Capital Structure and Hedging Demand with Incomplete Markets

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Abstract

Capital structure choices are the result of supply considerations, such as taxes, costly default, agency, and asymmetric information, as well as demand factors, among which investors’ hedging demand. The latter, which has received very little attention in the academic literature, is at the core of this paper. In a general equilibrium model with production and incomplete markets where households differ in their risk–sharing needs, ex–ante identical value–maximizing firms issue different securities, in order to cater to different groups of investors. We find that as the demand for hedging increases, corporates grow in size – to allow for greater precautionary saving – and issue more debt. How much more, depends on the availability of competing risk-sharing instruments, such as (government–issued) risk–free debt and derivatives. When capital structure is jointly shaped by demand and supply considerations – the latter, in the form of an asset–substitution problem – we find that (i) agency is relevant only when hedging demand is high and that (ii) larger investors’ risk–sharing needs lead to equilibria featuring greater aggregate risk.

Key words: Intermediation, Short–sales, Asymmetric Information, Agency, Makowski Criterion.

JEL Codes: D51, D52, D53, G32.

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1 Introduction

Over the roughly 60 years since the appearance in print of the Modigliani–Miller theorem, a large body of literature has investigated in detail the implications for firms’ decisions regarding their capital structure stemming from the violation of one or more of the sufficient conditions for that result. As a consequence, we now have a firm understanding of how capital structure is shaped by tax considerations, costly default, agency, and asymmetric information. We refer to such elements as supply considerations, as they are directly associated with the decisions of firms, as issuers of securities.

Much less attention, in our view, has been paid to what we dub demand considerations, i.e., circumstances that may lead different investors to develop an appetite for different types of securities, including those issued by firms. In this paper, we are first of all interested in understanding how cross-sectional variation in hedging needs drives capital structure choices.

Under complete markets, hedging opportunities available to investors are invariant to capital structure. Therefore we consider a general equilibrium framework with production and, crucially, incomplete markets. Investors are assumed to differ in terms of the correlation of their endowment with the single aggregate factor. Riskier investors – those with relatively higher correlation – display a higher willingness to pay for instruments, such as corporate debt, that are safer and allow so for better risk sharing.

We show that capital structure choices are shaped by the demand for hedging. As the fraction of wealth in the hands of risky investors increase, firms find it value-maximizing to cater to the greater hedging needs of investors by increasing leverage. Investment in physical capital also increases with hedging needs, as firms increase their size to allow for larger precautionary savings.

Furthermore, we show that the equilibrium capital structure is affected by the availability of alternative means which provide risk-sharing in the economy. We consider first the case of competing risk-sharing instruments such as (government-issued) risk-free debt and derivatives on the firms’ assets. In response to a larger volume of government debt or a decline in transaction costs in derivatives’ markets, both leverage and investment decline.

Generalizing then the model to allow firms to choose between technologies characterized by different loading on the risk factor, we show that, when hedging demand is large, ex-ante identical firms adopt different technologies and issue securities that cater to different sets of investors. In particular, firms might specialize on safe technologies to
better satisfy investors' hedging demand.

At last, but definitely not least, we explore the interaction between the demand considerations at the core of our work and a particular supply factor, i.e. agency between shareholders and bondholders. As it is common in the asset substitution literature, we let equityholders pick the firm’s loading on the risk factor in order to maximize their own value. The choice of loading not being observable, equityholders have clear incentives to maximize risk in order to profit from the increased upside, at the expense of bondholders.

With respect to the scenario with no agency considerations, where all choices are observable, risk is higher and leverage is lower. Qualitatively, this is the standard result that obtains in partial equilibrium models where, by construction, risk–sharing plays no role. In our framework, however, the choice of leverage has two drivers. One is the desire to lower debt in order to ameliorate the inefficiency caused by asset substitution. The other is the need to increase debt in order to improve risk–sharing. We portray a scenario where, with agency, the risk loading actually increases as the demand for hedging rises, while the opposite occurs in the scenario without agency.

The notion of competitive equilibrium in incomplete market economies with production is considered problematic in economics. It is only in rather special environments, where the firms’ production and capital structure decisions do not affect investors’ hedging possibilities, as in Diamond (1967) and Carceles-Poveda and Coen Pirani (2009), that the complete market analysis can be extended to incomplete market economies. Outside these environments, starting with the contributions of Dreze (1974), Grossman and Hart (1979) and Duffie and Shafer (1986), a large literature has dealt with the question of what is the appropriate objective function of the firm in economies with incomplete markets. Different objective functions have been proposed and results generally display unappealing theoretical properties, in particular the lack of unanimity of shareholders on the firms’ decisions. As a consequence, macroeconomic models with production and incomplete markets typically assume that firms’ equity is not traded, or that firms operate with a backyard technology and are managed by households. This is the case, for instance, in the Bewley–type economies surveyed by Heathcoate et al. (2009).

We show that the analysis of general equilibrium production economies with incomplete markets can be grounded on solid theoretical foundations, providing the basis for the integrated study of macroeconomics and corporate finance. Key is the requirement that firm value is defined on the basis of rational conjectures, as in Makowski (1983a) and Makowski (1983b). This literature seems to have somewhat overlooked these important
contributions.\footnote{For instance, Makowski is not cited in Dreze (1985) nor in the main later contributions to this literature, like Demarzo (1993), Kelsey and Milne (1996), Dierker and Dierker (2002), Bonnisseau and Lachiri (2004), Dreze et al. (2007), Carceles-Poveda and Coen Pirani (2009). When it is cited, as in Duffie and Shafer (1986), it is to a large extent disregarded. Makowski is not even cited in the main surveys of the GEI literature, as Grossman and Hart (1990) and Magill and Shafer (1991).}

When imposing rationality of pricing conjectures, firms’ beliefs about the prices of equity and debt in correspondence of any choice of investment and capital structure are given by the highest among the consumers’ marginal valuations for each liability. It is shown that shareholders unanimously support value maximization and hence equilibrium firms’ choices. Furthermore, in absence of agency frictions, equilibrium allocations are constrained efficient. Our analysis re-formulates and extends the earlier findings by Makowski\footnote{Indeed, Makowski (1983a) and Makowski (1983b) showed that if firms operate on the basis of rational conjectures, under the condition that agents cannot short-sell equity and under symmetric information, value maximization is unanimously supported by shareholders as the firm’s objective. He also showed that competitive equilibria are constrained Pareto optimal.} to economies with asymmetric information and agency frictions and provides a systematic study of the properties of competitive equilibria of these economies in a standard macro finance environment. It also shows that agency frictions generate an externality in the firms’ problem so that equilibria may be constrained inefficient.

The remainder of the paper is organized as follows. In Section 2 we introduce the model. Section 3 is dedicated to the characterization of the equilibrium investment and capital structure choices and to the analysis of the relation between such choices and hedging demand. In Section 4 we describe how ex-ante identical firms specialize in equilibrium, issuing securities that appeal to different investors. In Section 5, we bring together demand and supply consideration and describe how asset substitution and hedging demand shape firms’ leverage choices. The detailed discussion of various aspects of the notion of equilibrium, as long as the proofs of existence, welfare properties, and unanimity, are in Section 6. Section 7 concludes.

2 Benchmark economy

The economy lasts two periods, indexed by $t = 0, 1$. It is populated by a continuum of households of $I$ different types, each of them of unit mass, and a continuum of identical firms, also of unit mass. At each date, a single commodity is available for production and consumption. The economy is perfectly competitive and both firms and households take prices as given.

By investing $k$ units of commodity at $t = 0$, a firm obtains $e^\varepsilon f(k)$ units of output at $t = 1$, where $\varepsilon$ is a random variable, and $f$ is strictly increasing and strictly concave.
Investment is financed with equity (out of current dividends paid to equityholders) or by issuing debt in notional amount $B \geq 0$. When $e^\varepsilon f(k) < B$, the firm defaults and output is divided pro-rata among all bondholders. It follows that the conditional payoffs to equity and debt at date 1 are

$$d^e(k, B; \varepsilon) = \max \{e^\varepsilon f(k) - B, 0\},$$
$$d^B(k, B; \varepsilon) = \min \{1, e^\varepsilon f(k)/B\},$$

respectively.

As of $t = 0$, households of type $i = 1, \ldots, I$ are endowed with an equity stake $\theta^i_0$ in the firm and commodity in quantity $w^i_0$. The endowment at $t = 1$, denoted $w^i_1(\varepsilon)$, is stochastic and correlated with firms’ output.

Preferences are described by a strictly increasing, strictly quasi-concave, Von Neumann-Morgenstern utility function over random consumption sequences $\{c^i_0, c^i_1(\varepsilon)\}$:

$$U(c^i_0, c^i_1) \equiv u(c^i_0) + \beta \mathbb{E}[u(c^i_1)], \quad \beta > 0.$$

Trade in financial assets occurs at $t = 0$. Firms issue debt and household trade in both equity and debt. Let $\theta^i$ and $b^i$ denote agent $i$’s post-trade holdings of equity and bonds, respectively. Denote also the bond price as $p$ and the equity price as $q$. Finally, let $V$ be the initial market value of the firm, before date 0 dividends are paid. Then, the household optimization problem is as follows

$$\max_{c^i_0, \theta^i, b^i, c^i_1(\varepsilon)} \quad u(c^i_0) + \beta \mathbb{E}[u(c^i_1)]$$
$$\text{s.t.} \quad c^i_0 = w^i_0 + \theta^i_0 V - q\theta^i - pb^i,$n
$$c^i_1(\varepsilon) = w^i_1(\varepsilon) + \theta^i d^e(\varepsilon) + b^i d^b(\varepsilon) \quad \forall \epsilon,$n
$$\theta^i \geq 0, \quad b^i \geq 0.$$

The last two inequality constraints state that short-sales are not allowed.

The firm’s problem consists in choosing the pair $k, B$ that maximizes its market value:

$$V \equiv \max_{k, B} -k + q(k, B) + p(k, B)B,$$

Notice that in the above expression the value of equity and debt for any production and

\footnote{For simplicity, here we state the consumers’ choice problem for the case where all firms make the same production and financial choice, hence only one type of equity and debt is available for trade to consumers. We will see that identical firms may also end up making different choices in equilibrium, in which case different types of equities and bonds are available for trade; see Section 4.}
financing plan \( \{k, B\} \) is determined by the price conjectures \( q(k, B) \) and \( p(k, B) \). These specify the conjectures about the market valuation of the future yields of equity and debt for any possible choice of the firm.

At equilibrium, we shall require such conjectures to be rational, that is for any production and financing plan we have:

\[
q(k, B) = \max_i \mathbb{E} \left[ \frac{\beta u'(c^i_0)}{u'(c^i_0)} d^e(k, B) \right], \quad (3)
\]
\[
p(k, B) = \max_i \mathbb{E} \left[ \frac{\beta u'(c^i_1)}{u'(c^i_0)} d^b(k, B) \right]. \quad (4)
\]

This requirement prescribes that the conjectured prices of equity and debt, \( q(k, B) \) and \( p(k, B) \), associated with any pair \( \{k, B\} \), are equal to the respective highest marginal valuations across all consumers in the economy - of the securities’ payoffs associated with that pair. The consumers with the highest marginal valuation, for instance, for the payoff \( d^e(k, B) \) are in fact those willing to pay the most for equity when the firms choose \( k, B \).

Importantly, the price conjectures are determined taking as given the households’ marginal rates of substitution, independent of the firm’s decisions.

If the price conjectures corresponding to the plan chosen by firms in equilibrium are correct, that is equal to the market prices \( q \) and \( p \), it is immediate to verify that the rationality of the conjecture coincides with the agents’ Euler equations. Here we require that (3)-(4) hold true for all other feasible pairs, including out-of-equilibrium ones.

With complete financial markets, marginal rates of substitutions are equalized across households and hence a unique stochastic discount factor prices all possible payoffs of existing assets. This is no longer true when markets are incomplete. Rationality of the conjectures implies that the stochastic discount factor pricing equity might be different from the one pricing bonds, and both may vary for different plans \( \{k, B\} \). In Section 6 we will show that, when firms operate on the basis of rational conjectures, the objective of the firm is still well defined, as shareholders unanimously support firms’ choices.

**Definition 1 Competitive Equilibrium.** A competitive equilibrium consists of firms’ choices \( k, B \), firms’ value \( V \), asset prices \( q, p \), price conjectures \( p(k, B), q(k, B) \), as well as consumption choices \( (c^0_i, c^1_i(\varepsilon)) \) and portfolio choices \( (\theta^i, b^i) \) for each agent \( i = 1, \ldots, I \), such that (i) \( k, B \) attain the maximum in problem (2), (ii) \( V \) is the value of (2), (iii) \( (c^0_i, c^1_i(\varepsilon)) \) and \( (\theta^i, b^i) \) solve the consumer problem (1) for each agent \( i = 1, \ldots, I \), (iv) \( p(k, B) \) and \( q(k, B) \) are rational, that is satisfy (4), (3), (v) price conjectures and asset

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4This notion of rationality is the Makowski’s criterion for rational conjectures introduced in Makowski (1980) and Makowski (1983a).
payoffs at the equilibrium choices \(k, B\) satisfy the following consistency conditions:

\[ q = q(k, B), \quad p = p(k, B), \quad d^c = d^c(k, B), \quad d^b = d^b(k, B), \]

and (vi) markets clear, i.e.,

\[ \sum_i b^i \leq B, \quad \sum_i \theta^i \leq 1. \]

3 Hedging Demand and Capital Structure

We further specialize the economy introduced in the previous section by assuming that \(I = 2\) and the production function is \(f(k) = Ak^\alpha, \alpha \in (0, 1)\). Furthermore, we assume that the common shock to firms’ productivity—the aggregate shock—is Normally distributed, i.e., \(\varepsilon \sim N(\mu, \sigma^2)\), \(\sigma > 0\), and that the utility function is \(u(c) = \frac{c^{1-\psi}}{1-\psi}\). Finally, we assume that the endowment processes at \(t = 1\) has the form

\[ w^i_1 = e^{-\chi_i \mu - \frac{1}{2} \chi_i^2 \sigma^2 + \chi_i \varepsilon}, \]

where \(\chi_i \in [0, 1]\). This implies the following moments for the endowment process:

\[ \mathbb{E}(w^i_1) = 1, \quad Var(w^i_1) = e^{\chi_i^2 \sigma^2} - 1, \quad Cov(w^i_1, e^\varepsilon) = e^{\mu + \frac{1}{2} \sigma^2} \left(e^{\chi_i \sigma^2} - 1\right). \]

For \(\chi_i = 0\) the endowment is riskless. As \(\chi_i\) increases, so do the variance of the endowment and its covariance with the productivity shock. Initial equity ownership is uniformly distributed across household, i.e. \(\theta^i_0 = 1/2\) for \(i = 1, 2\).

Households differ in their initial endowment \(w^i_0\) and in their exposure to risk \(\chi_i\). To ensure that households are sufficiently different in their hedging needs, we set \(\chi_1 = 0\) and \(\chi_2 = 0.9\). Thus the endowment of type-1 households is riskless, while that of type-2 households is risky so that the latter will have a relatively high demand for hedging. Figure 1 illustrates the variation of two households’ endowments and firm productivity at \(t = 1\) over the (truncated) set of realizations of the innovation \(\varepsilon\), along with the density function for the latter.

With regard to the other parameter values, we will assume throughout that households have a relative risk aversion coefficient \(\psi = 3\). The span of control parameter is \(\alpha = 0.6\), \(A = 1\), while the parameters of the normal distribution are \(\mu = -0.025\) and \(\sigma = 0.3\).
3.1 Characterization

In this economy we find that both households hold equity in equilibrium, but type–2 households hold all the debt issued by firms. The reason is rather intuitive, since debt is relatively safer compared to equity and, as argued above, type–2 agents have a significantly higher hedging demand.

To better understand the properties of the levels of investment and debt issue chosen by firms it is useful to derive the necessary conditions for a solution to their choice problem (2). Since only type-2 households hold debt while both types hold equity, the necessary Condition for the optimal choice of capital $k$ is

$$1 = \beta A \alpha k^{\alpha - 1} \left[ \int_{\varepsilon^* (k, B)}^{\infty} u'(c_1^2) c^\varepsilon g(\varepsilon) d\varepsilon + \int_{-\infty}^{\varepsilon^* (k, B)} \beta u'(c_1^2) c^\varepsilon g(\varepsilon) d\varepsilon \right],$$

where $g(.)$ denotes the density of the normal distribution and $\varepsilon^* (k, B) \equiv \log \left( \frac{B}{k^\alpha} \right)$ is the lowest realization of the innovation $\varepsilon$ for which the firm is solvent. A little algebra allows to rewrite the above condition as

$$1 = \beta A \alpha k^{\alpha - 1} \left[ \text{cov} \left( \frac{u'(c_1^2)}{u'(c_0^2)}, c^\varepsilon \right) + \mathbb{E} \left( \frac{u'(c_1^2)}{u'(c_0^2)} \right) \mathbb{E} (c^\varepsilon) \right].$$

Note the first order condition is only necessary since in the present environment, where markets are incomplete and firms operate on the basis of rational conjectures as specified above, the firms’ problem is not convex.
In square brackets are two terms familiar in asset pricing models. The first is the covariance between household 2’s marginal rate of substitution and the innovation. The second is the inverse of the risk-free rate\(^6\) times the expected value of the innovation. The fact that the optimal level of \(k\) is increasing in both these terms makes it transparent that the investment policy is also shaped by household 2’s hedging needs.

Similarly the necessary condition for debt optimization is

\[
\beta \int_{\varepsilon^*(k,B)}^{+\infty} \frac{u'(c_1^1)}{u'(c_1^0)} g(\varepsilon) d\varepsilon = \beta \int_{-\infty}^{\varepsilon^*(k)} \frac{u'(c_1^2)}{u'(c_1^0)} e^\varepsilon g(\varepsilon) d\varepsilon.
\]

Since the firm may default, raising debt issuance transfers resources from shareholders to bondholders, but only in the states of nature where the firm is solvent. The above condition requires that, at the margin, such transfer has no effect on firm’s value. Because of market incompleteness, the Modigliani-Miller indeterminacy result does not hold: firms’ leverage is determinate because the marginal rates of substitution of the two types of investors are not equal. In other words, the choice of leverage is dictated by the firm’s incentive to cater to agent 2’s risk–sharing needs.

The extent to which firm’s choices are affected by consumers’ hedging demand can be better appreciated by examining how equilibrium leverage and investment vary with the distribution of the endowment at \(t = 0\) across the two types of agents. As the fraction of the endowment accruing to agent 2 rises, the latter’s saving motive and so her share of asset holdings will also increase. Since her \(t = 1\)’s endowment is positively correlated with the equity’s payoff, her desire to hedge aggregate risk will also increase.

For given total endowment at \(t = 0\), we compute equilibria for several values of the ratio \(w_2^0/(w_1^0+w_2^0)\). Investment, firms’ market value, debt issuance and leverage, computed as \(pB/k\), are depicted by red (solid) lines in Figure 2.

The black (dashed) and blue (dash-dot) lines refer instead to alternative scenarios where markets are complete and equity is the only asset, respectively. Debt or leverage are not reported in the complete market scenario since in that case firms’ capital structure is indeterminate.

As type-2 households grow wealthier, as argued above, the demand for hedging in the economy and hence for firms’ debt - a safer asset - will also increase. This drives up the price of corporate debt, making it optimal for firms to issue more debt and increase leverage. Investment also increases, reflecting the fact that the increase in the demand for hedging is accompanied by a higher level of precautionary savings. The firm caters to

\(^6\)Strictly speaking, this is the shadow price of an asset with riskless unit payoff (not available for trade in the market).
this by increasing its size, in line with our comments to equation (5).

Comparing our environment to the scenario where debt issuance is not allowed, we see that in the latter the equilibrium level of capital is uniformly higher. This is the case because in such situation the supply of hedging instruments is very limited and hence, faced with higher risk, type-2 agents can only respond to this by engaging in larger precautionary savings. In the complete market scenario the level of investment is instead lower, and the reason is specular.

As the fraction of wealth held by agent 2 declines towards 1/2, investment in our model converges towards the complete–market level. This is not surprising, since a larger fraction of wealth in the hands of agents with safe endowments implies a lower aggregate hedging demand and therefore a lower appetite for precautionary savings.

Figure 3 further corroborates our narrative. When equity is the only asset, type-2 agents’ consumption growth is uniformly higher, again reflecting the greater precautionary saving in this case. When hedging demand is relatively low (i.e., the fraction of wealth held by type-2 agents is low), the increase in the firms’ debt level is enough to bring consumption profiles close to those of the complete market scenario, in line with what we saw for capital. In contrast, when the fraction of initial endowment of the commodity held by type-2 households approaches 100%, both the mean and the variance of consumption growth for such agents exceed their complete markets counterparts.
Figure 4 illustrates the implications for assets returns. Since type-1 agents have always a strictly lower valuation for the risk-free asset, the risk–free rate is determined by the marginal valuation of type-2 agents. The risk–free rate declines as type–2 agents become richer, since the variance of their consumption growth increases, driving up hedging demand and the price of debt.

The pattern of excess equity returns – with respect to the risk free rate – with incomplete markets is primarily driven by firms’ leverage. In the scenario without debt equity is less risky and the excess return – the blue (dash-dot) line – lower simply because the firm is unlevered. Recall also that leverage increases as type-2 agents’ wealth increases, contributing to widening the gap between excess returns that obtain with incomplete markets and in the two alternative scenarios. Finally, corporate bond spreads increase, as the probability of default rises with leverage.

3.2 Supply of public debt

We now turn to analyze the effects of the introduction of a further hedging instrument. In addition to the corporate assets described above, household can now purchase risk-free (public) debt, available in fixed and exogenous supply. Results are shown in Figures 5 and 6. Both firms’ investment and leverage decrease as the supply of public debt increases. Crowding out is partial, as corporate debt declines by about 0.8 for each unit increase
in the supply of public debt. The lower leverage drives down excess equity returns. The improved risk–sharing opportunities afforded with debt reduce type–2 agents’ variance of consumption growth and therefore lead to a higher risk-free rate.

### 3.3 Short–Sales

In this section we introduce instead markets for derivatives on corporate debt,\footnote{Since in the environment considered in this section the only short–sale constraint which binds is the one on debt for type–1 agents, we consider here the case where derivatives are given by short and long positions on the firms’ debt. We also only outline in this section the key features of the equilibria with intermediated short-sales. But the general analysis of these equilibria is important from a theoretical standpoint; see Section 6 for a detailed discussion of this economy.} with the goal of understanding how the possibility of short selling corporate assets affects equilibrium capital structure decisions. To model short sales it is natural to introduce financial intermediaries who can issue claims corresponding to short positions on the firms’ debt. Both conceptually and in the practice of financial markets, in fact, a short position on firm’s debt is different from a simple negative holding of debt. It is a loan contract with a promise to repay an amount equal to the future value of debt and hence may be subject to costs, e.g. due to default risk.

We consider a simple environment where consumers shorting a derivative on the firm’s debt partly default and repay only a fraction \((1 - \delta) \in (0, 1)\) of the amount due on each
short position taken.\textsuperscript{8} Hence an intermediary who is intermediating $H$ units of this derivative (that is, issuing $H$ long and short positions), to ensure its ability to meet all its future obligations in the presence of this default risk, holds an amount $\gamma$ of debt as reserve, satisfying:

$$H \leq H(1 - \delta) + \gamma,$$

(7)

In this environment, originating short positions involve then a linear cost, given by the value of the debt which is needed to fully cover the shortfall in future revenue deriving from consumers’ default. To cover such cost intermediaries may charge a different price for long and short positions in the derivative issued. Let $p^+$ (resp. $p^-$) be the price at which long (resp. short) positions in the derivative issued by the intermediary are traded, while $p$ is still the price at which debt trades in the market. The intermediary chooses then the amount $H$ issued of long and short positions in the derivative and the amount $\gamma$ of debt held so as to maximize its profits at $t = 0$, equal to $(p^+ - p^-)H - p\gamma$, subject to the solvency constraint (16).

A solution to the intermediary’s problem exists and features a positive level of intermediation $H > 0$ provided the spread between the price of long and short positions on the derivative satisfies the following no arbitrage condition: $p^+ - p^- = \delta p$. In this case

\textsuperscript{8}We assume the default rate is the same and exogenously given for all consumers. This is just for simplicity.
the spread allows to fully recoup the default cost of intermediation. Intermediaries make zero profits in equilibrium and purchase, per unit of derivative issued, an amount $\delta$ of corporate debt which is just enough to cover the shortfall due to default. It thus follows that the total level of intermediation activity is limited by the amount of the firm’s outstanding debt, $B$, requested by intermediaries to operate. Two situations can then obtain in equilibrium.

In the first one intermediation is not constrained in equilibrium, i.e., $\delta H < B$ and the firm’s outstanding debt is partially held by consumers: in this case debt and long positions in the derivative trade at the same price, i.e., $p = p^+$. If instead $\delta H = B$ intermediation is constrained and the whole of the firm’s debt is held by intermediaries, debt sells at a premium over the long positions on the derivative claim, due to its additional value as reserve in the intermediation technology, i.e., $p \geq p^+$.

Figure 7 depicts the values of firms’ debt and investment and consumers’ portfolio holdings at an equilibrium with intermediated short sales for different values of the default rate $\delta$. At these equilibria intermediation is never constrained, type–1 investors acquire short positions on debt, while type–2 households purchase long positions.

We see that, when intermediation costs are relatively high, the volume of intermediation is negligible. For lower costs, in contrast, intermediation becomes more significant and the availability of derivatives considerably increases the supply of hedging opportu-
nities. As a consequence, firms optimally choose to lower their investment and leverage, a similar effect to the one we saw in the previous section with the supply of public debt.

![Graphs showing Derivatives as fraction of notional (percent), Debt Level, Fraction of equity held by Agent 1, Capital, Collateral (percent of bonds outstanding), and Probability of Default against various transaction costs.](image)

Figure 7: Short Sales and Capital Structure.

4 Specialization

When markets are incomplete, the rationality of the price conjectures implies, as already noticed in Section 2, that different production and financing plans are possibly evaluated using different stochastic discount factors, that is, by the marginal rates of substitution of different households. This means that the firm’s choice problem is not convex, giving rise to the possibility that in equilibrium firms specialize in different production and financial plans. This is not just a technical issue but reflects a fundamental implication of the rationality of firms’ conjectures: firms have an incentive to specialize their plans so as
Figure 8: Specialization

to better cater to the demands of different types of households. On the other hand, specialization may entail a distortion of the firm’s decisions and a smaller market for the assets issued so their price could be lower. This is why specialization does not always occur in equilibrium, as we saw in all the situations considered in Section 3.

To illustrate the possibility of specialization, we now consider an economy where firms can choose between the risky technology \( e^\varepsilon k^\alpha \) considered above and another, safer, technology. For simplicity, let the safer technology be entirely deterministic: \( A_w k^\alpha \). We assume the safe technology is less productive, i.e., \( A_w < \mathbb{E}(e^\varepsilon) \). In particular we set \( A_w = 1.75 \). The production function of the (representative) firm is then now

\[
F(k, \phi; \varepsilon) = \phi e^\varepsilon k^\alpha + (1 - \phi)A_w k^\alpha, \quad \phi \in \{0, 1\}.
\]

Although firms remain ex-ante identical, for some initial endowment distributions they end up specializing in equilibrium. That is, they make different choices so as to cater to the demands of different households.\(^9\) Figure 8 indicates that specialization occurs only when the demand for hedging is high. The red, solid lines repropose the equilibrium values under incomplete markets when the firms’ technology is the one described in Section 3. The blue

\(^9\)While the specification of the technology we adopt in this section introduces an additional element of non-convexity in the firm’s choice problem, we should note that this is not necessary in order to generate specialization. The fundamental non-convexity comes, as argued above, from the rationality requirement for price conjectures.
Figure 9: Choices of risky firms with specialization

(dash-dot) lines illustrate the equilibria that obtain with the current specification. When the fraction of wealth held by type–2 agents is relatively low, all firms choose the risky, more productive technology and hence the equilibrium allocation is the same as the one obtained above.

As the wealth is redistributed towards type–2 households, equilibria feature a non-zero fraction of firms choosing the safe technology. The cost of specialization is the lower expected return, the benefit is that - because of the lower variance - it allows to satisfy more effectively the different hedging demands of different types of households. Specialization arises when the demand for hedging by type 2 households is sufficiently large. With specialization a new asset becomes available to households, making markets endogenously more complete. As a consequence – see the top–right panel of Figure 8 – type–2 agents decrease their equity holdings in risky firms. As shown in Figure 9, such firms shrink in size and value, and they also reduce their leverage.

Figure 10 illustrates the impact of specialization on returns. The supply of risk-free assets by firms opting for the safe technology reduces the variance of type–2 agents’ consumption growth and their precautionary motive. As a result, the risk–free rate increases. Risky firms reduce their leverage, leading to lower excess return of equity, default probability, and corporate bond spreads.
We now generalize our environment by introducing an agency friction akin to the standard asset substitution problem pioneered by Jensen and Meckling (1976). The production function of the (representative) firm is the same as in Section 4. However, we let the firm choose any combination of the safe and the risky technology, i.e. the factor loading $\phi$ can take any number in $[0, 1]$:

$$F(k, \phi; \varepsilon) = \phi e^{\varepsilon} k^\alpha + (1 - \phi) A_w k^\alpha, \quad \phi \in [0, 1].$$

For given $k$, the higher $\phi$, the higher is output volatility.

While the levels of capital $k$ and debt $B$ remain observable to outside investors and are so still chosen to maximize firm value, we assume the choice of $\phi$ is not observable and is then taken by shareholders to maximize their benefits from holding equity. As a consequence, the objectives of shareholders and bondholders are no longer aligned, as the former prefer greater risk. Higher $\phi$ increases equity payoffs by increasing the importance of extreme realizations of the random variable $\varepsilon$, at the expense of bondholders’ value. We also assume that by choosing a value $\phi$ of the loading on the risky factor, firms incur a cost $^{10}$

$$C(\phi) = a_0 \phi + \frac{a_1}{1-\phi} \quad \text{at } t = 0, \quad \text{with } a_0, a_1 > 0.$$  

The presence of a convex cost also makes the choice of the loading a nonlinear optimization problem, thus avoiding the uninteresting scenario where the solution is always on a corner.

Figure 10: Specialization and asset returns.
The firm’s decision problem is then now

$$\max_{k,B,\phi} -k - C(\phi) + q(k, B, \phi) + p(k, B, \phi)B,$$

s.t. \( \phi \in \arg \max q(k, B, \phi) - C(\phi), \)

with price conjectures \( q(k, B, \phi) \), \( p(k, B, \phi) \) satisfying the same rationality condition as in the previous section. Condition (9) is the incentive constraint and reflects the fact that risk loading \( \phi \) is chosen by shareholders to maximize equity value.

Replacing \( q(k, B, \phi) \) with the expression obtained under the rationality of price conjectures, condition (9) becomes

$$\max_{\phi} \left[ \max_{i} \beta \int_{\epsilon^*}^{+\infty} \frac{u'(c_i)}{u'(c_0)} \left[ k^\alpha (\phi A e^\epsilon + (1 - \phi) A_w) - B g(\epsilon) d\epsilon \right] - C(\phi). \right]$$

The factor loading \( \phi \) then obtains as a solution of the following first order condition:

$$\beta k^\alpha \left[ \max_{i} \int_{\epsilon^*}^{+\infty} \frac{u'(c_i)}{u'(c_0)} [k^\alpha (\phi A e^\epsilon - A_0) - B g(\epsilon) d\epsilon ] - C'(\phi) = 0. \right]$$

Notice that the value of \( \phi \) depends on the levels of capital and debt chosen by the firm. Hence the firm, when choosing \( k \) and \( B \) solving problem (8) takes also into account how these levels impact the risk loading chosen by equityholders. As a consequence, the first order conditions obtained in Section 3.1, (5) and (6), have now to be modified to include new terms capturing the effects of marginal changes in \( k \) and \( B \) on the value of \( \phi \). Furthermore, the fact that both \( k \) and \( B \) influence the choice of the risk loading \( \phi \) generate an additional link between production and financing decisions, besides the direct effect of \( B \) and \( k \) on asset returns, due to the agency friction.

Differentiating further the expression on the left hand side of (10), we see that the sign of the effect of increasing debt on the risk loading \( \phi \) depends on the level of \( B \). When the debt level is not too high, in particular when \( B < A_w k^\alpha \) so that the firm is solvent with full loading on the safe technology, \( \phi \) increases when \( B \) increases. Thus higher leverage leads to higher loading on the risky factor. This is the case that corresponds to the standard asset substitution effect. In contrast, when \( B \) is high the sign of the effect is reversed. For the parameter values considered, in equilibrium \( B \) is never too high, so that \( \phi \) is increasing in \( B \).

\( \epsilon^*(k, B, \phi) \) is similarly defined as the threshold value of the innovation below which the firm is involvent. It is then easy to verify that the terms \([A e^{\epsilon^*} - A_w]\) and \( \frac{\partial \epsilon^*}{\partial \phi} \) have always the opposite sign. Hence, the first order conditions are also sufficient for an optimum provided the second derivative of the cost function \( C''(\phi) \) is large enough.
In Figure 11, blue (dash-dot) lines illustrate firms’ choices in equilibria again indexed by different distributions of initial wealth across the two types of households. The red (solid) lines, instead, refer to equilibria of an economy where $\phi$ is observable and firms choose $\{\phi, k, B\}$ so as to maximize their market value (that is, solve problem (8) without the constraint (9)).

We see that the loading on the risky factor is uniformly higher with agency. In contrast, both investment and debt are smaller with agency. These findings are the clear reflection of the incentive of shareholders to choose higher levels of risk loading to maximize the value of equity and of the role played by investment and debt to mitigate the agency problem. At the equilibrium allocation without agency, the incentive compatibility constraint (9) is violated. Hence firms need to modify their choices of risk loading, leverage, and investment. As we saw, a reduction in $B$ tends to reduce risk and the same is true for a reduction in $k$. It turns out that in our scenario firms primarily respond by increasing their risk loading choice, with a smaller adjustment in investment and debt issuance. Hence the overall effect, as already noted, is that equilibrium risk is actually greater under agency. Qualitatively, this is the standard effect of agency.

The more novel aspect of the results displayed in Figure 11 emerges when we also

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12 The equilibria are computed under the conjecture that the agent with the highest marginal valuation for equity, for all values of $\phi$, is the type-1 household, whose endowment is riskless and who always held equity in the equilibria considered in the previous sections.
consider the effects of the endowment distribution. The larger the endowment share of type–2 households (that is, the greater aggregate hedging demand), the wider is the gap with respect to the no–agency scenario. This finding highlight for the first time a non–trivial interaction between demand and supply considerations as determinants of firms’ leverage and investment decisions. As well established above, greater hedging needs call for more leverage. At the same time, the agency friction limits the increase in corporate debt and leads equity–holders to select greater risk.

Notice that, in a general equilibrium environment, market incompleteness is necessary for agency considerations to matter for the firms’ capital structure. Setting \( B = 0 \) in fact ensures that the incentive constraint (9) is always satisfied and, when markets are complete there is no benefit for having a positive level of corporate debt. Hence incentives can be provided to shareholders without any need to affect firms’ production choices and the risk allocation among agents.

The environment considered in these two sections allows to see how the analysis of competitive equilibria with incomplete markets may yield interesting implications for important issues in corporate finance. In the extant literature, it is often the case that the capital structure of firms is fully determined by its role in providing incentives to shareholders or managers, typically in partial equilibrium. In a general equilibrium, incomplete–market framework, however, the capital structure is also determined by the consumers’ willingness to pay for the different forms of corporate securities. In this paper, we have referred to this as hedging demand. This is important, because demand forces link capital structure choices to the allocation of risk in the economy, asset prices, and business fluctuations.

6 Production economies with incomplete markets and agency frictions

In this section we provide a systematic analysis of the properties of the competitive equilibria of the economy with incomplete markets and heterogeneous consumers introduced in Section 2, extended to account for agency frictions modeled along similar lines to Section 5. We consider here a more general specification of the representative firm’s production function:

\[
F(k, \phi; \varepsilon) = \phi f_1(k; \varepsilon) + (1 - \phi) f_2(k; \varepsilon), \quad \phi \in \Phi,
\]

where \( f_i(k; \varepsilon), \ i = 1, 2, \) represent two technologies, characterized by different levels of exposure to the aggregate productivity shock, that is, by different risk. Also, here we allow the effect of the shock to take an arbitrary form, without restricting attention to
multiplicative shocks. The risk loading can take values in the set $\Phi$, assumed to be a subset of $[0, 1]$.

For any possible choice of investment, risk loading and debt, the corresponding expression of the date 1 payoffs of equity and debt is then

$$d^e(k, \phi, B; \varepsilon) = \max \left\{ F(k, \phi; \varepsilon) - B, 0 \right\},$$
$$d^B(k, \phi, B; \varepsilon) = \min \left\{ 1, F(k, \phi; \varepsilon)/B \right\},$$

while the rational price conjectures are

$$q(k, \phi, B) = \max_i \mathbb{E} \left[ \frac{\beta u'(c_i^1)}{u'(c_0^i)} d^e(k, \phi, B) \right],$$
$$p(k, \phi, B) = \max_i \mathbb{E} \left[ \frac{\beta u'(c_i^1)}{u'(c_0^i)} d^b(k, \phi, B) \right].$$

The problem of the firm is as in (8), subject to the agency constraint (9) or, using the above expressions,

$$\max_{\phi} \left[ \max_i \mathbb{E} \left[ \frac{\beta u'(c_i^1)}{u'(c_0^i)} \max \left\{ F(k, \phi; \varepsilon) - B, 0 \right\} \right] \right] - C(\phi). \quad (11)$$

In all other respects the definition of a competitive equilibrium is the same as in Definition 1. For this economy we will show that competitive equilibria as defined has several desirable properties: i) existence of an equilibrium is ensured, ii) equilibrium allocations have well-defined welfare properties, and iii) shareholders unanimously support firms’ decisions.

6.1 A few remarks on the equilibrium concept

The key feature of the competitive equilibrium notion we consider consists in the formulation of the restriction imposed on firms’ price conjectures, the rationality condition.

Note that rational price conjectures are consistent with competitive (indeed Walrasian) markets: the consumers’ marginal rate of substitution used to determine the conjectures over the market valuation of debt and equity are taken as given, evaluated at the equilibrium consumption values and unaffected by the firm’s choice of $k, B, \phi$. In this sense each firm is price taker, is ”small” relative to the market, and we can think of each consumer as holding a negligible amount of shares of any given firm.

We claim this equilibrium notion is natural in competitive production economies. Before discussing the properties of equilibria, we argue here that this notion is equivalent
to two others adopted in the literature (in different environments).

**All markets open at market clearing prices.** Consider a specification where markets for all possible ‘types’ of equity and bonds are open: that is, equity and bonds corresponding to any possible value of $k', B' \text{ and } \phi'$ (consistent with the agency constraint) are available for trade to consumers at the prices $q(k', B', \phi')$, $p(k', B', \phi')$. It is immediate to see that all such markets - except the one corresponding to the firms’ equilibrium choice $k', B', \phi'$ - clear at zero trades. As a consequence, $q(k', B', \phi')$ and $p(k', B', \phi')$ correspond to the equilibrium prices of equity and bonds of a firm who were to “deviate” from the equilibrium choice and choose $k', B', \phi'$ instead. In this sense, we can say that rational conjectures impose a consistency condition on the out of equilibrium values of the equity and bonds price conjectures, that corresponds to a “refinement” somewhat analogous to subgame perfection.\textsuperscript{13}

**Prescott and Townsend equilibria.** Consider the equilibrium concept adopted by Prescott and Townsend (1984) for exchange economies with asymmetric information. In this concept prices depend both on observable and unobservable choices (or states) and this is sustained, drawing a parallel with mechanism design formulations of related problems relying on the Revelation Principle, by restricting admissible choices to those which are incentive compatible. This is analogous to what we do in the firm’s problem (8), where price conjectures also depend on the choice of the risk loading $\phi$, though this choice is not observable by outside investors, but the values of $\phi$ are restricted by the agency constraint (11). Via this constraint, the level of $\phi$ is determined by the observable choices of the firm, $k, B$. Hence price conjectures reflect the correct anticipation of the firm’s unobservable choices. Our equilibrium notion with agency frictions is thus equivalent to the notion proposed by Prescott and Townsend (1984), once extended to production economies with incomplete markets.

Another feature we should point out of the environment considered is the no short-sales assumption. It is evident from our analysis in the previous sections that the unlimited short-sales paradigm adopted by the General Equilibrium with Incomplete markets literature, while elegant and convenient, is incompatible with competitive equilibrium modeling in economies with production. With unrestricted short sales, e.g., of equity, a small firm can in fact have a large effect on the economy by choosing a production plan with cash flows which, when traded as equity, change the asset span and hence the admissible trades.

\textsuperscript{13} This is in line with what already observed in Section 2: under the consistency conditions (v), rationality is always satisfied for the prices associated with the firms’ equilibrium choices. Rationality of conjectures requires that the same property holds also for all out of equilibrium choices.
of all consumers, allocations and equilibrium prices. We should point out that, as we already saw in Section 3.3 and will see more in detail in Section 6.4, limited short-sales can be consistently introduced in our competitive economy with production, by allowing for derivatives on the firms’ assets.

6.2 Equilibrium properties

We now establish and discuss the stated properties of competitive equilibria.

Existence. As we noted in Section 4, the firms’ choice problem is not convex. To ensure the existence of an equilibrium we have then to allow for the possibility of asymmetric equilibria, where ex ante identical firms end up making different choices. The existence proof (in the Appendix) exploits the presence of a continuum of firms of the same type to convexify the firms’ choice problem.\footnote{The existence proof requires for simplicity that $\Phi$ is a discrete set and a natural regularity condition (spelled out in the Appendix) for the solutions of the agency constraint (11). But existence is also guaranteed when $\Phi$ is more generally a compact set if the first order approach is satisfied, that is, if the solution of (11) is unique and described by a continuous function.}

Proposition 1 A competitive equilibrium always exists.

Efficiency. The appropriate efficiency notion for our economy is constrained: attainable allocations are restricted not only by the limited set of financial assets that are available but also by the presence of agency frictions. More formally, a consumption allocation $(c^0_i, c^1_i(\varepsilon))_{i=1}^I$ is admissible if:\footnote{Again, we restrict notation for simplicity to symmetric allocations.}

1. it is feasible: there exists a production plan $k, \phi$ such that

\begin{align}
\sum_i c^0_i + k + C(\phi) &\leq \sum_i w^0_i \quad (12) \\
\sum_i c^1_i(\varepsilon) &\leq \sum_i w^1_i(\varepsilon) + F(k, \phi; \varepsilon);
\end{align}

2. it is attainable with the existing asset structure: that is, in addition to $k, \phi$ as specified in 1. there exists $B$ and, for each consumer’s type $i$, a pair $\theta^i, b^i \geq 0$ such that

\begin{align}
c^1_i(\varepsilon) = w^1_i(\varepsilon) + d^c(k, \phi, B; \varepsilon) \theta^i + d^b(k, \phi, B; \varepsilon)b^i; \quad (13)
\end{align}

3. it is incentive compatible: the risk loading $\phi$ satisfies (11).
We then say that a competitive equilibrium allocation is constrained Pareto efficient if we cannot find another admissible allocation which is Pareto improving.

**Proposition 2** In the absence of agency frictions competitive equilibria are constrained Pareto efficient. With agency frictions constrained efficiency may fail.

The case without agency frictions is the one where $\phi$ is also observable and hence firms are free to choose this variable to maximize their market value, without the constraint imposed by (11). In this situation the above result states that constrained efficiency always holds.

On the other hand competitive equilibria with agency frictions may be constrained inefficient. The reason is that the incentive constraint, given by condition (11), generates what is essentially a pecuniary externality. The values of the risk loading $\phi$ chosen by shareholders does not only depend on the level of investment $k$ and debt $B$ chosen by the firm, but also on the equilibrium stochastic discount factors, that is the the consumers’ marginal rate of substitution, which are used to determine the market value of equity for all possible values of $\phi$. These marginal rates of substitution are taken as given by the firm, but depend on consumption allocation, so that a change in this allocation may relax the constraint.

At the same time, we should point out that this is the only source of inefficiency in our economy. In all other respects, firms’ decisions are efficient and, as we show next, unanimously supported by shareholders.

**Unanimity.** In both the economies with and without the agency friction shareholders unanimously agree on the firm’s production and financing decisions, that is on the choice of $k, B, \phi$ which maximizes the firm’s market value, determined on the basis of rational price conjectures (subject, when $\phi$ is unobservable, to the agency constraint (11)):

**Proposition 3 (Unanimity)** Let $k, B, \phi$ be the firms’ choice at a competitive equilibrium and $(c^i_0, c^i_1(\varepsilon))_{i=1}^I$ be the consumption allocation. Then every agent $i$ holding a positive initial amount $\theta^i_0$ of equity of a firm will be made - weakly - worse off by any other possible admissible and incentive compatible choice of the firm ($k', B', \phi$ satisfying (11)).

The result follows from the fact that, as noticed in Section 6.1, the equilibrium allocation is the same as the one which would obtain if markets for all possible types of equity and bonds were open. Consequently, the unanimity result holds by the same argument as the one used to establish this property for Arrow-Debreu economies.
6.3 Relationship with the literature

The literature on incomplete markets with production has emphasized the problems concerning the specification of the firms’ objective function. These problems do not arise for the equilibrium notion we propose: as shown in the previous section, in the set-up typically considered in this literature (that is, with no agency frictions) both unanimity and constrained efficiency hold. The key difference lies in the specification of the firms’ price conjectures. It is useful then to compare the (Makoswki criterion for) rational conjectures we consider to the two main alternative specifications in the literature in the literature, the Dreze and the Grossman-Hart criteria, in the context of an economy without agency frictions.

Applied to our environment, the criterion proposed by Dreze (1974) for equity price conjectures is as follows:

\[
q(k, B, \phi) = E \left[ \sum_i \theta^i \frac{\beta u'(c_i^f)}{w'(c_i^0)} d^e(k, \phi, B) \right], \forall k, B, \phi
\]

It requires the conjectured price of equity for any plan \(k, B, \phi\) to equal - pro rata - the marginal valuation of the agents who in equilibrium are shareholders of the firm (that is, the agents who value the most the plan chosen by the firm in equilibrium and hence choose to buy equity). It does not however require that the firm’s shareholders are those who value the most any possible plan of the firm. Intuitively, the choice of a plan which maximizes the firm’s value with \(q(k, B, \phi)\) as in (14) corresponds to a situation in which the firm’s shareholders choose the plan which is optimal for them without contemplating the possibility of selling the firm in the market, to allow the buyers of equity to operate the plan they instead prefer. Equivalently, the value of equity for out of equilibrium production and financial plans is determined using the - possibly incorrect - conjecture that the agents who in equilibrium own the equity of a firm remain the firm’s shareholders also for any alternative production and financial plan.\(^{16}\)

Grossman and Hart (1979) propose an alternative criterion for price conjectures which,\(^{16}\)

\(^{16}\)It is then easy to see that any allocation constituting an equilibrium with rational conjectures is also an equilibrium under the Dreze criterion: all shareholders of a firm have in fact the same valuation for the firm’s production and financial plan and their marginal utility for any other possible plan is lower, hence a fortiori the chosen plan maximizes the weighted average of the shareholders’ valuations. But the reverse implication is not true, i.e., an equilibrium under the Dreze criterion is not always an equilibrium under rational conjectures.
when applied to the price of equity in our environment, requires:

\[ q(k, B, \phi) = E \left[ \sum \theta_0^i \frac{\beta u'(c_1^i)}{u'(c_0^i)} du(k, \phi, B) \right], \quad \forall k, B, \phi \]  

(15)

We can interpret this specification as describing a situation where the firm’s plan is chosen by the initial shareholders (i.e., those with some predetermined endowment of equity at the beginning of date 0) so as to maximize their welfare, again without contemplating the possibility of selling the equity to other consumers who value it more. According to this criterion, the value of equity for all production and financial plans is derived on the basis of the conjecture that the firm’s initial shareholders stay in control of the firm whatever is the plan.

In contrast, according to the Makowski criterion for rational conjectures each firm evaluates different production and financial plans using possibly different marginal valuations (that is, possibly different pricing kernels, but all still consistent with the consumers’ marginal rate of substitution at the equilibrium allocation). This is essential to ensure the unanimity of shareholders’ decisions and is a key difference with respect to Dreze (1974) and Grossman and Hart (1979), both of whom rely on the use of a single pricing kernel.\(^\text{17}\)

Turning then to asymmetric information and agency frictions, most of the competitive equilibrium concepts which have been proposed for production economies build on the one proposed by Prescott and Townsend (1984) for exchange economies, therefore exhibiting no traded equity.\(^\text{18}\) While Prescott and Townsend’s approach, rooted in mechanism design, is rather different from ours, which instead relies on the extension of rational conjectures to economies with asymmetric information, as we argued in Section 6.1 our equilibrium notion is indeed equivalent to the one of Prescott and Townsend once this is extended to economies with incomplete markets where firms rather than consumers face agency frictions.\(^\text{19}\) Nonetheless, interesting and important conceptual differences emerge between the properties of equilibria in the two environments.

While competitive equilibria are always constrained efficient in the exchange economies

17 This feature distinguishes also the equilibrium notion based on the Makowski rationality criterion from the several others proposed in the literature, including those applying elements from the theory of social choice and voting to model the control of shareholders over the firm’s decisions; see for instance Demarzo (1993), Boyarchenko (2004), Cres and Tvede (2005).


19 We do not discuss economies with adverse selection in this paper. We conjecture that the equilibrium concepts studied by Bisin and Gottardi (2006) have an equivalent reformulation in terms of equilibria with rational conjectures in economies with production along similar lines to those considered in the present paper.
with moral hazard considered by Prescott and Townsend, this is not the case in production economies, where agency frictions enter the firms’ choice problem, as we have shown in Proposition 2. The nature of the equilibrium concept considered plays no role in this, given the equivalence recalled above. Rather, the incentive constraint in the firm’s choice problem features a pecuniary externality, due to the presence of price conjectures needed to determine the market value of equity for any risk loading choice.\footnote{Prescott and Townsend also assume that markets are complete, while we do not. But whether markets are complete or not, and hence whether marginal rates of substitution are equalized or not across consumers, is not crucial for the welfare result. What is crucial is that these marginal rates of substitution enter the incentive constraint.}

An important implication of the welfare properties of production economies with agency frictions is that when equilibrium allocations are constrained inefficient, a Pareto improvement might be achieved by modifying the types of agents owning equity with respect to those who do so in equilibrium. Since the unanimity result in Proposition 3 always holds, even when equilibrium allocations are not constrained efficient, this misallocation of equity ownership is not a consequence of the lack of unanimity of shareholders (as it is instead the case for the equilibrium concepts adopting the Dreze or the Grossman-Hart criterion). It is rather a consequence of the externality affecting firms’ incentive constraints, which may turn out to be more severe when some types of agents are shareholders than when others are.

6.4 Short sales

We extend here the environment described in Section 6 by introducing intermediated short-sales, along the lines of Section 3.3. Now intermediaries can issue derivatives both on corporate debt and an equity. In both cases the origination of a derivative entails a cost, due to the fact that consumers taking a short position repay only a fraction \((1 - \delta)\) of the amount due. To ensure its own solvency, the intermediary must hold debt as collateral. The intermediary’s solvency constraint then requires that it holds an amount \(\gamma\) of debt to ensure its ability to meet all its future obligations:

\[
H \leq H(1 - \delta) + \gamma, \tag{16}
\]

To cover the cost of this collateral, intermediaries may charge a different price for long and short positions in the derivative issued. Let \(p^+\) (resp. \(p^-\)) be the price at which long (resp. short) positions in the derivative issued by the intermediary are traded, while \(p\) is still the price at which debt trades in the market. The intermediary chooses then the amount \(H\) issued of long and short positions in the derivative and the amount \(\gamma\) of debt...
held as a collateral, so as to maximize its total revenue at \( t = 0 \),

\[
\max_{H, \gamma \in \mathbb{R}^2_+} \left[ (p^+ - p^-)H^b - p\gamma^b + (q^+ - q^-)H^e - q\gamma^e \right],
\]

subject to the solvency constraints

\[
H^b \leq H(1 - \delta) + \gamma,
\]

(16).

A solution to the intermediary’s choice problem exists provided that

\[
p \geq \frac{p^+ - p^-}{\delta};
\]

and is characterized by \( \gamma = \delta H \) and \( H > 0 \) only if \( p = \frac{p^+ - p^-}{\delta} \).

Let \( h^i_+ \in \mathbb{R}_+ \) denote consumer \( i \)'s holdings of long positions in the derivative issued by intermediaries, and \( h^i_- \in \mathbb{R}_+ \) his holdings of short positions. The consumer’s choice problem consists in maximizing his expected utility subject to the budget constraints

\[
c^i_0 = w^i_0 + \theta^i_0 V - q^i \theta^i - pb^i - p^+ h^i_+ + p^- h^i_-
\]

(19)

\[
c^i_1(\varepsilon) = w^i_1(s) + R^e(\varepsilon)(h^i_+ + h^i_- - (1 - \delta)h^i_-) + R^e(\varepsilon)\theta^i
\]

(20)

and \( (\theta^i, b^i, h^i_+, h^i_-) \geq 0 \). Note that a fraction \( 1 - \delta \) of each agent’s short position is defaulted on.\(^{21}\)

The asset market clearing condition for debt becomes

\[
\gamma + \sum_{i \in I} b^i = B,
\]

and for the derivative security

\[
\sum_{i \in I} h^i_+ = \sum_{i \in I} h^i_- = H.
\]

The firm’s choice problem is the same as in Section 2. However, the condition imposing rational conjectures debt, \( q(k, B) \), has to be adjusted to reflect the fact that now intermediaries may also demand debt in the market:

\[
q(k, B) = \max \left\{ \max_i \mathbb{E}\left[ \frac{\text{MRS}^i(c^i(\varepsilon)) \text{R}^b(k, B; \varepsilon)}{\max_i \mathbb{E}[\text{MRS}^i(c^i(\varepsilon)) \text{R}^b(k, B; \varepsilon)] - \min_i \mathbb{E}[\text{MRS}^i(c^i(\varepsilon)) \text{R}^b(k, B; \varepsilon)]} \right] \right\}
\]

(21)

\(^{21}\)Default is modeled exogenously for simplicity. The simplest model would have \( 1 - \delta \) as the cutoff above which the intermediary would gain from enforcing a court ruling against the agent defaulting.
for all $k, B$.

The above expression states that the conjecture of a firm over the price of its debt when the firm chooses the plan $k, B$ equals the maximal marginal valuation of the corresponding debt’s cash flows among both intermediaries and consumers. The second term on the right hand–side of the above expression is in fact the intermediaries’ marginal valuation for debt and can be interpreted as the value of intermediation.

Since an appropriate amount of debt is needed to serve as collateral, the intermediary’s willingness to pay for debt $R^b(k, B; \varepsilon)$ is determined by the consumers’ marginal valuation for the corresponding derivative claims which can be issued.²² Hence the above specification of the debt price conjectures allows firms to take into account the effects of their decisions on the value of intermediation.

In all other respects, a competitive equilibrium of the economy with intermediation and short sales is defined along similar lines to Section 2.²³

The model of intermediation proposed in this section is admittedly quite stylized. We believe however it allows to capture in a simple way the relationship between the financial claims issued by firms and the intermediation process. The key feature is that the derivatives issues by intermediaries are backed by the claims issued by firms in two ways. First, the yields of these derivatives are pegged to the yield of the claims issued by firms; second, the intermediaries must hold some amount of these claims to back the derivatives issued. Hence part of the demand for the firms’ claims now also comes from intermediaries (as such claims enter as some sort of input in the intermediation technology).

Finally, we can provide the following simple characterization of the intermediation levels at equilibrium, which follows from (18):

**Proposition 4 (Intermediation)** In the economy with financial intermediation and short sales, at an equilibrium, either (i) $p = (p^+ - p^-)/\delta > p^+$ and intermediation is full (the whole amount of outstanding debt is purchased by intermediaries) or (ii) $p = q^+$ and intermediation is partial (some if not all the amount of outstanding debt is held by consumers).

At an equilibrium where intermediation is full, debt sells at a premium over the long positions on the derivative claim issued by the intermediary, due to its additional value

²²More precisely, the first term on the numerator of the second expression in (21) equals the consumers’ valuation for long positions in the derivative, the second one their valuation for short positions; dividing by $\delta$ yields the profits of intermediation, per unit of debt purchased.

²³The proofs of existence, constrained–Pareto efficiency and unanimity follow the same arguments as the results for the model presented in Section 2.
as input in the intermediation technology. Intermediaries in turn recoup the higher cost of debt through a sufficiently high spread $p^+ - p^-$ between the price of long and short positions on the derivative.

When intermediation is partial, debt and long positions in the derivative trade at the same price, intermediaries may not be active in equilibrium and the bid ask spread $p^+ - p^-$ is low (in particular, less or equal than $\delta p$).

7 Conclusions

In this paper we have provided an equilibrium foundation to the study of corporate finance by showing how a consistent definition of competitive equilibria can be provided in environments with production and incomplete financial markets. We have shown that, once firms are postulated to operate under rational conjectures, along the lines of Makowski (1983a) and Makowski (1983b), equilibria exist, ensure unanimity, and display appealing welfare properties.

We have shown that when households differ in their risk-sharing needs, ex-ante identical value-maximizing firms issue different securities, in order to cater to different groups of investors. As the demand for hedging increases, corporates grow in size – to allow for greater precautionary saving – and issue more debt. How much more, depends on the availability of competing risk-sharing instruments, such as (government-issued) risk-free debt and derivatives.

When capital structure is jointly shaped by demand and supply considerations – the latter, in the form of an asset-substitution problem – we find that (i) agency is relevant only when hedging demand is high and that (ii) larger investors’ risk-sharing needs lead to equilibria featuring greater aggregate risk.

The next step, which we leave for future work, consists in adapting the equilibrium concept and extending the analysis to the dynamic economies typically considered in macroeconomics and finance.
References


Appendix

Proof of Proposition 1

We only provide here an outline of the main steps. Since the firms’ choice problem is non convex, we allow for the possibility that firms undertake different production and financial plans in equilibrium. By Caratheodory’s Theorem, given the finite dimensionality of the sets where these variables lie, it is enough to consider the case where firms make at most a finite number \( N \) of different choices \( k^n, \phi^n, m^n, B_n \). As a consequence, we extend the consumers’ budget constraints (??)-(??) to allow for the possibility that they trade \( N \) different types of equity and bonds, with prices \( q^n, p^n \) and returns \( R^{e,n}(s), R^{b,n}(s) \).

Since short sales are not allowed, the consumers’ budget set is non empty, compact and convex for all \( p^n, q^n \gg 0 \), all \( R^{e,n}(s), R^{b,n}(s) \geq 0 \) and all \( V^n \geq 0 \), for \( n = 1, \ldots, N \). Under the assumptions made on individual preferences, consumers’ net demand (for the consumption good and the different types of bonds and equity) are then well behaved, continuous functions.

Let us turn then our attention to the firms’ problem (??). Whenever the first order approach is not satisfied and the map \( \phi(k, m, B; c(s)) \) is not single-valued and continuous, it is convenient to write the implementability constraint (??) in terms of the inverse map:

\[
 k, m, B \in \phi^{-1}(\phi; c(s)).
\]

We also impose here the following regularity condition, requiring that the above inverse map can be described by a set of functions

\[
 k, m, B \in \phi^{-1}(\phi; c(s)) \Leftrightarrow G(k, m, B; c(s), \phi) \leq 0,
\]

with \( G(.) \) assumed to be continuous in \( k, m, B, c(s) \) for all \( \phi \in \Phi \). Note that this condition is satisfied in natural environments, as for instance in the case of (??) and (??).

Let us partition the set \( N \equiv \{1, \ldots, N\} \) into equal-sized subsets \( N(\phi) \) for each \( \phi \in \Phi \). The firms’ choice problem can then be rewritten as

\[
 \max_{(k^n, m^n, B^n, \gamma^n)_{n \in N(\phi)}, \phi \in \Phi} \left[ \sum_{\phi \in \Phi} \sum_{n \in N(\phi)} \gamma^n \left( -k^n + \mathbb{E} \left[ \max_i MRS^i(c^i(s)) R^e(k^n, \phi, m^n, B^n; s) \right] \right) + \mathbb{E} \left[ \max_i MRS^i(c^i(s)) R^b(k^n, \phi, m^n, B^n; s) \right] B^n \right] \\
 s.t. \left\{ \begin{array}{l}
 \gamma \in \Delta^{N-1} \\
 G(k^n, m^n, B^n; c(s), \phi) \leq 0 \text{ for all } n \in N(\phi) \text{ and all } \phi
 \end{array} \right.
\]

where \( \gamma \in \Delta^{N-1} \) can be equivalently interpreted as the fraction of firms choosing each of the \( N \) plans, or the probability weights of the lottery over production and financial plans.
describing the choice of each firm\textsuperscript{24}. In the above expression of the firms’ problem we have also used condition M) to substitute for the equity and bond price conjectures and used (22) to rewrite the incentive constraint (??).

The objective function and the constraints of the firms’ problem (23) are continuous w.r.t. \((k, m, B, q)\) and \(c(s)\). Since the sets \(K, M, B\) are compact, the correspondence describing the solution of the firm’s problem (23) above is then non empty and upper hemicontinuous, for all \(c_0 \in (0, \max \{\sum_i w_i^j\}], \) \(\bar{c}_1(s) \in (0, \max \sum_i w_i^j(s)]\).

By a standard fixed point argument there exists so a value \(\bar{\phi}^n, \bar{k}^n, \bar{m}^n, \bar{B}^n, \bar{p}^n, \bar{q}^n, \bar{\gamma}^n, \bar{R}^{e,n}(s), \)
\(\bar{R}^{b,n}(s)\) for \(n = 1, \ldots, N\) and \(\bar{c}(s)\) such that: (a) \(\bar{k}^n, \bar{m}^n, \bar{B}^n, \bar{\gamma}^n\) for \(n = 1, \ldots, N\) solve the firms’ optimal choice problem (23) when the terms \(MRS^i\) appearing in the equity and bond price conjecture maps above are evaluated at \(\bar{c}(s)\), and \(n \in N(\phi)\) implies \(\bar{\phi}^n = \phi\), (b) for each \(i = 1, \ldots, I\), \(\bar{e}^i(s)\) is a solution of the choice problem of type \(i\) consumers at prices and returns \(\bar{p}^n, \bar{q}^n, \bar{V}^n, \bar{R}^{e,n}(s), \bar{R}^{b,n}(s), n = 1, \ldots, N\), satisfying the consistency condition C), (c) the market clearing conditions hold (for each type \(n\) of equity and bonds, the supply \(\bar{\gamma}^n\) equals consumers’ demand).

\section*{Proof of Proposition 2}

Suppose \(\hat{c}(s)\) is admissible and Pareto dominates the competitive equilibrium allocation \(\bar{c}(s)\). By the definition of admissibility a \(k, \hat{m}, \hat{B}\) and \(\bar{\phi}, \bar{\theta}, \bar{\beta}\) exists such that \(\hat{c}(s)\) satisfies (12), (13) and (??). The equilibrium consumption level \(\hat{c}(s)\) is the optimal choice of a type \(i\) consumer at the equilibrium prices \(\hat{q}, \hat{p}\) and returns \(\bar{R}^e(s) = R_e(\hat{k}, \hat{m}, \hat{B}; s), \bar{R}^b(s) = R_b(\hat{k}, \hat{m}, \hat{B}; s)\). As argued in Section 6.1, the consumer’s choice problem is analogous to one where any possible type of equity and bonds are available for trade, at the prices \(q(k, \phi, B, m)\), \(p(k, \phi, B, m)\) satisfying the Makowski criterion M) with \(\phi \in \phi(k, m, B; \hat{c}(s))\). When the map \(\phi(\cdot)\) only depends on \(k, m, B\), we have \(\hat{\phi} = \phi(\hat{k}, \hat{m}, \hat{B})\) and so we get:

\[
\hat{c}_0 + \hat{q} \hat{\theta}^i + \hat{p} \hat{\beta}^i \geq \hat{c}_0 + \hat{q} \hat{\theta}^i + \hat{p} \hat{\beta}^i ,
\]

where \(\hat{q} = q(\hat{k}, \hat{\phi}, \hat{m}, \hat{B}), \hat{p} = p(\hat{k}, \hat{\phi}, \hat{m}, \hat{B})\). Or, equivalently,

\[
\left[ -\hat{k} + \hat{q} + \hat{p} \hat{B} \right] \hat{\theta}_0 + \tau^i \geq \left[ -\hat{k} + \hat{q} + \hat{p} \hat{B} \right] \hat{\theta}_0 ,
\]

(24)

\textsuperscript{24}With the realizations of the lottery observed by consumers when choosing their portfolios.
for \( \tau^i \equiv \hat{c}_0^i + \hat{q} \theta^i + \hat{p} \theta^i = \left[ -\bar{k} + \bar{q} + \bar{p} \bar{B} \right] \theta^i_0 \). Since (24) holds for all \( i \), strictly for some \( i \), summing over \( i \) yields:

\[
\left[ -\bar{k} + \bar{q} + \bar{p} \bar{B} \right] + \sum_i \tau^i > \left[ -\bar{k} + \bar{q} + \bar{p} \bar{B} \right]
\]  \hspace{1cm} (25)

The fact that \( \bar{k}, \bar{m}, \bar{B} \) solves the firms’ optimization problem (??) in turn implies that:

\[-\bar{k} + \bar{q} + \bar{p} \bar{B} \geq -\hat{k} + \hat{q} + \hat{p} \hat{B}, \]

which, together with (25), yields:

\[\sum_i \tau^i > 0,\]

or equivalently:

\[\sum_i \hat{c}_0^i + \hat{k} > \sum_i \hat{w}_0^i,\]

a contradiction to (12) at date 0. ■

**Proof of Proposition 3**

Note that we can always consider a situation where, in equilibrium, each consumer holds at most a negligible amount of equity of any individual firm and so the effects on a consumer’s utility of alternative choices by a firm can then be evaluated using the consumer’s marginal utility. Let \( c(s) \) be the equilibrium consumption allocation. For any possible choice \( k', \phi', m', B' \) by a firm, with \( \phi' \in \phi(k', m', B'; c(s)) \), the (marginal) utility of a type \( j \) consumer if he holds the firm’s equity and debt is

\[-k' - W(k', \phi', m', B') + \max_i \mathbb{E} \left[ MRS^i(c^i(s))R^e(k', \phi', m', B'; s) \right] + \max_i \mathbb{E} \left[ MRS^i(c^i(s))R^b(k', \phi', m', B'; s) \right] B',\]

But this is always lower or equal than the agent’s utility if instead he sells the firm’s equity and bonds at the market price, evaluated on the basis of price conjectures satisfying M),

\[-k' - W(k', \phi', m', B') + \max_i \mathbb{E} \left[ MRS^i(c^i(s))R^e(k', \phi', m', B'; s) \right] + \max_i \mathbb{E} \left[ MRS^i(c^i(s))R^b(k', \phi', m', B'; s) \right] B',\]

and the latter is in turn lower than the corresponding expression if the firm adopts the equilibrium choice \( k, \phi, m, B \), since this choice solves problem (??). ■
Further details of the proof of Proposition ??

When (??) holds as equality only for consumer $i = 2$ we have $c_1^2(s) = w_1^2(s) + a_1(s)k^\alpha > c_1^1(s) = w_1^1(s)$, $c_0^2 = w_0 + V0.5 - q < c_0^1 = w_0 + V0.5$. For simplicity we assume here that the following symmetry condition also holds: $\mathbb{E}[MRS(c^2(s))a_1(s)k^\alpha] = \mathbb{E}[MRS(c^1(s))a_2(s)k^\alpha]$, for $c_1^1 = w_0 + V0.5 - q$, $c_1^1(s) = w_1^1(s) + a_2(s)k^\alpha$ for all $k, q, V > 0$.

For $\phi = 0$ to be an optimal choice for the firms, we must have in this case:

$$q = \mathbb{E}[MRS(c^2(s))a_1(s)k^\alpha] \geq \mathbb{E}[MRS(c^1(s))a_2(s)k^\alpha]$$

which contradicts the assumed symmetry condition, since

$$\mathbb{E}[MRS(c^1(s))a_2(s)k^\alpha] > \mathbb{E}[MRS(c^1(s))a_2(s)k^\alpha].$$

Consider next the case where $w_1^1(s) + a_2(s)k^\alpha$ and $w_1^2(s) + a_2(s)k^\alpha$ varies comonotonically with $a_1(s)$ for all $k \in K$ (a slightly stronger condition than the comononicity of $w_1^1(s), w_1^2(s)$ and $a_1(s)$). In this case we have

$$\mathbb{E}[MRS(c(s))a_2(s)k^\alpha] > \mathbb{E}[MRS(c(s))a_1(s)k^\alpha]$$

for all $k \in K, c_0$ and $c_1(s) = w_1^1(s) + \theta a_2(s)k^\alpha, i = 1, 2, \theta \in [0, 1]$, since $\text{Cov}(MRS(c(s)), a_2(s)) > 0 > \text{Cov}(MRS(c(s)), a_1(s))$. Hence in equilibrium both consumers’ types are only willing to buy equity of firms with full loading on factor $a_2(s)$.

Details on the Dierker, Dierker, and Grodal (2002) example

There are two types of consumers, with type 2 having twice the mass of type 1, and (non expected utility) preferences, respectively, $u^1(c_0^1, c_1^1(s_1), c_1^1(s_2)) = c_1^1(s_1)/\left(1 - (c_0^1)^{\frac{3}{10}}\right)^{\frac{10}{3}}$ and $u^2(c_0^2, c_1^2(s_1), c_1^2(s_2)) = c_0^2 + (c_1^2(s_2))^{1/2}$, endowments $w_0^1 = .95, w_0^2 = 1$ and $w_1^1(s) = w_1^2(s) = 0$ for all $s \in S$.

In this economy Dierker, Dierker and Grodal (2002) find a unique, symmetric Dreze equilibrium where all firms choose the same value of $k$ and $\phi \approx 0.7^{25}$ and this equilibrium is constrained inefficient. We show next that a symmetric competitive equilibrium, according to our definition in Section ??, does not exist. Given the agents’ endowments and preferences, both types of consumers buy equity in equilibrium. It is then easy to see that the firms’ optimality condition with respect to $\phi$ can never hold for an interior value

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25 The notion of Dreze criterion used by Dierker, Dierker, and Grodal to specify price conjectures differs from the Makowski criterion M) in two main respects: i) only the MRS of the consumers who in equilibrium are shareholders of the firms are considered to evaluate alternative production plans, and ii) these MRS are not constant but vary to take into account the effect of each plan on the agents’ consumption.
of $\phi$ nor for a corner solution. On the other hand, an asymmetric equilibrium exists, where a fraction $1/3$ of the firms choose $\phi^1 = 0.99$ and $k^1 = 0.3513$ and the remaining fraction chooses $\phi^2 = 2/3$ and $k^2 = 0.1667$, type 1 consumers hold only equity of the firms choosing $\phi^1$, $k^1$ and type 2 consumers only equity of the other firms. At this allocation, we have $\frac{\partial u^1}{\partial c^1}(s_1) = 1.0101$, $\frac{\partial u^2}{\partial c^2}(s_2) = 3$. Also, the marginal valuation of type 1 agents for the equity of firms choosing $\phi^2$, $k^2$ is 0.1122, thus smaller than the market value of these firms’ equity, equal to 0.1667, while the marginal valuation of type 2 agents for the equity of the firms choosing $\phi^1$, $k^1$ is 0.0105, smaller than the market value of these firms’ equity, equal to 0.3513. Therefore, at these values the firms’ optimality conditions are satisfied. It can then be easily verified that this constitutes a competitive equilibrium according to our definition and that the equilibrium allocation is constrained optimal.

A Parametric Example

Consumers have identical preferences described by $E(u(c_0, c_1(s))) = u(c_0) + E(u(c_1(s)))$, with $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$, for $\gamma = 2$. The state space is $S = \{1, 2, 3\}$ with $\pi(1) = \pi(2) = \pi(3) = \frac{1}{3}$. The production technology is as in (??), with $\alpha = 0.75$ and productivity shocks $a_1(s)$ and $a_2(s)$ taking values, respectively, $\{1, 2, 3\}$ and $\{1.1, 2, 2.9\}$. The second period endowments of type 1 and type 2 agents take values, respectively, $\{1, 2, 3\}$ and $\{1.1, 2, 2.9\}$, while in the first period they are endowed with $w^i_0 = w^i_1(2)$, $i = 1, 2$, units of the good and the same amount $\theta_0 = 0.5$ units of equity. Also, the utility cost of different choices of $\phi$ is $v^i(1) = -0.006$ and $v^i(0) = 0$, for all $i$.

The equilibrium values with and without the agency friction are reported in the following table.

In order to implement the same choice $\phi = 0$ the firm modifies its production and financial decisions together with the portfolio of the agent selected as manager (in particular, the manager’s compensation exhibits a higher amount of equity, (.6456), a lower one of debt (0) and also a lower consumption at date 0).

---

26Consider for instance $\phi = 0.99$. To have an equilibrium at this value the marginal valuation of equity for both consumers must be the same at $\phi = 0.99$, and higher than at any other values of $\phi$, but this second property clearly cannot hold for type 2 consumers.
Appendix A: Additional material

Characterization of the firms’ optimal capital structure conditions

Let $I^e$ (resp. $I^d$) denote the set of shareholders (resp. bondholders) of a firm and consider for simplicity the case where capital is the only input, that is the technology is given by $f(k, s)$.

**Proposition A. 1** If the optimal production and financing decisions of a firm are obtained\(^{27}\) at a level $B$ such that bonds are risk free, that is, $f(k; s) \geq B$ with probability 1, then all equity holders are also bond holders (while the reverse may not be true: $I^e \subseteq I^d$):

$$\max_{i \in I^e} \mathbb{E}MRS^i(c^i(s)) = \min_{i \in I^e} \mathbb{E}MRS^i(c^i(s)) = p = \max_i \mathbb{E}MRS^i(c^i(s))$$ (26)

and

$$\max_{i \in I^e} \mathbb{E} \left[ MRS^i(c^i(s))f_k(s) \right] = \min_{i \in I^e} \mathbb{E} \left[ MRS^i(c^i(s))f_k(s) \right] = 1;$$ (27)

In the situation described above all shareholders value equally the effect on the payoff of equity of an infinitesimal increase in the investment level $k$, and such value is always equal to the marginal cost of the investment.

**Proof of Proposition A. 1** Note first that

$$q(k, B + dB) = \max_i \mathbb{E}MRS^i(c^i(s)) \left[ f(k; s) - B - dB \right].$$

\(^{27}\)We focus here on the conditions concerning the investment level $k$ and capital structure $B$, ignoring those regarding $\phi$, which are straightforward.
Since for all \( i \notin I^c \), \( \mathbb{E} \text{MRS}^i(c^j(s)) [f(k; s) - B] < q(k, B) \), the max in the above expression is attained for some \( i \in I^c \) and hence

\[
q(k, B + dB) = q(k, B) + \max_{i \in I^c} \mathbb{E} \text{MRS}^i(c^j(s)) [-dB].
\]

The right and left derivative of \( q(k, B) \) with respect to \( B \) are then given by:

\[
\frac{\partial q}{\partial B_+} = -\min_{i \in I^c} \mathbb{E} \text{MRS}^i(c^j(s)) ; \quad \frac{\partial q}{\partial B_-} = -\max_{i \in I^c} \mathbb{E} \text{MRS}^i(c^j(s)) \tag{28}
\]

and may differ. Similarly the derivatives with respect to \( k \) are:

\[
\frac{\partial q}{\partial k_+} = \max_{i \in I^c} \mathbb{E} [\text{MRS}^i(c^j(s)) f_k(s)] ; \quad \frac{\partial q}{\partial k_-} = \min_{i \in I^c} \mathbb{E} [\text{MRS}^i(c^j(s)) f_k(s)] \tag{29}
\]

where \( f_k \) denotes the derivative of \( f \) with respect to \( k \).

The first order conditions when \( f(k, \phi, m; s) \geq B \) with probability 1 are:

\[
\frac{\partial V}{\partial B_+} = \frac{\partial q}{\partial B_+} + p \leq 0, \quad \frac{\partial V}{\partial k_+} = \frac{\partial q}{\partial k_+} - 1 \leq 0, \tag{30}
\]

\[
\frac{\partial V}{\partial B_-} = \frac{\partial q}{\partial B_-} + p \geq 0, \quad \frac{\partial V}{\partial k_-} = \frac{\partial q}{\partial k_-} - 1 \geq 0;
\]

Since (28) implies that \( \frac{\partial q}{\partial B_+} \geq \frac{\partial q}{\partial B_-} \), the above conditions (with respect to \( B \)) are equivalent to:

\[
\frac{\partial V}{\partial B_+} = \frac{\partial q}{\partial B_+} + p = \frac{\partial V}{\partial B_-} = \frac{\partial q}{\partial B_-} + p = 0,
\]

that is:

\[
\max_{i \in I^c} \mathbb{E} \text{MRS}^i(c^j(s)) = \min_{i \in I^c} \mathbb{E} \text{MRS}^i(c^j(s)) = p = \max_{i \in I^c} \mathbb{E} \text{MRS}^i(c^j(s))
\]

or (26) holds. Similarly, from (29) we see that \( \frac{\partial q}{\partial k_+} \geq \frac{\partial q}{\partial k_-} \), the above conditions (with respect to \( k \)) are equivalent to:

\[
\frac{\partial q}{\partial k_+} - 1 = \frac{\partial q}{\partial k_-} - 1 = 0,
\]

that is,

\[
\max_{i \in I^c} \mathbb{E} [\text{MRS}^i(c^j(s)) f_k(s)] = \min_{i \in I^c} \mathbb{E} [\text{MRS}^i(c^j(s)) f_k(s)] = 1
\]

or (27) holds, thus completing the proof of the proposition. \( \blacksquare \)

We study next the case where firms can default on their debt obligations, hence corporate debt is risky. Before stating the conditions for an optimum of the firms’ decision problem in the presence of risky debt, it is useful to introduce some further notation. Given a face value of debt equal to \( B \), let \( S^{\text{nd}} \) denote the collection of states in \( t = 1 \) for
which \( f(k; s) \geq B \) and by \( S^{nd} \) the lowest state in \( S^{nd} \), that is the state with the lowest realization of the technology shock for which the firm does not default. Conversely, denote \( S^d \) the collection of states in \( t = 1 \) for which \( f(k; s) < B \), i.e. the firm (partially) defaults on its debt.

**Proposition A. 2** If the optimal production and financing decisions of a firm are obtained at a level \( B \) such that bonds are risk free, the optimal investment and debt levels obtain either at an interior solution, where \( f(k; s^{nd}) > B \), with:

\[
p = \min_{i \in I^d} \mathbb{E} \left( \frac{MRS^i(c^i(s))}{B} \right) | s \in S^d \right) \text{Pr}\{s \in S^d\} + \min_{i \in I^d} \mathbb{E}(MRS^i(c^i(s)) | s \in S^{nd}) \text{Pr}\{s \in S^{nd}\} = \max_{i \in I^d} \mathbb{E} \left( \frac{MRS^i(c^i(s))}{B} \right) | s \in S^d \right) \text{Pr}\{s \in S^d\} + \max_{i \in I^d} \mathbb{E}(MRS^i(c^i(s)) | s \in S^{nd}) \text{Pr}\{s \in S^{nd}\}.
\]

and

\[
1 = \max_{i \in I^d} \mathbb{E} \left( \frac{MRS^i(c^i(s))f_k(k, s)}{s \in S^{nd}} \right) \text{Pr}\{s \in S^{nd}\} + \min_{i \in I^d} \mathbb{E}(MRS^i(c^i(s))f_k(k, s) | s \in S^{nd}) \text{Pr}\{s \in S^{nd}\} + \min_{i \in I^d} \mathbb{E}(MRS^i(c^i(s))f_k(k, s) | s \in S^d) \text{Pr}\{s \in S^d\}
\]

or at a corner solution, \( f(k; s^{nd}) = B \).

**Proof of Proposition A. 2** We first proceed to characterize the conditions for corner solutions.

**Claim 2** The conditions for an optimum at a corner, \( f(k; s^{nd}) = B \), are:

\[
\min_{i \in I^d} E_{s_0} \left\{ \frac{MRS^i(c^i(s_1))}{s_1 \in S^{nd}} \right\} \text{Pr}\{s_1 \in S^{nd}\} + \min_{i \in I^d} E_{s_0} \left( \frac{f(k; s_1)}{B} \right) | s_1 \in S^{nd} \right) \text{Pr}\{s_1 \in S^{nd}\} \geq p \geq \max_{i \in I^d} E_{s_0} \left\{ \frac{MRS^i(c^i(s_1))}{s_1 \in S^{nd}} \right\} \text{Pr}\{s_1 \in S^{nd}\} + \max_{i \in I^d} E_{s_0} \left( \frac{f(k; s_1)}{B} \right) | s_1 \in S^d \right) \text{Pr}\{s_1 \in S^d\}
\]
\[
\min_{i \in I^e} E_{s_0} \left\{ \frac{MRS^i(c^i(s_1)) f_k(k; s_1)}{B} \right\}_{s_1 \in S^{nd}} \left\{ s_1 \in S^{nd} \right\} \Pr\{s_1 \in S^{nd}\} + \\
\min_{i \in I^d} E_{s_0} \left\{ \frac{MRS^i(c^i(s_1)) f_k(k; s_1)}{B} \right\}_{s_1 \in S^{d}} \Pr\{s_1 \in S^{d}\} - \left[ \frac{1}{\epsilon} \right] \left\{ s_1 \in S^{nd} \right\} \Pr\{s_1 \in S^{nd}\}
\]

\[
1 - \max_{i \in I^e} E_{s_0} \left\{ \frac{MRS^i(c^i(s_1)) f_k(k; s_1)}{B} \right\}_{s_1 \in S^{nd}} \Pr\{s_1 \in S^{nd}\} - \left( \frac{1}{\epsilon} \right) \left\{ s_1 \in S^{nd} \right\} \Pr\{s_1 \in S^{nd}\}
\]

**Proof of Claim 1** Note first that, in this case, \( f(k; s_1^0) = B \). Denote by \( S^{nd}_{1} \subset S^{nd}_{1} \) the collection of states in \( t = 1 \) for which the firm does not default, after marginal deviations \( dB > 0 \) and/or \( dk < 0 \) (and similarly \( S^{d}_{1} \subset S^{d}_{1} \)). Evidently, for marginal deviations \( dB > 0 \) and/or \( dk < 0 \) the collection of such states is still given by \( S^{nd}_{1} \).

The partials of the price maps wrt to \( B \) are\(^{28}\)

\[
\frac{\partial q}{\partial B^+} = - \min_{i \in I^e} E_{s_0} \left\{ \frac{MRS^i(c^i(s_1)) f_k(k; s_1)}{B^2} \right\}_{s_1 \in S^{nd}} \Pr\{s_1 \in S^{nd}\}
\]

\[
\frac{\partial q}{\partial B^-} = - \max_{i \in I^e} E_{s_0} \left\{ \frac{MRS^i(c^i(s_1)) f_k(k; s_1)}{B^2} \right\}_{s_1 \in S^{nd}} \Pr\{s_1 \in S^{nd}\}
\]

and

\[
\frac{\partial p}{\partial B^+} = - \min_{i \in I^d} E_{s_0} \left\{ \frac{MRS^i(c^i(s_1)) f_k(k; s_1)}{B^2} \right\}_{s_1 \in S^{d}} \Pr\{s_1 \in S^{d}\}
\]

\[
\frac{\partial p}{\partial B^-} = - \max_{i \in I^d} E_{s_0} \left\{ \frac{MRS^i(c^i(s_1)) f_k(k; s_1)}{B^2} \right\}_{s_1 \in S^{d}} \Pr\{s_1 \in S^{d}\}
\]

\(^{28}\)Obviously, if \( S^{nd}_{1} \) is a singleton, the right derivative is equal to 0.
Analogously, the partials wrt to \( k \) are\(^{29} \)

\[
\frac{\partial q}{\partial k_+} = \max_{i \in I^d} E_{s_0} \left\{ MRS^i(c'(s_1)) f_k(k, s_1) \mid s_1 \in S^{nd} \right\} \Pr\{s_1 \in S^{nd}\}
\]

\[
\frac{\partial q}{\partial k_-} = \min_{i \in I^d} E_{s_0} \left\{ MRS^i(c'(s_1)) f_k(k, s_1) \mid s_1 \in S^{ndr} \right\} \Pr\{s_1 \in S^{ndr}\}
\]

and

\[
\frac{\partial p}{\partial k_+} = \max_{i \in I^d} E_{s_0} (MRS^i(c'(s_1)) \left[ \frac{f_k(k; s_1)}{B} \right] \mid s_1 \in S^d) \Pr\{s_1 \in S^d\} + \\
\frac{\partial p}{\partial k_-} = \min_{i \in I^d} E_{s_0} (MRS^i(c'(s_1)) \left[ \frac{f_k(k; s_1)}{B} \right] \mid s_1 \in S^{dr} \Pr\{s_1 \in S^{dr}\}
\]

So, if \( f(k; s_1) = B \), the FOCs wrt \( B \) are:

\[
\frac{\partial V}{\partial B_+} = \frac{\partial q}{\partial B_+} + \left( \frac{\partial p}{\partial B_+} B + p \right) = \\
- \min_{i \in I^d} E_{s_0} \left\{ MRS^i(c'(s_1)) \mid s_1 \in S^{ndr} \right\} \Pr\{s_1 \in S^{ndr}\} + \\
\left( - \min_{i \in I^d} E_{s_0} (MRS^i(c'(s_1)) \left[ \frac{f(k; s_1)}{B^2} \right] \mid s_1 \in S^d) \Pr\{s_1 \in S^d\} B + p \right) \leq 0 \\
\frac{\partial V}{\partial B_-} = \frac{\partial q}{\partial B_-} + \left( \frac{\partial p}{\partial B_-} B + p \right) = \\
- \max_{i \in I^d} E_{s_0} \left\{ MRS^i(c'(s_1)) \mid s_1 \in S^{nd} \right\} \Pr\{s_1 \in S^{nd}\} + \\
\left( - \max_{i \in I^d} E_{s_0} (MRS^i(c'(s_1)) \left[ \frac{f(k; s_1)}{B^2} \right] \mid s_1 \in S^d) \Pr\{s_1 \in S^d\} B + p \right) \geq 0
\]

which implies (33). Finally, the FOCs wrt \( k \) are:

\[
\frac{\partial V}{\partial k_+} = -1 + \frac{\partial q}{\partial k_+} + \left( \frac{\partial p}{\partial k_+} B \right) = \\
- 1 + \max_{i \in I^d} E_{s_0} \left\{ MRS^i(c'(s_1)) f_k(k, s_1) \mid s_1 \in S^{nd} \right\} \Pr\{s_1 \in S^{nd}\} + \\
\left( \max_{i \in I^d} E_{s_0} (MRS^i(c'(s_1)) \left[ \frac{f_k(k; s_1)}{B} \right] \mid s_1 \in S^d) \Pr\{s_1 \in S^d\} B \right) \leq 0 \\
\frac{\partial V}{\partial k_-} = -1 + \frac{\partial q}{\partial k_-} + \left( \frac{\partial p}{\partial k_-} B \right) = \\
- 1 + \min_{i \in I^d} E_{s_0} \left\{ MRS^i(c'(s_1)) f_k(k, s_1) \mid s_1 \in S^{ndr} \right\} \Pr\{s_1 \in S^{ndr}\} + \\
\left( \min_{i \in I^d} E_{s_0} (MRS^i(c'(s_1)) \left[ \frac{f_k(k; s_1)}{B} \right] \mid s_1 \in S^{dr} \Pr\{s_1 \in S^{dr}\} B \right) \geq 0
\]

which implies (34). Since now expectations in the terms on the two sides of the inequality

\(^{29}\)Obviously, if \( S^{nd} = \{s_1\} \) - is a singleton - the left derivative is equal to 0.
are taken over different sets, such condition is a little harder to interpret. In particular we can no longer say that all equity holders have the same valuation for the marginal productivity of capital in the no default states. Rather the condition imposes some relationship between the difference among equity holders and bond holders’ valuation for the marginal productivity of capital in the two situations \( S^d \) and \( S^{nd} \).

We also have to check in this case the optimality of \( k, B \) wrt joint deviations of \( B \) and \( k \). As before, without loss of generality, we can restrict our attention to changes of \( B \) and \( k \) such that \( f(k; s^1) = B \) keeps holding (the set of states for which default occurs does not change).

\[
\frac{\partial V}{\partial B} dB + \frac{\partial V}{\partial k} dk = \left[ \frac{\partial q}{\partial B} + \left( \frac{\partial p}{\partial B} B + p \right) \right] dB + \left[ -1 + \frac{\partial q}{\partial k} + \left( \frac{\partial p}{\partial k} B \right) \right] dk \leq 0,
\]

for \( dB = f_k(s^1)dk > 0 \); also,

\[
\frac{\partial V}{\partial B} dB + \frac{\partial V}{\partial k} dk = \left[ \frac{\partial q}{\partial B} + \left( \frac{\partial p}{\partial B} B + p \right) \right] dB + \left[ -1 + \frac{\partial q}{\partial k} + \left( \frac{\partial p}{\partial k} B \right) \right] dk \geq 0
\]

for \( dB = f_k(s^1)dk < 0 \). Substituting the expressions for the partials obtained above, we get

\[
\begin{align*}
&\left[ -\min_{i \in I^e} E_{s_0} \left\{ MRS^i(c^i(s_1)) \mid s_1 \in S^{nd} \right\} \Pr \{ s_1 \in S^{nd} \} - \min_{i \in I^d} E_{s_0} \left( MRS^i(c^i(s_1)) \frac{f(k; s_1)}{B} \right) \mid s_1 \in S^d \} \Pr \{ s_1 \in S^d \} + p \right] f_k(s^1) \\
&- 1 + \max_{i \in I^e} E_{s_0} \left\{ MRS^i(c^i(s_1)) f_k(k; s_1) \mid s_1 \in S^{nd} \right\} \Pr \{ s_1 \in S^{nd} \} + \\
&\max_{i \in I^d} E_{s_0} \left( MRS^i(c^i(s_1)) \frac{f_k(k; s_1)}{B} \right) \mid s_1 \in S^d \} \Pr \{ s_1 \in S^d \} \leq 0
\end{align*}
\]
or

\[
1 - \max_{i \in I^e} E_{s_0} \left\{ MRS^i(c^i(s_1)) f_k(k, s_1) \big| s_1 \in S^{nd} \right\} \Pr\{s_1 \in S^{nd}\} - \frac{f(k; s_1)}{B} \left| s_1 \in S^d \right\} \Pr\{s_1 \in S^d\} \geq \]

\[
\left[-\min_{i \in I^d} E_{s_0} \left\{ MRS^i(c^i(s_1)) \big| s_1 \in S^{nd} \right\} \Pr\{s_1 \in S^{nd}\} \right] + \frac{f(k; s_1)}{B} \left| s_1 \in S^d \right\} \Pr\{s_1 \in S^d\} + \]

\[
\max_i E_{s_0} (MRS^i(c^i(s_1))) \left| s_1 \in S^{nd} \right\} \Pr\{s_1 \in S^{nd}\} + \]

\[
E_{s_0} (MRS^i(c^i(s_1))) \left| s_1 \in S^d \right\} \Pr\{s_1 \in S^d\} \right] f_k(s_1^{nd})
\]

where the term on the lhs is nonnegative because of (34) and the one on the rhs is also nonnegative by construction. Analogously, substituting the expressions for the partial derivatives into the FOC for \(dB = f_k(s_1^{nd})dk < 0\) yields:

\[
\left[-\max_{i \in I^e} E_{s_0} \left\{ MRS^i(c^i(s_1)) \big| s_1 \in S^{nd} \right\} \Pr\{s_1 \in S^{nd}\} \right] + \frac{f(k; s_1)}{B} \left| s_1 \in S^d \right\} \Pr\{s_1 \in S^d\} + \]

\[
-1 + \min_{i \in I^d} E_{s_0} \left\{ MRS^i(c^i(s_1)) f_k(k, s_1) \big| s_1 \in S^{nd} \right\} \Pr\{s_1 \in S^{nd}\} + \]

\[
\min_{i \in I^d} E_{s_0} (MRS^i(c^i(s_1))) \frac{f_k(k; s_1)}{B} \left| s_1 \in S^d \right\} \Pr\{s_1 \in S^d\} \geq 0
\]

or

\[
\left[-\max_{i \in I^e} E_{s_0} \left\{ MRS^i(c^i(s_1)) \big| s_1 \in S^{nd} \right\} \Pr\{s_1 \in S^{nd}\} \right] + \frac{f(k; s_1)}{B} \left| s_1 \in S^d \right\} \Pr\{s_1 \in S^d\} + \]

\[
\geq 1 - \min_{i \in I^d} E_{s_0} \left\{ MRS^i(c^i(s_1)) f_k(k, s_1) \big| s_1 \in S^{nd} \right\} \Pr\{s_1 \in S^{nd}\} + \]

\[
\min_{i \in I^d} E_{s_0} (MRS^i(c^i(s_1))) \frac{f_k(k; s_1)}{B} \left| s_1 \in S^d \right\} \Pr\{s_1 \in S^d\}
\]

where the term on the lhs is nonnegative because of (33) and the one on the rhs is also
nonnegative as it immediately follows from (34). Putting (36) and (37) together,

\[
1 - \max_{i \in I^c} \left\{ MRS^i(c^i(s_1)) f_k(k; s_1) \middle| s_1 \in S^{nd} \right\} \Pr\{s_1 \in S^{nd}\} - \\
\max_{i \in I^d} E_{s_0} \left( MRS^i(c^i(s_1)) \frac{f_k(k; s_1)}{B} \middle| s_1 \in S^d \right) \Pr\{s_1 \in S^d\} \geq \\
\left[ - \min_{i \in I^c} E_{s_0} \left\{ MRS^i(c^i(s_1)) \middle| s_1 \in S^{nd} \right\} \Pr\{s_1 \in S^{nd}\} \right] \\
- \min_{i \in I^d} E_{s_0} \left( MRS^i(c^i(s_1)) \left[ \frac{f(k; s_1)}{B} \right] \middle| s_1 \in S^d \right) \Pr\{s_1 \in S^d\} + p \right] f_k(s_1^{nd}) \geq \\
\left[ - \max_{i \in I^c} E_{s_0} \left\{ MRS^i(c^i(s_1)) \middle| s_1 \in S^{nd} \right\} \Pr\{s_1 \in S^{nd}\} - \\
\max_{i \in I^d} E_{s_0} \left( MRS^i(c^i(s_1)) \left[ \frac{f(k; s_1)}{B} \right] \middle| s_1 \in S^d \right) \Pr\{s_1 \in S^d\} \right]
\]

Since

\[
- \min_{i \in I^c} E_{s_0} \left\{ MRS^i(c^i(s_1)) \middle| s_1 \in S^{nd} \right\} \Pr\{s_1 \in S^{nd}\} \\
- \min_{i \in I^d} E_{s_0}(MRS^i(c^i(s_1)) \left[ \frac{f(k; s_1)}{B} \right] \middle| s_1 \in S^d) \Pr\{s_1 \in S^d\} \geq \\
\max_{i \in I^c} E_{s_0} \left\{ MRS^i(c^i(s_1)) \middle| s_1 \in S^{nd} \right\} \Pr\{s_1 \in S^{nd}\} \\
- \max_{i \in I^d} E_{s_0}(MRS^i(c^i(s_1)) \left[ \frac{f(k; s_1)}{B} \right] \middle| s_1 \in S^d) \Pr\{s_1 \in S^d\} \geq
\]

and

\[
- \min_{i \in I^c} E_{s_0} \left\{ MRS^i(c^i(s_1)) f_k(k; s_1) \middle| s_1 \in S^{nd} \right\} \Pr\{s_1 \in S^{nd}\} - \\
\min_{i \in I^d} E_{s_0}(MRS^i(c^i(s_1)) \left[ \frac{f(k; s_1)}{B} \right] \middle| s_1 \in S^d) \Pr\{s_1 \in S^d\} \geq \\
\max_{i \in I^c} E_{s_0} \left\{ MRS^i(c^i(s_1)) f_k(k; s_1) \middle| s_1 \in S^{nd} \right\} \Pr\{s_1 \in S^{nd}\} \\
- \max_{i \in I^d} E_{s_0}(MRS^i(c^i(s_1)) \left[ \frac{f(k; s_1)}{B} \right] \middle| s_1 \in S^d) \Pr\{s_1 \in S^d\} \geq
\]
it must be that (35) holds, where recall that

\[
p = \max_i \left\{ E_{s_0} \left( MRS^i(c^i(s_1)) \right| s_1 \in S^{nd} \Pr\{s_1 \in S^{nd} \} + \right.
\]
\[
E_{s_0} \left( MRS^i(c^i(s_1)) \left[ \frac{f(k; s_1)}{B} \right] \right| s_1 \in S^d \Pr\{s_1 \in S^d \} \right\}
\]

This implies

\[
\min_{i \in I^c} E_{s_0} \left\{ MRS^i(c^i(s_1)) \right| s_1 \in S^{nd} \Pr\{s_1 \in S^{nd} \} = \right.
\]
\[
\max_{i \in I^c} E_{s_0} \left\{ MRS^i(c^i(s_1)) \right| s_1 \in S^{nd} \Pr\{s_1 \in S^{nd} \} \right.
\]
\[
\min_{i \in I^d} E_{s_0} \left\{ MRS^i(c^i(s_1)) \left[ \frac{f(k; s_1)}{B} \right] \right| s_1 \in S^d \Pr\{s_1 \in S^d \} = \right.
\]
\[
\max_{i \in I^d} E_{s_0} \left\{ MRS^i(c^i(s_1)) \left[ \frac{f(k; s_1)}{B} \right] \right| s_1 \in S^d \Pr\{s_1 \in S^d \} \right.
\]

and

\[
\min_{i \in I^c} E_{s_0} \left\{ MRS^i(c^i(s_1)) f_k(k, s_1) \right| s_1 \in S^{nd} \Pr\{s_1 \in S^{nd} \} = \right.
\]
\[
\max_{i \in I^c} E_{s_0} \left\{ MRS^i(c^i(s_1)) f_k(k, s_1) \right| s_1 \in S^{nd} \Pr\{s_1 \in S^{nd} \} \right.
\]
\[
\min_{i \in I^d} E_{s_0} \left\{ MRS^i(c^i(s_1)) \frac{f(k; s_1)}{B} \right| s_1 \in S^d \Pr\{s_1 \in S^d \} = \right.
\]
\[
\max_{i \in I^d} E_{s_0} \left\{ MRS^i(c^i(s_1)) \frac{f(k; s_1)}{B} \right| s_1 \in S^d \Pr\{s_1 \in S^d \} \right.
\]

Note that conditions (33), (34) and (35) can be alternatively stated as:

\[
\min_{i \in I^c} E_{s_0} \left\{ MRS^i(c^i(s_1)) \right| s_1 \in S^{nd} \Pr\{s_1 \in S^{nd} \} + \right.
\]
\[
\min_{i \in I^d} E_{s_0} \left\{ MRS^i(c^i(s_1)) \left[ \frac{f(k; s_1)}{B} \right] \right| s_1 \in S^d \Pr\{s_1 \in S^d \} \right\} \geq p \geq \right.
\]
\[
\max_{i \in I^c} E_{s_0} \left\{ MRS^i(c^i(s_1)) \right| s_1 \in S^{nd} \Pr\{s_1 \in S^{nd} \} + \right.
\]
\[
\max_{i \in I^d} E_{s_0} \left\{ MRS^i(c^i(s_1)) \left[ \frac{f(k; s_1)}{B} \right] \right| s_1 \in S^d \Pr\{s_1 \in S^d \} \right\}.
The debt price map is the presence of risky debt is given by

\[ f \quad \text{The statement only refers to the interior case: } f(k; s_1) > B. \] Here, the partials of the
price maps with respect to \( B \) are

\[
\frac{\partial q}{\partial B} = -\frac{\min_{i \in I^c} E_{s_0} \left\{ MRS^i(c^i(s_1)) \mid s_1 \in S^{nd} \right\} Pr\{s_1 \in S^{nd}\}}{B} \\
\frac{\partial q}{\partial B} = -\frac{\max_{i \in I^c} E_{s_0} \left\{ MRS^i(c^i(s_1)) \mid s_1 \in S^{nd} \right\} Pr\{s_1 \in S^{nd}\}}{B}
\]

and

\[
\frac{\partial p}{\partial B} = -\frac{\min_{i \in I^c} E_{s_0} (MRS^i(c^i(s_1)) \left[ \frac{f(k; s_1)}{B} \right] \mid s_1 \in S^d) Pr\{s_1 \in S^d\}}{B} \\
\frac{\partial p}{\partial B} = -\frac{\max_{i \in I^c} E_{s_0} (MRS^i(c^i(s_1)) \left[ \frac{f(k; s_1)}{B} \right] \mid s_1 \in S^d) Pr\{s_1 \in S^d\}}{B}
\]

Analogously, the partials with respect to \( k \) are

\[
\frac{\partial q}{\partial k} = \frac{\max_{i \in I^c} E_{s_0} \left\{ MRS^i(c^i(s_1))f_k(k, s_1) \mid s_1 \in S^{nd} \right\} Pr\{s_1 \in S^{nd}\}}{B} \\
\frac{\partial q}{\partial k} = \frac{\min_{i \in I^c} E_{s_0} \left\{ MRS^i(c^i(s_1))f_k(k, s_1) \mid s_1 \in S^{nd} \right\} Pr\{s_1 \in S^{nd}\}}{B}
\]

and

\[
\frac{\partial p}{\partial k} = \frac{\max_{i \in I^d} E_{s_0} (MRS^i(c^i(s_1)) \left[ \frac{f_k(k; s_1)}{B} \right] \mid s_1 \in S^d) Pr\{s_1 \in S^d\}}{B} \\
\frac{\partial p}{\partial k} = \frac{\min_{i \in I^d} E_{s_0} (MRS^i(c^i(s_1)) \left[ \frac{f_k(k; s_1)}{B} \right] \mid s_1 \in S^d) Pr\{s_1 \in S^d\}}{B}
\]

So, if \( f(k; s_1^{nd}) > B \), the FOCs with respect to \( B \) are:

\[
\frac{\partial V}{\partial B} = \frac{\partial q}{\partial B} + \left( \frac{\partial p}{\partial B} B + p \right) = \\
- \frac{\min_{i \in I^c} E_{s_0} \left\{ MRS^i(c^i(s_1)) \mid s_1 \in S^{nd} \right\} Pr\{s_1 \in S^{nd}\}}{B} \\
+ \left( - \frac{\min_{i \in I^d} E_{s_0} (MRS^i(c^i(s_1)) \left[ \frac{f_k(k; s_1)}{B} \right] \mid s_1 \in S^d) Pr\{s_1 \in S^d\} + p \right) \leq 0
\]

\[
\frac{\partial V}{\partial B} = \frac{\partial q}{\partial B} + \left( \frac{\partial p}{\partial B} B + p \right) = \\
- \frac{\max_{i \in I^c} E_{s_0} \left\{ MRS^i(c^i(s_1)) \mid s_1 \in S^{nd} \right\} Pr\{s_1 \in S^{nd}\}}{B} \\
+ \left( - \frac{\max_{i \in I^d} E_{s_0} (MRS^i(c^i(s_1)) \left[ \frac{f_k(k; s_1)}{B} \right] \mid s_1 \in S^d) Pr\{s_1 \in S^d\} + p \right) \geq 0
\]
which implies

\[ p = \max_{i} E_{s_0} \left( MRS^{i}(c^{i}(s_1)) \mid s_1 \in S^{nd} \right) \Pr\{s_1 \in S^{nd}\} + \]

\[ E_{s_0} \left( MRS^{i}(c^{i}(s_1)) \left[ \frac{f(k; s_1)}{B} \right] \mid s_1 \in S^{d} \right) \Pr\{s_1 \in S^{d}\} \]

and (31). On the other hand, the FOCs with respect to \( k \) give:

\[ \frac{\partial V}{\partial k} = -1 + \frac{\partial q}{\partial k} + \left( \frac{\partial p}{\partial k} B \right) = \]

\[ = -1 + \max_{i \in I^c} E_{s_0} \left\{ MRS^{i}(c^{i}(s_1)) f_k(k, s_1) \mid s_1 \in S^{nd} \right\} \Pr\{s_1 \in S^{nd}\} + \]

\[ \max_{i \in I^d} E_{s_0} \left( MRS^{i}(c^{i}(s_1)) \left[ \frac{f_k(k; s_1)}{B} \right] \mid s_1 \in S^{d} \right) \Pr\{s_1 \in S^{d}\} \leq 0 \]

\[ \frac{\partial V}{\partial k} = -1 + \frac{\partial q}{\partial k} + \left( \frac{\partial p}{\partial k} B \right) = \]

\[ = -1 + \min_{i \in I^c} E_{s_0} \left\{ MRS^{i}(c^{i}(s_1)) f_k(k, s_1) \mid s_1 \in S^{nd} \right\} \Pr\{s_1 \in S^{nd}\} + \]

\[ \min_{i \in I^d} E_{s_0} \left( MRS^{i}(c^{i}(s_1)) \left[ \frac{f_k(k; s_1)}{B} \right] \mid s_1 \in S^{d} \right) \Pr\{s_1 \in S^{d}\} \geq 0 \]

which implies (32). This completes the proof of Proposition 2. ■

Equilibria with Short sales when intermediation costs are negligible

In Section ?? we established the existence of an equilibrium with intermediated short sales for all levels \( \delta > 0 \) of the intermediation cost (capturing the default rate on short positions). It is then of interest to investigate the properties of these equilibria as we let \( \delta \) go to 0. Clearly the spread \( \max_{i \in I^c} \mathbb{E} \left[ MRS^{i}(c^{i}(s))R^c(k, \phi, m, B; s) \right] - \min_{i \in I^c} \mathbb{E} \left[ MRS^{i}(c^{i}(s))R^c(k, \phi, m, B; s) \right] \) must go to zero, since \( q(k, \phi, m, B) \) is bounded above for all \( k, \phi, m, B \) and all \( \delta > 0 \), total resources being finite. We conjecture therefore that the limit of the competitive equilibria with short sales as \( \delta \to 0 \) exists, as all variables lie in a compact set.

The previous observation also implies that the marginal valuation for all possible production and financial plans is equalized across all consumers, as in an environment where unlimited short sales are allowed and markets are complete (or a spanning property holds for all admissible production and financial plans of firms). In the limit as \( \delta \to 0 \) not only all possible markets, corresponding to all possible choices \( k, \phi, m, B \), are open, as in the case without short sales, but a larger set of markets are open and active, to ensure the equalization of agents’ marginal rates of substitution.
Short sales with endogenous default

We extend here the analysis of Section ?? by examining the case where the consumers’ default rate, rather than being exogenous and state and type invariant, is optimally chosen by consumers, and may depend therefore on the state s as well as the type i of the consumer. We show in what follows the required changes in the model. The specification of the intermediation activity and the structure of markets is clearly more complicated, still the main results on unanimity and optimality remain valid.

Since consumers’ loans are non-collateralized, we follow Dubey et al. (2005) in introducing a utility penalty $\xi^i$ for a type $i$ consumer per unit defaulted in any state s, for all $i, s$. It is convenient to assume here that preferences are additively separable over time, so that they take the following form:

$$u^i_0(c^i_0) + \mathbb{E}[u^i_1(c^i(s)) - \xi^i \delta^i_s[f(k, \phi; s) - B]]$$

(38)

where $\delta^i_s$ is the default rate of consumer $i$ in state $s$. Given this feature of consumers’ preferences, the optimal default level in each state $s$ for consumer $i$ is obtained by maximizing (38) with respect to $(\delta^i_s)_s$ subject to the date 1 budget constraint (20), where $\delta$ is replaced by $\delta^i_s$. It is immediate to see that the solution is a well defined map $\delta^i_s(\theta^i, \lambda^i_+, \lambda^i_-, \lambda^i_\theta, B)$ for all $s$ and $\theta^i, \lambda^i_+, \lambda^i_-$, and for any given $k, \phi, B$.

Thus the default rate in any state $s$ on the loans granted to consumers via the sale of short positions depends not only on the type $i$ of the consumer but also on his overall portfolio holdings. We consider then the case where both the consumer’s type and his portfolio holdings are observable by his trading partners. The loan contract offered by intermediaries is so an exclusive contract and the price depends both on the consumer’s type and portfolio, $q^i_\theta, \lambda^i_+, \lambda^i_-, \lambda^i_\theta, B$, as well as, obviously, on the return structure of the underlying equity. Hence the budget constraint faced by consumers at date 0 is now

$$c^i_0 = w^i_0 + [-k + q + p B] \theta^i_0 - q \theta^i - p b^i - q^i \lambda^i_+ + q^i_\theta \lambda^i_\theta, \lambda^i_+, \lambda^i_-, \lambda^i_\theta$$

(39)

An intermediary who is intermediating $m$ units of the derivative by selling the short positions to consumers of type $i$, with portfolio $(\theta^i, \lambda^i_+, \lambda^i_-)$, faces a default rate on its loans equal to $\delta^i_s(\theta^i, \lambda^i_+, \lambda^i_-)$. As a consequence, the shortfall in its revenue at date 1 is:

$$[(f(k, \phi; s) - B) \delta^i_s(\theta^i, \lambda^i_+, \lambda^i_-)] m.$$

(40)

We consider still the case where only equity, an asset that is ‘safe’ as it is in positive net supply and backed by real claims, is used to hedge the consumers’ default risk. To be able
to fully meet the shortfall in (40) due to consumers’ default, the intermediary must hold at least
\[ \max_s \delta_s^i(\theta^i, \lambda^i_+, b^i, \lambda^i_-) m \]
units of equity. The total date 0 revenue of the intermediary is then:
\[ \max_m \left( q^+ - q^-_{i,\theta^i, \lambda^i_+, b^i, \lambda^i_-} - q \left( \max_s \delta_s^i(\theta^i, \lambda^i_+, b^i, \lambda^i_-) \right) \right) m \]  
(41)

The intermediary’s choice problem consists in the choice of the amount \( m \) to issue of each type \( i, \theta^i, \lambda^i_+, b^i, \lambda^i_- \) of derivative so as to maximize its profits, that is its revenue at date 0. Note that the intermediation technology still exhibits constant returns to scale, hence a solution exists provided
\[ q \geq \frac{q^+ - q^-_{i,\theta^i, \lambda^i_+, b^i, \lambda^i_-}}{\max_s \delta_s^i(\theta^i, \lambda^i_+, b^i, \lambda^i_-)}; \]
and is characterized by \( m(i, \theta^i, \lambda^i_+, b^i, \lambda^i_-) > 0 \) only if \( q = \frac{q^+ - q^-_{i,\theta^i, \lambda^i_+, b^i, \lambda^i_-}}{\max_s \delta_s^i(\theta^i, \lambda^i_+, b^i, \lambda^i_-)} \).

The main difference with respect to the reduced form model is then the fact that the market for derivative claims is differentiated according to consumers’ types and portfolio choices. This has the following implications for the consumers’ optimization problem and the market clearing conditions.

Consumer \( i \) chooses his portfolio \( \theta^i, \lambda^i_+, b^i, \lambda^i_- \) so as to maximize
\[ u^i_0(c^i_0) + \mathbb{E} \left\{ u^i_1 \left[ w^i(s) + b^i + (f(k, \phi; s) - B)(\theta^i + \lambda^i_+ - \lambda^i_- (1 - \delta^i(\theta^i, \lambda^i_+, b^i, \lambda^i_-))) \right] - \xi^i \delta^i(\theta^i, \lambda^i_+, b^i, \lambda^i_-) \right\} \]
subject to the budget constraint (39), given the asset prices \( q, q^+, p \) and \( q^-_{i,\theta^i, \lambda^i_+, b^i, \lambda^i_-} \) and the default map \( \delta^i(\cdot) \) obtained as above. Let \( \tilde{\theta}^i, \tilde{\lambda}^i_+, \tilde{b}^i, \tilde{\lambda}^i_- \) denote the consumer’s optimal choice in equilibrium. The asset market clearing conditions are then
\[ \sum_i m(i, \tilde{\theta}^i, \tilde{\lambda}^i_+, \tilde{b}^i, \tilde{\lambda}^i_-) \left[ \max_s \delta_s^i(\theta^i, \lambda^i_+, b^i, \lambda^i_-) \right] + \sum_i \tilde{\theta}^i = 1 \]
for equity, and
\[ \tilde{\lambda}^i_- = m(i, \tilde{\theta}^i, \tilde{\lambda}^i_+, \tilde{b}^i, \tilde{\lambda}^i_-) \text{ for each } i \]
\[ 0 = m(i, \theta^i, \lambda^i_+, b^i, \lambda^i_-) \text{ for each } i, (\theta^i, \lambda^i_+, b^i, \lambda^i_-) \neq (\tilde{\theta}^i, \tilde{\lambda}^i_+, \tilde{b}^i, \tilde{\lambda}^i_-) \]
\[ \sum_i m(i, \tilde{\theta}^i, \tilde{\lambda}^i_+, \tilde{b}^i, \tilde{\lambda}^i_-) = \sum_i \tilde{\lambda}^i_+ \]
for the derivative security.
The consistency condition $M$) on the firms’ equity conjectures must also be properly modified to reflect the different specification of the value of intermediation in the present context:

$$M') \quad q(k, \phi, B) = \max \left\{ \max_{i,\theta^i,\lambda^i_+,\lambda^i_-} E \left[ MRS^i(c^i(s))(f(k, \phi; s) - B) \right] \right\},$$

$$\forall k, \phi, B$$

where $q^-(i, \theta^i, \lambda^i_+, \lambda^i_-; k, \phi, B, \bar{U}^i)$ is constructed as follows. For any $k, \phi, B$ and $i, \theta^i, \lambda^i_+, \lambda^i_-$, set $q^-(i, \theta^i, \lambda^i_+, \lambda^i_-; k, \phi, B, \bar{U}^i)$ as the value of $q^-$ that satisfies the following equation:

$$\bar{U}^i = u_0^i(\bar{w}_0^i + [-k + q + \bar{p} \bar{B}] \theta_0 - q \theta^i - \bar{p} \bar{b}^i - q^+ \lambda^i_+ + q^- \lambda^i_-) +$$

$$E \left[ u_1^i \left[ w^i(s) + \bar{b}^i + (f(\bar{k}, \bar{\phi}; s) - \bar{B})(\theta^i + \lambda^i_+ - \lambda^i_-) \right] \left[ f(k, \phi; s) - B \right] \left[ 1 - \delta^i_s(\theta^i, \lambda^i_+, \lambda^i_-; k, \phi, B) \right] \left[ \lambda^i_- \left( f(k, \phi; s) - B \right) \right] \right]$$

$$= \bar{U}^i$$

where $\bar{U}^i$ denotes the utility level of type $i$ consumers at the equilibrium choices $\bar{\theta}^i, \bar{\lambda}^i_+, \bar{\lambda}^i_-$ and the map $\delta^i_s(\theta^i, \lambda^i_+, \lambda^i_-; k, \phi, B)$ is similarly obtained by maximizing the expected utility term on the right hand side of the above expression with respect to $\delta^i_s$. That is, $q^-(i, \theta^i, \lambda^i_+, \lambda^i_-; k, \phi, B, \bar{U}^i)$ identifies the maximal willingness to pay in equilibrium of consumer $i$ for a short position equal to $\lambda^i_-$ in the firm with plan $k, \phi, B$, when the rest of his portfolio is given by $\theta^i, \lambda^i_+, \lambda^i_-$. At these prices consumers are indifferent between choosing the equilibrium portfolio $\bar{\theta}^i, \bar{\lambda}^i_+, \bar{\lambda}^i_-$ and any other portfolio with a short position $\lambda^i_-$ in the equity of a firm with plan $k, \phi, B$.

An important difference with respect to the previous analysis is the fact that here the price of short positions is no longer defined at the margin. This is due to the exclusive nature of loan contracts corresponding to short positions. Also, at the same prices intermediaries are indifferent between issuing the derivatives traded in equilibrium and any other derivative on equity of firms with plan $k, \phi, B$, such that $q = \max_{i,\theta^i,\lambda^i_+,\lambda^i_-} \left( MRS^i(c^i(s))(f(k, \phi; s) - B) - q^-(i, \theta^i, \lambda^i_+, \lambda^i_-; k, \phi, \bar{B}, \bar{U}^i) \right)$.

The unanimity and constrained optimality properties still hold in this environment. The argument again is very similar and relies on the the fact that, given the above specification of the intermediation technology and the price conjectures, the model is equivalent to a setup where the markets for all types of equity and all types of corresponding derivatives are available for trade. The notion of completeness here also requires the exclusivity of the loan contracts associated to short positions, so that the market for all types of

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[30] In the specification of $q^-(i, \theta^i, \lambda^i_+, \lambda^i_-; k, \phi, B, \bar{U}^i)$ we have implicitly assumed that all the long positions of a consumer are in the assets corresponding to the firms’ equilibrium choices. This is with no loss of generality and to avoid excessive notational complexities.
derivatives can also be differentiated according to the type of a consumer and the level of his trades.