The interdependence of bank capital and liquidity *

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Abstract

We build a global game model to analyze the interdependent effects of bank capital and liquidity on the likelihood of solvency- and liquidity-driven crises. We show that changes in the level of bank capitalization always reduce the likelihood of solvency-driven crises; while changes in the level of liquidity of its portfolio always increase it. The effects on liquidity-driven crises are more mixed and depend on the initial level of bank capitalization and the liquidity of its portfolio. Improving capitalization is beneficial, unless the bank is little capitalized and/or holds an illiquid portfolio. Improving the level of liquidity of bank portfolio reduces the likelihood of a crisis only if the bank is characterized by intermediate level of capitalization and portfolio liquidity. We then derive some implications in terms of the design of optimal capital and liquidity regulation. The main insight is that capital and liquidity requirements cannot be set independently of bank capitalization and portfolio liquidity. Moreover, when properly designed, capital and liquidity regulation are perfect substitutes in restoring stability.

Keywords: illiquidity, insolvency, capital and liquidity regulation

JEL classifications: G01, G21, G28

*The views expressed here are the authors’ and do not reflect those of the ECB or the Eurosystem.
1 Introduction

The 2007-2009 financial crisis was a milestone for financial regulation, leading to significant reforms to the existing capital regulation and the introduction of a new set of liquidity requirements. In particular, banks have been required to hold higher capital buffers to reduce their exposure to solvency-driven crises and, at the same time, to increase their liquidity holdings to reduce liquidity and maturity mismatch and so the risk of liquidity-driven crises. The introduction of a new set of liquidity requirements, namely the Liquidity coverage ratio (LCR) and the Net stable funding ratio (NSFR), as complements to the existing and improved capital-based regulation, have fuelled the discussion in both the academic and policy arena about the interaction between these various regulatory tools, their possible complementarities and contrasting effects on financial stability.

Bank (il)liquidity and (in)solvency are closely intertwined concepts and often difficult to tell apart when a crisis manifests. On the one hand, liquidity-driven crises can spur solvency issues; on the other hand, fears about bank solvency may precipitate liquidity problems. Furthermore, when a crisis is underway and a bank faces a large outflow of funds, it becomes very difficult to assess the ultimate source of these withdrawals, which, in turn, may limit policymakers’ ability to intervene effectively. It is precisely this close link between solvency- and liquidity-driven crises that motivates the discussion about the joint effects that capital and liquidity regulations may have on bank stability.

To visualize the issue, consider a simple bank balance as in Table 1. Bank stability depends negatively on its leverage (i.e., \( \frac{E}{L+I} \)), as well as on the proportion of illiquid assets (i.e., \( \frac{I}{L+I} \)): A bank with a larger share of short-term funding and a lower proportion of liquid assets is more likely to be subject to a run. In this case, increasing equity (\( E \)), while keeping constant the asset side, has a similar effect on stability as increasing the proportion of liquid assets (\( L \)), while keeping the liability side constant. This means that, in this example, capital and liquidity can be seen as substitutes in terms of their effects on stability.

<table>
<thead>
<tr>
<th>Assets</th>
<th>Liabilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>Liquid assets (L)</td>
<td>Short-term debt (D)</td>
</tr>
<tr>
<td>Illiquid assets (I)</td>
<td>Equity (E)</td>
</tr>
</tbody>
</table>

Table 1: A simplified bank balance sheet

Does this equivalence between capital and liquidity also hold in more realistic contexts? What are the effects of a change in the level of bank capitalization and portfolio liquidity on bank stability? Are these effects different depending on whether crises are solvency- or liquidity-driven? All these questions are key in
light of the recent regulatory reforms, as answering them represent a first step in assessing whether capital and liquidity requirements are substitutes or complements in terms of their effects on bank stability and so in characterizing an optimal regulatory mix.

To tackle these questions, we build a two-period model where banks raise funds by issuing debt and equity, and invest in a risky portfolio consisting of both liquid and illiquid assets, whose final return depends on the fundamentals of the economy. The portfolio composition shapes the trade-off between intermediate and final date portfolio returns, whereby a higher proportion of liquid assets in the portfolio leads to a higher (safe) return at the interim date, but to a lower (risky) return at the final date. This determines bank available resources and, together with the bank capital structure, affects the likelihood of a bank failure. In our model, bank default can be driven by both solvency and liquidity considerations and the probability of each type of crisis is determined endogenously using the global-game methodology. At the interim date, each debt holder receives an imperfect signal regarding bank portfolio return and, based on this signal, decides whether to roll over or to withdraw his debt claim depending on which of these two actions gives him the highest payoff. Those payoffs depend on the fundamentals of the economy, as well as on the expectation about the proportion of debt holders rolling over.

As standard in the global game literature (see e.g., Goldstein and Pauzner, 2005), the equilibrium outcome is that bank failures occur when fundamentals of the economy are below a unique threshold and take the form of a massive withdrawals of funds by debt holders at the interim date. Within the range where they occur, crises can be classified into solvency-driven crises and liquidity-driven crises. The former happen at the lower part of the crisis region where the signal on the fundamentals is so low that not rolling over the debt claim at the interim date is a dominant strategy for debt holders. The latter result from a coordination failure among debt holders, in that each of them does not roll over out the self-fulfilling belief that others will do the same.

Both the probability of a solvency-driven crisis and that of liquidity-driven one depend on the level of capitalization of the bank and the liquidity of its portfolio. Thus, our model delivers a first set of results about the differential effects that a change in the level of bank capitalization and in the liquidity of its portfolio have on these probabilities. An increase in the level of bank capitalization always reduces the likelihood of a solvency-driven crisis, while an increase in the level of liquidity of bank portfolio always increases it. The effects on the likelihood of a liquidity-driven crisis are more involved, as they depend on the initial level of capitalization and portfolio liquidity of the bank. For banks that are very little capitalized and/or hold
very illiquid portfolios, both an increase in capital and liquidity leads to an increase in the probability of liquidity-driven crises. The reason is that those banks are already very close to fail and so the change in the level of capitalization and portfolio liquidity has mostly the effect of increasing debt holders’ repayment in the event of a failure at the expenses of the equity holders. For banks that have intermediate levels of capital and/or liquidity, both higher capitalization and increased portfolio liquidity produce a beneficial effect on the stability in that they both reduce the occurrence of liquidity-driven crises. The effects of a change in the level of capitalization and in portfolio liquidity on stability differ for banks with an already high level of capitalization and/or portfolio liquidity: While increasing capital continues to have a beneficial effect, increasing liquidity leads to more instability. The reason is that the already high level of capitalization and/or portfolio liquidity makes these banks unlike to fail. A further increase in capital further strengthens this, while increasing liquidity has the drawback of decreasing the profitability of bank portfolio, thus negatively impacting bank net worth at the final date and so its solvency.

Building on this comparative statics exercise, we analyze bank’s choices regarding capital structure and portfolio liquidity in an unregulated equilibrium. We show that banks always choose intermediate level of capitalization and/or portfolio liquidity. On the one hand, this means that banks are always in the range where marginal increases in capital and in portfolio liquidity are both beneficial to stability. On the other hand, it also implies that banks choose to be still exposed to liquidity-driven crises. While banks internalize the cost in terms of foregoing profit associated with a failure, the higher cost of equity financing relative to debt and the negative impact that liquidity has on the profitability of their portfolio induce them to choose a capital structure and a level of portfolio liquidity consistent with the occurrence of liquidity-driven crises. Such choice is optimal from the perspective of an individual bank, but not for the economy as a whole. The reason is that liquidity-driven crises are inefficient as they induce the premature liquidation of the bank portfolio, when, if let to continue, a bank would have generated a higher return to be shared between the bank and its financiers.

The inefficiency spurred by the occurrence of liquidity-driven crises calls for public intervention in the form of capital and liquidity regulation. By requiring banks to hold a minimum level of capital and/or portfolio liquidity, regulation could constrain their choice and so reduce the occurrence of inefficient liquidity-driven crises. To analyze the effectiveness of regulation and whether capital- and liquidity-based instruments are either complements or substitutes in their effect on stability, we analyze two different regulatory interventions. First, we consider the possibility for a regulator to set either a capital or a liquidity requirement which
leaves banks completely free to choose the unregulated variable. This regulatory intervention resembles the introduction of a simple (unweighted) leverage ratio or an ad hoc increase in bank liquidity in that the requirement set by the regulator is independent of the bank balance sheet situation. Then, we consider the possibility for the regulator to use requirements resembling more closely those embedded in the Basel III regulation, like a risk-weighted capital ratio, the liquidity coverage ratio and the net stable funding ratio. Those differ from the first type of regulation as these requirements prescribe banks to hold a certain amount of capital and/or liquidity that depend on the bank balance sheet situation and so on the level of the unregulated variable chosen by the bank.

We show that the first type of regulatory intervention always fails to prevent the occurrence of liquidity-driven crises, whether it takes the form of either a capital or a liquidity requirement. Furthermore, the two tools are not equivalent in terms of their effects on stability, leading possibly to different equilibrium allocations. On the contrary, any of the second type of regulatory tools can be effective in eliminating liquidity-driven crises and so improve efficiency. Furthermore, capital and liquidity requirements lead to the same equilibrium allocation and are essentially equivalent in improving stability and efficiency. This means that, in our framework, capital and liquidity regulation emerge as perfectly substitutable tools, so that only one of them is needed to improve the unregulated market equilibrium. This result crucially hinges on two elements. First, each requirement specifies a ratio between the level of bank capitalization and portfolio liquidity rather than a single value and so indirectly constrains banks also in the choice of the unregulated variable. Second, liquidity-driven crises are spurred by a combination of too high debt financing and too little liquidity holdings, thus, similarly to the simple example illustrated in Table 1, preventing them could be achieved by using either a capital-based instrument or a liquidity-based one.

In summary, two main insights emerge from the analysis of the effects of capital and liquidity on bank stability. On the one hand, these effects can be very different depending on the source of instability, i.e., whether a crisis is solvency- or liquidity-driven, and on the bank balance sheet situation, namely on bank capital structure and liquidity holdings. On the other hand, it highlights that, when adequately designed, both capital and liquidity requirements serve effectively the purpose of eliminating inefficient liquidity-driven crises.

Our analysis of the impact of capital and liquidity on bank stability is conducted in a framework where the only source of inefficiency of the unregulated market equilibrium is the premature liquidation of bank portfolio. In doing this, we disregard other possible sources of inefficiencies that may motivate the use of
capital and liquidity regulation, such as, for example, a moral hazard problem on the side of bank managers spurred by the inability or unwillingness of debt holders to monitor the managers’ behaviour and so prevent excessive risk-taking incentives. It is possible that extending the model further to account for these issues will introduce additional and possible different roles for capital and liquidity, thus making the two no longer substitutable in terms of their effect on stability. Also, while the remuneration to debt and equity are both endogenous in our framework, the cost of equity is pinned down by the (exogenous) return of an outside investment opportunity only available to equity holders. While keeping the analysis tractable, this may prevent to capture additional implications about efficiency. Despite these limitations, our framework, to the best of our knowledge, is the first one that allows disentangling the different effects that capital and liquidity have on both solvency- and liquidity-driven crises and so studying the interactions and optimal design of different regulatory requirements.

The analysis of our paper provides a step towards understanding the interaction between capital and liquidity requirements, their effect on bank stability and the design of an optimal regulatory mix.

A number of recent papers have looked at the role and implications of the newly introduced liquidity regulation, also in connection with capital requirements (see e.g., Walther, 2015; Calomiris, Heider and Hoerova, 2015; Vives, 2014 and König, 2015). Among those papers, the closest ones are Vives (2014) and König (2015) as they also use the global game model to study the implications of capital and liquidity on the probability of a banking crisis. Both papers build on the bank run model developed by Rochet and Vives (2004) and perform a comparative statics exercise on the run threshold. Vives (2014) finds that both capital and liquidity have a beneficial effect on stability. Similarly to our paper, König (2015) shows that the effect of liquidity is more mixed. Its beneficial effect on stability must be balanced out with the fact that liquid assets earn lower returns on average than less liquid ones, thus hurting bank’s ability to repay investors when no run manifests and so stability. Relative to those papers, we have a richer payoff structure for debt holders that reveals an additional detrimental effect of both capital and liquidity on bank stability arising when banks that have low level of capitalization and portfolio liquidity. Furthermore, we endogenize bank capital structure and portfolio liquidity, as well as the remuneration of both equity and debt. This allows us to highlight the inefficiency of the market equilibrium and to characterize optimal regulation.

Among the new literature on joint capital and liquidity regulation, the closest paper to ours is Kashyap, Tsomocos and Vardoulakis (2017). We share with them the scope of the analysis, namely the analysis of optimal regulation, and some features of the model, like the use of global games to pin down bank default
probability. Despite these differences, our paper differs from them in terms of results. First, we show that the effect of increases in capital and liquidity do not unambiguously lead to reduction in the bank default probability. This is not the case in their framework where each type of regulation leads to a reduction in the crisis probability (both solvency- and liquidity-driven ones). Second, in our paper capital and liquidity regulation, if adequately designed, are substitutes in achieving constrained efficiency. Finally, our partial equilibrium approach allows us to obtain analytical results about optimal regulation, which is not the case in their general equilibrium framework.

As emerged in our analysis, having an endogenous probability of both solvency- and liquidity-driven crises and being able to disentangle the various effects of a change in bank capitalization and portfolio liquidity is key to evaluate the effectiveness of different regulatory interventions and so characterize the optimal regulation. The ability to endogenize a crisis probability and to distinguish between solvency- and liquidity-driven ones relies on the use of global games as in the literature originating with Carlsson and van Damme (1993) (see Morris and Shin, 2003 for a survey on the theory and application of global games). Our paper is particularly close to two contributions in this literature. First, it extends the rollover game in Eisenbach (2017) by adding equity and considering a richer payoff structure for debt holders. Second, it shares the same technical challenge of characterizing the existence of a unique equilibrium in a context in which there are no global strategic complementarities with Goldstein and Pauzner (2005).

The paper proceeds as follows. Section 2 presents the baseline model. Section 3 shows the effect that changes in capital and liquidity have on the probability of solvency- and liquidity-driven crises. Section 4 characterized the unregulated equilibrium and highlights its inefficiencies. Section 5 analyze different regulatory intervention. Section 6 contains conclusion. All proofs are contained in the appendix.

2 The model

2.1 The setup

There are three dates (t = 0, 1, 2), a continuum [0, 1] of banks and numerous investors. Investors are risk-neutral and are endowed with one unit of resources at date 0 and nothing thereafter.

At date 0, banks raise one unit of funds from investors in the form of short-term debt and equity. We denote as k and 1 − k the proportion of funds raised as equity and debt, respectively. In exchange for their funds, debt holders are promised a (gross) interest rate r ≥ 1 if they withdraw their investment at date 1
and \( r^2 \) at date 2, if the bank is solvent. Otherwise, if the bank is not able to repay the promised interest rate either at date 1 or 2, debt holders receive a share of the bank available resources. In assuming that the interest rate on debt is compounded, we follow Eisenbach (2017). Similarly to his framework, this constrains the choice of the interest rate in the interim period and, as a result, bank’s liability structure choice becomes crucial in determining the occurrence of crises and whether they are efficient or not.\(^1\)

Unlike debt holders, equity holders do not receive anything if the bank fails and are willing to provide funds to the bank in exchange for the (gross) return \( r^E \). To capture the higher cost of equity financing, we assume that equity holders have an outside investment opportunity worth \( \rho > \int_0^1 R(\theta) \, d\theta > 1 \). Thus, they are willing to provide funds to the bank only if they expect to receive at least \( \rho \), which then represents the cost of capital for the banks. The assumption that equity financing is more costly than debt financing for banks is common in the banking literature (see e.g., Hellmann, Murdock, and Stiglitz, 2000; Repullo, 2004; Allen, Carletti, and Marquez, 2011). In the literature, such an assumption hinges on various hypotheses, ranging from the existence of segmented debt and equity markets (see e.g., Allen, Carletti, and Marquez, 2015), to a different tax treatment between equity and debt,\(^2\) to the existence of costs associated with the issuance of outside equity (see Harris, Opp, and Opp, 2017).

At date 0, banks invest one unit of funds in a risky portfolio consisting of both liquid and illiquid assets. For each unit invested at date 0, the portfolio returns \( \ell \leq 1 \) at date 1 and \( R(\theta)(1-\alpha\ell) \) at date 2, with \( 0 \leq \alpha \leq 1, R'(\theta) > 0 \) and \( E_\theta[R(\theta)(1-\alpha\ell)] > 1 \), so that investing in the portfolio dominates storing the funds. The variable \( \ell \) is a measure of the liquidity of bank portfolio. A liquidity-return trade off is present: the more liquid a bank portfolio, the lower its expected return at date 2. Besides liquidity, the return of bank portfolio at date 2 also positively depends on the fundamental of the economy \( \theta \), with \( \theta \sim U[0,1] \).\(^3\)

The state of the economy \( \theta \) is realized at the beginning of date 1, but is not publicly observed until date 2. After \( \theta \) is realized at date 1, each debt holder receives a private signal \( s_i \) of the form

\[
s_i = \theta + \varepsilon_i, \tag{1}
\]

\(^1\)Importantly, such specification is equivalent to one in which the bank promises \( r_1 \) and \( r_2 \) to debtholders withdrawing at date 1 and 2, respectively, under the assumption that rates cannot be negative. In this case, bank would choose \( r_1 \geq 1 \) and the bank’s choice of its liability structure would play a key role for the likelihood of crises.

\(^2\)In most jurisdictions, the cost of debt is tax-deductible, while dividends are not. Schepens (2016) shows that a reduction in the tax discrimination between debt and equity financing leads to a significant increase in bank capital ratios.

\(^3\)Similarly to Goldstein and Pauzner (2005), we assume that there exists a level of the state of the economy \( \bar{\theta} \) such that for any \( \theta > \bar{\theta} \) the bank portfolio is safe and returns \( R(1)(1-\alpha\ell) \) both at date 1 and 2. Throughout the analysis, we assume that \( \theta \rightarrow 1 \).
where \( \varepsilon_i \) are small error terms that are independently and uniformly distributed over \([-\varepsilon, +\varepsilon]\). Based on this signal, debt holders decide whether to withdraw their investment at date 1 or roll it over until date 2. The signal gives them information about bank solvency and so its ability to repay the promised interest rate.

The timing of the model is as follows. Each bank chooses the terms of the debt contract \( r \), the capital structure \( \{k, 1-k\} \) and the level of portfolio liquidity \( \ell \) at date 0 so as to maximize its expected profit. At date 1, after receiving the private signal about the state of the fundamentals \( \theta \), each debt holder decides whether to withdraw at date 1 or roll the debt over until date 2. At date 2, the bank portfolio return realizes and all claims are paid, if the bank is solvent. The model is solved backwards.

### 3 Capital, liquidity and bank fragility

In this section, we analyze how capital and liquidity, as well as their interaction, affect bank exposure to solvency- and liquidity-driven crises.

A bank can fail both at date 1 and date 2 and default can be triggered by both liquidity and solvency considerations. Denoting as \( n \) the proportion of debt holders not rolling over the debt claim at the interim date, a bank defaults at date 1 when the liquidation proceeds \( \ell \) are not enough to repay \( r \) to all \( n \) withdrawing investors. Formally, this is the case when

\[
\ell < (1-k)nr. \tag{2}
\]

When a bank defaults at date 1, all \( (1-k) n \) withdrawing debt holders receive a pro-rata share \( \frac{\ell}{(1-k)n} \), while the remaining \( (1-k)(1-n) \) debt holders and the \( k \) equity holders receive nothing.

Similarly, a bank defaults at date 2 if the return of its portfolio is not enough to repay the interest rate \( r^2 \) to the \( (1-k)(1-n) \) investors who have decided to roll over their debt claim. Formally, this is the case when

\[
R(\theta)(1-\alpha\ell) \left[ 1 - \frac{(1-k)nr}{\ell} \right] - (1-k)(1-n)r^2 < 0, \tag{3}
\]

where \( \frac{(1-k)nr}{\ell} \) is the fraction of the bank portfolio liquidated at date 1 to repay the \( (1-k)n \) debt holders that have decided not to roll over their debt claim. When a bank defaults at date 2, equity holders receive nothing, while the \( (1-k)(1-n) \) debt holders receive a pro-rata share of the bank’s available resources, as given by \( \frac{R(\theta)(1-\alpha\ell)[1-\frac{(1-k)nr}{\ell}]}{(1-k)(1-n)} \).

From (2) and (3), it emerges that both the default at date 1 and 2 are affected by bank capitalization
and the liquidity of its portfolio. Furthermore, the state of the economy $\theta$ directly affects the possibility of a bank default at date 2, while only indirectly that at date 1. At date 2, a bank defaults either because of a low realization of $\theta$ or because of a large liquidation of funds at date 1, or due to a combination of the two. Anticipating the possibility of a default at date 2, debt holders choose not to rollover their debt claim at the interim date, thus precipitating a crisis. In this sense, as we will show in detail below, the fundamental $\theta$ indirectly affects the occurrence of a default at date 1.

In what follows, we refer to solvency-driven crises as bank failures that are triggered only by debt holders’ expectations of a low realization of the state of the economy $\theta$. We refer, instead, to liquidity-driven crises as bank failures that are triggered by debt holders’ fear that other investors would not roll over their debt claim, thus forcing the bank to liquidate its portfolio at date 1. We characterize the probability of each crisis and its properties in turn below.

### 3.1 Solvency-driven crises

Solvency-driven crises are triggered exclusively by debt holders’ fear of a low realization of the state of the economy $\theta$ at date 2. A solvency-driven crisis occurs when a bank resources are not enough to repay the interest rate $r_2$ at date 2 even when all debt holders rollover their claim. Formally, this corresponds to the range of fundamental $\theta < \theta(k, \ell)$, where $\theta(k, \ell)$ solves

\[ R(\theta) (1 - \alpha\ell) - (1 - k) r_2^2 = 0. \] (4)

The interval $[0, \theta(k, \ell))$ identifies the range of values of the state of the economy $\theta$ where a banking crisis is fully driven by solvency considerations and $\theta(k, \ell)$ represents the probability of a solvency-driven crisis. From (4), it is easy to see that $\theta(k, \ell)$ depends on the interest rate $r$ offered to investors at date 1, as well as on the level of bank capitalization $k$ and the liquidity of its portfolio $\ell$. We have the following results.

**Proposition 1** The threshold $\theta(k, \ell)$ is increasing in $r$ and $\ell$ and is decreasing in $k$, i.e., $\frac{\partial \theta(k, \ell)}{\partial r} > 0$, $\frac{\partial \theta(k, \ell)}{\partial \ell} > 0$ and $\frac{\partial \theta(k, \ell)}{\partial k} < 0$.

The proposition highlights the key role that capital and liquidity play for the emergence of solvency-driven crises and shows that they have opposite effects. Capital $k$ has a beneficial effect on bank exposure to solvency-driven crises so that better capitalized banks are less likely to be insolvent. This is the case because capital provides a buffer to the bank to withstand potential low realization of the fundamental $\theta$, thus limiting
the states where the bank is unable to repay its debt at date 2. By contrast, liquidity has a detrimental effect on bank solvency. This is due to the negative impact that liquidity has on bank profitability, that is on the date 2 (per unit) portfolio return $R(\theta)(1 - \alpha \ell)$. As we will see in detail below, choosing a more liquid portfolio (i.e., with a higher $\ell$) allows banks to better hedge against large withdrawals by debt holders, but entails a lower portfolio return at date 2, thus increasing the likelihood of bank insolvency. Finally, as standard in banking models, the threshold $\theta(k, \ell)$ increases with the interest rate $r$.

In the discussion above, we have referred to solvency-driven crises as those triggered by a low realization of the fundamental $\theta$, without specifying anything about the timing of such crises. In our framework, solvency-driven crises may materialize both as a failure at date 2 or as a massive withdrawals by debt holders at date 1, i.e., a bank run (Figure 1). The latter is the case for any $\theta < \theta_1(k, \ell)$, with $\theta_1(k, \ell) \leq \theta(k, \ell)$. When the fundamentals of the economy are very low (i.e., $\theta < \theta_1(k, \ell)$), not rolling over the debt claim at date 1 is a dominant strategy for each debt holder. This is the case because debt holders not only expect the bank not to be able to repay the interest rate $r^2$ at date 2, but also that the pro-rata share of the bank available resources is less than $r$. Thus, withdrawing at date 1 is optimal irrespective of what others do. Formally, the threshold $\theta_1(k, \ell)$ is the solution to

$$R(\theta)(1 - \alpha \ell) - (1 - k) r = 0, \quad (5)$$

and exhibits the same properties as the threshold $\theta(k, \ell)$ as illustrated in Proposition 1.

Insert Figure 1

### 3.2 Liquidity-driven crises

Besides insolvency, a banking crisis can be also driven by liquidity considerations. Even when the state of the economy $\theta$ is higher than $\theta_1(k, \ell)$, debt holders may have the incentive not to rollover their debt claims at the interim date as they fear that others would do the same. Their concern is that a large number of withdrawals at date 1 would force a massive liquidation of bank portfolio at date 1, thus depleting bank available resources at date 2 and triggering a default.

The signal $s_i$ plays a key role for debt holders’ withdrawal decision and so for the occurrence of liquidity-driven crises. The reason is that the signal provides information on both $\theta$ and other debt holders’ actions.
When the signal is high, a debt holder attributes a high posterior probability to the event that the bank portfolio yields a high return and, at the same time, he infers that the other debt holders have also received a high signal. This overall lowers his belief about the likelihood of a bank failure and, as a result, also his own incentive to withdraw at date 1. Conversely, when the signal is low, a debt holder has a high incentive not to roll over the debt, as he attributes a high likelihood to the possibility that the return of the bank portfolio is low and that the other investors withdraw their debt claim at date 1. As a result, in the region for \( \theta \geq \theta_1 (k, \ell) \) an investor’s decision about whether to roll over or not its debt claim at date 1 depends on the realization of \( \theta \) as well as on his beliefs regarding the other debt holders’ actions. To see this, we specify an investor’s payoff from withdrawing at date 1 and that from rolling the claim over until date 2.

A debt holder’s payoff at date 1 is given by

\[
\pi_1 = \begin{cases} 
    r \geq 1 & \text{if } 0 \leq n < \tilde{n} \\
    \frac{r}{(1-k)n} & \text{if } \tilde{n} \leq n \leq 1
\end{cases}
\]  

(6)

where \( \tilde{n} = \frac{\ell}{(1-k)r} \) denotes the proportion of investors not rolling over the debt at the intermediate date such that the bank is on the brick of default at date 1, that is \( \ell = (1 - k)n \). At date 1, a debt holder obtains \( r \geq 1 \) as long as the value of bank portfolio at date 1 \( \ell \) is enough to repay \( r \) to all \( (1 - k)n \) withdrawing debt holders. Otherwise, the bank is forced to liquidate the entire portfolio and each debt holder receives a pro-rata share of bank portfolio value equal to \( \frac{\ell}{(1-k)n} \).

Consider now a debt holder’s payoff at date 2. It is given by

\[
\pi_2 = \begin{cases} 
    r^2 \geq r & \text{if } 0 \leq n < \hat{n} (\theta) \\
    \frac{R(\theta)(1-\alpha\ell)[1-(1-k)nr]}{(1-k)(1-n)} & \text{if } \hat{n} (\theta) \leq n \leq \Pi \\
    0 & \text{if } \Pi \leq n \leq 1
\end{cases}
\]  

(7)

where \( \hat{n} (\theta) \) corresponds to the solution to \( R(\theta)(1-\alpha\ell)[1-(1-k)nr] = (1-k)(1-n)r^2 \) and so denotes the proportion of investors not rolling over the debt at date 1 pushing the bank at the brick of default at date
2. The threshold \( \hat{n}(\theta) \) is then equal to
\[
\hat{n}(\theta) = \frac{R(\theta)(1 - \alpha \ell) - (1 - k)r^2}{(1 - k)r \left[ \frac{R(\theta)(1 - \alpha \ell)}{\ell} - r \right]}. \tag{8}
\]

At date 2, a debt holder obtains \( r^2 \) as long as the bank is solvent, otherwise he obtains a pro-rata share of bank available resources. Whether the bank is solvent or not, it depends on the realization of \( \theta \), as well as, on the proportion \( n \) of debt holders not rolling over at date 1, as they determine both the value of bank portfolio as well as that of the bank liabilities. As the proportion \( n \) of debt holders withdrawing at the interim date 1 increases, a debt holder’s incentive to run and withdraw at date 1 also increases, even if not monotonically. Specifically, a debt holder’s incentive to roll over his debt claim until at date 2 does not monotonically decreases with \( n \) because, when \( n \) is very large (i.e., \( n > \hat{n} \)), the more debt holders do not roll over at date 1, the lower a debt holder’s payoff from withdrawing at date 1. This implies that the model exhibits the property of one-sided strategic complementarity, as defined in Goldstein and Pauzner (2005). Furthermore, it is important to notice that the payoff from withdrawing at date 2 \( \pi_2 \) is decreasing in the fraction of debt holders withdrawing at date 1 \( n \) if and only if the following condition holds
\[
\ell < (1 - k)r \tag{9}
\]
In this case, the bank resources constraint at date 2 becomes more binding as \( n \) increases, thus leading to a higher risk of default for the bank at date 2.\footnote{The condition (9) follows from \(-R(\theta)(1 - \alpha \ell)(1 - \frac{(1-k)r}{\ell}) + (1 - k)r^2\), as for any \( \theta > \hat{\theta}(k, \ell) \), \( R(\theta)(1 - \alpha \ell) > (1 - k)r^2 \). In words, this condition states that more withdrawals at date 1 increase the risk that the bank defaults at date 2 as long as the value of bank portfolio at date 1 \( \ell \) is not enough to repay \( r \) to all \((1 - k)\) debt holders. We show that this condition always holds when solving for the bank’s optimal choice of \( k, \ell \) and \( r \).}

**Proposition 2** The model has a unique threshold equilibrium in which debt holders withdraw their debt claim at date 1 if they observe a signal below the threshold \( s^*(k, \ell) \) and roll it over above. At the limit, as \( \varepsilon \to 0 \), \( s^*(k, \ell) \to \theta^*(k, \ell) \), with \( \theta^*(k, \ell) \) being the solution to
\[
\int_0^{\hat{n}(\theta)} r^2dn + \int_{\hat{n}(\theta)}^{\hat{n}} \frac{R(\theta)(1 - \alpha \ell) \left[ 1 - \frac{(1-k)r}{\ell} \right]}{(1 - k)(1 - n)}dn - \int_0^{\hat{n}} rdn - \int_{\hat{n}}^{1} \frac{\ell}{(1 - k)n}dn = 0 \tag{10}
\]
The proposition states that in the interval for \( \theta \geq \theta(k, \ell) \), debt holders’ rollover decision is driven by the fear that others will not roll over, thus reducing a bank available resources at date 2 and, in turn, their repayment. When \((1 - k) r > \ell\), a debt holder has the incentive to withdraw when he expects a large proportion of investors to do the same, as otherwise he faces the risk of not receiving anything. In other words, in the range \( \theta_1(k, \ell) \leq \theta < \theta^*(k, \ell) \) banks fail at date 1 and their defaults are driven by the fear of other debt holders’ withdrawals and the associated illiquidity they cause. Thus, we refer \([\theta_1(k, \ell), \theta^*(k, \ell)]\) as the range of fundamentals \( \theta \) where liquidity-driven crises occur and to \( \theta^*(k, \ell) \) as the probability of liquidity-driven crises (see Figure 2).

3.2.1 How capital and liquidity affect liquidity-driven crises

As the threshold of solvency-driven crises \( \theta(k, \ell) \) and \( \theta_1(k, \ell) \), also \( \theta^*(k, \ell) \) is affected by the terms of the debt contract \( r \), by the bank capitalization \( k \), as well as the level of liquidity of its portfolio \( \ell \). Capital and liquidity affect the likelihood of liquidity-driven crises differently. The following lemma illustrates the channels through which a change in the level of bank capitalization and portfolio liquidity affect the likelihood of a liquidity-driven crisis, respectively.

**Lemma 1** The sign of the effect of capital \( k \) on \( \theta^*(k, \ell) \) (i.e., \( \frac{\partial \theta^*(k, \ell)}{\partial k} \)) is equal to the sign of

\[
\frac{1}{(1 - k)^2} \left[ - \int_{\hat{n}(\theta^*)}^{\pi} R(\theta^*) \frac{(1 - \alpha \ell)}{1 - n} dn + \ell \int_{\hat{n}(\theta^*)}^{1} \frac{1}{n} dn \right] \tag{11}
\]

while that of portfolio liquidity \( \ell \) (i.e., \( \frac{\partial \theta^*(k, \ell)}{\partial \ell} \)) corresponds to the sign of

\[
\frac{1}{(1 - k)} \left[ \int_{\hat{n}(\theta^*)}^{\pi} \frac{R(\theta^*)}{1 - n} dn - \int_{\hat{n}(\theta^*)}^{\pi} \left( \frac{R(\theta^*)}{1 - n} \right) \frac{n r}{(1 - n) \ell^2} dn + \int_{\hat{n}(\theta^*)}^{1} \frac{1}{n} dn \right]. \tag{12}
\]

The lemma shows that changes in the capital and liquidity positions of the bank affect the likelihood of a liquidity-driven crisis through different channels. The effect of capital on the probability of a liquidity-driven crisis is twofold. On the one hand, an increase in the bank capitalization has a beneficial effect on stability as it increases the pro-rata share received by debt holders at date 2 in the range \([\hat{n}(\theta), \pi]\). This effect is captured by the first term in (11) and leads to an increase in debt holders’ incentives to roll over. On the other hand, an increase in bank capitalization also increases the pro-rata share received by debt holders at date 1, when
the bank defaults, as captured by the second term in (11). This term, thus, captures the detrimental effect that capital may have on stability. Such effect, which may seem at odd with common wisdom, captures the crowding out effect associated with the strengthening of bank capitalization: Increasing capital is a way for the bank to increase the proceeds for debt holders at date 1 at the expenses of equity holders who are wiped out when it defaults.

The effect of liquidity on the likelihood of a liquidity-driven crisis is more involved. First, as captured by the first term in (12), an increase in bank portfolio liquidity translates into a lower (per unit) portfolio return at date 2 and, in turn, into a lower pro-rata share for the debt holders in the range \([\tilde{n}(\theta), \pi]\). This is destabilizing as it increases debt holders’ incentive to withdraw at date 1. An increase in liquidity \(\ell\) has an additional detrimental effect on the likelihood of a liquidity-driven crisis because, similarly to capital, it increases the pro-rata share received by debt holders when the bank defaults at date 1. More liquidity implies that the return of bank portfolio at date 1 is higher, thus increasing the payoffs from not rolling over the debt claim, as it is captured by the third term in (12). Similarly to the case of capital, such effect could be interpreted as the crowding out associated with the increase in the liquidity of bank portfolio: Investing in a more liquid portfolio allows the bank to increase the proceeds for early withdrawing debt holders at the expenses of those who roll over the debt and the equity holders, which are fully wiped out in the event of a bank failure at date 1. Finally, liquidity has a beneficial effect on stability, as it is captured by the second term in (12). This term captures the positive effect that an increase in liquidity has on the pro-rata shares received by debt holders at date 2 in the range \([\tilde{n}(\theta), \pi]\). A more liquid portfolio implies that the bank needs to liquidate fewer units of its portfolio at date 1 to meet debt holders’ withdrawals, thus leaving more resources for those who roll their debt claim over to date 2. This is the commonly recognized beneficial effect of liquidity and the rationale behind the newly introduced liquidity regulation.

The overall effect of capital and liquidity on the threshold \(\theta^*(k, \ell)\) depends on which of the various effects illustrated above dominates. The following proposition shows that this crucially depends on the initial level of capitalization and portfolio liquidity of the bank. We have the following result.

**Proposition 3** The threshold \(\theta^*(k, \ell)\) decreases with the level of capital \(k\) for any \(k \in [\bar{k}(\ell), 1]\) and increases otherwise, \(i.e., \frac{\partial \theta^*(k, \ell)}{\partial k} < 0\) if \(k \geq \bar{k}(\ell)\) and \(\frac{\partial \theta^*(k, \ell)}{\partial k} < 0\) if \(k < \bar{k}(\ell)\). The threshold \(\theta^*(k, \ell)\) decreases with bank portfolio liquidity \(\ell\) for any \(k \in (\bar{k}(\ell), \bar{K}(\ell))\) and increases otherwise \(i.e., \frac{\partial \theta^*(k, \ell)}{\partial \ell} < 0\) if \(\bar{k}(\ell) < k < \bar{K}(\ell)\) and \(\frac{\partial \theta^*(k, \ell)}{\partial \ell} > 0\) if \(k < \bar{k}(\ell)\) and \(k > \bar{K}(\ell)\). The thresholds \(k^{\max}(\ell), \bar{k}(\ell)\) and \(\bar{K}(\ell)\) are defined in
The proposition, which is also illustrated in Figure 3a and 3b, shows that a bank capitalization and the liquidity of its portfolio jointly matter in determining the effect that an increase in capital and liquidity has on bank stability. In other words, whether a change in the bank capital structure increases or decreases the likelihood of a bank failure it crucially depends on the initial level of capitalization and the liquidity position of the bank. Symmetrically, whether a change in the bank portfolio liquidity has a beneficial or detrimental impact on stability depends on the level of capitalization of the bank, as well as the initial level of portfolio liquidity.

Let's first consider the effect of capital on the occurrence of liquidity-driven crises. In Lemma 1, we have shown that an increase in the level of bank capitalization $k$ has two effects on the threshold $\theta^*(k, \ell)$. On the one hand, more capital $k$ increases debt holders’ payoff at date 2, thus increasing their incentive to roll over. On the other hand, an increase in the level of capitalization of the bank increases its ability to repay debt holders when it fails at date 1 (i.e., the pro-rata share $\frac{\ell}{(1-k)n}$ in the case of a default at date 1 increases with $k$). Thus, by increasing a debt holder’s payoff at date 1 when $n > \pi$, as given in (6), an improvement in bank capitalization may translate in a higher probability of liquidity-driven crises. As stated in the proposition, the former dominates when both capital and liquidity are sufficiently high (i.e., $k > \bar{k}(\ell)$), while is dominated by the latter when either the level of capitalization of the bank or its portfolio liquidity or both are low (i.e., $k < \bar{k}(\ell)$).

The intuition behind the result is as follows. When capital is scarce and/or the bank portfolio is not very liquid, a bank default at the interim date is already very likely, since even if the proportion of debt holders withdrawing at date 1 is small, this may force the bank into a massive liquidation of its portfolio at date 1. Thus, in this case, debt holders attach a larger weight to the payoff at date 1 than to the one at date 2, which means that the increase in the debt holders’ payoff at date 2 is dominated by that of the payoff at date 1. As a result, liquidity-driven crises become more likely after an increase in capital. The opposite is true for banks that are already well capitalized and/or have high portfolio liquidity. In those banks, a default at date 1 is unlikely and so the debt holders significantly benefit from the increase in the date 2 payoff due to an increase in capital. In the extreme case, when $k > k_{\text{max}}(\ell)$, only solvency-driven crises occur.\footnote{The curve $k_{\text{max}}(\ell)$ corresponds to the solution to $\ell = (1-k) r$. In the region above $k_{\text{max}}(\ell)$, there is no strategic complementarity between debt holders’ actions, since the bank does not need to liquidate more than one unit of its portfolio for each withdrawing debt holder. As a result only liquidity-driven crises occur. Formally, this means that as $\ell \to (1-k) r$, $\theta^*(k, \ell) \to \bar{\theta}_1(k, \ell)$.}
case, then the effect of an increase in the level of capital is always beneficial as shown in Proposition 1.

An interesting implication of Proposition 3 is that capital helps the banks that need it the least. Capital improves stability precisely for those banks facing a lower probability of liquidity-driven crises due to their better capital and liquidity positions, while it has a destabilizing effect for poorly capitalized banks and those holding very illiquid portfolios.

Consider now the effect of liquidity \( \ell \) on the likelihood of a banking crisis. The proposition shows that it crucially depends on bank capital position, as well as initial portfolio liquidity. An increase in \( \ell \) is detrimental for stability for both very capitalized with very liquid portfolios and little capitalized banks with illiquid portfolios, while it is beneficial for banks that have an intermediate level of capitalization and portfolio liquidity.

The intuition is simple but subtle and relies on the various effects that liquidity has on debt holders' incentives to rollover illustrated in Lemma 1. First, liquidity positively affects debt holders' payoff at date 1 in the case of a run. This effect is very strong when banks are not well capitalized and/or have very little liquidity, as it implies that they are close to a failure. On the contrary, it vanishes as the bank approaches the level of capitalization \( k^{\text{max}}(\ell) \), as at this level only solvency-driven crises matter. Second, liquidity affects debt holders' date 2 payoff. The effect is twofold. On the one hand there is a detrimental effect of liquidity on the bank's asset return at date 2, thus increasing debt holders' incentive to withdraw their debt claim at date 1. On the other hand, there is a beneficial effect of liquidity on the bank solvency constraint at date 2 due to more contained liquidation, thus increasing debt holders' incentives to roll over. The former effect dominates the latter when capital and portfolio liquidity are high, while the opposite is true for lowly capitalized banks with illiquid portfolios. Combining all these effects together implies that improving bank portfolio liquidity is detrimental for stability for banks with either strong or weak capital positions and is beneficial only for banks holding an intermediate level of capital so that a default at date 1 is not too likely, but at the same time, liquidity is an issue for the bank.

Insert Figure 3a and 3b

An important implication of Proposition 3 concerns the benefits of an increase in portfolio liquidity for banks characterized by low levels of capital and liquidity and, thus, very exposed to default. For those banks, an increase in liquidity, if not sufficiently large, is likely to be detrimental. The reason is that the
debt holders expect that this additional liquidity would be used to increase the repayment to bank creditors in the event of a failure. As a result, the increase in liquidity further undermines debt holders’ incentives to roll over and so precipitates the bank into a crisis. Importantly, this would also hold in the case of the injection of emergency liquidity by a LOLR, which does not affect bank’s portfolio returns and so does not give rise to a negative impact on profitability.

To sum up, the analysis of the properties of the threshold $\theta^*(k, \ell)$ shows that the same increase in capital or liquidity may have very different effects on stability for weakly, moderately or strongly capitalized banks, as well as for banks with low, moderate or high portfolio liquidity. In other words, it highlights how important is the interaction between capital and liquidity to assess their effect on stability. As we will show in details below, this is crucial for the bank choice of capital and liquidity as well as for designing and evaluating capital and liquidity regulation.

4 Market equilibrium

In this section, we characterize a bank date 0 decisions about capital $k$, portfolio liquidity $\ell$, interest rate on debt $r$ and equity remuneration $r^E$. In doing this, we assume that debtholders’ outside opportunity is storage. A bank chooses each of these variables so as to maximize its expected profit as given by:

$$\max_{k, \ell, r, r^E} \int_{\theta^*(k, \ell)}^{1} \left[ R(\theta) \left( 1 - \alpha \ell \right) - (1 - k) r^2 - k r^E \right] d\theta$$

subject to

$$IR^D : \int_{0}^{\theta^*(k, \ell)} \frac{\ell}{(1 - k)} d\theta + \int_{\theta^*(k, \ell)}^{1} r^2 d\theta \geq 1$$

and

$$IR^E : \int_{\theta^*(k, \ell)}^{1} r^E d\theta \geq \rho,$$

where $IR^D$ and $IR^E$ represent the participation constraint of debt and equity holders, respectively.

The choice of $r^E$ is straightforward. Since $r^E$ does not enter in the expression for $\theta^*(k, \ell)$ and since it has a negative impact on bank profit, each bank chooses the lowest possible $r^E$ conditional on equity holders to participate (i.e., $IR^E$ is binding). Moreover, by comparing $IR^D$ and $IR^E$, it is easy to see that $r^E > r^2$, as equity holders are wiped out when a crisis occurs (i.e., when $\theta < \theta^*(k, \ell)$) and $\rho > 1$. Given the solution
for $r^E$, we can then rewrite the bank problem as follows:

$$\max_{k,\ell,r} \int_{\theta^*(k,\ell)}^{1} [R(\theta) (1 - \alpha \ell) - (1 - k) r^2] d\theta - k \rho$$

(16)

subject to

$$\text{IR}^D: \int_0^{\theta^*(k,\ell)} \frac{\ell}{1 - k} d\theta + \int_{\theta^*(k,\ell)}^{1} r^2 d\theta \geq 1.$$  

The problem is quite complex given that $k$, $\ell$ and $r$ affect the bank profits both directly and through their effect on $\theta^* (k, \ell)$. Thus, we proceed in steps. First, for given $k$ and $\ell$ characterize the choice of $r$. Then, we characterize the choice of $k$ and $\ell$. Regarding the choice of $r$, we have the following result.

**Proposition 4** The equilibrium interest rate $r^*$ is given by

$$r^* = \begin{cases} 
1 & \text{if } \ell \geq (1 - k) \\
\arg \min \theta^* (k, \ell) & \text{if } \ell < (1 - k)
\end{cases},$$

where the $\arg \min \theta^* (k, \ell)$ corresponds to the solution to

$$\frac{\partial \theta^* (k, \ell)}{\partial r} = 0.$$

For given $k$ and $\ell$, the equilibrium interest rate $r^*$ corresponds to the lowest possible rate conditional on debt holders providing the fund and the bank not defaulting. When $\ell \geq (1 - k)$, debt holders receive a payoff greater than 1 when the bank defaults at date 1 and thus any $r \geq 1$ would induce them to provide funds to the bank at date 0. Choosing $r^* = 1$ is optimal in this case, as it is the lowest rate bank can offer and, at the same time, it implies that the relevant crisis threshold becomes $\theta_1 (k, \ell) < \theta^* (k, \ell)$. The case when $\ell < (1 - k)$ is similar, although (14) implies that $r^* > 1$. In this case, banks choose $r$ that minimizes the probability of a bank failure $\theta^* (k, \ell)$. The reason is twofold. First, banks only make profits when they do not default (i.e., for $\theta > \theta^* (k, \ell)$). Second, they need to compensate debt holders for the loss in terms of expected repayment they suffer in the event of a banking crisis by offering them a higher promised interest rate $r$ (i.e., $r$ must increase with $\theta^* (k, \ell)$ for debt holders to provide funds to the bank at date 0).

The equilibrium interest rate corresponds to the solution to $\frac{\partial \theta^* (k, \ell)}{\partial r} = 0$ as $\theta^* (k, \ell)$ is a convex function of $r$. The effect of a change in $r$ on $\theta^* (k, \ell)$ is twofold. On the one hand, for a given probability of a bank failure, a higher $r$ reduce debt holders’ withdrawal incentives, as they expect to receive a higher repayment.
On the other hand, a higher $r$ makes it less likely for the bank not to default and so to repay the promised repayment. When $r$ is small, the probability to receive the promised interest rate is still high, thus the former effect dominates the latter and, as a result, $\theta^*(k, \ell)$ decreases with $r$. The opposite is true when $r$ is large. In this case debt holders are unlikely to receive the promised interest rate and so, despite the higher promised $r$, choose not to roll over the debt claim. This implies that $\theta^*(k, \ell)$ increases with $r$ when $r$ is large.

**Corollary 1** The partial effects of capital and liquidity on $\theta^*(k, \ell)$ derived in Proposition 3 also represent the respective total effects (i.e., $\frac{\partial \theta^*(k, \ell)}{\partial k} = \frac{d \theta^*(k, \ell)}{dk}$ and $\frac{\partial \theta^*(k, \ell)}{\partial \ell} = \frac{d \theta^*(k, \ell)}{d\ell}$).

The result of the corollary follows directly from Proposition 4. Since the equilibrium interest rate is either $r^* = 1$ or the solution to $\frac{\partial \theta^*(k, \ell)}{\partial r} = 0$, the indirect effects that capital $k$ and portfolio liquidity $\ell$ have on the run threshold $\theta^*(k, \ell)$ via the change they induce in $r$ are equal to zero. Thus, the total effects of a change in $k$ and $\ell$ on the likelihood of a liquidity-driven crisis equal the partial effects characterized in Lemma 1.

Having characterized the choice of the interest rate $r$, we now move to the analysis of bank optimal capital structure and portfolio liquidity. The solution for $k$ and $\ell$ are such that $\theta^*(k, \ell) < 1$ at the equilibrium choice of $k$ and $\ell$. This maximizes banks’ profit as crises do not always occur. We have the following results.

**Proposition 5** The optimal capital structure $k^*$ and portfolio liquidity $\ell^*$ solve

\[
\int_{\theta^*(k, \ell)}^{1} r^2 d\theta - \rho - \frac{\partial \theta^*(k, \ell)}{\partial k} \left[ R(\theta^*(k, \ell)) (1 - \alpha \ell) - (1 - k) r^2 \right] = 0, \tag{17}
\]

and

\[
- \int_{\theta^*(k, \ell)}^{1} R(\theta) \alpha - \frac{\partial \theta^*(k, \ell)}{\partial \ell} \left[ R(\theta^*(k, \ell)) (1 - \alpha \ell) - (1 - k) r^2 \right] = 0. \tag{18}
\]

In equilibrium, liquidity-driven crises always occur as $k^*$ and $\ell^*$ are such that $(1 - k)r > \ell$ holds.

In choosing its capital structure $k$, a bank trades off the marginal benefit of capital with its marginal cost. The former, as represented by the last term in (17), is the gain in expected profits $\left[ R(\theta^*(k, \ell)) (1 - \alpha \ell) - (1 - k) r^2 \right]$ induced by a lower probability of a liquidity-driven crisis, as measured by $\frac{\partial \theta^*(k, \ell)}{\partial k}$. The latter, as captured by the first two terms in (17), is the increase in funding cost $\int_{\theta^*(k, \ell)}^{1} r^2 d\theta - \rho$ associated with an increased reliance on equity financing.\footnote{The inequality $\int_{\theta^*(k, \ell)}^{1} r^2 d\theta - \rho < 0$ follows from the fact that in equilibrium equity holders’ participation constraint, as given in (15), binds and $r^2 < r^E$. Thus, $\rho = \int_{\theta^*(k, \ell)}^{1} r^E d\theta > \int_{\theta^*(k, \ell)}^{1} r^2 d\theta$.}
The choice of portfolio liquidity $\ell$ also trades off marginal benefit and cost. Similarly to the choice of capital, the former is captured by the last term in (18) and represents the gain in expected profits $[R(\theta^* (k, \ell)) (1 - \alpha \ell) - (1 - k) r^2]$ due to the reduced probability of a liquidity-driven crisis, as measured by $\frac{\partial \theta^* (k, \ell)}{\partial k}$. The latter, instead, corresponds to the first term in (18) and captures the detrimental effect that an increase in liquidity has on bank portfolio return.

At the optimum, banks never choose $k^*$ and $\ell^*$ in range where $\frac{\partial \theta^* (k, \ell)}{\partial k} > 0$ and $\frac{\partial \theta^* (k, \ell)}{\partial \ell} > 0$ as they could do better by reducing capital and/or liquidity. Furthermore, they choose $k^*$ and $\ell^*$ so that the inequality $(1 - k)r > \ell$ holds even if this entails liquidity-driven crises. The reason is that when $k^*$ and $\ell^*$ are such that $(1 - k)r = \ell$, the cost of a bank default in terms of reduced expected profits approaches zero.\(^7\) This implies that the marginal benefit of increasing either $k$ or $\ell$ in terms of lower crisis probability also approaches zero, while the marginal cost in terms of higher funding costs and reduced portfolio return are positive. Thus, banks have no incentives to increase $k^*$ and $\ell^*$ up to the point where $(1 - k)r = \ell$ holds and find it optimal, instead, to choose lower levels for $k^*$ and $\ell^*$, even though these foster strategic complementarity in debt holders’ action and so are consistent with the occurrence of liquidity-driven crises. In other words, in a framework in which there was no difference between the cost of equity and debt financing and/or liquidity did not have a negative impact on bank portfolio returns, banks would optimally choose a capital structure and portfolio liquidity preventing the occurrence of liquidity-driven crises and leaving only efficient solvency-driven ones.\(^8\)

**Corollary 2** The equilibrium pair $\{k^*, \ell^*\}$ always lies in the region bounded by the curves $k(\ell)$ and $\bar{k}(\ell)$.

The corollary identifies the pairs $\{k^*, \ell^*\}$ candidates to be the optimal level of capitalization and portfolio liquidity. In equilibrium, banks choose always a combination of debt and equity financing $k > 0$ and portfolio liquidity $\ell > 0$ so that the equilibrium pair $\{k^*, \ell^*\}$ corresponds to a point in the region between $k(\ell)$ and $\bar{k}(\ell)$. The reason is that, outside of this region, $\frac{\partial \theta^* (k, \ell)}{\partial \ell} > 0$ and banks could achieve higher profits by reducing $\ell$ for any given level of $k$.

\(^7\)When $(1 - k)r = \ell$, $\theta^* (k, \ell) \rightarrow \theta_1 (k, \ell)$ and at $\theta = \theta_1 (k, \ell)$ banks make zero profits.

\(^8\)This is what happens in Eisenbach (2017) where there is no difference in costs between long- and short-term debt financing and the liquidation value of bank’s investment does not negatively affect its date 2 return. In his framework, banks choose to raise short-term debt to induce the premature liquidation of their investment and the amount they choose is the one that leads only to efficient crises.
4.1 Inefficiency of the market equilibrium

The possibility for debt holders not to roll over their debt claim until date 2, thus forcing the bank to liquidate its portfolio at date 1, is the source of inefficiency of the market economy characterized above. To see this, denote as $TO$ the total output generated in the market equilibrium. This corresponds to the sum of bank profit, debt holders’ and equity holders’ payoffs and is, thus, equal to:

$$TO = \int_{\theta^*}^{1} \left[ R(\theta) (1 - \alpha \ell) - (1 - k) r^2 \right] d\theta - k\rho + (1 - k) \int_{0}^{\theta^*} \frac{\ell}{1 - k} d\theta + (1 - k) \int_{\theta^*}^{1} r^2 d\theta + k\rho = \int_{0}^{\theta^*} \ell d\theta + \int_{\theta^*}^{1} R(\theta) (1 - \alpha \ell) d\theta. \quad (19)$$

When the bank is forced into early liquidation by a massive withdrawal of funds by debt holders at date 1, the resources produced in the economy corresponds to the portfolio liquidation value $\ell$. Such liquidation is inefficient for any $\theta > \theta^E$, where $\theta^E$ is the solution to

$$R(\theta) (1 - \alpha \ell) = \ell, \quad (20)$$

as, if able to continue, the bank portfolio would generate the higher return $R(\theta) (1 - \alpha \ell)$ at date 2. It is important to notice that the the threshold $\theta^E < \theta(k, \ell) < \theta^*(k, \ell)$. This inequality follows directly from the comparison of (20), (4) and (10). Furthermore, when $\ell = (1 - k) r$, $\theta^E = \theta_1(k, \ell)$, as it can be seen by comparing (20) with (5). We denote then as total loss $TL$, the loss in term of total output spurred by a liquidity-driven crisis at date 1, with

$$TL = \int_{\theta^E}^{\theta^*(k, \ell)} \left[ R(\theta) (1 - \alpha \ell) - \ell \right] d\theta. \quad (21)$$

From (21), it is easy to see that the higher $\theta^*(k, \ell)$, the larger the loss in total output $TL$.

As shown in Proposition 5, banks’ choice of $k$ and $\ell$ spurs the occurrence of liquidity-driven crises (i.e., for any $\theta < \theta^*(k, \ell)$), so that the market equilibrium always entails an inefficient loss of output in the range $\left[ \theta^E, \theta^*(k, \ell) \right)$. The ultimate source of the inefficient output loss is the fact that banks, being protected by limited liability, do not internalize the loss of total output suffered by the economy as a whole in the event of a crisis and, as a result, choose a combination of $k$ and $\ell$ that does not prevent the occurrence of liquidity-driven crises. This implies a wedge between the level of bank capitalization and portfolio liquidity.
chosen by banks and those that a social planner would choose. To show that, the following proposition characterized the constrained efficient allocation corresponding to the allocation of a social planner choosing $k^{SP}$ and $\ell^{SP}$ so as to maximize total output in (19) under the constraint that both debt holders and equity holders provide funds at date 0. We have the following result.

**Proposition 6** The social planner chooses \( \{k^{SP}, \ell^{SP}\} \) solving

\[
\begin{align*}
    k^{SP} &= 1 - \ell^{SP}, \\
    \int_0^{\theta_E} \ell d\theta + \int_{\theta_E}^1 \theta R(\theta) (1 - \alpha \ell) d\theta - (1 - \ell) \rho = 0.
\end{align*}
\]

(22)

The social planner allocation entails either $k^{SP} > k^*$ or $\ell^{SP} > \ell^*$ or both and only efficient crises occur.

The social planner chooses bank capitalization $k$ and bank portfolio liquidity $\ell$ so that liquidity-driven crises are eliminated and only efficient ones occur. Any pair \( \{k, \ell\} \) guaranteeing that $(1 - k)r = \ell$ is satisfied would achieve this, as, when $(1 - k)r = \ell$ holds, $\theta^* \rightarrow \theta_1(k, \ell) = \theta^E$. The second equation in (22) guarantees that the resources produced in equilibrium are enough to provide equity holders with a repayment equivalent to their outside investment opportunity $\rho$, so that they are willing to provide funds to the bank administrated by the social planner. The result that $k^{SP} > k^*$ and/or $\ell^{SP} > \ell^*$, which follows directly from Proposition 5, hinges on the fact that the benefits and costs of increasing $k$ and $\ell$ are different for the social planner and the banks. In particular, the social planner finds it optimal to increase $k$ to the point that liquidity-driven crises are eliminated, as this allows to maximize total output, even though it leads to lower profits for banks.

5 Regulatory intervention

One way to improve the market equilibrium derived in Section 4 and so to reduce the inefficiency it entails is through capital and liquidity regulation. By requiring banks to increase their level of capitalization and portfolio liquidity, a regulator could restore efficiency and achieve the social planner allocation. In what follows, we analyze whether this is indeed the case, whether capital and liquidity regulation are equally good in achieving this, and what the key features of an optimal regulatory intervention should have. In particular, we are interested to see whether the efficient allocation can be achieved by a single regulation or it necessarily requires a combination of capital and liquidity requirements.

To do this, we modify the timing of the model as follows. At date 0, banks raise funds and invest, while the regulator set the regulatory requirement for capital and/or liquidity. Then, given the requirements,
banks chooses their level of capitalization $k$ and portfolio liquidity $\ell$ and the interest rate on debt $r$. At date 1, debt holders decide whether to roll over or not their debt claim based on the signal they receive. At date 2, the portfolio return realizes and banks pay their investors if solvent.

### 5.1 A single regulatory requirement

We first analyze the case of a regulator having the possibility to set a single regulatory requirement, either capital or liquidity. In this case, banks face the constraint that one of their choice variable must not fall below the regulatory requirement, i.e., either $k \geq k^{\text{reg}}$ or $\ell \geq \ell^{\text{reg}}$. However, banks are free to choose the other variable so as to maximize their expected profits. We show that a regulator with a single regulatory tool fails to eliminate inefficient liquidity-driven crises and so cannot achieve the constrained efficient allocation. The following proposition formally presents this argument.

**Proposition 7** If only one regulatory tool is available, inefficient liquidity-driven crises are not eliminated.

The regulator chooses the optimal regulatory requirement in order to maximize total output. In doing this, it takes into account that banks will respond to the regulation by choosing the non-regulated variable so to maximize their expected profits. As shown in Proposition 5, banks choose $\{k^*, \ell^*\}$ so that liquidity-driven crises occur in equilibrium. Formally, this means that $\{k^*, \ell^*\}$ always satisfies $\ell < (1 - k)r$. This continues to be the case also when regulation is in place, as illustrated in Figure 4a and 4b. In other words, only one regulatory requirement does not allow the social planner to enforce $\ell = (1 - k)r$ to hold, as it still leaves the possibility for banks to freely choose the non-regulated variable.

Forcing banks to increase their level of capitalization reduces the likelihood of liquidity-driven crises for given portfolio liquidity. However, once the likelihood of liquidity-driven crises is reduced, banks have the incentives to reduce their portfolio liquidity so as to increase the (per unit) portfolio return at date 2. The case of liquidity regulation is symmetric. Forcing banks to increase their liquidity holdings, reduces the likelihood of liquidity-driven crises and, thus, it decreases the beneficial effects of equity financing. As a result, banks finds it optimal to choose to raise funds cheaply through debt. In this context, anticipating banks’ responses to regulation, the best that the regulator can do is to choose the regulatory requirement such that, given banks’ choice, the likelihood of liquidity-driven crises is minimized.

Insert Figure 4a and 4b
5.2 Coordinated capital and liquidity requirements

The key feature of the regulatory intervention described in the previous section is that the capital and liquidity requirements leave banks the possibility to freely adjust their capitalization and portfolio liquidity, respectively, while still comply to the regulation. While these requirements resemble real world regulation, as, for example, a simple (unweighted) leverage ratio or an ad hoc increase in bank liquidity, most of the regulatory instruments that are embedded in the Basel III regulation do not have this feature. Risk-weighted capital requirements (RWC) prescribe banks to have a level of capitalization adequate to the riskiness of their portfolio. To the extent that liquid assets are less risky than less liquid ones, RWC links the amount of capital banks are required to hold to the liquidity of their portfolio. Similarly, the newly introduced liquidity coverage ratio (LCR), which prescribe banks to hold enough liquid assets to cover its short-term obligations and the net stable funding ratio (NSFR), requiring banks to hold a certain amount of stable funding (i.e., equity in the context of this model) to finance its long-term assets, link the amount of liquidity that banks are required to hold in portfolio to their level of capitalization.

In this section, we analyze this type of requirements and their effectiveness in achieving the constrained efficient allocation. To do so, we assume that the regulator imposes either a capital requirement \( k \geq k^{reg}(\ell) \) or a liquidity requirement \( \ell \geq \ell^{reg}(k) \). Intuitively, even when the regulator is constrained to use only one requirement, either capital or liquidly one, this should be better than the corresponding one analyzed in the previous section, as it constraints more bank choice of the unregulated variable. The question is whether any of this requirement allows to achieve the constrained efficient allocation and whether capital-based instruments are better than liquidity-based ones. We have the following result.

**Proposition 8**  If the regulator can set capital and liquidity requirement as a function of the non-regulated variable, the constrained efficient allocation is achieved.

i) Optimal capital regulation features the minimum requirement \( k \geq k^R \), with \( k^R = 1 - \ell^* \);

ii) Optimal liquidity regulation features the minimum requirement \( \ell \geq \ell^R \), with \( \ell^R = (1 - k^*) \).

In both cases, the equilibrium with regulation entails \( r^* = 1 \) and \( k^* = k^{SP} \) and \( \ell^* = \ell^{SP} \).

The proposition shows that the constrained efficient allocation can be achieved with either capital or liquidity regulation, as long as the regulator can condition the capital(liquidity) requirement to the level of liquidity(capital) chosen by banks. Regulation require banks to satisfy \( k \geq 1 - \ell \) and, as a result, liquidity-driven crises do not occur in equilibrium. When this is the case, as it was the case for the social planner, it
is optimal to choose the lowest $k$ and $\ell$ as to minimize the cost of funding and the negative effect of liquidity on bank portfolio return.

There are two important insights from the proposition. First, capital and liquidity requirements are perfect substitutes in that either regulatory tool, if adequately designed, allows to eliminate inefficient liquidity-driven crises and, in turn, to achieve the constrained efficient allocation. Second, in order to work properly, capital and liquidity requirements must be fine tuned to each other, in that banks with different level of portfolio liquidity should be required to be differently capitalized and, symmetrically, banks with different levels of capitalization should be required to hold a different proportion of liquid assets in their portfolio. In other words, effective capital and liquidity regulation prescribe banks to hold a minimum ratio $\frac{k}{\ell}$. In this respect, the optimal regulation characterized in Proposition 8 closely resembles real world regulation as risk-weighted capital ratio (RWC), liquidity coverage ratio (LCR) or net stable funding ratio (NSFR). To see this, consider a simplified bank balance sheet as illustrated in Figure 5, we can specify the risk-weighted capital ratio as requiring $k \geq \beta (1 - \ell)$, the liquidity coverage ratio requiring $\ell \geq \zeta (1 - k)$ and the net stable funding ratio requiring $k \geq \sigma (1 - \ell)$, with $\beta$, $\zeta$ and $\sigma$ being the coefficient set by the regulators. Thus, the requirements above can be easily rearranged as a a ratio between $k$ and $\ell$ and are so consistent with the regulatory intervention analyzed in Proposition 8.

6 Concluding remarks

In this paper we develop a model where both solvency- and liquidity-driven crises occur and both banks’ and debt holders’ decisions are endogenously determined. In our framework, banks choose inefficient levels of capitalization and portfolio liquidity, which in turn lead to liquidity-driven crises. This creates the scope for public intervention in the form of capital and liquidity regulation. We show that the design of these requirements matters a lot for their effectiveness in preventing inefficient liquidity-driven crises and that the optimal regulation features capital and liquidity requirements depending, respectively, on the bank portfolio liquidity and bank capitalization.

The paper offers a convenient framework to evaluate the implications of capital and liquid regulation because it allows to endogeneize the probability of both solvency- and liquidity-driven crises and to account for the different effects that changes in bank capital structure and portfolio liquidity have on them. Thus, this framework allows to analyze the interaction between capital and liquidity requirements and so the optimal
regulatory mix. The analysis shows that, if adequately designed, capital and liquidity requirements are perfect substitutes in removing liquidity-driven crises and so restore efficiency.

7 References


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8 Appendix

Proof of Proposition 1: Denote as \( f(\theta, r, k, \ell) = 0 \) the condition pinning down the threshold \( \hat{\theta}(k, \ell) \) as given in (4). By using the implicit function theorem, we have that

\[
\frac{\partial \hat{\theta}(k, \ell)}{\partial r} = -\frac{\frac{\partial f(\theta, r, k, \ell)}{\partial r}}{\frac{\partial f(\theta, r, k, \ell)}{\partial k}}, \quad \frac{\partial \hat{\theta}(k, \ell)}{\partial k} = -\frac{\frac{\partial f(\theta, r, k, \ell)}{\partial k}}{\frac{\partial f(\theta, r, k, \ell)}{\partial \ell}} \quad \text{and} \quad \frac{\partial \hat{\theta}(k, \ell)}{\partial \ell} = -\frac{\frac{\partial f(\theta, r, k, \ell)}{\partial \ell}}{\frac{\partial f(\theta, r, k, \ell)}{\partial \ell}}.
\]

The denominator \( \frac{\partial f(\theta, r, k, \ell)}{\partial \ell} = R(\hat{\theta}(k, \ell))(1 - \alpha \ell) > 0 \) as \( R'(<0) > 0 \). Thus, the signs of \( \frac{\partial \hat{\theta}(k, \ell)}{\partial r}, \frac{\partial \hat{\theta}(k, \ell)}{\partial k} \) and \( \frac{\partial \hat{\theta}(k, \ell)}{\partial \ell} \) are equal to the opposite sign of the respective numerators. We start from \( \frac{\partial \hat{\theta}(k, \ell)}{\partial r} \). Deriving (4) with respect to \( r \), we obtain

\[
\frac{\partial f(\theta, r, k, \ell)}{\partial r} = -2 (1 - k) r < 0.
\]

It follows that \( \frac{\partial \hat{\theta}(k, \ell)}{\partial r} > 0 \). Similarly, we have that

\[
\frac{\partial f(\theta, r, k, \ell)}{\partial k} = R(\hat{\theta}(k, \ell))(1 - \alpha \ell) > 0 \quad \text{and} \quad \frac{\partial f(\theta, r, k, \ell)}{\partial \ell} = -R(\hat{\theta}(k, \ell)) \alpha < 0,
\]

which imply \( \frac{\partial \hat{\theta}(k, \ell)}{\partial k} < 0 \) and \( \frac{\partial \hat{\theta}(k, \ell)}{\partial \ell} > 0 \). Thus, the proposition follows. \( \Box \)

Proof of Proposition 2: The proof follows closely that in Goldstein and Pauzner (2005) since the model also exhibits the property of one-sided strategic complementarity. We start by characterizing a region where the state of the economy \( \theta \) takes extremely high values and we refer to this region as the upper dominance region. The upper dominance region of \( \theta \) corresponds to the range \([\bar{\theta}, 1) \) in which fundamentals are so good that all debt holders roll over at date 1. As in Goldstein and Pauzner (2005), we construct this region by assuming that, in this range, the investment is safe and returns \( R(1)(1 - \alpha \ell) \) both at date 1 and 2. Given that \( r < r^2 < R(1)(1 - \alpha \ell) \), this ensures that the bank does not have to liquidate more units than the \( n \) debt holders not rolling over at date 1. Then, when an investor receives a signal such that he believes that the fundamental \( \theta \) are in the upper dominance region, he is certain to receive the promised payment \( r^2 \), irrespective of his beliefs on other debt holders’ action and so he does not have any incentive to withdraw early.

The upper dominance region is the mirror image of the range \([0, \bar{\theta}(k, \ell)] \) characterized in the text, where the values of \( \theta \) are so low that not rolling over the debt claim at date 1 is a dominant strategy. Besides these extreme ranges of values of the state of the economy \( \theta \), a debt holder’s rollover decision depends on what he expects the other investors do and so on the signal he receives.

Assume that debt holders behave accordingly to a threshold strategy, that is each debt holder withdraw at date 1 if he receive a signal below \( s^*(k, \ell) \) and rolls over otherwise, the fraction of debt holders not rolling over the debt claim \( n \) is equal to the probability of receiving a signal below \( s^*(k, \ell) \). Given that debt holders’
signal are independent and uniformly distributed in the range $[\theta - \varepsilon, \theta + \varepsilon]$, $n(s^*(k, \ell), \theta)$ is equal to

$$
n(s^*(k, \ell), \theta) = \begin{cases} 
1 & \text{if } \theta \leq s^*(k, \ell) - \varepsilon \\
\frac{s^*(k, \ell) - \theta + \varepsilon}{2\varepsilon} & \text{if } s^*(k, \ell) - \varepsilon \leq \theta \leq s^*(k, \ell) + \varepsilon \\
0 & \text{if } \theta \geq s^*(k, \ell) + \varepsilon
\end{cases}
$$

When $\theta$ is lower than $s^*(k, \ell) - \varepsilon$, all $(1-k)$ debt holders receive a signal below $s^*(k, \ell)$ and so all debt holders withdraw at date 1. On the contrary, when $\theta$ is very high, that is $\theta \geq s^*(k, \ell) + \varepsilon$, all $(1-k)$ debt holders receive a signal above $s^*(k, \ell)$ and, as a result, decide to rollover their debt claim. In the intermediate range of fundamental, when $s^*(k, \ell) - \varepsilon \leq \theta \leq s^*(k, \ell) + \varepsilon$, there is a partial run, in that only some debt holders withdraw at date 1. The proportion of those not rolling over their debt claim decreases linearly with $\theta$, as fewer investors observe a signal below the threshold $s^*(k, \ell)$.

Denote as $\Delta(s_i, n(s^*(k, \ell), \theta))$, a debt holder’s expected utility differential between rolling over the debt claim until date 2 and withdrawing it at date 1 when all agents are assumed to behave accordingly to the same threshold strategy $s^*(k, \ell)$. We have

$$
\Delta(s_i, n(s^*(k, \ell), \theta)) = \frac{1}{2\varepsilon} \int_{s_i - \varepsilon}^{s_i + \varepsilon} (\pi_2 - \pi_1) \, d\theta,
$$

with $\pi_2$ and $\pi_1$ as given by (7) and (6), respectively. The following lemma states a few properties of the function $\Delta(s_i, n(s^*(k, \ell), \theta))$.

**Lemma 2** i) The function $\Delta(s_i, n(s^*(k, \ell), \theta))$ is continuos in $s_i$; ii) for any $a > 0$, $\Delta(s_i + a, n(s^*(k, \ell), \theta) + a)$ is non-decreasing in $a$; iii) $\Delta(s_i, n(s^*(k, \ell), \theta))$ is strictly increasing in $a$ if there is a positive probability that $n < \pi$ and $\theta < \overline{\theta}$.

**Proof of Lemma 2:** The proof follows Goldstein and Pauzner (2005). The function $\Delta(.)$ is continuos in $s_i$, as $s_i$ only changes the limits of integration in the formula for $\Delta(s_i, n(s^*(k, \ell), \theta))$. To show that the function $\Delta(s_i, n(s^*(k, \ell), \theta))$ is non-decreasing in $a$, we need first to show that $(\pi_2 - \pi_1)$ is non-decreasing in $\theta$. As $\theta$ increases, we have two effects. First, a higher $\theta$ implies that $R(\theta)$ is also higher, thus increasing the date 2 payoff in the range $\hat{n}(\theta) < n \leq \pi$. Second, a change in $\theta$ affects the threshold $\hat{n}(\theta)$ as follows.

$$
\frac{\partial \hat{n}(\theta)}{\partial \theta} = R'(\theta)(1 - \alpha \ell) \left[ \frac{R(\theta)(1 - \alpha \ell) r}{\ell} - r^2 \right] - \left[ R(\theta)(1 - \alpha \ell) - (1-k) r^2 \right] \frac{r}{\ell} =
$$

$$
= \frac{R'(\theta)(1 - \alpha \ell)}{(1-k) \left[ \frac{R(\theta)(1 - \alpha \ell) r}{\ell} - r^2 \right]^2} r^2 \left[ \frac{r}{\ell} (1-k) - 1 \right] > 0,
$$

since $R'(\theta) > 0$ and $(1-k) r > \ell$. Thus, since the interval $[0, \hat{n}(\theta)]$ where the utility differential $\pi_2 - \pi_1$ is the highest becomes larger, while the range $(\pi, 1]$ is unaffected by a change in $\theta$, the date 2 payoff also increases so that the utility differential $\pi_2 - \pi_1$ is non-decreasing in $\theta$. This also implies that $\Delta(s_i, n(s^*(k, \ell), \theta))$ is non-decreasing in $a$, as when $a$ increases, debt holders see the same distribution of $n$ but expects $\theta$ to be...
larger. The function \( \Delta (s_i, n (s^* (k, \ell), \theta)) \) is strictly increasing in \( n \) since when \( n < \pi \) and \( \theta < \bar{\theta} \), \( \pi_2 - \pi_1 \) is strictly increasing in \( \theta \).

Since the rest of the proof follows closely that in Goldstein and Pauzner (2005) we omit it here and only specify the condition pinning down the threshold \( s^* (k, \ell) \). A debt holder who receives the signal \( s^* (k, \ell) \) is indifferent between rolling over the debt claim until date 2 and withdrawing it at date 1. The threshold \( s^* (k, \ell) \) can be computed as follows. First, we denote as \( v (\theta, n) \) the utility differential between rolling over the debt claim until date 2 and withdrawing it at date 1. This is equal to the difference \( \pi_2 - \pi_1 \). Thus, using (7) and (6), we obtain

\[
v (\theta, n) = \begin{cases} 
 r^2 - r & \text{if } 0 \leq n \leq \hat{n} (\theta) \\
 \frac{R (\theta) (1 - \alpha \ell) \left( 1 - \frac{(1 - k) n r}{\ell} \right)}{(1 - k) (1 - n)} - r & \text{if } \hat{n} (\theta) < n \leq \pi \\
 0 - \frac{\ell}{(1 - k) n} & \pi < n \leq 1 
\end{cases}
\]  

(24)

Since when receiving the threshold signal a debt holder is indifferent between rolling the debt until date 2 and withdrawing it at date 1, the threshold \( s^* (k, \ell) \) is equal to the solution to

\[
f (\theta, k, \ell) = \int_0^{\hat{n} (\theta)} r^2 d n + \int_{\hat{n} (\theta)}^{\pi} R (\theta) (1 - \alpha \ell) \left( 1 - \frac{(1 - k) n r}{\ell} \right) d n - \int_0^{\hat{n}} r d n - \int_{\hat{n}}^{1} \frac{\ell}{(1 - k) n} d n = 0,
\]

(25)

where from (23), we obtain \( \theta (n) = s^* (k, \ell) + \varepsilon - 2 \varepsilon \frac{n}{1 - k} \). At the limit, when \( \varepsilon \to 0 \), \( \theta (n) \to s^* (k, \ell) \) and we denote it as \( \theta^* (k, \ell) \). Then, the proposition follows.

**Proof of Lemma 1:** We compute the effect of capital and liquidity on \( \theta^* (k, \ell) \) (i.e., \( \frac{\partial \theta^* (k, \ell)}{\partial k} \) and \( \frac{\partial \theta^* (k, \ell)}{\partial \ell} \)) by using the implicit function theorem as follows:

\[
\frac{\partial \theta^* (k, \ell)}{\partial k} = - \frac{\frac{\partial f (\theta^*, k, \ell)}{\partial k}}{\frac{\partial f (\theta^*, k, \ell)}{\partial \theta}} \quad \text{and} \quad \frac{\partial \theta^* (k, \ell)}{\partial \ell} = - \frac{\frac{\partial f (\theta^*, k, \ell)}{\partial \ell}}{\frac{\partial f (\theta^*, k, \ell)}{\partial \theta}},
\]

with \( f (\theta^*, k, \ell) \) is the equation pinning down \( \theta^* (k, \ell) \), as defined in (25). The denominator \( \frac{\partial f (\theta^*, k, \ell)}{\partial \theta} \) is given by

\[
\frac{\partial f (\theta^*, k, \ell)}{\partial \theta} = \int_{\hat{n} (\theta)}^{\pi} R (\theta^*) (1 - \alpha \ell) \left( 1 - \frac{(1 - k) n r}{\ell} \right) d n > 0
\]

since the derivatives of the extremes of the integrals cancel out. Thus, the sign of \( \frac{\partial \theta^* (k, \ell)}{\partial k} \) and \( \frac{\partial \theta^* (k, \ell)}{\partial \ell} \) are equal to the opposite sign of \( \frac{\partial f (\theta^*, k, \ell)}{\partial k} \) and \( \frac{\partial f (\theta^*, k, \ell)}{\partial \ell} \), respectively.

We start from \( \frac{\partial f (\theta^*, k, \ell)}{\partial k} \). Deriving (25) with respect to \( k \) and multiplying it by \(-1\), we obtain the expression in (11) since the derivatives of the extremes of integrals cancel out.

Similarly, deriving (25) with respect to \( \ell \) and multiplying it by \(-1\), we obtain the expression in (12), as
Proof of Proposition 3: The proof proceeds in steps and builds on the results derived in the Lemma 1.

First, denote as \( k^{\text{max}}(\ell) \) the solution to \((1-k)r = \ell \). It is easy to see that \( k^{\text{max}}(\ell) \) decreases with \( \ell \), \( k^{\text{max}}(0) = 1 \) and \( k^{\text{max}}(1) = 1 - \frac{1}{r} \). When \( k \to k^{\text{max}}(\ell) \), the threshold \( \theta^*(k, \ell) \to \underline{b}_1(k, \ell) \). To see this, we can rearrange the expression in (25) as follows:

\[
\int_0^{\tilde{n}(\theta)} \left[ \min \left\{ \rho^2, \frac{R(\theta)(1 - \alpha\ell) \left[ 1 - \frac{(1-k)nr}{\ell} \right]}{(1-k)(1-n)} \right\} - r \right] dn + \int_{\tilde{n}(\theta)}^{\bar{n}} \left[ \frac{R(\theta)(1 - \alpha\ell) \left[ 1 - \frac{(1-k)nr}{\ell} \right]}{(1-k)(1-n)} - r \right] dn - \int_{\bar{n}}^1 \frac{\ell}{(1-k)n} dn,
\]

with \( \tilde{n}(\theta) = \frac{R(\theta)(1 - \alpha\ell) - (1-k)r}{(1-k)\left( \frac{R(\theta)}{R(\theta + (1-\alpha)\ell)} - 1 \right) r} \). When \( k \to k^{\text{max}}(\ell) \), \( \underline{n} \to \tilde{n}(\theta) \to 1 \) and the expression above pinning down \( \underline{n} \) simplifies to

\[
\int_0^1 \left[ \min \left\{ \rho^2, \frac{R(\theta)(1 - \alpha\ell) \left[ 1 - \frac{(1-k)nr}{\ell} \right]}{(1-k)(1-n)} \right\} - r \right] dn = 0.
\]

Since \( r > 1 \) and, thus, \( r^2 > r \), \( \theta^*(k, \ell) \) solves \( \frac{R(\theta)(1 - \alpha\ell)}{(1-k)} - r = 0 \), which is equivalent to the equation pinning down \( \underline{b}_1(k, \ell) \), as given in (5).

Second, we rearrange the expression in (11) as follows:

\[
R(\theta^*)(1 - \alpha\ell) \log \left[ \frac{1 - \underline{n}}{1 - \tilde{n}(\theta^*)} \right] = -\ell \log \underline{n}.
\]

The first term is negative since \( \underline{n} > \tilde{n}(\theta^*) \) and so \( \frac{1 - \underline{n}}{1 - \tilde{n}(\theta^*)} < 1 \), while the second one is positive since \( \underline{n} < 1 \). Using \( \underline{n} = \frac{\ell}{(1-k)r} \) and \( \tilde{n}(\theta^*) = \frac{R(\theta^*)(1 - \alpha\ell) - (1-k)r^2}{(1-k)\left( \frac{R(\theta^*)}{R(\theta^* + (1-\alpha)\ell)} - 1 \right) r} \), after a few manipulations, the expression above can be rewritten as follows:

\[
\log \left[ \left( 1 - \frac{\ell r}{R(\theta^*) (1 - \alpha\ell)} \right) R(\theta^*) (1 - \alpha\ell) \right] - \log \left[ \left( \frac{\ell}{(1-k)r} \right)^\ell \right] = 0.
\]

Denote as \( \tilde{k}(\ell) \) the solution to \( \log \left[ \left( 1 - \frac{\ell r}{R(\theta^*) (1 - \alpha\ell)} \right) R(\theta^*) (1 - \alpha\ell) \right] - \log \left[ \left( \frac{\ell}{(1-k)r} \right)^\ell \right] = 0 \). The threshold \( \tilde{k}(\ell) \) is then equal to

\[
\tilde{k}(\ell) = 1 - \frac{\ell}{r} (\Lambda) \frac{R(\theta^*) (1 - \alpha\ell)}{r},
\]

where \( \Lambda = \left( 1 - \frac{\ell r}{R(\theta^*) (1 - \alpha\ell)} \right) \). It is easy to see that \( \tilde{k}(\ell) < k^{\text{max}}(\ell) \) for any \( \ell > 0 \), since \( k^{\text{max}}(\ell) = 1 - \frac{\ell}{r} < 1 \) and \((\Lambda) \frac{R(\theta^*) (1 - \alpha\ell)}{r} > 1 \). Furthermore, from (29), it follows that \( \tilde{k}(\ell) \to 1 \), when \( \ell \to 0 \) and that \( \tilde{k}(\ell) = 0 \)
requires \( \ell > 0 \). The expression in (28) is decreasing in \( k \), as its derivative with respect to \( k \) is equal to

\[
- \frac{\ell}{(1-k)r} > 0.
\]

Thus, it follows that the expression in (28), and, as a result \( \frac{\partial f(\theta^*, k, \ell)}{\partial k} \), is negative for \( k \geq \bar{k}(\ell) \), and positive for \( k < \bar{k}(\ell) \). As a result, \( \frac{\partial \theta^*(k, \ell)}{\partial k} < 0 \), for \( k \geq \bar{k}(\ell) \), and \( \frac{\partial \theta^*(k, \ell)}{\partial k} > 0 \) for \( k < \bar{k}(\ell) \).

Consider now the effect of liquidity \( \ell \) on \( \theta^*(k, \ell) \). The expression (12) determining the sign of \( \frac{\partial \theta^*(k, \ell)}{\partial k} \) (i.e., \( -\frac{\partial f(\theta^*, k, \ell)}{\partial k} \)) can be rearranged as follows, after adding and subtracting \( \frac{1}{(1-k)r} \int_{\hat{n}(\theta^*')} R(\theta^*) \, dn \):

\[
\frac{\partial f(\theta^*, k, \ell)}{\partial \ell} = \frac{1}{(1-k)\ell} \left[ \int_{\hat{n}(\theta^*')} R(\theta^*) (1-\alpha \ell) \, dn - \ell \int_{\hat{n}(\theta^*')} R(\theta^*) \frac{1}{1-n} \, dn + \int_{\hat{n}(\theta^*')} R(\theta^*) \left( 1 - \frac{(1-k)nr}{\ell} \right) \, dn \right].
\]

Since, from (11), we have that \( \frac{\partial f(\theta^*, k, \ell)}{\partial k} = \frac{1}{(1-k)^2} \left[ \int_{\hat{n}(\theta^*')} R(\theta^*) (1-\alpha \ell) \, dn - \ell \int_{\hat{n}(\theta^*')} R(\theta^*) \frac{1}{1-n} \, dn \right] \), we can write

\[
\frac{\partial f(\theta^*, k, \ell)}{\partial \ell} = (1-k) \frac{\partial f(\theta^*, k, \ell)}{\partial k} - \frac{1}{(1-k)\ell} \int_{\hat{n}(\theta^*')} R(\theta^*) (1-\alpha \ell) \, dn =
\]

\[
= \frac{1}{\ell} \left[ (1-k) \frac{\partial f(\theta^*, k, \ell)}{\partial k} - \frac{1}{(1-k)\ell} \int_{\hat{n}(\theta^*')} R(\theta^*) (1-\alpha \ell) \, dn \right].
\]

From (30), then, it is easy to see that when \( k \leq \bar{k}(\ell) \), \( \frac{\partial f(\theta^*, k, \ell)}{\partial k} < 0 \), as \( \frac{\partial \theta^*(k, \ell)}{\partial k} \leq 0 \). This implies that \( \frac{\partial \theta^*(k, \ell)}{\partial \ell} > 0 \). Furthermore, since for any \( k \geq k_{\max}(\ell) \), the relevant threshold is \( \theta_1(k, \ell) \) and, from (5), it holds \( \frac{\partial \theta_1(k, \ell)}{\partial \ell} > 0 \), we have that an increase in liquidity has a detrimental effect on stability for \( k \leq \bar{k}(\ell) \) and \( k \geq k_{\max}(\ell) \).

Consider now the range \( (\bar{k}(\ell), k_{\max}(\ell)) \). We want to show that in this range there are levels of bank capitalization \( k \) for which increasing liquidity leads to a lower probability of panic-driven runs, i.e., \( \frac{\partial f(\theta^*, k, \ell)}{\partial \ell} < 0 \) for some \( k \in (\bar{k}(\ell), k_{\max}(\ell)) \). To do this, we need to show that there exists a range of \( k \) where the expression in the square bracket in (30) is positive, that is

\[
(1-k) \frac{\partial f(\theta^*, k, \ell)}{\partial k} = \frac{1}{(1-k)^2} \int_{\hat{n}(\theta^*')} R(\theta^*) (1-\alpha \ell) \, dn > 0.
\]

with \( \frac{\partial f(\theta^*, k, \ell)}{\partial k} > 0 \) in the range \( k \in (\bar{k}(\ell), k_{\max}(\ell)) \).

First, denote \( k^T(\ell) \) as the lowest level of \( k \) for which \( 1 - \frac{(1-k)nr}{\ell} = 0 \). Since the expression \( 1 - \frac{(1-k)nr}{\ell} \) is decreasing in \( n \) and increasing in \( k \), \( k^T(\ell) = 1 - \frac{\ell}{\hat{n}(\theta^*)r} \). It follows that for any \( k \geq k^T(\ell) \) the second term in (31) is non negative. Notice that \( k^T(\ell) \to 1 \) when \( \ell \to 0 \) and \( k^T(\ell) = 0 \) requires \( \ell > 0 \). Since \( k_{\max}(\ell) = 1 - \frac{\ell}{r} \) and \( \hat{n}(\theta^*) < 1 \), it holds that \( k^T(\ell) < k_{\max}(\ell) \). Furthermore, by evaluating \( 1 - \frac{(1-k)nr}{\ell} \)
at $k = \bar{k}(\ell)$, where $\bar{k}(\ell)$ is defined as in (29), we obtain

$$1 - n \left( \frac{R(\theta^*(1-\alpha \ell))}{r} \right) = \frac{R(\theta^*(1-\alpha \ell))}{r} - n.$$  

From (31), the greater value that $n$ can take is $\overline{n} = \frac{\ell}{1-k} r$ and, as shown above, $\left( \frac{R(\theta^*(1-\alpha \ell))}{r} \right) = \frac{\ell}{1-k}$ if $k = \bar{k}(\ell)$. It follows that $(\Lambda) \left( \frac{R(\theta^*(1-\alpha \ell))}{\ell r} - n > 0 \right.$ for $n \in (\overline{n}, \pi)$, which, in turn, implies that $k^T(\ell) > \bar{k}(\ell)$. Since $\frac{\partial f(\theta, k, \ell)}{\partial k} > 0$ for any $k > \bar{k}(\ell)$, the inequality in (31) holds at $k = k^T(\ell)$ so that $\frac{\partial f(\theta, k, \ell)}{\partial k} > 0$ and $\frac{\partial \theta^*(k, \ell)}{\partial k} < 0$. Given that $\bar{k}(\ell) < k^T(\ell) < k^{\text{max}}(\ell)$ and $\frac{\partial \theta^*(k, \ell)}{\partial k} > 0$ for $k \leq \bar{k}(\ell)$ and $k \geq k^{\text{max}}(\ell)$, by continuity, there must exist two thresholds $\tilde{k}(\ell) \in (\bar{k}(\ell), k^T(\ell))$ and $\check{k}(\ell) \in (k^T(\ell), k^{\text{max}}(\ell))$, such that $\frac{\partial f(\theta, k, \ell)}{\partial k} < 0$ for $\check{k}(\ell) < k < \tilde{k}(\ell)$ and $\frac{\partial f(\theta, k, \ell)}{\partial k} > 0$ for $\tilde{k}(\ell) < k < \check{k}(\ell)$. Thus, the proposition follows. □

**Proof of Proposition 4:** We distinguish two cases depending on whether $\ell < (1-k)$ holds or not.

Consider, first the case $\ell \geq (1-k)$. In this case, the debt holders expect to receive a pro-rata share greater than 1 when the bank defaults at date 1. Thus, for any $r \geq 1$, the inequality in (14) holds. It follows that banks optimally choose $r^* = 1$ so to minimize their funding cost.

Consider now the case where $\ell < (1-k)$. In this case, an interest rate $r$ strictly greater than 1 is needed for (14) to hold. As above the interest rate is a cost for the bank and so it would like to choose the lowest possible $r$ conditional on being solvent and on debt holders to provide funds at date 0. From (14) holding with equality, we can obtain

$$\int_{\theta^*(k, \ell)}^{1} (1-k) r^2 d\theta = (1-k) - \int_{0}^{\theta^*} \ell d\theta.$$

Substituting it into the expression in (16), we can rearrange the expression for bank profit as follows:

$$\int_{\theta^*(k, \ell)}^{1} R(\theta) (1-\alpha \ell) d\theta + \int_{0}^{\theta^*} \ell d\theta - (1-k) - k\rho.$$

Deriving the expression above with respect to $r$, after a few manipulations, we obtain the first order condition

$$- \frac{\partial \theta^*(k, \ell)}{\partial r} \left[ R(\theta^*(k, \ell)) (1-\alpha \ell) - \ell \right].$$  

(32)

The expression above is zero in two cases: if $[R(\theta^*(k, \ell)) (1-\alpha \ell) - \ell] = 0$ and if $\frac{\partial \theta^*(k, \ell)}{\partial r} = 0$. The bracket $[R(\theta^*(k, \ell)) (1-\alpha \ell) - \ell]$ is equal to zero when $r = \frac{\ell}{1-\ell}$, as, in this case, $\theta^*(k, \ell) \rightarrow \theta_1(k, \ell)$. However, given that $\ell < (1-k)$, and so $\frac{\ell}{1-\ell} < 1$, the inequality in (14) would not hold. Thus, it follows that the choice of $r^*$ depends on the sign of $\frac{\partial \theta^*(k, \ell)}{\partial r}$.
Rearrange the condition (10) pinning down $\theta^* (k, \ell)$ as follows:

$$
r^2 \Pr (n < \n (\theta)) + \frac{R (\theta) (1 - \alpha \ell)}{(1 - k) (1 - n)} \left[ 1 - \frac{(1-k)nr}{\ell} \right] [1 - \Pr (n < \n (\theta))] = 
$$

$$
= \int_0^n rdn - \int_0^1 \frac{\ell}{(1-k)n} dn, 
$$

(33)

where $\n (\theta)$ is as in (8). After a few manipulations, the expression in (33) can be rearranged as follows:

$$
r^2 - \frac{R (\theta)(1-\alpha \ell)[1-(1-k)nr]}{(1-k)(1-n)} \Pr (n < \n (\theta)) - 1 = 0. 
$$

(34)

From (8), it is easy to see that

$$
\frac{\partial \n (\theta)}{\partial r} = -2r (1-k) \frac{R (\theta)(1-\alpha \ell)(1-k)}{(1-k)[1-n]} - \n (\theta) \frac{R (\theta)(1-\alpha \ell)(1-k)}{R (\theta)(1-\alpha \ell)r - r^2} < 0.
$$

Thus the higher $r$, the lower $\Pr (n < \n (\theta))$, with $\Pr (n < \n (\theta)) \to 0$ when $r$ becomes very large. Denote as $A_1$ the other term in (34), that is

$$
A_1 = \frac{r^2 - \frac{R (\theta)(1-\alpha \ell)[1-(1-k)nr]}{(1-k)(1-n)}}{\int_0^n rdn - \int_0^1 \frac{\ell}{(1-k)n} dn - \frac{R (\theta)(1-\alpha \ell)[1-(1-k)nr]}{(1-k)(1-n)}}.
$$

Both the numerator and denominator of $A_1$ increase with $r$ as

$$
\frac{\partial A_1}{\partial r} < 0. 
$$

However, the numerator increases more given that

$$
2r > \frac{\partial}{\partial r} \left[ \int_0^n rdn - \int_0^1 \frac{\ell}{(1-k)n} dn \right],
$$

thus implying that

$$
\frac{\partial A_1}{\partial r} > 0.
$$

Taking together $\frac{\partial A_1}{\partial r} > 0$, $\frac{\partial \Pr (n < \n (\theta))}{\partial r} < 0$ and $\Pr (n < \n (\theta)) \to 0$ as $r$ gets very large, it follows that $\theta^* (k, \ell)$ is a convex function of $r$, whose minimum corresponds to the solution to $\frac{\partial \theta^* (k, \ell)}{\partial r} = 0$. Thus, the equilibrium $r^*$, corresponding to the solution to the expression in (32) equal to zero, is the solution to $\frac{\partial \theta^* (k, \ell)}{\partial r} = 0$ and the proposition follows. □

**Proof of Corollary 1:** The proof is straightforward and follows directly from the result of Proposition 4. The total effect of capital and liquidity on the threshold $\theta^* (k, \ell)$ are equal to

$$
\frac{d \theta^* (k, \ell)}{dk} = \frac{\partial \theta^* (k, \ell)}{\partial k} + \frac{\partial \theta^* (k, \ell)}{\partial r} \frac{dr^*}{dk},
$$

for capital and

$$
\frac{d \theta^* (k, \ell)}{d\ell} = \frac{\partial \theta^* (k, \ell)}{\partial \ell} + \frac{\partial \theta^* (k, \ell)}{\partial r} \frac{dr^*}{d\ell},
$$

34
for liquidity. Consider first the case when \( \ell < (1 - k) \). In this case, \( \frac{\partial \theta^*(k, \ell)}{\partial r} \bigg|_{r=r^*} = 0 \). Thus, it follows that

\[
\frac{d\theta^*(k, \ell)}{dk} = \frac{\partial \theta^*(k, \ell)}{\partial k} \quad \text{and} \quad \frac{d\theta^*(k, \ell)}{d\ell} = \frac{\partial \theta^*(k, \ell)}{\partial \ell}.
\]

Consider now the case when \( \ell \geq (1 - k) \). In this case, \( r^* = 1 \) and so \( \frac{dr^*}{dr} = \frac{dr^*}{d\ell} = 0 \), which, in turn, implies that \( \frac{d\theta^*(k, \ell)}{dk} = \frac{\partial \theta^*(k, \ell)}{\partial k} \quad \text{and} \quad \frac{d\theta^*(k, \ell)}{d\ell} = \frac{\partial \theta^*(k, \ell)}{\partial \ell} \). Thus, the corollary holds. \( \square \)

**Proof of Proposition 5:** By deriving (16) with respect to \( k \) and \( \ell \), we obtain the expression in (17) and (18), respectively. To show that liquidity-driven always occur in equilibrium, we evaluate (17) at \( k = k_{\text{max}}(\ell) \) and (18) at \( \ell = (1 - k) r \) and show that they are negative. For \( k = k_{\text{max}}(\ell) \) and \( \ell = (1 - k) r \), \( \theta^*(k, \ell) = \theta_1(k, \ell) \) and bank makes zero profits. Thus, the expressions in (17) and (18) simplify to

\[
\int_{\theta^*}^{1} r^2 d\theta - \rho < 0, \quad (35)
\]

because \( \rho = \int_{\theta^*}^{1} r^2 E \) and

\[
-\int_{\theta^*}^{1} R(\theta) \alpha < 0, \quad (36)
\]

respectively. Thus, the proposition follows. \( \square \)

**Proof of Corollary 2:** The bank would never choose a equilibrium pair \( \{k^*, \ell^*\} \) being in a region whether \( \frac{\partial \theta^*(k, \ell)}{dk} < 0 \) and \( \frac{\partial \theta^*(k, \ell)}{d\ell} > 0 \), as it could unambiguously achieve higher expected profits by reducing the crisis probability through a reduction in \( \ell \), while keeping the same level of capital \( k \). This means that a pair \( \{k^*, \ell^*\} \) in the region bounded by \( k(\ell) \) and \( \bar{k}(\ell) \) is indeed a candidate equilibrium.

Using a similar argument, \( \{k^*, \ell^*\} = \{0, 0\} \) could also be a candidate equilibrium when the high cost of capital and liquidity (i.e., \( \rho \) and \( \alpha \)) constrain the bank in the region below \( k(\ell) \), where \( \frac{\partial \theta^*(k, \ell)}{dk} > 0 \) and \( \frac{\partial \theta^*(k, \ell)}{d\ell} > 0 \). However, if \( \{k^*, \ell^*\} = \{0, 0\} \), banks would make zero profits because when \( \ell \to 0 \), \( \theta^*(k, 0) \to 1 \). Thus, the bank could to unambiguously better by choosing positive \( \{k^*, \ell^*\} \) and the corollary follows. \( \square \)

**Proof of Proposition 6:** The social planner chooses \( k^{SP}, \ell^{SP} \) and \( r^{SP} \) so as to maximize (19), subject to debt holders’ and equity holders’ participation constraints, as given in (14) and (15). To do this, it sets \( k^{SP} = (1 - \ell) r \) and, as a consequence, \( r^{SP} = 1 \) follows, and the inefficiency associated with liquidity-driven crises is removed (i.e., \( \theta^*(k, \ell) \to \theta_1(k, \ell) = \theta^E \)). The choice of \( \ell^{SP} \) is fully driven by equity holders’ participation constraint. As \( \rho > \int_{\theta^E}^{1} R(\theta) d\theta \), the social planner cannot set \( \ell^{SP} = 0 \) and \( k^{SP} = 1 \), which would maximize total output implying \( \theta^E = 0 \). Thus, it chooses the lowest \( \ell \) consistent with both debt and equity holders providing funds at date 0 that is \( \ell^{SP} \) solves

\[
\int_{0}^{\theta^E} \ell d\theta + \int_{\theta^E}^{1} R(\theta) (1 - \alpha \ell) d\theta = \rho (1 - \ell) + \ell. \quad (37)
\]

The inequalities \( k^{SP} > k^* \) and/or \( \ell^{SP} > \ell^* \) follows from the fact that the social planner allocation corresponds to a point on the curve \( k_{\text{max}}(\ell) \), as defined in the proof of Proposition 3, while the market equilibrium
\( \{k^*, \ell^*\} \) always lies in the region bounded between \( k(\ell) \) and \( \ell(\ell) \), as shown in Corollary 2. The proposition follows. \( \square \)

**Proof of Proposition 7:** Consider first capital regulation, in the form of a minimum requirement \( k^R \). The bank problem is like the one characterized in (13)-(15), with the addition of the constraint \( k \geq k^R \). The determination of \( r^E \) and \( r^* \) is as in the baseline model, bank’s choice of portfolio liquidity is still given by the solution to (18), while that of capital is the maximum between \( k^R \) and the solution to (17). The same argument as in the proof of Proposition 5 implies that bank never chooses \( \ell^* \) such that liquidity-driven runs are prevented. The proof in the case of liquidity regulation is analogous. Thus, the proposition follows. \( \square \)

**Proof of Proposition 8:** Consider first a regulatory intervention only targeting bank capitalization. The regulator prescribes banks to have the level of capital \( k \geq k^R \), where \( k^R \) need to satisfy \( \ell = (1 - k) r \). From the proof of Proposition 3, we know that when \( \ell = (1 - k) r \), the relevant threshold becomes \( \theta^E_1 (k, \ell) = \theta^E \). This means that banks choose \( k^*, \ell^* \) and \( r^* \) so as to maximize their expected profits, which are given by

\[
\int_{\theta^E}^{1} \left[ R(\theta) (1 - \alpha\ell) - (1 - k) r^2 \right] d\theta - k\rho
\]

subject to

\[
\int_{\theta^E}^{1} \frac{\ell}{(1 - k)} d\theta + \int_{\theta^E}^{1} r^2 d\theta = 1,
\]

and

\[
\int_{\theta^E}^{1} \left[ R(\theta) (1 - \alpha\ell) - (1 - k) r^2 \right] d\theta - k\rho > 0.
\]

The choice of \( r \) is straightforward since \( \ell = (1 - k) r \) implies that \( \frac{\ell}{(1 - k)} > 1 \) and thus there is no risk for which debt holders need to be compensated for and banks optimally set \( r^* = 1 \).

Deriving the expression in (38) with respect to \( k \) and \( \ell \), we obtain:

\[-\frac{\partial \theta^E}{\partial k} \left[ R(\theta^E) (1 - \alpha\ell) - (1 - k) r^2 \right] + \int_{\theta^E}^{1} r^2 d\theta - \rho < 0, \tag{39}\]

and

\[-\frac{\partial \theta^E}{\partial \ell} \left[ R(\theta^E) (1 - \alpha\ell) - (1 - k) r^2 \right] - \int_{\theta^E}^{1} \alpha R(\theta) d\theta < 0, \tag{40}\]

respectively.

Both expressions in (39) and (40) are negative since the first term is zero, given that banks are covered by limited liability, and \( \rho > \int_{\theta^E}^{1} R(\theta) d\theta > \int_{\theta^E}^{1} r^2 d\theta \). As a result, banks would choose \( k^* = k^R \). Substituting \( k^* = 1 - \ell \) in (38) and deriving it with respect to \( \ell \), we obtain

\[-\int_{\theta^E}^{1} [\alpha R(\theta) + 1] d\theta + \rho < 0.
\]

Thus, the equilibrium \( \ell \) corresponds to the lowest possible \( \ell \) that is consistent with the bank making non-negative profits. It follows that \( \ell^* \) solves
Comparing the expression in (41) with that in (37), it easy to see that they are equal so it follows that $\ell^* = \ell^{SP}$ and, in turn, $k^* = k^{SP}$.

The case in which the regulator can only set liquidity requirements is similar. In this case, banks are required to have the level of portfolio liquidity $\ell \geq \ell^R$, where $\ell^R$ need to satisfy $\ell = (1 - k) r$. Using the same arguments as above, banks choose $\ell^* = \ell^R$ and $k^*$ as the solution to

$$\int_{\theta^E}^{1} [R(\theta) (1 - \alpha \ell) - \ell] d\theta - (1 - \ell) \rho = 0$$

As above comparing (42) with the corresponding expression in (37), that is

$$\int_{\theta^E}^{1} [R(\theta) (1 - \alpha (1 - k)) - (1 - k)] d\theta - k \rho = 0$$

it easy to see that they are equal, thus implying that $k^* = k^{SP}$ and, in turn, $\ell^* = \ell^{SP}$. Thus, the proposition follows. □
Figure 1: Solvency-driven crises. The figure illustrates the occurrence of solvency-driven crises. When each debtholder expects the others to roll over the debt claim at date 2, a bank failure occurs only when the fundamental of the economy are so low that the bank cannot repay the promised repayment (i.e., when $\theta < \overline{\theta}(k, \ell)$). In the range where they occur, solvency-driven crises can either take the form of a failure at date 2, as it is the case in the range $\underline{\theta}_1(k, \ell) < \theta < \overline{\theta}(k, \ell)$ or of a bank run at date 1, as it is the case in the range $0 < \theta < \underline{\theta}_1(k, \ell)$. The latter occurs because, when the fundamental of the economy are very low, each debtholder expects not only that the bank’s resources are not enough to repay the promised repayment $r^2$ at date 2, but also that the pro-rata share received in the case of a bank’s failure is lower than the repayment $r$ obtained at date 1. Thus, not rolling over the debt claim at date 1 is a dominant strategy in the range $\underline{\theta}_1(k, \ell) < \theta < \overline{\theta}(k, \ell)$. 

<table>
<thead>
<tr>
<th>Solvency-driven crises materialize as a run at date 1</th>
<th>Solvency-driven crises materialize as a failure at date 2</th>
<th>No Solvency-driven crises occur</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0$</td>
<td>$\underline{\theta}_1(k, \ell)$</td>
<td>$1$</td>
</tr>
</tbody>
</table>
Figure 2: Liquidity-driven crises. The figure illustrates the occurrence of liquidity-driven crises. In the range for $\theta < \theta^*(k, \ell)$ debt holders choose not to roll over their debt claim, thus forcing the bank to default at date 1. In the region in which a bank default occurs, crises can be distinguished into liquidity-driven ones (i.e., for $\theta_1(k, \ell) < \theta < \theta^*(k, \ell)$) and solvency-driven ones (i.e., for $0 < \theta < \theta_1(k, \ell)$). The former are due to debt holders’ fear that others would not roll over their debt claim, while the latter are exclusively driven by debt holders’ expectation of a low realization of the fundamental of the economy $\theta$. 
Figure 3a: Effect of capital on financial stability

Figure 2b: The figure shows that the effect of capital on financial stability depends on the initial level of bank capitalization $k$ and on the liquidity of bank portfolio $\ell$. Capital has a detrimental effect on stability (i.e., $\frac{\partial \theta^*(k, \ell)}{\partial k} > 0$) when the level of bank capitalization is low, as it is the case for $k < \bar{k}(\ell)$. For higher values of bank capitalization an increase in capital reduces the probability of a run (i.e., $\frac{\partial \theta^*(k, \ell)}{\partial k} < 0$) as it is the case for $k > \bar{k}(\ell)$. 
Figure 3b: Effect of liquidity on financial stability. The figure shows that the effect of liquidity on financial stability depends on the initial level of capital $k$ and on the liquidity of bank portfolio $\ell$. Liquidity has a detrimental effect on stability (i.e., $\frac{\partial \theta^*(k, \ell)}{\partial \ell} > 0$) when the level of bank capitalization and the liquidity of bank portfolio are either very low or very high, as it is the case for $k > \overline{k}(\ell)$ and $k < \underline{k}(\ell)$. For intermediate values of bank capitalization and portfolio liquidity, as it is the case for $\underline{k}(\ell) < k < \overline{k}(\ell)$ an improvement in banks' liquidity position reduces the probability of a run (i.e., $\frac{\partial \theta^*(k, \ell)}{\partial \ell} < 0$).
Figure 4a: Capital regulation. The figure shows the consequence of an increase in the level of bank capitalization required by the regulator. As the regulator prescribes an increase in the level of capitalization from $k_0$ to $k_1$, banks respond to such a requirement by choosing a lower level of portfolio liquidity (i.e., $\ell_1 < \ell_0$), so that the equilibrium $\{k_1, \ell_1\}$ is below the curve $k^{\text{max}}(\ell)$ and inefficient liquidity-driven crises still occur.
Figure 4b: Liquidity regulation. The figure shows the consequence of an increase in the level of portfolio liquidity required by the regulator. As the regulator prescribes an increase in the liquidity of bank portfolio from $\ell_0$ to $\ell_1$, banks respond to such a requirement by choosing a lower level of capitalization (i.e., $k_1 < k_0$), so that the equilibrium $\{k_1, \ell_1\}$ is below the curve $k^{\text{max}}(\ell)$ and inefficient liquidity-driven crises still occur.
Figure 5: Simplified bank balance sheet. The figure illustrates a simplified bank balance sheet where the asset side is normalized to 1. The bank raise one unit of funds by issuing short-term debt $(1 - k)$ and raising equity $k$. Those funds are invested in a portfolio consisting of both liquid and illiquid assets, in the proportion of $\ell$ and $(1 - \ell)$, respectively.

<table>
<thead>
<tr>
<th>Assets</th>
<th>Liabilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>Liquid assets $(\ell)$</td>
<td>Short-term debt $(1 - k)$</td>
</tr>
<tr>
<td>Illiquid assets $(1 - \ell)$</td>
<td>Equity $(k)$</td>
</tr>
</tbody>
</table>