for now let \( \beta \) be a placeholder for either root. Plugging in this \( \beta \), we’re left with

\[ 0 = -h(1 - \tau)\beta x^{\beta - 1} w - (\mu x + h(1 - \tau))x^\beta w' + \frac{\sigma^2}{2} x^2 [2\beta x^{\beta - 1} w' + x^\beta w''] \]

Multiply through by \( x^{-\beta + 2} \):
0 = -h(1 - \tau) \beta x w - (\mu x + h(1 - \tau)) x^2 w' + \frac{\sigma^2}{2} [2\beta x^3 w' + x^4 w'']

Finally, conduct a second change of variables to \( z = -\frac{2h(1-\tau)}{\sigma^2 x} \) and \( f(z) = w(x) \). Then \( w' = f' \frac{2h(1-\tau)}{\sigma^2 x^3} \) and \( w'' = -2f' \frac{2h(1-\tau)}{\sigma^2 x^4} + f'' \frac{4h(1-\tau)^2}{\sigma^4 x^2} \), where combining implies

\[
\frac{w''}{x} = -2w' + f'' \frac{4h(1-\tau)^2}{\sigma^4 x^4}
\]

Plugging in,

\[
0 = -h(1 - \tau) \beta x f - (\mu x + h(1 - \tau)) \frac{2h(1-\tau)}{\sigma^2} f' + \frac{\sigma^2}{2} \left[ f'' \frac{4h(1-\tau)^2}{\sigma^4} + 2(\beta - 1) x f' \frac{2h(1-\tau)}{\sigma^2} \right]
\]

Multiplying by \( -\frac{1}{h(1-\tau)x} \) and rearranging,

\[
0 = \beta f + (\frac{2\mu}{\sigma^2} - z) f' + z f''
\]

which is Kummer’s ODE, with general solutions

\[
f(z) = M(-\beta, -2(\beta - 1) + \frac{2\mu}{\sigma^2}, z)
\]

\[
f(z) = U(-\beta, -2(\beta - 1) + \frac{2\mu}{\sigma^2}, z)
\]

Thus for either root \( \beta = \kappa, \gamma \), and either solution \( f \) to Kummer’s ODE, we get a solution

\[
A_3 x^\beta f \left( \frac{-2h(1-\tau)}{\sigma^2 x} \right)
\]

B.2 Applying Boundary Conditions

Since \( x = \frac{R}{\delta} \), and \( R \) can go negative, \( x \) starts out large and positive, then declines over time. We conjecture that the option is exercised before \( x \) goes negative (which we verify shortly). Then we should be looking for a solution on a positive domain of \( x \). Also, as \( x = \frac{R}{\delta} \to \infty \), the cost of exercise is large and the payoff is small, so the value should converge to 0. We now impose this condition.

As \( x \to \infty \), \( M(a, b, -\frac{2h(1-\tau)}{\sigma^2 x}) \) converges to \( M(a, b, 0) = 1 \). Thus \( x^\beta M \) works as a solution for the negative root \( \gamma \), but will not work for the positive root \( \kappa \) since then \( x^\beta \to \infty \).

As \( x \to \infty \), applying the positive root \( \beta = \kappa \) we get \( U(-\beta, -2(\beta - 1) + \frac{2\mu}{\sigma^2}, 0) \) is a finite constant. But then \( x^\beta U \) goes to infinity. Applying the negative root \( \beta = \gamma \), \( U(-\beta, -2(\beta - 1) + \frac{2\mu}{\sigma^2}, z) \) explodes faster than \( x^\beta \) goes to 0, violating the boundary condition.

In conclusion, the homogeneous solution must take the form
\( v(x) = A_3 x^\gamma M(-\gamma, -2(\gamma - 1) + \frac{2\mu}{\sigma^2}, \frac{-2h(1 - \tau)}{\sigma^2 x}) \)

for some positive constant \( A_3 \).

### B.3 Finishing the Value Function

Finally, we must add back in the risk of liquidation to the single agent optimization. Recall after the change of variables, the HJB may be written

\[
(r + \iota - \mu)v = - (\mu x + h(1 - \tau))v' + \frac{\sigma^2}{2} x^2 v'' + \iota (\zeta - x)
\]

As discussed above, the only solution to the homogeneous equation satisfying the right boundary condition is

\[
v(x) = A_3 x^\gamma M(-\gamma, -2(\gamma - 1) + \frac{2\mu}{\sigma^2}, \frac{-2h(1 - \tau)}{\sigma^2 x})
\]

where \( \gamma \) is the negative root of

\[
0 = \left[ -(r + \iota - \mu) - \mu \beta + \frac{\sigma^2}{2} \beta (\beta - 1) \right]
\]

The relevant particular solution including the last nonhomogeneous term is

\[
\frac{\iota \zeta + \frac{h(1 - \tau)\iota}{r + \iota}}{r + \iota - \mu} - \frac{\iota x}{r + \iota},
\]

leading to a solution

\[
v(x) = A_3 x^\gamma M(-\gamma, -2(\gamma - 1) + \frac{2\mu}{\sigma^2}, \frac{-2h(1 - \tau)}{\sigma^2 x}) + \frac{\iota \zeta + \frac{h(1 - \tau)\iota}{r + \iota}}{r + \iota - \mu} - \frac{\iota x}{r + \iota}
\]

For some positive constant \( A_3 \). At exercise, the firm receives \( \theta \delta - R \) so this should smooth paste on \( \theta - x \). Conjecturing exercise occurs at a lower barrier \( \bar{x} \):

\[
v(\bar{x}) = \theta - \bar{x}
\]

\[
v'(\bar{x}) = 1
\]

Note that \( \frac{d}{dz} M(a, b, z) = \frac{a}{b} M(a + 1, b + 1, z) \), so

\[
v'(x) = A_3 x^\gamma x^{\gamma - 1} M(-\gamma, -2(\gamma - 1) + \frac{2\mu}{\sigma^2}, \frac{-2h(1 - \tau)}{\sigma^2 x}) + A_3 x^\gamma \frac{-\gamma}{-2(\gamma - 1) + \frac{2\mu}{\sigma^2}} M(-\gamma + 1, -2(\gamma - 1) + \frac{2\mu}{\sigma^2} + 1, \frac{-2h(1 - \tau)}{\sigma^2 x}) - \frac{\iota x}{r + \iota}
\]

Solving for this \( v \), we have \( V(\delta, R) = \delta v(\frac{R}{\delta}) \) for \( \delta \leq \frac{R}{\bar{x}} \) and \( V(\delta, R) = \theta \delta - R \) for \( \delta \geq \frac{R}{\bar{x}} \). The optimal trigger is exercise when \( \frac{R}{\delta} \leq \bar{x} \), or when \( \delta \geq \frac{R}{\bar{x}} \).
B.4 Proof of Proposition 2

Define an operator $A$ that maps smooth functions $V$ of $\delta, R$ to

$$-h(1-\tau)\delta V_R + \delta \mu V_\delta + \frac{\sigma^2}{2} \delta^2 V_{\delta\delta} + \iota [\zeta \delta - R - V]$$

By construction, $V(\delta, R) = \delta v\left(\frac{R}{\delta}\right)$ is smooth since it smooth pastes at $\delta = \frac{R}{\bar{x}}$. Also by construction, $AV = rV$ for $\delta \leq \frac{R}{\bar{x}}$. For $\delta \geq \frac{R}{\bar{x}}$ we have $V = \theta \delta - R$, so in this region

$$AV = h(1-\tau)\delta + \delta \mu \theta + \iota (\zeta - \theta) \delta = [h(1-\tau) + \mu \theta + \iota (\zeta - \theta)]\delta$$

and thus in this region, $-rV + AV \leq 0$ if and only if

$$-r(\theta \delta - R) + [h(1-\tau) + \mu \theta + \iota (\zeta - \theta)]\delta \leq 0$$

$$\iff \frac{h(1-\tau) + \mu \theta + \iota (\zeta - \theta) - r \theta}{r} \delta \leq -R$$

$$\iff -\frac{h(1-\tau) + \mu \theta + \iota (\zeta - \theta) - r \theta}{r} \geq \bar{x}$$

which is guaranteed by the first condition of proposition 2,

$$-\frac{h(1-\tau) + \mu \theta + \iota (\zeta - \theta) - r \theta}{r} \geq \bar{x}$$

By construction of $V$, we have $V(\delta, R) = \theta \delta - R$ when $\delta \geq \frac{R}{\bar{x}}$, and by condition 2 of proposition 2,

$$V(\delta, R) = \delta v\left(\frac{R}{\delta}\right) \geq \delta(\theta - x) = \delta \theta - R$$

so putting this together, under the conditions of proposition 2, our candidate value function is smooth and satisfies the variational inequality

$$\max(\theta \delta - R - V, -rV + AV) = 0$$

Next, we show that there is a constant $C$ such that $x \geq \bar{x} \Rightarrow v(x) \leq C$. To see this, we can use the fact that $\gamma < 0$ to write $M(-\gamma, -2(\gamma - 1) + \frac{2\mu}{\sigma^2}, -\frac{2h(1-\tau)}{\sigma^2 x}) = M(a, b, z)$ in its integral representation

$$M(a, b, z) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 e^{zu}u^{a-1}(1-u)^{b-a-1}du$$

For $u \in [0, 1]$ we have $e^{z_1u} \leq e^{z_2u}$ for $z_1 \leq z_2$ and everything is positive, so $M(a, b, z)$ is positive and monotonically increasing in $z$. Then $M(-\gamma, -2(\gamma - 1) + \frac{2\mu}{\sigma^2}, -\frac{2h(1-\tau)}{\sigma^2 x})$ is monotonically increasing in $x$ for $x \geq \bar{x}$. Since the expression converges to $M(a, b, 0) = 1$ as $x \to \infty$, it follows
that \( M \in (0, 1) \) for \( x \geq \bar{x} \). Then since \( \gamma < 0 \), we have \( x^\gamma M \leq x^\gamma \bar{x}^\gamma \) for \( x \geq \bar{x} \). It follows that

\[
v(x) = A_3 x^\gamma M(-\gamma, -2(\gamma - 1) + \frac{2\mu}{\sigma^2}, -\frac{2h}{\sigma^2 x}) + \frac{\zeta + \frac{h(1-\tau)v}{r+1}}{r+1} - \frac{\bar{x}}{r+1} \leq A_3 \bar{x}^\gamma + \frac{\zeta + \frac{h(1-\tau)v}{r+1}}{r+1} - \frac{\bar{x}}{r+1} = C
\]

whenever \( x \geq \bar{x} \). Using this constant \( C \), there exists a constant \( B \) such that \( V(\delta, R) \leq B \delta - R + R_0 \) for all \( \delta \) and \( R \leq R_0 \). For \( \frac{R}{\delta} \leq \bar{x} \) this holds for any \( B \geq \theta \) since \( V = \theta \delta - R \). For \( \frac{R}{\delta} \geq \bar{x} \), we have \( V(\delta, R) = \delta v(\frac{R}{\delta}) \leq C \delta \) by the previous part, and \( R \leq R_0 \), so \( V(\delta, R) + R \leq C \delta + R_0 \). Thus setting \( B = \max(\theta, C) \) satisfies this.

We are now ready to finish the verification. Fix \( \delta_0, R_0 \) and define \( Y_t = 1(t < T_c) e^{-rT} V(\delta_t, R_t) + 1(t \geq T_c) e^{-rT}(\zeta \delta_t - R_t) \). By Ito’s lemma, for \( t < T_c \),

\[
Y_t = e^{-rT} V(\delta_t, R_t) = V(\delta_0, R_0) + \int_0^t e^{-r\sigma} [-rV(\delta_s, R_s) + \mathcal{A}V(\delta_s, R_s)] ds + M_t
\]

for a local martingale \( M_t \) with \( M_0 = 0 \). Since \( V \) satisfies the variational inequality,

\[
Y_t = e^{-rT} V(R_t, \delta_t) \leq V(R_0, \delta_0) + M_t
\]

Where \( V \) is bounded below by \(-R_0 \) so \( M_t \) is a supermartingale. Take an arbitrary stopping time \( T \) and let \( T_n \) be a sequence of stopping times increasing to \( T \wedge T_c \). Applying optional sampling\(^{46}\) for the bounded below supermartingale \( M_t \) at \( T_n \):

\[
\mathbb{E}^{\delta_0, R_0}[Y_{T_n}] \leq V(\delta_0, R_0)
\]

Since \( V(\delta, R) \geq \theta \delta - R \), it follows that

\[
\mathbb{E}^{\delta_0, R_0}[e^{-rT_n} 1(T_n < T_c)(\theta \delta_{T_n} - R_{T_n}) + e^{-rT_n} 1(T_n > T_c)(\zeta \delta_{T_n} - R_{T_n})] \leq V(\delta_0, R_0)
\]

Taking \( n \) to infinity and using the bound \( V(\delta, R) \leq B \delta - R + R_0 \) along with the lemma of appendix A to apply dominated convergence,

\[
\mathbb{E}^{\delta_0, R_0}[e^{-rT} 1(T < T_c)(\theta \delta_T - R_T) + e^{-rT} 1(T > T_c)(\zeta \delta_T - R_T)] \leq V(\delta_0, R_0)
\]

Now, define \( T_R = \inf\{ t : \frac{R}{\delta_t} \leq \bar{x} \} \) and fix \( R_0, \delta_0 \) such that \( T_R > 0 \).\(^{47}\) Then by definition of \( V \), \(-rV + \mathcal{A}V = 0 \) for \( t < T_R \), so applying Ito’s lemma as before gives

\[
Y_t = e^{-rT} V(R_t, \delta_t) = V(R_0, \delta_0) + M_t
\]

\(^{45}\)For example, \( T_n = \max(0, T \wedge T_c - \frac{1}{n}) \)

\(^{46}\)By an application of Fatou’s lemma, optional sampling for bounded below supermartingales holds for arbitrary stopping times.

\(^{47}\)If \( T_R = 0 \), the following conclusion is immediate.
let \( Q_n \) be a sequence of stopping times increasing to \( T_{\mathcal{R}} \wedge T_c \), let \( \tau_n \) be the localizing sequence of stopping times for the local martingale \( M_t \), and let \( T_n = Q_n \wedge \tau_n \wedge n \). Applying optional sampling,

\[
\mathbb{E}^{\delta_0, R_0}[Y_{T_n}] = V(\delta_0, R_0)
\]

Taking \( n \) to infinity and using the bound \( V(\delta, R) \leq B\delta - R + R_0 \) along with the lemma of appendix A to apply dominated convergence,

\[
\mathbb{E}^{\delta_0, R_0}[Y_{T_{\mathcal{R}} \wedge T_c}] = \mathbb{E}^{\delta_0, R_0}[1(T_R < T_c) e^{-rT_R} V(\delta_{T_R}, R_{T_R}) + 1(T_R \geq T_c) e^{-rT_c}(\zeta_{T_c} - R_{T_c})] = V(\delta_0, R_0)
\]

where the penultimate equality follows from the definition of \( T_{\mathcal{R}} \) and \( V \), completing the verification.

### C Proof of Proposition 3, Calculating Equilibrium

**Proof of Proposition 3:** The proof proceeds in two steps. First, we show it is without loss of generality to assume the offer strategy is \( \omega_i(\delta, R) = V^j(\delta, R, i) \) is optimal. If there were an alternate strategy \((\hat{\omega}, \hat{A}, \hat{O})\) that performed strictly better than the proposed strategy where for some \( \delta, R, i, j \), \( \hat{\omega}_i(\delta, R) > V^j(\delta, R, i) \), then another strategy \((\hat{\omega}, \hat{A}, \hat{O})\) does even better by setting \( \hat{\omega} = V^j(\delta, R, i) \) in cases. If there were an alternate strategy \((\hat{\omega}, \hat{A}, \hat{O})\) that performed strictly better than the proposed strategy where for some \( \delta, R, i, j \), \( \hat{\omega}_i(\delta, R) < V^j(\delta, R, i) \), then another strategy \((\hat{\omega}, \hat{A}, \hat{O})\) does just as well where \( \hat{\omega}_i(\delta, R) = V^j(\delta, R, i) \) and those cases are removed from the offer region. Therefore, when we consider profitable deviations, it is sufficient to consider deviations of \( \hat{A}, \hat{O} \) where the alternate offer function is still \( \omega_i(\delta, R) = V^j(\delta, R, i) \).

Second, we show that the equilibrium time \( T \) solves

\[
\sup_{T_i \in \mathcal{F}_{\delta, R, s}} \mathbb{E}^{(\delta, R, s)}[1(T_i < T_c) e^{-rT_i} J_i(\delta_{T_i}, R_{T_i}, s_{T_i}) + 1(i = d)1(T_c < T_i) e^{-rT_c}(\zeta_{T_c} - R_{T_c})] = \mathbb{E}^{(\delta_0, R_0)}[1(T_i < T_c) e^{-rT_i} J_i(\delta_{T_i}, R_{T_i}, s_{T_i}) + 1(i = d)1(T_c < T_i) e^{-rT_c}(\zeta_{T_c} - R_{T_c})]
\]

with associated value function \( V^i(\delta, R, s) \). Since each player tries to optimize this quantity subject to constraints imposed by the opponent’s strategy, and the equilibrium time \( T \) solves the unconstrained problem, this implies each player acts optimally in the MPE.

To show this, define \( N_t = 1(t > T_c) \) and for notational convenience define an operator \( \mathcal{H}_s \) mapping appropriately differentiable functions \( f(\delta, R, s, N) \) to$^{48}$

$^{48}$The fact that \( N_t \) does not transition from 1 to 0 is irrelevant.
\[-h(1 - \tau)\delta f_R(\delta, R, s, N) + \mu \delta f_\delta(\delta, R, s, N) + \frac{\sigma^2 \delta^2}{2} f_\delta(\delta, R, s, N) + \lambda_s[f(\delta, R, s^i, N) - f(\delta, R, s, N)] + i[f(\delta, R, s, 1) - f(\delta, R, s, 0)]\]

Fix \(N_0 = 0\). Defining \(U^i(\delta, R, s, 0) = V^i(\delta, R, s)\) and \(U^i(\delta, R, s, 1) = 1(i = d)(\zeta \delta - R)\), by construction \(U^i\) solves

\[-rU^i + \mathcal{H}_sU^i = 0\]

except possibly when \((\delta, R, s) \in O_1^s \times \{i\}\). By proposition 1, in this case we have \(U^i + U^j = V(\delta, R)\). Also by construction, when \((\delta, R, s) \in O_1^s \times \{i\}\) we have \(-rU^j + \mathcal{H}_sU^j = 0\) and \(-rV(\delta, R) + \mathcal{H}_sV(\delta, R) \leq 0\) by proposition 2.\(^{49}\) By the linearity of \(\mathcal{H}_s\), it follows that

\[-r + \mathcal{H}_s]U^i = \leq 0\]

For \(t < T_c\), applying Ito’s lemma for semimartingales (see, for example, Duffie (2010)) to \(U^i\) gives

\[e^{-rt}U^i(\delta_t, R_t, s_t, N_t) = U^i(\delta_0, R_0, s_0, N_0) + \int_0^t e^{-rs}[(-r + \mathcal{H}_s)U^i(\delta_s, R_s, s_s, N_s)]ds + M_t\]

for a local martingale \(M_t\) with \(M_t = 0\). Applying an identical argument to that used in the proof of proposition 2 (appendix B.4) gives that for an arbitrary stopping time \(T\),

\[\mathbb{E}^{\delta_0, R_0, s_0, 0}[U^i(\delta_{T \wedge T_c}, R_{T \wedge T_c}, s_{T \wedge T_c}, N_{T \wedge T_c})]\]

\[= \mathbb{E}^{\delta_0, R_0, s_0}[1(T < T_c)e^{-rT}V^i(\delta_T, R_T, s_T) + 1(i = d)1(T_c < T)e^{-rT_c}(\zeta \delta_{T_c} - R_{T_c})]\]

\[\leq U^i(\delta_0, R_0, s_0, 0) = V^i(\delta_0, R_0, s_0)\]

For the equilibrium time \(T\), we have \(t < T\) implies\(^{50}\) \((\delta_t, R_t, s_t) \notin O_1^s \times \{i\}\) which implies \((-r + \mathcal{H}_s)U^i = 0\), so an argument identical to that used in the proof of proposition 2 (appendix B.4) gives that

\[\mathbb{E}^{\delta_0, R_0, s_0, 0}[U^i(\delta_{T \wedge T_c}, R_{T \wedge T_c}, s_{T \wedge T_c}, N_{T \wedge T_c})]\]

\[= \mathbb{E}^{\delta_0, R_0, s_0}[1(T < T_c)e^{-rT}V^i(\delta_T, R_T, s_T) + 1(i = d)1(T_c < T)e^{-rT_c}(\zeta \delta_{T_c} - R_{T_c})]\]

\[= U^i(\delta_0, R_0, s_0, 0) = V^i(\delta_0, R_0, s_0)\]

Note that \(V(\delta, R) \geq \theta \delta - R\), combined with proposition 1, implies that

---

\(^{49}\) When applying \(\mathcal{H}_s\), we view \(V\) as a trivial function of \(s\) that equals \(\zeta \delta - R\) when \(N = 1\).

\(^{50}\) The conclusion is trivial when \(T = 0\).
\[ J_i(\delta, R, s) = 1(s = i)[\theta R - V^j(\delta, R, i)] + 1(s = j)V^i(\delta, R, j) \]

\[ \leq 1(s = i)[V(\delta, R) - V^j(\delta, R, i)] + 1(s = j)V^i(\delta, R, j) = V^i(\delta, R, s) \]

so \( V^i(\delta, R, s) \geq J_i(\delta, R, s) \), and by definition \( V^i(\delta_T, R_T, s_T) = J_i(\delta_T, R_T, s_T) \). Plugging this in completes the proof.

### C.1 Equilibrium Value Functions

Following appendix B, general solutions of the homogeneous equation

\[ (r + \iota - \mu)v = - (\mu x + h(1 - \tau))v' + \frac{\sigma^2}{2}x^2v'' \]

take the form

\[ Gx^\beta f\left(-\frac{2h(1 - \tau)}{\sigma^2 x}\right) \]

for a constant \( G \), where \( f \) is either of the solutions to Kummer’s ODE:

\[ f(z) = M(-\beta, -2(\beta - 1) + \frac{2\mu}{\sigma^2}, z) \]

\[ f(z) = U(-\beta, -2(\beta - 1) + \frac{2\mu}{\sigma^2}, z) \]

and \( \beta \) is either root of

\[ 0 = [- (r + \iota - \mu) - \mu \beta + \frac{\sigma^2}{2} \beta (\beta - 1)] \]

First, consider general solutions in the range \( x \leq 0 \). As \( x \to -\infty \), each player’s value function should be bounded above by \( \theta - x \), such that \( V^i = \delta v \) is bounded above by the value of immediate exercise with all proceeds going to one player. It turns out that none of the general solutions satisfy this with \( G \neq 0 \), so the general solution in this region is 0.

Next, consider general solutions in the range \( x \in [0, \bar{x}] \). As \( x \to 0 \) from above, \( z = -\frac{2h(1 - \tau)}{\sigma^2 x} \to -\infty \), and except for some devious corner cases \( M(a, b, z) \) is asymptotically proportional to \((-z)^{-a}\). Thus for either the positive or negative root \( \beta \), the product

\[ Gx^\beta M(-\beta, -2(\beta - 1) + \frac{2\mu}{\sigma^2}, z) \]

is finite at \( x = 0, z = -\infty \). The Tricomi U function, evaluated at a negative \( z \), is complex valued and cannot be multiplied by a constant \( Z \) to have all real values, so we rule this function out. Thus the general solution in this region is

\[ \text{51 We state this without proof, but by showing the conditions of proposition 3 are met the final solution must be the value function.} \]

\[ \text{52 Again, all that matters in the end is that the conditions of proposition 3 are met.} \]
\[ G_1 x^\gamma M(-\gamma, -2(\gamma - 1) + \frac{2\mu}{\sigma^2}, z) + G_2 x^\kappa M(-\kappa, -2(\kappa - 1) + \frac{2\mu}{\sigma^2}, z) \]

where \( \kappa (\gamma) \) is the positive (negative) root \( \beta \) of the above quadratic.

We are now ready to solve the system of equations (29)-(32) characterizing the value functions for \((\delta, R) \notin O^*\). Recall these are

\[
\begin{align*}
\hat{r} E(\delta, R, e) &= \mathcal{L} E(\delta, R, e) + \lambda_e [E(\delta, R, d) - E(\delta, R, e)] + \iota [0 - E(\delta, R, e)] \quad (68) \\
\hat{r} E(\delta, R, d) &= \mathcal{L} E(\delta, R, d) + \lambda_d [E(\delta, R, e) - E(\delta, R, d)] + \iota [0 - E(\delta, R, d)] \quad (69) \\
\hat{r} D(\delta, R, e) &= \mathcal{L} D(\delta, R, e) + \lambda_e [D(\delta, R, d) - D(\delta, R, e)] + \iota [\zeta \delta - R - D(\delta, R, e)] \quad (70) \\
\hat{r} D(\delta, R, d) &= \mathcal{L} D(\delta, R, d) + \lambda_d [D(\delta, R, e) - D(\delta, R, d)] + \iota [\zeta \delta - R - D(\delta, R, d)] \quad (71)
\end{align*}
\]

where

\[
\mathcal{L} f = \delta \mu f \delta + \frac{\sigma^2}{2} \delta^2 f \delta - (1 - \tau) h \delta f_R
\]

Start with equity values: letting \( \hat{r} = r + \iota \) and rearranging (29,30), we can use the linearity of the operator \( \mathcal{L} \) to write

\[
\begin{bmatrix}
\hat{r} + \lambda_e & -\lambda_e \\
-\lambda_d & \hat{r} + \lambda_d
\end{bmatrix}
\begin{bmatrix}
E(\delta, R, e) \\
E(\delta, R, d)
\end{bmatrix}
= \mathcal{L}
\begin{bmatrix}
E(\delta, R, e) \\
E(\delta, R, d)
\end{bmatrix}
\]

The matrix

\[
\begin{bmatrix}
\hat{r} + \lambda_e & -\lambda_e \\
-\lambda_d & \hat{r} + \lambda_d
\end{bmatrix}
\]

has eigendecomposition

\[
\begin{bmatrix}
\hat{r} + \lambda_e & -\lambda_e \\
-\lambda_d & \hat{r} + \lambda_d
\end{bmatrix}
= \begin{bmatrix} 1 & 1 \\ -\lambda_d & \hat{r} + \lambda_d \end{bmatrix}^{-1}
\begin{bmatrix}
\hat{r} & 0 \\
0 & \hat{r} + \lambda_e + \lambda_d
\end{bmatrix}
\begin{bmatrix} 1 & 1 \\ -\lambda_d & \hat{r} + \lambda_d \end{bmatrix}
\]

Define

\[
\begin{bmatrix}
\hat{E}(\delta, R, e) \\
\hat{E}(\delta, R, d)
\end{bmatrix}
= \begin{bmatrix} 1 & 1 \\ -\lambda_d & \hat{r} + \lambda_d \end{bmatrix}^{-1}
\begin{bmatrix}
E(\delta, R, e) \\
E(\delta, R, d)
\end{bmatrix}
\]

Then \( \hat{E} \) follows the delinked system of HJBs

\[
\begin{bmatrix}
\hat{r} & 0 \\
0 & \hat{r} + \lambda_e + \lambda_d
\end{bmatrix}
\begin{bmatrix}
\hat{E}(\delta, R, e) \\
\hat{E}(\delta, R, d)
\end{bmatrix}
= \mathcal{L}
\begin{bmatrix}
\hat{E}(\delta, R, e) \\
\hat{E}(\delta, R, d)
\end{bmatrix}
\]

Define

\[
\xi(x, \gamma) = x^\gamma M(-\gamma, -2(\gamma - 1) + \frac{2\mu}{\sigma^2}, -\frac{2h(1 - \tau)}{\sigma^2 x})
\]

As before, let \( \gamma \) be the negative root of
\[
0 = \frac{1}{2} \frac{\sigma^2}{\beta} (\beta - 1) - \mu \hat{\beta} + \mu 
\]

and let \( \nu \) be the negative root of

\[
0 = \frac{1}{2} \frac{\sigma^2}{\beta} (\beta - 1) - \mu \hat{\beta} + \mu + \lambda \theta \delta 
\]

Then as shown in appendix B.3, these equations have general solutions \( \hat{E}(\delta, R, e) = K_1 \delta \xi (\frac{R}{\gamma}, \gamma) \) and \( \hat{E}(\delta, R, d) = K_2 \delta \xi (\frac{R}{\gamma}, \nu) \) for some constants \( K_1, K_2 \). Multiplying by \[
\begin{bmatrix} 1 & 1 \\ 1 & -\frac{\lambda d }{\lambda e} \end{bmatrix}
\]
delivers

\[
\begin{bmatrix}
E(\delta, R, e) \\
E(\delta, R, d)
\end{bmatrix} = \begin{bmatrix} K_1 \delta \xi (\frac{R}{\gamma}, \gamma) + K_2 \delta \xi (\frac{R}{\gamma}, \nu) \\
K_1 \delta \xi (\frac{R}{\gamma}, \gamma) - \frac{\lambda d }{\lambda e} K_2 \delta \xi (\frac{R}{\gamma}, \nu) \end{bmatrix}
\]

Given these value functions, we can define \( D(\delta, R, s) = V(\delta, R) - E(\delta, R, s) \), and by linearity of the operator \( \mathcal{L} \),

\[
(r - \mathcal{L}) D(\delta, R, s) = (r - \mathcal{L})(V(\delta, R) - E(\delta, R, s)) = (r - \mathcal{L}) V(\delta, R) - (r - \mathcal{L}) E(\delta, R, s)
\]

\[
= \iota [\delta - R - V(\delta, R)] - [\lambda_s [E(\delta, R, s') - E(\delta, R, s)] - \iota E(\delta, R, s)]
\]

\[
= \iota [\delta - R - (V(\delta, R) - E(\delta, R, s))] + \lambda_s [V(\delta, R) - E(\delta, R, s') - (V(\delta, R) - E(\delta, R, s))]
\]

\[
= \iota [\delta - R - D(\delta, R, s)] + \lambda_s [D(\delta, R, s') - D(\delta, R, s)]
\]

So \( D(\delta, R, s) \) satisfies (31, 32) as desired. Thus we have determined the value functions for \( (\delta, R) \notin O^* \) up to two constants \( K_1, K_2 \). Now, in the region where \( (\delta, R) \in O^* \), we will solve for the receivers value functions \( E(\delta, R, d) \) and \( D(\delta, R, d) \). Recall these must satisfy the HJBs

\[
r E(\delta, R, d) = \mathcal{L} E(\delta, R, d) + \lambda_d [E(\delta, R, e) - E(\delta, R, d)] + \iota [0 - E(\delta, R, d)]
\]

\[
r D(\delta, R, e) = \mathcal{L} D(\delta, R, e) + \lambda_e [D(\delta, R, d) - D(\delta, R, e)] + \iota [\delta - R - D(\delta, R, e)]
\]

and since offers are made in equilibrium in this region, \( E(\delta, R, e) = \theta \delta - R - D(\delta, R, e) \) and \( D(\delta, R, d) = \theta \delta - R - E(\delta, R, d) \). Plugging this in and rearranging, this is

\[
\begin{bmatrix}
\hat{r} + \lambda_e & \lambda_e \\
\lambda_d & \hat{r} + \lambda_d
\end{bmatrix}
\begin{bmatrix}
D(\delta, R, e) \\
E(\delta, R, d)
\end{bmatrix} = \mathcal{L}
\begin{bmatrix}
D(\delta, R, e) \\
E(\delta, R, d)
\end{bmatrix} + \begin{bmatrix} \lambda_e & \lambda_d \\ \lambda_e & \lambda_d \end{bmatrix} (\theta \delta - R) + \begin{bmatrix} \iota & 0 \\ 0 & \iota \end{bmatrix} (\delta - R)
\]

The matrix

\[
\begin{bmatrix}
\hat{r} + \lambda_e & \lambda_e \\
\lambda_d & \hat{r} + \lambda_d
\end{bmatrix}
\]

has eigendecomposition

\[
\begin{bmatrix}
\hat{r} + \lambda_e & \lambda_e \\
\lambda_d & \hat{r} + \lambda_d
\end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & \frac{\lambda d }{\lambda e} \end{bmatrix}
\begin{bmatrix} \hat{r} & 0 \\ 0 & \hat{r} + \lambda_e + \lambda_d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & \frac{\lambda d }{\lambda e} \end{bmatrix}^{-1}
\]

62
Define

\[
\begin{bmatrix}
\hat{D}(\delta, R, e) \\
\hat{E}(\delta, R, d)
\end{bmatrix}
= \begin{bmatrix}
1 & 1 \\
-1 & \frac{\lambda_d}{\lambda_e}
\end{bmatrix}^{-1}
\begin{bmatrix}
D(\delta, R, e) \\
E(\delta, R, d)
\end{bmatrix}
\]

Then \(\hat{E}, \hat{D}\) follow the delinked system of HJBs

\[
\begin{bmatrix}
\hat{r} & 0 \\
0 & \hat{r} + \lambda_e + \lambda_d
\end{bmatrix}
\begin{bmatrix}
\hat{D}(\delta, R, e) \\
\hat{E}(\delta, R, d)
\end{bmatrix}
= \mathcal{L}
\begin{bmatrix}
\hat{D}(\delta, R, e) \\
\hat{E}(\delta, R, d)
\end{bmatrix}
+ \begin{bmatrix}
1 & 1 \\
-1 & \frac{\lambda_d}{\lambda_e}
\end{bmatrix}^{-1}
\begin{bmatrix}
\lambda_e \\
\lambda_d
\end{bmatrix}
\begin{bmatrix}
\theta \delta - R \\
\theta \delta - R + \frac{\omega \delta (\zeta \delta - R)}{\lambda_e + \lambda_d}
\end{bmatrix}
\]

Let \(\kappa\) be the positive root of

\[
0 = [-{}(\hat{r} - \mu) - \mu \beta + \frac{\sigma^2}{2} \beta (\beta - 1)]
\]

and let \(\phi\) be the positive root of

\[
0 = [-{}(\hat{r} + \lambda_e + \lambda_d - \mu) - \mu \beta + \frac{\sigma^2}{2} \beta (\beta - 1)]
\]

Then by the previous discussion, in the region where \(\delta \geq \frac{R}{\delta} \) and \(R \geq 0\) (so \(x \in [0, \bar{x}]\)), the homogeneous ODEs associated with this system (i.e., ignoring \(\delta, R\) terms) have general solutions

\[
\hat{D}(\delta, R, e) = K_3 \delta \xi \left(\frac{R}{\delta}, \gamma\right) + K_4 \delta \xi \left(\frac{R}{\delta}, \kappa\right)
\]

\[
\hat{E}(\delta, R, d) = K_5 \delta \xi \left(\frac{R}{\delta}, \nu\right) + K_6 \delta \xi \left(\frac{R}{\delta}, \phi\right)
\]

Calculation: given constants \(q, c, d\), one can show the particular solution to

\[
(q - \mu)v = -(\mu x + h(1 - \tau))v' + \frac{\sigma^2}{2} x^2 v'' + cx + d
\]

takes the form

\[
v = \frac{c}{q} x + \frac{\left(-\frac{ch(1-\tau)}{q}\right) + d}{q - \mu}
\]
\[ V = \delta v = \frac{c}{q} R + \left( -\frac{eh(1-\tau)}{q} + d \right) \delta \]

After carrying out the matrix multiplication, the relevant parameters for \( \hat{E}(\delta, R, d) \) are \( c = -\lambda_e - \frac{\lambda_e}{\lambda_c + \lambda_d} \delta, \) \( d = \lambda_d \theta + \frac{\lambda_e}{\lambda_c + \lambda_d} i \zeta, \) \( q = \hat{r} + \lambda_d + \lambda_e, \) while for \( \hat{D}(\delta, R, e) \) they are \( c = \frac{-\lambda_e}{\lambda_c + \lambda_d} \delta, \) \( d = \frac{\lambda_e}{\lambda_c + \lambda_d} i \zeta, \) \( q = \hat{r}. \) Plugging this in, the relevant particular solutions are

\[
\hat{D} = \frac{-\lambda_d}{\lambda_c + \lambda_d} \hat{r} R + \left( \frac{\lambda_d}{\lambda_c + \lambda_d} \frac{h(1-\tau)}{\hat{r} - \mu} + \frac{\lambda_d}{\lambda_c + \lambda_d} i \zeta \right) \delta
\]

\[
\hat{E} = -\lambda_e - \frac{\lambda_e}{\lambda_c + \lambda_d} \hat{r} R + \left( \frac{\lambda_e + \lambda_d}{\hat{r} + \lambda_d + \lambda_e} \theta + \frac{\lambda_e}{\lambda_c + \lambda_d} i \zeta \right) \delta
\]

and multiplying by \( \begin{bmatrix} 1 & 1 \\ -1 & \frac{\lambda_d}{\lambda_e} \end{bmatrix} \) gives

\[
\begin{bmatrix}
D(\delta, R, e) \\
E(\delta, R, d)
\end{bmatrix} = \begin{bmatrix}
K_3 \delta \xi(\frac{R}{\hat{r}}, \gamma) + K_4 \delta \xi(\frac{R}{\hat{r}}, \kappa) \\
K_5 \delta \xi(\frac{R}{\hat{r}}, \nu) + K_6 \delta \xi(\frac{R}{\hat{r}}, \phi)
\end{bmatrix} + \begin{bmatrix}
\frac{-\lambda_d}{\lambda_c + \lambda_d} \hat{r} R + \left( \frac{\lambda_d}{\lambda_c + \lambda_d} \frac{h(1-\tau)}{\hat{r} - \mu} + \frac{\lambda_d}{\lambda_c + \lambda_d} i \zeta \right) \delta \\
-\lambda_e - \frac{\lambda_e}{\lambda_c + \lambda_d} \hat{r} R + \left( \frac{\lambda_e + \lambda_d}{\hat{r} + \lambda_d + \lambda_e} \theta + \frac{\lambda_e}{\lambda_c + \lambda_d} i \zeta \right) \delta
\end{bmatrix}
\]

where

\[
\begin{bmatrix}
c_2 \\
c_4
\end{bmatrix} = \begin{bmatrix}
1 & 1 \\
-1 & \frac{\lambda_d}{\lambda_e}
\end{bmatrix} \begin{bmatrix}
\frac{-\lambda_d}{\lambda_c + \lambda_d} \hat{r} \\
-\lambda_e - \frac{\lambda_e}{\lambda_c + \lambda_d} \hat{r}
\end{bmatrix}
\]

\[
\begin{bmatrix}
c_1 \\
c_3
\end{bmatrix} = \begin{bmatrix}
1 & 1 \\
-1 & \frac{\lambda_d}{\lambda_e}
\end{bmatrix} \begin{bmatrix}
\frac{\lambda_d}{\lambda_c + \lambda_d} \frac{h(1-\tau)}{\hat{r} - \mu} + \frac{\lambda_d}{\lambda_c + \lambda_d} i \zeta \\
\frac{\lambda_e + \lambda_d}{\hat{r} + \lambda_d + \lambda_e} \theta + \frac{\lambda_e}{\lambda_c + \lambda_d} i \zeta
\end{bmatrix}
\]

The Confluent Hypergeometric function adds one more complication: we need a different solution for the off equilibrium region where \( \delta \leq \frac{R}{\hat{r}} \) and \( R < 0. \) Luckily, we already showed the only general solution satisfying the boundary conditions is 0, so in this region

\[
\begin{bmatrix}
D(\delta, R, e) \\
E(\delta, R, d)
\end{bmatrix} = \begin{bmatrix}
c_1 \delta + c_2 \hat{R} \\
c_3 \delta + c_4 \hat{R}
\end{bmatrix}
\]
for the same constants \( c_1 - c_4 \). We thus have 6 unknowns, and we require \( E(\delta, R, e) \) and \( E(\delta, R, d) \) to value match and smooth paste at \( \frac{R}{\sigma^2} = \bar{x} \) (already calculated) and at \( \frac{R}{\sigma^2} = 0 \) due to the confluent hypergeometric function. Recall in the exercise region, \( E(\delta, R, e) = \theta \delta - R - D(\delta, R, e) \), so imposing VM and SP for \( D(\delta, R, e) \) at \( \frac{R}{\sigma^2} = 0 \) is sufficient and necessary for VM and SP of \( E(\delta, R, e) \). These conditions are easiest to impose by switching back to \( x = \frac{R}{\sigma^2} \).\(^{53}\)

\[
0 = \begin{bmatrix}
K_3 \xi(0, \gamma) + K_4 \xi(0, \kappa) + K_5 \xi(0, \nu) + K_6 \xi(0, \phi) \\
-K_3 \xi(0, \gamma) - K_4 \xi(0, \kappa) + \frac{Na}{\bar{x}}[K_5 \xi(0, \nu) + K_6 \xi(0, \phi)]
\end{bmatrix}
\]

\[
0 = \begin{bmatrix}
K_3 \xi'(0, \gamma) + K_4 \xi'(0, \kappa) + K_5 \xi'(0, \nu) + K_6 \xi'(0, \phi) \\
-K_3 \xi'(0, \gamma) - K_4 \xi'(0, \kappa) + \frac{Na}{\bar{x}}[K_5 \xi'(0, \nu) + K_6 \xi'(0, \phi)]
\end{bmatrix}
\]

We verify two of these are redundant,\(^{54}\) and thus these are actually equivalent to two restrictions:

\[
\begin{bmatrix}
K_3 \xi(0, \gamma) + K_4 \xi(0, \kappa) + K_5 \xi(0, \nu) + K_6 \xi(0, \phi) \\
-K_3 \xi(0, \gamma) - K_4 \xi(0, \kappa) + \frac{Na}{\bar{x}}[K_5 \xi(0, \nu) + K_6 \xi(0, \phi)]
\end{bmatrix} = \begin{bmatrix}
K_3 \xi'(0, \gamma) + K_4 \xi'(0, \kappa) + K_5 \xi'(0, \nu) + K_6 \xi'(0, \phi) \\
-K_3 \xi'(0, \gamma) - K_4 \xi'(0, \kappa) + \frac{Na}{\bar{x}}[K_5 \xi'(0, \nu) + K_6 \xi'(0, \phi)]
\end{bmatrix}
\]

and then VM and SP at \( \bar{x} \):

\[
\begin{bmatrix}
\theta - \bar{x} - [K_3 \xi(\bar{x}, \gamma) + K_4 \xi(\bar{x}, \kappa) + K_5 \xi(\bar{x}, \nu) + K_6 \xi(\bar{x}, \phi)] \\
-K_3 \xi(\bar{x}, \gamma) - K_4 \xi(\bar{x}, \kappa) + \frac{Na}{\bar{x}}[K_5 \xi(\bar{x}, \nu) + K_6 \xi(\bar{x}, \phi)]
\end{bmatrix} + \begin{bmatrix}
-c_1 - c_2 \bar{x} \\
c_3 + c_4 \bar{x}
\end{bmatrix} = \begin{bmatrix}
K_1 \xi(\bar{x}, \gamma) + K_2 \xi(\bar{x}, \nu) \\
K_1 \xi'(\bar{x}, \gamma) - \frac{Na}{\bar{x}}K_2 \xi'(\bar{x}, \nu)
\end{bmatrix}
\]

\[
\begin{bmatrix}
-1 - [K_3 \xi'(\bar{x}, \gamma) + K_4 \xi'(\bar{x}, \kappa) + K_5 \xi'(\bar{x}, \nu) + K_6 \xi'(\bar{x}, \phi)] \\
-K_3 \xi'(\bar{x}, \gamma) - K_4 \xi'(\bar{x}, \kappa) + \frac{Na}{\bar{x}}[K_5 \xi'(\bar{x}, \nu) + K_6 \xi'(\bar{x}, \phi)]
\end{bmatrix} + \begin{bmatrix}
-c_2 \\
c_4
\end{bmatrix} = \begin{bmatrix}
K_1 \xi'(\bar{x}, \gamma) + K_2 \xi'(\bar{x}, \nu) \\
K_1 \xi'(\bar{x}, \gamma) - \frac{Na}{\bar{x}}K_2 \xi'(\bar{x}, \nu)
\end{bmatrix}
\]

This is a linear system which is easily solved for \( K_1 - K_6 \), once one notes that

\[
\xi'(x, \gamma) = \gamma x^{\gamma-1} M(-\gamma, -2(\gamma - 1) + \frac{2\mu}{\sigma^2}, \frac{-2h(1 - \tau)}{\sigma^2 x})
\]

\[
+ x^{\gamma} \frac{-\gamma}{-2(\gamma - 1) + \frac{2\mu}{\sigma^2}} \frac{2h(1 - \tau)}{\sigma^2 x^2} M(-\gamma + 1, -2(\gamma - 1) + \frac{2\mu}{\sigma^2} + 1, \frac{-2h(1 - \tau)}{\sigma^2 x})
\]

**D  Period 1 Decision to Liquidate or Enter Chapter 11**

First, we provide a proof of Proposition 4, which gives the solution to the problem of optimally entering Chapter 11.

---

\(^{53}\)Note we arrive at these equalities by first subtracting \( c_1 \delta + c_2 R \) or \( c_3 \delta + c_4 R \) from both sides.

\(^{54}\)Specifically, since \( \xi \) converges as \( x \to 0 \) (\( z \to -\infty \)), its derivative must converge to zero. This can be shown directly with the asymptotic properties of the Confluent Hypergeometric function.
D.1 Proving Proposition 4

The conditions of proposition 4 guarantee that

\[
\max(\mathcal{E}(\delta) - B - E^B(\delta), -rE^B(\delta) + D E^B(\delta) + (1 - \tau)(\delta - C_0)) = 0
\]

where

\[
D f(\delta) = f'(\delta)\mu\delta + f''(\delta)\frac{\sigma^2}{2}\delta^2
\]  

(74)

Since the candidate \(E^B\) is smooth, applying Ito’s lemma to \(e^{-rt}E^B(\delta_t)\) delivers

\[
e^{-rt}E^B(\delta_t) = E^B(\delta_0) + \int_0^t e^{-rs}[-rE^B(\delta_s) + DE^B(\delta_s)]ds + M_t
\]

for a local martingale \(M_t\) with \(M_0 = 0\). By the variational inequality,

\[
E^B(\delta_0) + \int_0^t e^{-rs}[-rE^B(\delta_s) + DE^B(\delta_s)]ds + M_t \leq E^B(\delta_0) + \int_0^t e^{-rs}(-(1 - \tau)(\delta_s - C_0)]ds + M_t
\]

implying

\[
e^{-rt}E^B(\delta_t) + \int_0^t e^{-rs}[(1 - \tau)(\delta_s - C_0)]ds \leq E^B(\delta_0) + M_t
\]

Let \(\tau_n\) be the sequence of localizing stopping times for \(M_t\), let \(T\) be an arbitrary stopping time, and let \(Q_n = T \wedge \tau_n \wedge n\). Then we can apply optional sampling to write

\[
E^0[\mathbb{E}^{-rQ_n} E^B(\delta_{Q_n}) + \int_0^{Q_n} e^{-rs}[(1 - \tau)(\delta_s - C_0)]ds] \leq E^B(\delta_0) + \mathbb{E}^{\delta_0}[M_{Q_n}] = E^B(\delta_0) + M_0 = E^B(\delta_0)
\]

Since \(\delta^\psi \to 0\) as \(\delta \to \infty\), there exist constants \(k^0, k^1\) such that \(E(\delta) \leq k^0 + k^1 \delta\). Also, it is clear that

\[
|\int_0^t e^{-rs}[(1 - \tau)(\delta_s - C_0)]ds| \leq (1 - \tau) \int_0^\infty e^{-rs} \delta_s ds + \frac{C_0}{r}
\]

where the right side is integrable. Thus we can apply the lemma of appendix A to use dominated convergence, and clearly \(Q_n \to T\) as \(n \to \infty\), so

\[
E^0[e^{-rT} E^B(\delta_T) + \int_0^T e^{-rs}[(1 - \tau)(\delta_s - C_0)]ds] \leq E^B(\delta_0)
\]

By the variational inequality, this implies

\[
E^0[e^{-rT}(\mathcal{E}(\delta_T) - B) + \int_0^T e^{-rs}[(1 - \tau)(\delta_s - C_0)]ds] \leq E^B(\delta_0)
\]
Applying the same argument for the optimal $T_B$, and using the fact that $\delta \geq \delta_B$ implies $-r E^B(\delta) + D E^B(\delta) + (1-\tau)(\delta - C_0) = 0$ by the construction of $E^B$, we get
\[
\mathbb{E}^\delta [e^{-rT_B}(\mathcal{E}(\delta_{T_B}) - B) + \int_0^{T_B} e^{-r_s}[(1-\tau)(\delta_s - C_0)]ds] = E^B(\delta_0)
\]
completing the proof.

D.2 Liquidation vs Chapter 11 Renegotiation

First, recall that asymptotically $\mathcal{E}(\delta) = E(\delta, R_0, e)$ approaches $(\theta - c_1)\delta - (1 - c_2)R_0$. Thus a sufficient condition for assumption 1 is that $(\theta - c_1) < \frac{(1-\tau)\rho}{r-\mu}$.

In period 1, the firm can decide to liquidate and receive 0 or enter chapter 11 and receive $\mathcal{E}(\delta) - B = E(\delta, R_0, e) - B$. Specifically, they solve
\[
E_0(\delta) = \sup_{T_B, T_L \in \mathcal{F}_T} \mathbb{E}^\delta \left[ \int_0^{T_B \wedge T_L} e^{-rt}(1-\tau)(\delta_t - C_0)dt + 1(T_B < T_L)e^{-rT_B}[\mathcal{E}(\delta_{T_B}) - B] \right] \tag{75}
\]

Note that choosing stopping times $T_B, T_L$ is equivalent to choosing $T = T_B \wedge T_L$ and whether to liquidate or enter chapter 11 at time $T$. The latter decision is of course trivial since the firm will always choose the larger of $E(\delta_T, R_0, e) - B$ and 0. Thus (75) can be rewritten equivalently as
\[
E_0(\delta) = \sup_{T \in \mathcal{F}_T} \mathbb{E}^\delta \left[ \int_0^T e^{-rt}(1-\tau)(\delta_t - C_0)dt + e^{-rT}g(\delta_T) \right] \tag{76}
\]

where $g(\delta) = \max(0, \mathcal{E}(\delta) - B)$. Further, we can define the Ito process $G_t = \int_0^t e^{-rs}(1-\tau)(\delta_s - C_0)ds$ and $\hat{g}(G, \delta, t) = G + \frac{C_0}{r} + e^{-rt}g(\delta) \geq 0$ write this as
\[
\hat{E}_0(\delta, G, t) = \sup_{T \in \mathcal{F}_T} \mathbb{E}^{\delta,G,T} [\hat{g}(G_T, \delta_T, T)] \tag{77}
\]
which exists by Øksendal (2003) Theorem 10.1.9. It is clear from inspection that $E_0(\delta) = \hat{E}_0(\delta, 0, 0) - \frac{C_0}{r}$. We can thus define, for any fixed $C_0$, the exercise region $S(C_0) = \{\delta : E_0(\delta) = g(\delta)\}$.

**Proof of Proposition 5:** By assumption 1, there exists $\delta(C)$ such that $\frac{1-\tau}{r-\mu} - \frac{(1-\tau)C}{r} > g(\delta')$ for all $\delta' > \delta(C)$. Then it cannot be that $E_0(\delta') = g(\delta')$, or else deviating to $T = \infty$ would produce a reward greater than the maximal reward, a contradiction. Thus $\delta(C) = \{\delta : E_0(\delta) = g(\delta)\}$ is finite. Suppose that $\mathcal{E}(\delta(C)) > B$. Then, again by Theorem 10.1.9, if we define $T = \inf\{t : \delta_t \leq \delta(C)\}$, for $\delta > \delta(C)$ we have
\[
E_0(\delta) = \mathbb{E}^\delta \left[ \int_0^{T} e^{-rt}(1-\tau)(\delta_t - C_0)dt + e^{-rT}g(\delta_T) \right]
\]

\[
= \mathbb{E}^\delta \left[ \int_0^{T} e^{-rt}(1-\tau)(\delta_t - C_0)dt + e^{-rT}(\mathcal{E}(\delta(C)) - B) \right]
\]
Since $\delta_B(C)$ maximizes this by proposition 4, it must be that $\bar{\delta}(C) = \delta_B(C)$. Finally, suppose that $E(\bar{\delta}(C)) \leq B$. Then for $\delta > \bar{\delta}(C)$ we have

$$E_0(\delta) = E[\int_0^T e^{-rt}(1 - \tau)(\delta_t - C_0)dt + e^{-rT}g(\delta_T)]$$

$$= E[\int_0^T e^{-rt}(1 - \tau)(\delta_t - C_0)dt]$$

and again since $\delta_L(C)$ maximizes this, it must be that $\bar{\delta}(C) = \delta_L(C)$.

Given the existence of $\bar{\delta}(C)$, we may apply a standard formula for the first hitting time of a geometric Brownian motion to write the value function explicitly for $\delta > \bar{\delta}(C)$:

$$E_0(\delta) = (1 - \tau)[\frac{\delta}{r - \mu} - \frac{C}{r}] + (\frac{\delta}{\bar{\delta}(C)})^r[g(\bar{\delta}(C)) - (1 - \tau)[\frac{\bar{\delta}(C)}{r - \mu} - \frac{C}{r}]]$$

It will be helpful to define

$$I(\delta, x) = (1 - \tau)[\frac{\delta}{r - \mu} - \frac{C}{r}] + (\frac{\delta}{x})^r[g(x) - (1 - \tau)[\frac{x}{r - \mu} - \frac{C}{r}]]$$

and note that $E_0(\delta) = \sup_{x \leq \delta} I(\delta, x)$.

Finally, before we prove proposition 6, the following lemma is useful.

**Lemma D.1** $E(\delta)$ diverges to infinity as $\delta \to \infty$.

**Proof of Lemma:** Using the notation of appendix C, we first prove $c_2 > -1$. Multiply the expression for $c_2$ by $\hat{r}(\hat{r} + \lambda_e + \lambda_d)$:

$$-\hat{r} \frac{\lambda_d}{\lambda_d + \lambda_e} - \lambda_d\hat{r} - \frac{\lambda_e}{\lambda_d + \lambda_e} - \hat{r} = -\hat{r} - \lambda_d\hat{r} - \lambda_e\hat{r}$$

$$> -\hat{r} - \lambda_d\hat{r} - \lambda_d\hat{r} - \lambda_e\hat{r} = -\hat{r}(\hat{r} + \lambda_d + \lambda_e)$$

Since $c_2 > -1$, it must be that $c_1 < \theta$. If not, then asymptotically debt value approaches $c_1\delta + c_2R > \theta\delta - R$, the reorganized firm value. This is a contradiction. Therefore, in the limit as $\delta \to \infty$, we have $E(\delta) = E(\hat{\delta}, R_0, e) = (\theta - c_1)\delta - (1 - c_2)R_0$ starts to increase in $\delta$ and thus also goes to infinity.

**Proof of Proposition 6:** First, we note that if $\hat{C} > C$, then $S(C) \subset S(\hat{C})$. As a result, $\bar{\delta}(C)$ must be weakly increasing in $C$.

---

55 Specifically, we have $E^B \leq E_0$, so if $\delta_B$ were different from $\bar{\delta}(C)$ then $E^B$ could be improved to $E_0$ by deviating to $T$.

56 The value function corresponding to $\hat{C}$ is clearly weakly smaller than that corresponding to $C$. If there were a point $y \in S(C)$ such that $y \notin S(\hat{C})$, then by definition of $S(C), S(\hat{C})$ the value function corresponding to $\hat{C}$ would be strictly larger at $y$ (since the payoff is independent of $C, \hat{C}$) which is a contradiction.
Next, we show that \( \tilde{\delta} \) diverges to infinity. Suppose by contradiction this weren’t the case: there exists \( K \) such that \( \tilde{\delta}(C) \leq K \) for some large enough \( C \). For any \( \epsilon > 0 \), we can pick \( \delta_0 > K \) arbitrarily high so that \( (\frac{\delta_0}{K})^\gamma \leq \epsilon \). For arbitrary \( C \), the value function as above must be

\[
E_0(\delta_0) = (1 - \tau)\left[\frac{\delta_0}{r - \mu} - \frac{C}{r}\right] + \left(\frac{\delta_0}{\tilde{\delta}(C)}\right)^\gamma [g(\tilde{\delta}(C)) - (1 - \tau)\left(\frac{\tilde{\delta}(C)}{r - \mu} - \frac{C}{r}\right)]
\]

By continuity, \( g(\delta) - \frac{(1 - \tau)d}{\tau - \mu} \) attains a maximum \( H \) on the compact set \([0, K]\), so

\[
E_0(\delta_0) \leq (1 - \tau)\left[\frac{\delta_0}{r - \mu} - \frac{C}{r}\right] + \left(\frac{\delta_0}{\tilde{\delta}(C)}\right)^\gamma[H + \frac{C(1 - \tau)}{r}]
\]

\[
\leq (1 - \tau)\left[\frac{\delta_0}{r - \mu} - \frac{C}{r}\right] + \epsilon\max(H, 0) + \frac{C(1 - \tau)}{r}
\]

Now, send \( C \) to infinity, keeping \( \delta_0 \) constant and taking \( \epsilon < 1 \). This clearly becomes negative, a contradiction. Then \( \tilde{\delta}(C) \) increases to infinity as \( C \to \infty \), and by the previous lemma \( \mathcal{E}(\delta) \) increases to infinity as \( \delta \to \infty \), so there must exist some \( \tilde{C} \) such that \( \mathcal{E}(\tilde{\delta}(C)) > B \) whenever \( C > \tilde{C} \). Since \( \mathcal{E}(0) = 0 \), if \( \tilde{\delta}(C) \) is continuous in \( C \) then we can take \( \tilde{C} \) to satisfy \( \mathcal{E}(\tilde{\delta}(C)) = B \).

The remainder of the proof shows \( \tilde{\delta}(C) \) is continuous in \( C \). By the analysis of appendix C, as \( \delta \to \infty \) we have \( \mathcal{E}(\delta) \) converges to an affine function \( a\delta + b \) with \( 0 < a \). By Assumption 1, \( a < \frac{(1 - \tau)}{\tau - \mu} \).

Then there exists a constant \( d \) such that \( \mathcal{E}(\delta) < a\delta + d \), so

\[
\frac{(1 - \tau)C}{r - \mu} - \frac{(1 - \tau)\delta}{r - \mu} + \mathcal{E}(\delta) - B < \frac{(1 - \tau)C}{r - \mu} - \delta\left(\frac{1 - \tau}{r - \mu} - a\right) + d
\]

Then defining \( \phi(C) = \left(\frac{(1 - \tau)}{r - \mu} - a\right)^{-1}d + \left(\frac{(1 - \tau)}{r - \mu} - a\right)^{-1}\frac{(1 - \tau)C}{r - \mu} \), we have for any \( C \) that \( \delta \geq \phi(C) \) implies

\[
\frac{(1 - \tau)C}{r - \mu} - \frac{(1 - \tau)\delta}{r - \mu} + \mathcal{E}(\delta) - B < 0
\]

In this case, for \( \delta > \phi(C) \), it must be that the value function \( E_0(\delta) \) is defined by a lower threshold \( \tilde{\delta}(C) \) with \( \tilde{\delta}(C) < \delta \), since exercise for \( \delta \geq \phi(C) \) is strictly suboptimal. We are now ready to show, for any \( \tilde{C} \), that \( \tilde{\delta}(C) \) is continuous on \([0, \tilde{C}] \). To see this, fix \( \delta > \phi(\tilde{C}) \). Pick some arbitrary \( C \in [0, \tilde{C}] \) and let \( E_0^C(\delta) \) be the associated value function. As above,

\[
E_0^C(\delta) = \sup_{x \leq \delta} I^C(\delta, x)
\]

since we just showed the value function has a lower threshold \( x \leq \phi(C) \), it must be that this equals

\[
= \sup_{x \leq \phi(C)} I^C(\delta, x)
\]

This is a parameterized constrained optimization where the objective \( F(x, C) \) is continuous,
and the correspondence $C \to [0, \phi(C)]$ is clearly continuous and compact valued. Applying Berge’s Theorem, the correspondence $C \to Z(C) = \{x : F(x, C) = F^*(x, C)\}$ mapping $C$ to the set of optimal $x$ corresponding to $C$ is upper hemicontinuous. Since any $x \in Z(C)$ must be in $S$ (the region where $E_0(x) = g(x)$), we have $\bar{\delta}(C) \geq \sup Z(C)$, but if $\bar{\delta}(C) > \sup Z(C)$ then the firm is not acting optimally by definition of $Z(C)$, so $\bar{\delta}(C) = \sup Z(C)$. A standard argument shows the supremum of the image of an upper hemicontinuous correspondence is continuous, completing the proof.\footnote{For any $\epsilon, C$, by upper hemicontinuity there exists $\epsilon_1$ such that $|y - C| < \epsilon_1$ implies $|z - u| < \frac{\epsilon}{2}$ for any $z \in Z(C), u \in Z(y)$. Pick $z \in Z(C)$ with $|\sup(Z(C)) - z| < \frac{\epsilon}{2}$ and $u \in Z(y)$ with $|\sup(Z(y)) - u| < \frac{\epsilon}{2}$ so by triangle inequality $|\sup(Z(y)) - \sup(Z(C))| < \epsilon$.}