Valuing Private Equity Investments Strip by Strip

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Abstract

We propose a new valuation method for private equity investments. The first step is to construct a cash-flow replicating portfolio for the private investment, using cash-flows on listed equity and fixed income instruments. The second step is to value the replicating portfolio using a flexible asset pricing model that accurately prices the systematic risk in equity and fixed income strips. The method delivers time series for the expected return on PE funds as well as a measure of the realized risk-adjusted profit of a PE investment. We apply the method to real estate, infrastructure, energy, and corporate private equity funds, and compare the results to standard valuation approaches.

JEL codes: G24, G12
1 Introduction

Private equity investments have risen in importance over the past twenty-five years, relative to public equity. Private funds account for $4.7 trillion in assets under management, of which real estate funds comprise $800 billion.\(^1\) Large institutional investors now allocate substantial fractions of their portfolios to such alternative investments. For example, the celebrated Yale University endowment has a portfolio weight of over 50% in alternative investments. Pension funds and sovereign wealth funds have also ramped up their allocations to alternatives. As the fraction of overall wealth that is held in the form of private investment grows, so does the importance of developing appropriate valuation methods. The non-traded nature of the assets and their infrequent cash-flows makes this a challenging problem.

As with any investment, the price of a private equity (PE) investment equals the present discounted value of its cash-flows. The initial PE cash-flows are negative as the general partner (GP, the fund manager) calls in the capital from the limited partners (LPs, investors). Once the assets are deployed, they may throw off some positive cash-flows that can be distributed back to the LPs. The bulk of the returns received by investors, however, occur towards the end of the investment period when the GP sells the investments, and distributes the proceeds net of fees back to the LPs. We adopt the standpoint of the LP, and consider the key question to be one of calculating the investor’s profit, taking into account the risk of these cash-flows. Industry practice is to report the ratio of distributions to capital contributions (alongside the internal rate of return). However, this ratio neither takes into account time value of money nor the riskiness of the cash-flows. While the former problem is easily fixed, the latter problem is much more difficult, and the focus of our work.

We propose a novel methodology that centers on the nature and the timing of cash-flow risk for PE investments. It proceeds in two steps. In a first step, we estimate the exposure of the PE fund’s cash-flows to the cash-flows a set of liquid, publicly listed securities. In our application, we consider pay-offs to Treasury bonds, the stock market, and the publicly-traded real estate (REIT) and infrastructure markets; but the method easily accommodates additional publicly-traded factors. By stripping the sequence of PE cash-flows into individual cash-flows by horizon, and estimating their exposure to maturity-matched listed securities separately, our method decomposes the risk of a PE cash-flow into its different horizon components, as well as by vintage and investment

\(^1\)Estimates from Preqin.
In a second step, we use a flexible, no-arbitrage asset pricing model that prices the term structure of Treasury bonds, stocks, REITs, and infrastructure stocks. It postulates the main sources of systematic risk and estimates the prices of risk that the market assigns to these risk exposures. With those risk prices in hand, we can price strips, which are claims to a single cash-flow at each date and at each horizon in the bond, stock, REIT, and infrastructure market, as in Lettau and Wachter (2011); van Binsbergen, Brandt, and Koijen (2012). We use the shock price elasticities of Borovička and Hansen (2014) to understand how risk prices change with horizon in the model. The model closely matches the time series of bond yields across maturities, stock price-dividend ratios, as well as stock and bond risk premia.

Combining the replicating portfolio of strips obtained from the first step, with the asset pricing model from the second step; we obtain the expected return on the PE investment. We begin with a one factor model (bonds only) before presenting our main four factor model (incorporating bonds, stocks, REITs, and infrastructure markets). Our approach estimates the risk exposure of PE funds to the factor payoffs at different horizons and vintages. Our estimates reflect the systematic risk of each PE cash-flow over the life of the investment. We break out this expected return into its various horizon components, and, at each horizon, into its exposures to the various traded risk factors. We also examine the expected return of PE investments over horizon and vintages. The approach helps us understand how the expected return on PE investments changes with the state of the economy. With expected returns, variance, and covariances with traded securities in hand, the approach opens the door to optimal portfolio analysis with alternative investments.

The second objective of the analysis is to do performance analysis on PE funds. We isolate the component of PE cash-flows that is not due to factor exposure. The discounted sum of this idiosyncratic cash-flow component over the life of the investment reflects the asset selection skill of the fund manager. We label it the risk-adjusted profit. Under the null hypothesis that the asset pricing model is correct and that the PE manager has no asset selection skill, the risk-adjusted profit is zero.

One of our key findings is that the risk-adjusted profit from PE investments is centered around zero, but with a large cross-sectional variation. Using either a one or four-factor model, we find some evidence for the profitability of traditional Buyout funds, though the magnitude of the average profitability is low. Debt Funds and Restructuring funds also
have positive (though very small) risk-adjusted profits; we find little evidence for positive excess returns in other categories such as Venture Capital, Infrastructure, and Real Estate. The intuition for this result is that PE cash-flows are well-spanned by public markets, and so their cash-flows are well-replicated ex-post through an estimated synthetic portfolio. However, we also find substantial cross-sectional variation in the profitability of funds. In the time-series, we generally find that more recent vintages perform worse than those in the 1990s. Despite the high apparent profitability of recent funds, our approach would suggest that these funds deliver little excess profit given their factor exposure and the performance of those publicly traded factors.

While performance evaluation in private equity is still often expressed as an internal rate of return or a ratio of distributions to capital committed, several important papers have incorporated risk into the analysis. The public market equivalent (PME) approach of Kaplan and Schoar (2005) compares the private equity investment to an appropriate public market benchmark with the same magnitude and timing of cash-flows. Sorensen and Jagannathan (2015) assess the PME approach from a SDF perspective. Korteweg and Nagel (2016) propose a generalized PME approach that relaxes the assumption that the beta of PE funds to the market is one. This is particularly important in their application to venture capital funds. These approaches avoid making strong assumptions on the return-generating process of the PE fund, because they work directly with the cash-flows. Our approach contributes to this strand of the literature.

Several other papers have estimated beta exposures of PE funds with respect to the stock market, particularly for categories of buyout and venture capital. These include Gompers and Lerner (1997); Ewens, Jones, and Rhodes-Kropf (2013); Peng (2001); Woodward (2009). This literature has generally estimated stock market exposures of buyout funds above one, and even higher estimates for Venture Capital funds. Our work contributes to this literature estimating the risk exposure of PE funds by allowing for a flexible estimation approach across horizon and vintage; and in estimating fund exposures to a more expansive basket of publicly listed securities. We also allow for our risk exposure estimates to differ by category, and examine a broader set of PE categories than typically examined in this literature. Finally, we connect the systematic risk exposures of funds to a rich asset pricing model, which allows us to estimate risk-adjusted profits and time-varying expected returns.

\textsuperscript{2}See for example Cochrane (2005) and Korteweg and Sorensen (2010). Much of the literature assumes linear beta-pricing relationships, e.g., Ljungqvist and Richardson (2003), Driessen, Lin, and Phalippou (2012).
Like Korteweg and Nagel (2016), we estimate a stochastic discount factor (SDF) from public securities. Our SDF contains additional risk factors, but more importantly, richer risk price dynamics. Those dynamics are central for generating realistic risk premia on bond and stock strips, which are the building blocks of our PE valuation method. The SDF model extends earlier work by Lustig, Van Nieuwerburgh, and Verdelhan (2013) who value a claim to aggregate consumption to help guide the construction of consumption-based asset pricing models. The asset pricing model combines a vector auto-regression model for the state variables as in Campbell (1991, 1993, 1996) with a no-arbitrage model for the (SDF) as in Duffie and Kan (1996); Dai and Singleton (2000); Ang and Piazzesi (2003). The SDF model needs to encompass the sources of aggregate risk that the investor has access to in public securities markets and that PE funds are exposed to. The question of performance evaluation then becomes whether, at the margin, PE funds add value to a portfolio that already contains these traded assets.

In complementary work, Ang, Chen, Goetzmann, and Phalippou (2017) filter a time series of realized private equity returns using Bayesian methods. They then decompose that time series into a systematic component, which reflects compensation for factor risk exposure, and an idiosyncratic component (alpha). While our approach does not recover a time series of realized private equity returns, it does recover a time series of expected private equity returns. At each point in time, the asset pricing model can be used to revalue the replicating portfolio for the PE fund. Since it does not require a difficult Bayesian estimation step, our approach is more flexible in terms of number of factors as well as the factor risk premium dynamics.\(^3\)

We use data from Preqin on all private equity funds with non-missing cash-flow information that were started between 1990 and 2009. Cash-flow data until September 2017 are used in the analysis. Our sample includes 2,065 funds in seven investment categories. The largest categories are Private Equity and Venture Capital. We are particularly interested in the categories Real Estate and Infrastructure, but report results for all categories. Our results for VC funds have implications for the returns on entrepreneurial activity (Moskowitz and Vissing-Jorgensen, 2002). The main text reports results for these four categories and relegates the results for the other three to the appendix. Like in other papers in the literature, the PE data (here, taken from Preqin) are usually subject to some degree of selection bias.

Throughout, we emphasize results for four large investment categories: Buyout funds,

\(^3\)Other important methodological contributions to the valuation of private equity include Driessen, Lin, and Phalippou (2012), Sorensen, Wang, and Yang (2014), and Metrick and Yasuda (2010).
Venture Capital Funds, Real Estate funds, and Infrastructure funds. This work contributes to a large literature on performance evaluation in private equity funds, such as Kaplan and Schoar (2005), Cochrane (2005), Korteweg and Sorensen (2017), Harris, Jenkinson, and Kaplan (2014), Phalippou and Gottschalg (2009), Robinson and Sensoy (2011), and many other papers cited above. Most of this literature focuses on Buyout and Venture Capital funds, though recent work in valuing privately-held real estate assets includes Peng (2016) and Sagi (2017). Ammar and Eling (2015) have studied infrastructure investments. This literature has found mixed results regarding PE fund excess returns and persistence of performance, depending on the dataset and period in question. Relative to this literature, we emphasize that private funds risk exposure is well-spanned across a broad universe of tradable assets. Our replicating portfolio approach therefore results a relatively low estimate of risk-adjusted profits for PE funds (for several categories, negative) compared with the literature.

The rest of the paper is organized as follows. Section 2 describes our methodology. Section 3 sets up and solves the asset pricing model. Section 4 presents the main results on the risk-adjusted profits and expected returns of PE funds. Section 5 concludes. The appendix discusses several extensions.

2 Methodology

Private equity investments are finite-horizon strategies, typically around fifteen years in duration. In the initial years, the fund manager (GP) calls in the capital the investors (LPs) have committed and starts deploying that capital by investing in projects (for example, an office building). There may be intermediate cash-flows resulting from the operations of the assets (for example, the net operating income from the office building). Towards the end of the life of the fund, assets are sold by the GP and the proceeds (after fees) distributed to the investors. The cash-flow profile thus has an initial period of negative cash-flows, an intermediate period of small positive cash-flows, and a final period of large positive cash-flows. These cash-flows are risky, and understanding (and pricing) the nature of the risk in these cash-flows is the key question in this paper. Denote this sequence of net-of-fees cash-flows for fund $i$ by $\{X^i_{t+h}\}_{h=0}^H$. Time $t$ is the inception date of the fund (vintage year), while $h$ (for horizon) indicates the number of years since inception. The maximum horizon $H$ is the last year of the fund. Time is in years. Monthly cash-flows are aggregated at annual frequency.
2.1 Two-Step Approach

In a first step, we find the replicating portfolio for each of the PE cash-flows. We treat distributions to the investor separately from capital calls/takedowns. Specifically, we treat capital calls as short bond positions since the investor commits that money regardless of the outcome. Distributions, in contrast, are risky payouts. For the distribution, we build a replicating portfolio of risk-free and risky securities. All PE cash-flow data are reported for a $1 investor commitment.

Let the systematic component of cash-flow \( X_{t+h} \) be denoted by \( \beta^i_{t+h} Y_{t+h} \), where the dimension of \( \beta^i_{t+h} \) is \( 1 \times K \) and the dimension of \( Y_{t+h} \) is \( K \times 1 \). The first element of \( Y_{t+h} \) is a constant equal to 1. This is the cash-flow on a zero-coupon U.S. Treasury bond that pays off $1 at time \( t+h \). All other elements of \( Y_{t+h} \) denote risky cash-flow realizations at time \( t+h \). They are the payoffs on “zero coupon equity” or “dividend strips” (Lettau and Wachter, 2011; van Binsbergen, Brandt, and Koijen, 2012). They pay one (risky) cash-flow at time \( t+h \) and nothing at any other date. We scale the risky dividend at \( t+h \) by the cash-flow at time \( t \). For example, a risky cash-flow of \( Y_{t+h} = 1.05 \) implies that there was a 5% realized cash-flow growth rate between periods \( t \) and \( t+h \). This scaling gives the strips a “face value” around 1, just like the zero coupon bond, and makes the units of the various elements of \( \beta \) comparable in magnitude. Consider the projection of PE cash-flow at time \( t+h \) on the cash-flows of the risk-free and risky strips:

\[
X^i_{t+h} = \beta^i_{t+h} Y_{t+h} + e^i_{t+h}.
\]  

(1)

where \( e \) denotes the idiosyncratic cash-flow component, orthogonal to \( Y_{t+h} \). The vector \( \beta^i_{t+h} \) describes how many units of each strip are in the replicating portfolio for the fund cash-flows. Denote the distribution payoffs, betas and idiosyncratic cash-flows with a “+” superscript and the corresponding capital takedowns with a “-” superscript. When both types of cash-flows are defined as positive numbers, \( \beta^i_{t+h} = \beta^{i,+}_{t+h} - \beta^{i,-}_{t+h} \) where the b superscript refers to the bond position. The idiosyncratic payoff is \( e^i_{t+h} = e^{i,+}_{t+h} - e^{i,-}_{t+h} \). We estimate equation (1) separately for distribution and capital takedown payoffs, including only the zero-coupon bond payoff in \( Y \) in the latter case.

In the second step, we use our asset pricing model, spelled out in the next section, to price the zero coupon bond and equity strips. Denote the \( K \) price vector for horizon \( h \) by \( P_{t,h} \). The first element of this price vector is the price of the zero coupon bond which we denote by \( P^S_1 (h) \). Let the one-period stochastic discount factor (SDF) be \( M_{t+1} \), then the
h-period SDF is:

\[ M_{t+h}^h = \prod_{k=0}^{h} M_{t+k}. \]

The first place where we use the asset pricing model is in making sure that the replicating portfolio for the PE fund is budget feasible. Since one dollar is available to buy a portfolio of strips, exactly one dollar must be spent. The replicating portfolio positions \( \beta_{t+h}^i \), estimated from equation (1), need not satisfy this requirement. Therefore, we define a vector of scaled long portfolio positions \( q_{t+h}^i \) that costs exactly one dollar to buy

\[ q_{t+h}^i = \frac{\beta_{t+h}^i}{\sum_{h=0}^{H} \beta_{t+h}^i P_{t+h}} \Rightarrow \sum_{h=0}^{H} q_{t+h}^i P_{t+h} = \$1. \]

In other words, each element in the \( KH \times 1 \) vector \( q_{t+h}^i P_{t+h} \) is a portfolio weight, and these portfolio weights sum to 1. We also rescale the capital takedowns, which we treat as short positions in the bond, to make sure that exactly $1 is deployed:

\[ q_{t+h}^{i,b} = \frac{\beta_{t+h}^{i,b}}{\sum_{h=0}^{H} \beta_{t+h}^{i,b} P_{t+h}^S(h)} \Rightarrow \sum_{h=0}^{H} q_{t+h}^{i,b} P_{t+h}^S(h) = \$1. \]

The replicating portfolio for the capital takedowns has a zero position in the risky strips, so that \( q_{t+h}^i = q_{t+h}^{i,+} \) for those. For the bond strips, we have \( q_{t+h}^{i,b} = q_{t+h}^{i,b,+} - q_{t+h}^{i,b,-} \). With the budget feasible replicating portfolio in hand, we redefine the idiosyncratic component of fund cash-flows as \( v^i \):

\[ v_{t+h}^i = X_{t+h}^i - q_{t+h}^{i} Y_{t+h}. \]

Since the strip prices change over time, each vintage has its own rescaling.

Under the null of the asset pricing model, the expected present discounted value of fund cash-flows (including the capital takedowns) is zero:

\[ \mathbb{E}_t \left[ \sum_{h=0}^{H} M_{t+h}^h X_{t+h}^i \right] = \mathbb{E}_t \left[ \sum_{h=0}^{H} M_{t+h}^h q_{t+h}^{i} Y_{t+h} \right] = \sum_{h=0}^{H} q_{t+h}^{i} P_{t+h} = 0, \tag{2} \]

where the first equality follows from the fact that the idiosyncratic cash-flow component is uncorrelated with the SDF since all priced cash-flow shocks are included in the vector \( Y \).

\(^4\)For example, a PE fund may only call $0.50 of the $1 that was committed by the investors, and our rescaling will put that fund on the same footing with the funds that call the entire $1.
The second place where we use the asset pricing model is to calculate the expected return of the PE investment over the life of the investment. It is the expected return on the replicating portfolio of strips:

$$\mathbb{E}_t \left[ R^i \right] = \sum_{h=0}^{H} w_{t,h}^i \mathbb{E}_t \left[ R_{t,h} \right]$$  \hspace{1cm} (3)

where $w_{t,h}^i = q_{t,h}^i P_{t,h}^i$ are the portfolio weights in the replicating portfolio. Since the expected return on the PE fund is in excess of the capital takedowns, it is best understood as an expected excess return, or risk premium. The expression decomposes the risk premium into compensation for exposure to the various risk factors, horizon by horizon. The model delivers the strip risk premia at each date and horizon.

We are interested in performance evaluation of PE funds. We would like to quantify the investor’s profit on a particular PE investment taking into account its riskiness. This ex-post realized profit is the second main object of interest. Under the maintained assumption that all the relevant sources of systematic risk are captured by the replicating portfolio, the PE cash-flows consist of one component that reflects compensation for risk and a risk-adjusted profit equal to the discounted value of the idiosyncratic cash-flow component:

$$profit^i = \sum_{h=0}^{T} P_{t,h}^s v_{t+h}^i$$  \hspace{1cm} (4)

Since the idiosyncratic cash-flow components are orthogonal to the priced cash-flow shocks, they are to be discounted at the risk-free interest rate. The null hypothesis is that $\mathbb{E}[profit^i] = 0$. A fund with strong asset selection skills should have a positive risk-adjusted profit.

The fund’s horizon is endogenous because it is correlated with the success of the fund. As noted by Korteweg and Nagel (2016), this endogeneity does not pose a problem as long as cash-flows are observed. “Even if there is an endogenous state-dependence among cash-flows, the appropriate valuation of a payoff in a certain state is still the product of the state’s probability and the SDF in that state.” When calculating our profit measure, we exclude vintages after 2010, for which we are still missing a substantial fraction of the cash-flows.

We first discuss the relationship of our approach to other well-known approaches when valuing PE cash-flows. The rest of this section discusses implementation issues.
2.2 Connection to GPME and PME

Korteweg and Nagel (2016) define their realized GPME measure for fund $i$ as:

$$GPME^i = \sum_{h=0}^{H} M_{t+h}^h X_{t+h}^i$$

$$= \sum_{h=0}^{H} M_{t+h}^h \left\{ q_{t+h}^i Y_{t+h} + \varphi_{t+h}^i \right\}$$

$$= \text{profit}^i + \sum_{h=0}^{H} M_{t+h}^h \left\{ q_{t+h}^i (Y_{t+h} - \mathbb{E}_t[Y_{t+h}]) \right\}$$

(5)

(6)

If the SDF model is correct, $\mathbb{E}_t[GPME^i] = 0$. The difficulty with computing (5) is that it contains the realized SDF which is highly volatile. In KN’s implementation, $(\Pi_{k=0}^{h} M_{t+k}^k) = \exp(0.088h - 2.65 \sum_{k=0}^{h} r_{t+k}^m)$. If $h=10$, and the realized market return over the 10 year period is 100%, the realized SDF is 0.17. If the stock return is 30%, the SDF is 1.08. Using our SDF, which is more volatile than one considered in KN, this approach leads to unrealistically low PE valuations, on average. Our risk-adjusted profit avoids using the realized SDF and instead relies on strip prices, which are expectations of SDFs multiplied by cash-flows.

A second, related difference between the two approaches is that the realized GPME can be high (low) because the factor payoffs $Y_{t+h}$ are unexpectedly high (low); see equation (6). Our risk-adjusted profit measure takes out this unexpected systematic cash-flow component. It removes a “market timing” component of performance that is due to taking risk factor exposure. Our measure focuses on performance attributable to “asset selection,” usually referred to as “stock picking” in the mutual fund literature. A simple PME approach that invests a dollar in, say, the stock market and defines the outperformance of the PE investment as the difference between the discounted value of the PE payoffs and the stock market payoffs with the same payoff timing similarly subtracts out this market timing component.

Third, the main advantage of our approach relative to both PME and GPME approaches is that it accommodates heterogeneity in systematic risk across PE funds, since different funds are allowed to have different replicating portfolios. In the standard PME approach, the market beta of each fund is trivially the same and equal to 1. In the GPME approach, PE funds are allowed to have a market beta that differs from 1, but the beta is the same across funds.
Appendix B.3 provides more detail on the KN approach and more discussion on the points of differentiation.

2.3 Identifying and Estimating Cash-Flow Betas

The replicating portfolio must be rich enough that it spans all priced (aggregate) sources of risk, yet it must be parsimonious enough that its exposures can be estimated with sufficient precision. Allowing every fund in every category and vintage to have its own unrestricted cash-flow beta profile leads to parameter proliferation, and lack of identification. We impose cross-equation restrictions to aid identification.

**One-factor Model** We start with a simple model in which all private equity cash-flows are assumed to be bond-like. The only risk that is priced is interest rate risk. We refer to this as the one-factor model.

The empirical model assumes that the cash-flows \( X \) of all funds \( i \) in the same category \( c \) and vintage \( t \) have the same cash-flow betas at each horizon \( h \):

\[
X_{i}^{c,t+h} = q_{i}^{c,t} + v_{i}^{c,t,h} = a_{i}^{c} b_{h}^{c} + v_{i}^{c,t,h}. \tag{7}
\]

The bond position \( q_{i}^{c,t} \) is the product of a vintage effect \( a_{i}^{c} \) and a horizon effect \( b_{h}^{c} \). We estimate (7) using a random effects model with vintage times horizon fixed effects, category by category. The estimation equally weights all funds. Identification is achieved both from the cross-section and from the time series. The vintage effects are normalized to be 1 on average across vintages. The vintage effects shift the b-profiles up and down in parallel fashion. They allow for some vintages to have higher average cash-flows (more bond exposure) than other vintages.

With the cash-flow betas in hand, we can calculate the expected return on the PE investment as a weighted average of the expected returns on the bond strips per equation (3). We can also revalue the PE investment at any point in the life of the investment by applying the appropriate zero coupon bond prices at that time. Finally, we can calculate the risk-adjusted profit per equation (4), using the residuals from equation (7).

**Four-factor Model** Our main model is a four-factor model where we add to the bond factor a stock market factor, a traded real estate factor, and a traded infrastructure factor. This is the model we solve in section 3.
The empirical model assumes that the cash-flows $X$ of all funds $i$ in the same category $c$ and vintage $t$ have the same vector of cash-flow betas at each horizon $h$. The betas on the bond, stock, real estate, and infrastructure factors are allowed to be different, and to shift in different ways across vintages:

$$X_{t+h}^{i \in c} = \alpha_{t+h}^b + \alpha_{t+h}^m Y_{t+h}^m + \alpha_{t+h}^{reit} Y_{t+h}^{reit} + \alpha_{t+h}^{infra} Y_{t+h}^{infra} + \nu_{t+h}^{i \in c}$$

We estimate the vintage effects $(a_1, a_2, a_3, a_4)$ and the horizon profiles $(b_1, b_2, b_3, b_4)$ using a random effects model which contains vintage times horizon times cash-flows ($Y$) as dependent variables, category by category. As explained above, the horizon profiles were rescaled to be budget feasible and the vintage effects average to 1. One can test the null hypothesis that the vintage effects are the same across risky assets: $H_0 : a_{1,t}^c = a_{2,t}^c = a_{3,t}^c = a_{4,t}^c$. If the null cannot be rejected, imposing this condition may lead to efficiency gains in estimation.

One may be able to further enrich the model by allowing additional heterogeneity. One may want to split an investment category into additional subcategories. For example, real estate strategies are often subdivided into opportunistic, value-add, core plus, and core funds. Infrastructure could be divided into greenfield and brownfield, etc. One may be able to estimate a firm shifter as firms often have several funds in the data set.

### 3 Asset Pricing Model

The second main challenge is to price the replicating portfolio. If the only sources of risk were the risks inherent in the term structure of interest rates, this step would be straightforward. After all, on each date $t$, we can easily infer the prices of zero-coupon bonds of all maturities $j$ from the yield curve at time $t$. We would then multiply the zero coupon bond price at time $t$ for a bond of maturity $j$ with the private equity cash-flow at time $t+j$, and sum across $j$. However, interest rate risk is not the only and (arguably) not even the main source of risk in the cash-flows of private equity funds. If overall stock market risk were the only other source of aggregate risk, then we could use price information from dividend strips. Those prices can either be inferred from options and stock markets (van Binsbergen, Brandt, and Koijen, 2012) or observed directly from dividend strip futures markets (van Binsbergen, Hueskes, Koijen, and Vrugt, 2013). However, the
available time series is too short for our purposes. Moreover, there are no dividend strip data for publicly listed real estate or infrastructure assets. These are important additional traded factors we wish to include in our analysis, given our special interest in real estate and infrastructure funds. Finally, we do not observe expected returns on those strips, and would need to obtain those from the model anyway. For those reasons, we rely on a more standard asset pricing model instead to obtain the time series of $P_{t,t+j}$.

The asset pricing model must correctly price the assets whose cash-flows $Y$ are used in the cash-flow replication. This requires a rich enough model. We propose to use a reduced-form asset pricing model rather than a structural model that starts from preferences, since it is more important that we price the replicating portfolio of publicly traded assets correctly than that we understand the deeper sources of macro-economic risk that underly the prices of stocks and bonds. Our approach builds on Lustig et al. (2013), who price a claim to aggregate consumption and study the properties of the price-dividend ratio of this claim, the wealth-consumption ratio.

As emphasized by Korteweg and Nagel (2016), the objective is not to test the asset pricing model, but rather to investigate whether a potential PE investment adds value to an investor who already has access to securities whose sources of risk are captured by the SDF.

### 3.1 Setup

#### 3.1.1 State Variable Dynamics

We assume that the $N \times 1$ vector of state variables follows a Gaussian first-order VAR:

$$z_t = \Psi z_{t-1} + \Sigma^1 \varepsilon_t,$$

with shocks $\varepsilon_t \sim i.i.d. \mathcal{N}(0, I)$ whose variance is the identity matrix. The companion matrix $\Psi$ is a $N \times N$ matrix. The vector $z$ is demeaned. The covariance matrix of the innovations to the state variables is $\Sigma$; the model is homoscedastic. We use a Cholesky decomposition of the covariance matrix, $\Sigma = \Sigma^1 \Sigma^1'$, which has non-zero elements only on and below the diagonal. The Cholesky decomposition of the residual covariance matrix allows us to interpret the shock to each state variable as the shock that is orthogonal to the shocks of all state variables that precede it in the VAR. We discuss the elements of the state vector, and their ordering, in detail below.

For now, we note that the (demeaned) one-month bond nominal yield is one of the
elements of the state vector: \( y_t^s(1) = y_0^s(1) + e_{yn}^t z_t \), where \( y_0^s(1) \) is the unconditional average yield and \( e_{yn} \) is a vector that selects the element of the state vector corresponding to the one-month yield. Similarly, the (demeaned) inflation rate is part of the state vector: \( \pi_t = \pi_0 + e_{\pi}^t z_t \) is the (log) inflation rate between \( t - 1 \) and \( t \). Lowercase letters denote logs.

### 3.1.2 SDF

We specify an exponentially affine stochastic discount factor (SDF), similar in spirit to the no-arbitrage term structure literature (Ang and Piazzesi, 2003). The nominal SDF \( M_{t+1}^s = \exp(m_{t+1}^s) \) is conditionally log-normal:

\[
\begin{align*}
m_{t+1}^s &= -y_t^s(1) - \frac{1}{2} \Lambda_t' \Lambda_t - \Lambda_t' \varepsilon_{t+1},
\end{align*}
\]

The real SDF is \( M_{t+1} = \exp(m_{t+1}) = \exp(m_{t+1}^s + \pi_{t+1}) \); it is also conditionally Gaussian. The innovations in the vector \( \varepsilon_{t+1} \) are associated with a \( N \times 1 \) market price of risk vector \( \Lambda_t \) of the affine form:

\[
\Lambda_t = \Lambda_0 + \Lambda_1 z_t,
\]

The \( N \times 1 \) vector \( \Lambda_0 \) collects the average prices of risk while the \( N \times N \) matrix \( \Lambda_1 \) governs the time variation in risk premia. We specify the restrictions on the market price of risk vector below. Asset pricing in this model amounts to estimating the market prices of risk in \( \Lambda_0 \) and \( \Lambda_1 \).

Proposition 1 in the appendix shows that nominal bond yields of maturity \( \tau \) are affine in the state variables:

\[
y_t^s(\tau) = \frac{-A^s(\tau)}{\tau} + \frac{-B^s(\tau)}{\tau} z_t.
\]

The scalar \( A^s(\tau) \) and the vector \( B^s(\tau) \) follow ordinary difference equations that depend on the properties of the state vector and of the market prices of risk. It is easy to calculate the real interest rate, real bond risk premia, and inflation risk premia on bonds of various maturities.

The present-value relationship says that the price of a stock equals the present-discounted value of its future cash-flows. By value-additivity, the price of the stock, \( P_t^m \), is the sum of the prices to each of its future cash-flows. These future cash-flow claims are the so-called dividend strips or zero-coupon equity (Wachter, 2005). Dividing by the current dividend
where \( P_d^t(\tau) \) denotes the price of a \( \tau \)-period dividend strip divided by the current dividend. Appendix A proves that the log price-dividend ratio on dividend strips are affine in the state vector and shows how to compute the coefficients recursively. If dividend growth were unpredictable and its innovations carried a zero risk price, then dividend strips would be priced like real zero-coupon bonds. The dividend strips’ dividend-price ratios would equal yields on real bonds (with the coupon adjusted for dividend growth \( \mu_d \)). In this special case, all variation in the price-dividend ratio would reflect variation in the real yield curve. In fact, the dynamics of real bond yields only account for a small fraction of the variation in the price-dividend ratio, implying large prices of risk associated with shocks to stock returns that are orthogonal to shocks to bond yields. Since the log price-dividend ratio on the stock market is an element of the state vector, it is affine in the state vector by assumption. Equation (12) shows the present-value relationship. It imposes a restriction on the coefficients \( A^m(\tau) \) and \( B^{mt}(\tau) \) which we impose in the estimation. An analogous restriction holds and is imposed for traded real estate and infrastructure returns.

3.2 Estimation

3.2.1 State Vector Elements

Conceptually, we want to include in the state vector prices or returns on enough traded asset so that we span all sources of aggregate risk that are relevant for the pricing of private equity investments. Given our interest in real estate funds, infrastructure funds, energy funds, and corporate private equity funds, our benchmark state vector is:

\[
z_t = [CP_t, y_t^S(1), \pi_t, y_t^S(60) - y_t^S(1), p_d^m, \Delta d_t^m, p_d^{reit}, \Delta d_t^{reit}, p_d^{infra}, \Delta d_t^{infra}].
\]

The first four elements represent the bond market variables in the state, the next six represent the stock market variables. The state contains in order of appearance: the
Cochrane and Piazzesi (2005) factor (CP), the nominal short rate (yield on a one-month Treasury bill), realized inflation, the spread between the yield on a five-year Treasury note and a one-month Treasury bill, the log price-dividend ratio on the CRSP stock market, the log real dividend growth rate on the CRSP stock market, the log price-dividend ratio on the NAREIT All Equity REIT index of publicly listed real estate companies, the corresponding log real dividend growth rate on REITs, the log price-dividend ratio on a listed infrastructure index, and the corresponding log real dividend growth rate of infrastructure stocks. This state vector is observed at monthly frequency from 1990.01 until 2016.12 (324 observations), the period for which we have information on private equity funds. Monthly dividends are adjusted to remove the seasonal component. The VAR is estimated by OLS in the first stage of the estimation.

3.2.2 Market Prices of Risk

The state vector contains traded asset prices/returns, with the exception of the CP factor, the inflation rate, and the three price-dividend ratios. The prices of risk associated with the first, third, fifth, seventh, and ninth elements (rows) of $\Lambda_0$ ($\Lambda_1$) are correspondingly set to zero. We include the CP factor because it is a well-known predictor of bond returns, helping to pin down the dynamics of bond risk premia. The price-dividend ratios are well known predictors of the corresponding equity returns, helping to pin down equity risk premia on the overall stock market, real estate, and infrastructure. Moreover, the price-dividend ratios are useful to impose the present-value relationship. Inflation needs to be included to be able to go between real and nominal SDFs.

The five non-zero elements of $\Lambda_0$ pin down the average level of the short-rate, the average slope of the yield curve, and the unconditional risk premium on stocks, real estate, and infrastructure. The five non-zero rows of $\Lambda_1$ are chosen to match the dynamics of the short rate, the dynamics of the slope of the yield curve, and the conditional risk premia on bonds, the stock market, REITs, and infrastructure stocks. We impose some zero restrictions on the companion matrix $\Psi$ which imply zero restrictions on elements of $\Lambda_1$. We refer the reader to the appendix for the details. There are 32 non-zero elements of $\Lambda_1$ to be estimated, for a total of 37 market price of risk parameters.

We use the following moments to estimate the market price of risk parameters. First, we match the time-series of nominal bond yields for maturities of one month, three months, one year, three years, five years, ten years, twenty years, and thirty years. Second, we match the time-series of log price-dividend ratios on stocks, real estate, and infrastruc-
ture. The model-implied price-dividend ratios are built up from 4,800 monthly dividend strips according to equation (11). Thus, we impose the present-value relationship for all three stock prices. Third, we impose that the time-series of the CP factor in the model, constructed in identical fashion to its empirical counterpart, match that counterpart. This gives \((8 + 3 + 1) \times T = 12 \times 324\) moments. Fourth, we impose that the risk premium in the model matches that in the VAR, both its unconditional average and its dependence on the state variables, for stocks, REITs, and infrastructure. This provides 29 additional restrictions. Fifth, we impose good deal bounds on the standard deviation of the log SDF in the spirit of Cochrane and Saa-Requejo (2000). In total, we have 354 moments to estimate 37 parameters. Thus, the estimation is (massively) over-identified.

### 3.2.3 Model Fit

Figure 1 plots the bond yields on bonds of maturities 1 month, 3 months, 1 year, 3 years, 5 years, and 10 years. Those are the most relevant horizons for the private equity cash-flows. The model matches the time series of bond yields in the data closely.\(^5\)

The top panels of Figure 2 show the model’s implications for the average nominal (left panel) and real (right panel) yield curves at longer maturities. These yields are well behaved. The bottom left panel shows that the model matches the dynamics of the nominal bond risk premium, here the expected excess return on five-year nominal bonds, quite well. The bottom right panel shows a decomposition of the nominal bond yield on a five-year bond into the five-year real bond yield, annual expected inflation inflation over the next five years, and the five-year inflation risk premium. On average, the 3.9% nominal bond yield is comprised of a 1.2% real yield, a 2.4% expected inflation rate, and a 0.3% inflation risk premium. The graph shows that the importance of these components fluctuates over time. The inflation risk premium has been shrinking over time, consistent with the findings in the term structure literature.

Figure 3 shows the equity risk premium, the expected excess return, in the left panels and the price-dividend ratio in the right panels. The top row is for the overall stock market, the middle row for REITs, and the bottom row for infrastructure stocks. The dynamics of the risk premia in the data are dictated by the VAR. The model chooses the market prices of risk to fit these risk premium dynamics as closely as possible.\(^6\) The price-

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\(^5\)The model also closely matches yields on bonds of maturities 2-, 4-, 7-, 20-, and 30-years. These yields are not shown to save space.

\(^6\)The monthly risk premia are annualized by multiplying them by 12 for presentational purposes only. We note that the VAR does not restrict risk premia to turn negative. We observe a few periods where risk
Figure 1: Dynamics of the Nominal Term Structure of Interest Rates

The figure plots the observed and model-implied 1-, 3-, 12-, 36-, 60-, and 120-month nominal bond yields.

dividend ratios in the model are formed from the price-dividend ratios on the strips of maturities ranging from 1 month to 4800 months, as explained above. The figure shows a tight fit for both risk premia and equity price levels.

3.3 Temporal Pricing of Risk

Zero-Coupon Bond and Zero-Coupon Equity Prices The first key output from the model, and input in the private equity valuation exercise, is a nominal bond price for zero-coupon bonds with maturities ranging from one to approximately 120 months. The second key output from the model, and input in the private equity valuation exercise, is the price for dividend strips with maturities ranging from one to 120 months. We scale this premia are as low as -1% per month. The price-dividend ratio is also annualized by multiplying the current month’s dividend by 12. In the data, monthly dividends are seasonally adjusted.
Figure 2: Long-term Yields and Bond Risk Premia

The top panels plot the average bond yield on nominal (left panel) and real (right panel) bonds for maturities ranging from 1 month to 1200 months (100 years). The bottom left panel plots the nominal bond risk premium in model and data. The bottom right panel decomposes the model’s five-year nominal bond yield into the five-year real bond yield, the five-year inflation risk premium and the five-year real risk premium.

price by the current dividend. Figure 4 plots the time series for prices of nominal zero-coupon bonds and dividend strips on the overall stock market, REITs, and infrastructure stocks. We plot three maturities: one-month, five-years, and ten-years. We use these prices to value the replicating portfolio private equity cash-flows in our main valuation equation (3).

The model generates rich patterns in the temporal pricing of risk. Figure 5 plots the average risk premium on nominal zero coupon bond yields (top left panel) and on dividend strips (other three panels) of various maturities ranging from 1 month until 180 months. Those are the relevant maturities for the private equity investments. Risk premia on nominal bonds are increasing from 0 to 5% in the first panel. The second panel shows the risk premia on dividend strips on the overall stock market. The strip risk premium curve is downward sloping in maturity, consistent with the empirical findings of van Binsbergen, Brandt, and Koijen (2012), van Binsbergen, Hueskes, Koijen, and Vrugt (2013), and van Binsbergen and Koijen (2017). Short-maturity strips have risk premia that are al-
most 3% points higher than long-maturity stock market dividend strips. Interestingly, the average term structure of dividend strip risk premia for REITs is steeply upward sloping (bottom left panel), with risk premia growing from below 5% to almost 9% per annum. The risk premium curve on public infrastructure assets is hump-shaped at short horizons and fairly flat thereafter with risk premia of only around 4.5% per annum (bottom right panel). The difference in the level and the horizon-dependence of the three strip risk premia will generate differences in the risk premia on private equity investments if their cash-flows display differential exposure to the traded asset cash-flows.

**Shock Exposure and Shock-Price Elasticities**  
Borovička and Hansen (2014), building on earlier work by Hansen and Scheinkman (2009), provide a *dynamic value decomposition*, the asset pricing counterpart to an impulse response function. It allows a researcher to
Figure 4: Zero Coupon Bond Prices and Dividend Strip Prices

The figure plots the model-implied prices on zero-coupon Treasury bonds in the first panel, and price-dividend ratios for dividend strips on the overall stock market, REIT market, and infrastructure sector in the next three panels, for maturities of 12 months, 60 months, and 120 months.

decompose the risk premium an investor requires for exposure to a shock into the product of the exposure (shock exposure elasticity) and the risk price (shock price elasticity), horizon by horizon. Appendix C applies their analysis to our VAR setting.

Figure 6 plots the shock-exposure elasticities of dividend growth on the market (blue), dividend growth of REITs (red), and dividend growth of infrastructure stocks (green) to a one-standard deviation shock to the CP factor (top left), to the short rate (top middle), to the slope factor (top right), to the dividend growth rate on the market (bottom left), to the dividend growth rate on REITs (bottom middle), and to the dividend growth rate on infrastructure (bottom right). Since our private equity cash-flows are linear combinations of the dividends to stocks, REITs, and infrastructure, these are the relevant shock exposure elasticities. We notice that cash-flow stocks to the overall market are associated with positive cash-flow responses of infrastructure stocks and vice versa. REIT cash-flows fall when market cash-flows rise especially at longer horizons. A positive shock to the level of interest rates pushes cash-flows of the overall market and infrastructure stocks up, while
Figure 5: Zero Coupon Bond Prices and Dividend Strip Prices

The figure plots the model-implied average risk premia on nominal zero-coupon Treasury bonds in the first panel, and on dividend strips on the overall stock market, REIT market, and infrastructure sector in the next three panels, for maturities ranging from 1 to 180 months.

it pushes the cash-flows of REITs down. The opposite is true for shocks to the slope of the yield curve and CP shocks.

Figure 7 plots the shock-price elasticities to a one-standard deviation shock to the CP factor (top left), to the short rate (top middle), to the slope factor (top right), to the dividend growth rate on the market (bottom left), to the dividend growth rate on REITs (bottom middle), and to the dividend growth rate on infrastructure (bottom right). These shock price elasticities are a property of the (cumulative) stochastic discount factor process. They quantify the implied market compensation for horizon-specific risk exposures. In our case, these risk compensations are extracted from a rich menu of observed asset prices matched by a reduced-form asset pricing model. The price of level risk is negative while that of slope and CP risk is positive, consistent with Kojien, Lustig, and Van Nieuwerburgh (2017). All cash-flow shocks naturally have positive risk prices since increases in cash-flow growth are good shocks to the representative investor. All those shocks’ risk prices are downward sloping in the horizon.

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The figure plots the shock-exposure elasticities of dividend growth on the market, dividend growth of REITs, and dividend growth of infrastructure shocks to a one-standard deviation shock to the CP factor (top left), to the short rate (top middle), to the slope factor (top right), to the dividend growth rate on the market (bottom left), to the dividend growth rate on REITs (bottom middle), and to the dividend growth rate on infrastructure (bottom right).
Figure 7: Shock Price Elasticities

The figure plots the shock-price elasticities to a one-standard deviation shock to the CP factor (top left), to the short rate (top middle), to the slope factor (top right), to the dividend growth rate on the market (bottom left), to the dividend growth rate on REITs (bottom middle), and to the dividend growth rate on infrastructure (bottom right). The shocks whose risk prices are plotted are the VAR innovations $\Sigma^{1/2} \varepsilon$. 
4 Expected Returns and Risk-adjusted Profits on Private Equity Funds

In this section, we combine the cash-flow exposures from section 2 with the asset prices from section 3 to obtain risk-adjusted profits on private equity funds.

4.1 Summary Statistics

Our fund data cover the period January 1990 until September 2017. The data source is Preqin. We group private equity funds into seven categories: Buyout (LBO), Venture Capital (VC), Real Estate (RE), Infrastructure (IN), Fund of Funds (FF), Debt Funds (DF), and Restructuring (RS). Our FF category contains the Preqin categories Fund of Funds, Hybrid Equity, and Secondaries. The Buyout category is commonly referred to as Private Equity, whereas we use the PE label to refer to the combination of all investment categories.

We include all funds with non-missing cash-flow information. We group funds also by their vintage, the year in which they first appear in the data set. The last vintage we consider in the analysis is the 2010 vintage. Table 1 reports the number of funds and the aggregate AUM in each vintage-category pair. In total, we have 2,065 funds in our analysis. There is clear business cycle variation in when funds get started as well as in their size. Buyouts are the largest category, followed by Venture Capital.

Figure 8 shows the average cash-flow profile in each category, pooling all vintages together and equally weighting them. We combine all monthly cash-flows into one annual cash-flow for each fund. Year zero is the calendar year in which the first capital call takes place. The last bar is for year 15. For the purposes of this figure and for cash-flow beta estimation, we include all observed cash-flows until the end of the sample. Thus, the 2010 vintage funds only have cash-flows through year seven, which is 2017. Cash-flows arriving after year 15 are included in the last year under a separately highlighted color. The cash-flows follow a J-pattern with capital take downs in the initial years and distributions in the later years. While a majority of distribution cash-flows occur between years 5 and 10, there are still meaningful cash-flow distributions until years 15 and even beyond. This is especially true for Venture Capital funds.

Figure 9 zooms in on the four investment categories of most interest to us: Buyout, Venture Capital, Real Estate, and Infrastructure. The figure shows the average cash-flow profile for each vintage. Since there are few RE and IN funds prior to 2000, we start the latter two panels with vintage year 2000. The figure shows that there is substantial
Table 1: Summary Statistics

Panel A: Fund Count

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<th>Vintage</th>
<th>Debt Fund</th>
<th>Fund of Funds</th>
<th>Infrastructure</th>
<th>Buyout</th>
<th>Real Estate</th>
<th>Restructuring</th>
<th>Venture Capital</th>
<th>Total</th>
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Panel B: Fund AUM ($m)

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<th>Infrastructure</th>
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<td>983</td>
<td>5,367</td>
<td>5,446</td>
<td>32,855</td>
<td>6,269</td>
<td>2,580</td>
<td>7,596</td>
<td>61,096</td>
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<tr>
<td>2005</td>
<td>4,195</td>
<td>23,651</td>
<td>6,353</td>
<td>101,375</td>
<td>24,074</td>
<td>5,830</td>
<td>12,807</td>
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<td>9,632</td>
<td>36,156</td>
<td>9,726</td>
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<td>40,943</td>
<td>22,928</td>
<td>27,976</td>
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<td>44,287</td>
<td>40,483</td>
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<tr>
<td>2008</td>
<td>9,096</td>
<td>37,478</td>
<td>27,228</td>
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<td>25,036</td>
<td>27,301</td>
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<tr>
<td>2009</td>
<td>1,600</td>
<td>10,931</td>
<td>10,508</td>
<td>41,661</td>
<td>8,876</td>
<td>11,503</td>
<td>6,590</td>
<td>91,669</td>
</tr>
<tr>
<td>2010</td>
<td>6,120</td>
<td>20,624</td>
<td>19,486</td>
<td>30,346</td>
<td>20,778</td>
<td>12,055</td>
<td>16,811</td>
<td>126,220</td>
</tr>
</tbody>
</table>
variation in cash-flows across vintages, even within the same investment category. This variation will allow us to identify vintage effects. Appendix Figures A.5 shows cash-flow profiles for the remaining categories.

The figure also highlights that there is a lot of variation in cash-flows across calendar years. Venture Capital funds in the mid- to late-1990s vintages realized very high average cash-flows in 2000. Since the stock market also had very high payouts in the year 2000, this type of variation will help the model identify a high stock market beta for VC funds. This is an important distinction with other methods, such as the PME, which assume constant risk exposure and so would attribute high cash flow distributions in this period only to excess returns.
Figure 9: Cash-flows by Vintage

Panel A: Buyout

Panel B: Venture Capital

Panel C: Real Estate

Panel D: Infrastructure
4.2 One-factor Model

We start with a discussion of the one-factor model, which assumes that the only risk that is priced is interest rate risk. It is straightforward to estimate and provides a useful point of comparison for our benchmark four-factor model. The estimation chooses parameters to match the average fund cash flows, for each category-vintage pair. Crucially, the resulting positions in bonds of various maturities are then scaled down (or up) to ensure that the replicating portfolio of bonds does not cost too much (too little). The high cash flows of a particular PE vintage may not be achievable/replicable with a budget-feasible bond portfolio, but only with a budget infeasible one. This will result in high errors $v$ and high risk-adjusted profits.

Figure 10 show the estimated horizon effects $\hat{b}_h$ in the left panels and the vintage effects $a_t$ in the right panels. Each row is for one of our four main investment categories. The one-factor model explains 40% of the variation in cash-flows for RE, 33% for LBO, 31% for IN, and 24% for VC. Appendix Figure A.10 contains the same figure for other fund categories. There is substantial variation in the temporal distribution of cash-flows across categories of funds, both for the vintage effects and the horizon effects. The left panels show that VC funds have the highest average cash-flows in years 8 and later, as well as the slowest capital takedown. Buyout fund payoffs peak earliest at 7 years and drop more sharply than for VC as well as real estate and infrastructure funds. The plotted coefficients are the positions that the replicating portfolio holds in bond strips (zero coupon bonds) of the various horizons.

The right panels show how these exposure profiles from the left panel are shifted up or down for each vintage. On average, these vintage effects are 1. Values below one reflect lower bond risk, or lower average cash-flows, while values above 1 reflect vintages with above-average risk. The early 1990s vintages had higher average cash-flows, or higher bond market risk. For VC funds but not for the other categories, we estimate a high vintage effect around the year 2000. Real estate funds started in 2004-07 had average cash-flows in line with the historical average, while the replicating portfolio for the 2008-2010 vintages had lower cash-flows.\(^7\)

Appendix Figure A.7 also reports the cross-sectional standard deviation of the idiosyncratic payoff components $v_{i,c,t+h'} \in c_t+h$, horizon by horizon. It provides a metric of the amount of cash-flow variation of individual funds around the category average, as well as

\(^7\)For these last three vintages, many funds have not yet reached their terminal date. Their vintage effect is thus estimated off the first 9-7 years of cash-flow data.
how that variation depends on the horizon. Since the one-factor model is arguably missing some key risk factors, the one-factor idiosyncratic risk measure also captures factor exposures. Buyout funds have the lowest idiosyncratic risk, around 15% cross-sectional standard deviation, followed by real estate and infrastructure around 20%. VC funds have a dispersion as high as 80%. The dispersion tends to have a hump-shaped pattern in horizon, with idiosyncratic risk peaking at around 5 years.

**Expected Return** The asset pricing model provides the expected return on each zero-coupon bond. With the replicating portfolio of zero-coupon bonds in hand, we can calculate the expected return on PE funds in each investment category as in equation (3). The left panels of Figure 11 plot the total expected return over the life of the investment, broken down into its horizon components, and averaged across all vintages. The pattern of expected returns typically follows the similar “J-curve” of beta exposures and underlying cash flows. We notice that VC and IN funds are more “backloaded” than LBO and RE funds. Panels on the right plot the time-series of the expected return. Variation in the state variables of the VAR drive variation in the expected return of each zero-coupon bond, and therefore in the expected return of the replicating portfolio of the PE fund. These graphs plot annualized (rather than total) expected returns on the average PE fund over time.

**Performance Evaluation** Next, we turn to performance evaluation in the one-factor model. The left panels of Figure 15 plots the histogram of risk-adjusted profits, pooling all vintages. The right panels plot the average risk-adjusted profit for each vintage. Appendix Figure A.12 reports on the remaining investment categories. If the only risk considered is interest rate risk, the average buyout fund has delivered a small positive risk-adjusted profit of 1.1%. The average VC fund has made a small risk-adjusted loss of 0.9%, and similarly for the average real estate -0.5%) and infrastructure fund (-0.3%). More strikingly, there is large variation in profit across funds in the same category. Some funds gain 50 cents or more per dollar of committed capital while others lose 50 cents or more.

The right panels show interesting time series variation in average profits. Risk-adjusted profits are high for LBO and VC funds in the early 1990s. RE and IN funds have the highest profits for vintages in the early 2000s, when there also is a second peak for LBOs. Maybe surprisingly, the VC profits are not unusually high for the 1999-2001 vintages. The high factor exposures for that vintage are sufficient to explain the high average payouts.
of these vintages. Also interesting is that the 2005-07 RE vintages performed poorly. The most recent vintages of PE funds we consider tend to have poor average performance, especially in VC, but also in IN and LBO. The 2010 RE vintage average profit is zero.\textsuperscript{8}

\textsuperscript{8}The profit for these vintages is calculated as the discounted value of the idiosyncratic cash-flow components $v^t$ that are available through the end of the sample. Implicitly, the assumption is that the non-systematic cash-flow component of the average fund in those vintages will be zero in the remaining years for which no cash-flow data are available yet. In other words, the poor performance is not simply due to missing cash-flow information.
Figure 10: Replicating Portfolio Exposure by Feature

Panel A: Buyout

Panel B: Venture Capital

Panel C: Real Estate

Panel D: Infrastructure
Figure 11: Expected Return

Panel A: Buyout

Panel B: Venture Capital

Panel C: Real Estate

Panel D: Infrastructure

Avg Expected Return is: 0.088

Avg Expected Return is: 0.096

Avg Expected Return is: 0.094

Avg Expected Return is: 0.077
4.3 Four-factor Model

Next, we turn to our main results for the four factor model. Figure 13 show the estimated horizon effects $\hat{b}_h$ in the left panels and the vintage effects $\hat{a}_t$ in the right panels. The appendix contains the same plot for all other categories. Our model illustrates a rich estimation of factor risk exposure by horizon. The overall $R^2$ model fit for each category is comparable to the one-factor model, but we estimate exposures specific to each fund category, horizon, and factor. The bond factor exposures feature negative values during the call stage, before steadily rising. We find that the Buyout category loads most heavily on a stock market factor, while the Real Estate fund category loads heavily on a REIT factor. Venture Capital has a strong loading on a REIT factor in addition to the infrastructure factor, paired with a short position in the market factor across horizons—perhaps because the REIT factor has a strong pro-cyclical component as reflected in VC distributions.

Appendix Figure A.7 also shows the cross-sectional standard deviation of idiosyncratic cash-flows by horizon for the four-factor model. The idiosyncratic risk profile in the four-factor model lies below the one for the one-factor model.

Expected Return The asset pricing model provides the expected return on each zero-coupon bond and each stock, reit, and infra dividend strip. With the replicating portfolio in hand, we can calculate the expected return on PE funds in each investment category as in (3). Figure 14 breaks down the expected return by horizon, stacking the contribution from each security exposure. Each panel is for one investment category.9

We find high expected returns for Infrastructure, driven mostly by the long exposure to a market factor. Real Estate and Venture Capital have a strong expected return component driven by REITs, while the dominant component in expected returns for Buyout funds is driven by the market factor at most horizons. The different patterns across horizons illustrates the value of breaking down PE returns strip-by-strip; while the variation across categories illustrates that private funds load very differently on risk exposures depending on industry focus. In the time series, we generally observe falling expected returns over vintages. These differences suggest limitations of existing measures in fund performance evaluation, such as the PME, which do not account for different risk exposures of different funds. Appendix Figure A.12 does the same analysis for the remaining three categories.

9For ease in estimation, we estimate a full four-factor model only for LBO and VC. For the remaining categories, we estimate a three-factor model excluding the real estate factor (Real Estate Funds) or the
Performance Evaluation  Figure 15 plots the histogram of risk-adjusted profits for the four-factor model, pooling all vintages, in the left panels. The right panels plot the average risk-adjusted profit for each vintage. The rows contain LBO, VC, RE, and IN funds. The appendix contains the same figures for the remaining investment categories. Like in the one-factor model, average profits are close to zero with considerable variation around them. Average profits in the later vintages are lower in the four- than in the one-factor model, underscoring the importance of incorporating the traded equity risk factors. Infrastructure funds in particular seem to have lower risk-adjusted returns (average profits of -2% in the four-factor case, relative to -0.3% in the one-factor model). The cross-sectional dispersion of profits is also more tightly concentrated around zero in these estimates, suggesting a tighter fit using the replicating portfolio.

An interesting pattern in our results is that both the one and four-factor estimation series suggest positive profits for the Buyout category, and negative profits for Venture Capital. Our estimates would suggest that apparent high returns in the VC sector primarily reflect high loadings on risk factors, and therefore high expected returns, as opposed to abnormal returns beyond the yield of a replicating portfolio. The 2009 and 2010 vintage real estate funds fare better under the four-factor model than under the one-factor model and show substantial average risk-adjusted profit of 2.5%. Infrastructure funds, in contrast, have delivered negative risk-adjusted profits in the recent vintages.
Figure 12: Risk-Adjusted Profits by Category

Panel A: Buyout

Panel B: Venture Capital

Panel C: Real Estate Funds

Panel D: Infrastructure Funds
Figure 13: Replicating Portfolio Exposure by Feature: Multi-Factor Model

**Panel A: Buyout**

*Factor Exposure by Horizon*

*Factor Exposure by Vintage*

**Panel B: Venture Capital**

*Factor Exposure by Horizon*

*Factor Exposure by Vintage*

**Panel C: Real Estate**

*Factor Exposure by Horizon*

*Factor Exposure by Vintage*

**Panel D: Infrastructure**

*Factor Exposure by Horizon*

*Factor Exposure by Vintage*
Figure 14: Expected Return

**Panel A: Buyout**

Expected Return by Horizon and Risk Exposure

![Graph showing expected return by horizon and risk exposure for Buyout.](image)

Avg Expected Return is: 0.109

Expected Return by Vintage and Risk Exposure

![Graph showing expected return by vintage and risk exposure for Buyout.](image)

**Panel B: Venture Capital**

Expected Return by Horizon and Risk Exposure

![Graph showing expected return by horizon and risk exposure for Venture Capital.](image)

Avg Expected Return is: 0.107

Expected Return by Vintage and Risk Exposure

![Graph showing expected return by vintage and risk exposure for Venture Capital.](image)

**Panel C: Real Estate**

Expected Return by Horizon and Risk Exposure

![Graph showing expected return by horizon and risk exposure for Real Estate.](image)

Avg Expected Return is: 0.083

Expected Return by Vintage and Risk Exposure

![Graph showing expected return by vintage and risk exposure for Real Estate.](image)

**Panel D: Infrastructure**

Expected Return by Horizon and Risk Exposure

![Graph showing expected return by horizon and risk exposure for Infrastructure.](image)

Avg Expected Return is: 0.165

Expected Return by Vintage and Risk Exposure

![Graph showing expected return by vintage and risk exposure for Infrastructure.](image)
Figure 15: Risk-Adjusted Profits by Category: Multi-Factor Model

Panel A: Buyout

Panel B: Venture Capital

Panel C: Real Estate Funds

Panel D: Infrastructure Funds
5 Conclusion

We provide a novel valuation method for private equity cash-flows that decomposes a private equity cash-flow at each horizon into a systematic component that reflects exposure to traded securities paying a cash-flow at that same date (strips). A state-of-the-art no-arbitrage asset pricing model estimates prices and expected returns for these strips, fitting the time series bond yields and stock prices closely. The method delivers a budget-feasible replicating portfolio of strips, whose expected return equals the expected return on the PE fund. Taking out the systematic component of the PE cash-flow delivers a risk-adjusted profit measure, capturing the asset selection skill of the fund GP.

Our method extends existing fund evaluation techniques by considering more closely the factor risk exposures of arbitrary cash flows at different horizons, start years, and categories. While our methodology is generalizable to the valuation of other non-traded cash flow streams, our main focus is evaluating the performance of private funds. We extend the literature on PE performance by generating risk exposures and replicating portfolios across a range of publicly traded assets: risk-free bonds, stocks, REITs, and infrastructure-listed assets across multiple horizons. We consider not only the Buyout and Venture Capital categories, which have been the focus of much of the literature, but also alternative private fund categories such as Real Estate funds, Infrastructure funds, Debt funds, Fund of Funds, and Restructuring.

We find that the average private equity fund generates little, if any, outperformance across most of the categories we consider. This is because private fund cash-flows can be replicated, at least ex-post, through a basket of publicly traded equivalents. However, we also document rich heterogeneity across horizons, in the cross-section, and in the time-series in terms of fund performance and expected returns. Fund performance exhibits considerable variation, and generally trends downward in our sample.
References


A Appendix: Asset Pricing Model

A.1 Risk-free rate

The real short yield \( y_t(1) \), or risk-free rate, satisfies 
\[ E_t[\exp\{m_{t+1} + y_t(1)\}] = 1. \]
Solving out this Euler equation, we get:

\[
y_t(1) = y_t^s(1) - E_t[\pi_{t+1}] - \frac{1}{2} e'_{\pi} \Sigma e_{\pi} + e'_{\pi} \Sigma z_{t+1} \]
\[
y_0(1) = y_0^s(1) - \frac{1}{2} e'_{\pi} \Sigma e_{\pi} + e'_{\pi} \Sigma \Lambda_0.
\]

where we used the expression for the real SDF

\[
m_{t+1} = m_t^s + \pi_{t+1}
\]
\[
= -y_t^s(1) - \frac{1}{2} \Lambda_{t+1} - \Lambda_{t+1}^e_{t+1} - \pi_0 + e'_{\pi} \Sigma \pi_{t+1} + e'_{\pi} \Sigma \Lambda_0
\]
\[
= -y_t(1) - \frac{1}{2} e'_{\pi} \Sigma e_{\pi} + e'_{\pi} \Sigma \Lambda_0 - \left( \Lambda'_{t+1} - e'_{\pi} \Sigma \Lambda_0 \right) e_{t+1}
\]

The real short yield is the nominal short yield minus expected inflation minus a Jensen adjustment minus the inflation risk premium.

A.2 Nominal and real term structure

Proposition 1. Nominal bond yields are affine in the state vector:

\[
y_t^s(\tau) = -\frac{A^s(\tau)}{\tau} - \frac{B^s(\tau)}{\tau} z_{t+1},
\]

where the coefficients \( A^s(\tau) \) and \( B^s(\tau) \) satisfy the following recursions:

\[
A^s(\tau + 1) = -y_0^s(1) + A^s(\tau) + \frac{1}{2} \left( B^s(\tau) \right)' \Sigma \left( B^s(\tau) \right)' - \left( B^s(\tau) \right)' \Sigma \Lambda_0, \quad \text{(A.3)}
\]
\[
\left( B^s(\tau + 1) \right)' = \left( B^s(\tau) \right)' \Psi - e'_{y_{\tau}} - \left( B^s(\tau) \right)' \Sigma \Lambda_1, \quad \text{(A.4)}
\]

initialized at \( A^s(0) = 0 \) and \( B^s(0) = 0 \).

Proof. We conjecture that the \( t + 1 \)-price of a \( \tau \)-period bond is exponentially affine in the
state:
\[ \log(p^s_{t+1}(\tau)) = A^s(\tau) + \left(B^s(\tau)\right)'z_{t+1} \]

and solve for the coefficients \( A^s(\tau + 1) \) and \( B^s(\tau + 1) \) in the process of verifying this conjecture using the Euler equation:

\[
P^s_t(\tau + 1) = E_t[\exp\{m^s_{t+1} + \log(p^s_{t+1}(\tau))\}]
\]
\[
= E_t[\exp\{-y^s_0(1) - \frac{1}{2}(\Lambda_t'(\Lambda_t - A^s(\tau) + \left(B^s(\tau)\right)'z_{t+1})])]
\]
\[
= \exp\{-y^s_0(1) - e'_y z_t - \frac{1}{2}(\Lambda_t'\Lambda_t + A^s(\tau) + \left(B^s(\tau)\right)'\Psi z_t)\} \times
\]
\[
E_t\left[\exp\{-A^s(\tau) + \left(B^s(\tau)\right)'\Sigma_t^1\} \right].
\]

We use the log-normality of \( \epsilon_{t+1} \) and substitute for the affine expression for \( \Lambda_t \) to get:

\[
P^s_t(\tau + 1) = \exp\left\{-y^s_0(1) - e'_y z_t + A^s(\tau) + \left(B^s(\tau)\right)'\Psi z_t + \frac{1}{2}\left(B^s(\tau)\right)'\Sigma_t^1\left(B^s(\tau)\right)
- \left(B^s(\tau)\right)'\Sigma_t^{1/2}(\Lambda_0 + \Lambda_t z_t)\right\}.
\]

Taking logs and collecting terms, we obtain a linear equation for \( \log(p_t(\tau + 1)) \):

\[
\log\left(p^s_t(\tau + 1)\right) = A^s(\tau + 1) + \left(B^s(\tau + 1)\right)'z_t,
\]

where \( A^s(\tau + 1) \) satisfies (A.3) and \( B^s(\tau + 1) \) satisfies (A.4). The relationship between log bond prices and bond yields is given by \(- \log\left(p^s_t(\tau)\right) / \tau = y^s_t(\tau)\).

Define the one-period return on a nominal zero-coupon bond as:

\[
r^b_{t+1}(\tau) = \log\left(p^s_{t+1}(\tau)\right) - \log\left(p^s_t(\tau + 1)\right)
\]

The nominal bond risk premium on a bond of maturity \( \tau \) is the expected excess return corrected for a Jensen term, and equals negative the conditional covariance between that bond return and the nominal SDF:

\[
E_t\left[r^b_{t+1}(\tau) - y^s_t(1) + \frac{1}{2}V_t\right] = -\text{Cov}_t\left[m^s_{t+1}r^b_{t+1}(\tau)\right]
\]
\[
= \left(B^s(\tau)\right)'\Sigma_t^{1/2}\Lambda_t
\]
Real bond yields, $y_t(\tau)$, denoted without the $ superscript, are affine as well with coefficients that follow similar recursions:

$$A(\tau + 1) = -y_0(1) + A(\tau) + \frac{1}{2}(B(\tau)')\Sigma(B(\tau)) - (B(\tau))'\Sigma^{1/2}\left(\Lambda_0 - \Sigma^{1/2}e_\pi\right)$$

$$B(\tau + 1)' = -e_{yn}' + (e_\pi + B(\tau))'(\Psi - \Sigma^{1/2}\Lambda_1).$$

(A.5)  

(A.6)

For $\tau = 1$, we recover the expression for the risk-free rate in (A.1)-(A.2).

### A.3 Stock Market

We define the real return on equity as $R_{t+1}^m = \frac{P_{t+1}^m + D_{t+1}^m}{P_t^m}$, where $P_t^m$ is the end-of-period price on the equity market. A log-linearization delivers:

$$r_{t+1}^m = \kappa_0^m + \Delta d_{t+1}^m + \kappa_1^m p_{d_{t+1}}^m - p_{d_{t}}^m.$$

(A.7)

The unconditional mean log real stock return is $r_0^m = E[r_t^m]$, the unconditional mean dividend growth rate is $\mu^m = E[\delta d_{t+1}^m]$, and $p_{d^m} = E[p_{d_{t}}^m]$ is the unconditional average log price-dividend ratio on equity. The linearization constants $\kappa_0^m$ and $\kappa_1^m$ are defined as:

$$\kappa_1^m = \frac{e_{pd^m}}{e_{pd^m} + 1} < 1 \text{ and } \kappa_0^m = \log\left(\frac{e_{pd^m}}{e_{pd^m} + 1}\right) - \frac{e_{pd^m}}{e_{pd^m} + 1}p_{d^m}.$$

(A.8)

Our state vector $z$ contains the (demeaned) log real dividend growth rate on the stock market, $\delta d_{t+1}^m - \mu^m$, and the (demeaned) log price-dividend ratio $p_{d^m} - p_{d^m}$. Where $e_{pd}$ ($e_{divm}$) is a selector vector that has a one in the fifth (sixth) entry, since the log pd ratio (log dividend growth rate) is the fifth (sixth) element of the VAR.

We define the log return on the stock market so that the return equation holds exactly, given the time series for $\{\Delta d_{t}^m, p_{d_{t}}^m\}$. Rewriting (A.7):

$$r_{t+1}^m - r_0^m = \left[(e_{divm} + \kappa_1^m e_{pd})'\Psi - e_{pd}'\right]z_t + \left(e_{divm} + \kappa_1^m e_{pd}\right)'\Sigma^{1/2}\varepsilon_{t+1}.$$

$$r_0^m = \mu^m + \kappa_0^m - p_{d^m}(1 - \kappa_1^m).$$
The equity risk premium is the expected excess return on the stock market corrected for a Jensen term. By the Euler equation, it equals minus the conditional covariance between the log SDF and the log return:

\[
1 = E_t \left[ M_{t+1} \frac{P_m^{t+1} + D_m^{t+1}}{P_m^t} \right] = E_t \left[ \exp \{ m_{t+1}^S + \tau_{t+1} + r_{t+1}^m \} \right]
\]

\[
= E_t \left[ \exp \left\{ -y_t^S(1) - \frac{1}{2} \Lambda_t' \Lambda_t + \tau_t + r_{t+1}^m + (e_{divm} + \kappa_1^m e_{pd})' z_{t+1} - e_{pd}' z_t \right\} \right]
\]

\[
= \exp \left\{ -y_t^S(1) - \frac{1}{2} \Lambda_t' \Lambda_t + \tau_t + r_{t+1}^m + (e_{divm} + \kappa_1^m e_{pd} + e_\pi)' \Psi - e_{pd}' e_\pi^t \right\} z_t \times E_t \left[ \exp \left\{ -\Lambda_t' \epsilon_{t+1} + (e_{divm} + \kappa_1^m e_{pd} + e_\pi)' \Sigma^{1/2} \epsilon_{t+1} \right\} \right]
\]

\[
= \exp \left\{ \frac{1}{2} (e_{divm} + \kappa_1^m e_{pd} + e_\pi)' \Sigma (e_{divm} + \kappa_1^m e_{pd} + e_\pi) - (e_{divm} + \kappa_1^m e_{pd} + e_\pi)' \Sigma^{1/2} \Lambda_t \right\}
\]

Taking logs on both sides delivers:

\[
r_0^m + \tau_0 - y_0^S(1) + \left[ (e_{divm} + \kappa_1^m e_{pd} + e_\pi)' \Psi - e_{pd}' e_\pi \right] z_t = \frac{1}{2} \left[ (e_{divm} + \kappa_1^m e_{pd} + e_\pi)' \Sigma (e_{divm} + \kappa_1^m e_{pd} + e_\pi) - (e_{divm} + \kappa_1^m e_{pd} + e_\pi)' \Sigma^{1/2} \Lambda_t \right]
\]

(A.9)

The left-hand side is the expected excess return on the stock market, corrected for a Jensen term, while the right-hand side is the negative of the conditional covariance between the (nominal) log stock return and the nominal log SDF. We refer to the left-hand side as the equity risk premium in the data, since it is implied directly by the VAR. We refer to the right-hand side as the equity risk premium in the model, since it requires knowledge of the market prices of risk.

Note that we can obtain the same expression using the log real SDF and log real stock return:

\[
E_t \left[ r_{t+1}^m \right] - y_t(1) + \frac{1}{2} V_t \left[ r_{t+1}^m \right] = -\text{Cov}_t \left[ m_{t+1}, r_{t+1}^m \right]
\]

\[
r_{t+1}^m - y_0(1) + \left[ (e_{divm} + \kappa_1^m e_{pd} + e_\pi)' \Psi - e_{pd}' e_\pi^t - e_{\pi}' e^{1/2} \Lambda_1 \right] z_t
\]

\[
+ \frac{1}{2} \left( e_{divm} + \kappa_1^m e_{pd} \right)' \Sigma (e_{divm} + \kappa_1^m e_{pd}) = \left( e_{divm} + \kappa_1^m e_{pd} \right)' \Sigma^{1/2} \left( \Lambda_t - \left( \Sigma^{1/2} \right)' e_\pi \right)
\]

47
We combine the terms in $\Lambda_0$ and $\Lambda_1$ on the right-hand side and plug in for $y_0(1)$ from (A.2) to get:

$$r_0^m + \pi_0 - y_0^S(1) + \frac{1}{2} e^T \Pi e + \frac{1}{2} (e_{divm} + \kappa_1 e_{pd})' \Sigma (e_{divm} + \kappa_1 e_{pd}) + e^T \Pi \left( e_{divm} + \kappa_1 e_{pd} \right) + \left[ (e_{divm} + \kappa_1 e_{pd} + e_{\pi})' \Pi - e_{pd}' - e_{yi} \right] z_t = (e_{divm} + \kappa_1 e_{pd})' \Sigma^{1/2} \Lambda_t + e_{\pi}' \Sigma^{1/2} \Lambda_0 + e_{\pi}' \Sigma^{1/2} \Lambda_1 z_t$$

This recovers equation (A.9).

### A.4 Dividend Strips

**Proposition 2.** Log price-dividend ratios on dividend strips are affine in the state vector:

$$p^d_i(\tau) = \log \left( P_i^d(\tau) \right) = A^m(\tau) + B^m(\tau) z_t,$$

where the coefficients $A^m(\tau)$ and $B^m(\tau)$ follow recursions:

$$A^m(\tau + 1) = A^m(\tau) + \mu_m - y_0(1) + \frac{1}{2} \left( e_{divm} + B^m(\tau) \right)' \Sigma (e_{divm} + B^m(\tau))$$

$$- (e_{divm} + B^m(\tau))' \Sigma^{1/2} \left( \Lambda_0 - \Sigma^{1/2} e_{\pi} \right),$$

$$B^m(\tau + 1)' = (e_{divm} + e_{\pi} + B^m(\tau))' \Pi - e_{yi}' - (e_{divm} + e_{\pi} + B^m(\tau))' \Sigma^{1/2} \Lambda_1, \quad \text{(A.11)}$$

initialized at $A^m(0) = 0$ and $B^m(0) = 0$.

**Proof.** We conjecture the affine structure and solve for the coefficients $A^m(\tau + 1)$ and $B^m(\tau + 1)$ in the process of verifying this conjecture using the Euler equation:

$$P_i^d(\tau + 1) = E_t \left[ M_{t+1} P_{t+1}^d(\tau) \frac{D_{t+1}^m}{D_t^m} \right] = E_t \left[ \exp \{ m_{t+1}^S + \pi_{t+1} + \Delta d_{t+1}^m + p_{t+1}^d(\tau) \} \right]$$

$$= E_t \left[ \exp \{-y_0^S(1) - \frac{1}{2} \Lambda_t' \Lambda_t - \Lambda_t' \varepsilon_{t+1} + \pi_0 + e_{\pi}' z_{t+1} + \mu_m + e_{divm}' z_{t+1} + A^m(\tau) + B^m(\tau)' z_{t+1} \} \right]$$

$$= \exp \{-y_0^S(1) - e_{yi}' z_t - \frac{1}{2} \Lambda_t' \Lambda_t + \pi_0 + e_{\pi}' \Psi z_t + \mu_m + e_{divm}' \Psi z_t + A^m(\tau) + B^m(\tau)' \Psi z_t \}$$

$$\times E_t \left[ \exp \{ -\Lambda_t' \varepsilon_{t+1} + (e_{divm} + e_{\pi} + B^m(\tau))' \Sigma^{1/2} \varepsilon_{t+1} \} \right].$$

We use the log-normality of $\varepsilon_{t+1}$ and substitute for the affine expression for $\Lambda_t$ to get:

$$P_i^d(\tau + 1) = \exp \{-y_0^S(1) + \pi_0 + \mu_m + A^m(\tau) + \left[ (e_{divm} + e_{\pi} + B^m(\tau))' \Pi - e_{yi}' \right] z_t$$
Taking logs and collecting terms, we obtain a log-linear expression for \( p_t^d (\tau + 1) \):

\[
p_t^d (\tau + 1) = A^m (\tau + 1) + B^m (\tau + 1)' z_t,
\]

where:

\[
A^m (\tau + 1) = A^m (\tau) + \mu_m - y_0^s (1) + \pi_0 + \frac{1}{2} (e_{divm} + e_{\pi} + B^m (\tau))' \Sigma (e_{divm} + e_{\pi} + B^m (\tau))
\]

\[- (e_{divm} + e_{\pi} + B^m (\tau))' \Sigma^{\frac{1}{2}} \Lambda_0,
\]

\[
B^m (\tau + 1)' = (e_{divm} + e_{\pi} + B^m (\tau))' \Psi - e'_y n - (e_{divm} + e_{\pi} + B^m (\tau))' \Sigma^{\frac{1}{2}} \Lambda_1.
\]

We recover the recursions in (A.10) and (A.11) after using equation (A.2). □

Like we did for the stock market as a whole, we define the strip risk premium as:

\[
E_t \left[ r_{t+1}^{d,m} - y_t^s (1) + \frac{1}{2} V_t \left[ r_{t+1}^{d,m} \right] \right] = - \text{Cov}_t \left[ m_{t+1}^s, r_{t+1}^{d,m} \right] = (e_{divm} + e_{\pi} + B^m (\tau))' \Sigma^{\frac{1}{2}} \Lambda_t
\]

The risky strips for REITs and infrastructure are defined analogously.

## B Special case: Stock Market and Short Rate Only

### B.1 Setup Simplified Model

A special case of our SDF methodology is a model that only prices the short rate and the stock market return. Korteweg and Nagel (2016) consider such a model. The state variable dynamics and the SDF remain unchanged and given by (9) and (10). The state variable for that model is simpler and only contains four elements:

\[
z_t = [y_t^s (1), \pi_t, p d_t^m, r_t^m]'.
\]

The model will match the dynamics of the nominal short rate by virtue of the fact that the short rate is included in the state vector. Pricing the stock return results in the restrictions
implied by equation (A.9). Specifically, matching the level and the dynamics of the equity risk premium pins down the (only element of the) constant market price of risk \( \Lambda_0 \) in equation (A.12) and the four elements (fourth row) of the time-variation matrix \( \Lambda_1 \) in equation (A.13):

\[
\begin{align*}
\tau_0^m + \pi_0 - y_0^5(1) + \frac{1}{2} (e_{rm} + e_{\pi})' \Sigma (e_{rm} + e_{\pi}) &= (e_{rm} + e_{\pi})' \frac{1}{2} \Lambda_0 \\
(e_{rm} + e_{\pi})' \Psi - e_{\gamma m}' &= (e_{rm} + e_{\pi})' \frac{1}{2} \Lambda_1
\end{align*}
\]

Note that this approach generalizes Korteweg and Nagel (2016) in that we (a) not only price the T-bill and the stock market on average, but point by point, and (b) allow for time variation in the equity risk premium. Specifically, we allow for the nominal short rate and the price-dividend ratio to forecast future stock returns, as they do in the data. Equations (A.12) and (A.12) constitute five equations in five unknowns so that the system is exactly identified. Moreover, the five market prices of risk can be solved by hand, given the VAR companion matrix \( \Psi \) and covariance matrix \( \Sigma \). In contrast, in our general model we estimate the market prices of risk matching a much larger set of moments, including the time series of yields of multiple maturities, the Cochrane-Piazzesi factor, the price-dividend ratio on the stock market, the expected return and price-dividend ratio for REITs and infrastructure stocks, etc.

**B.2 Results Simplified Model**

Solving equations (A.12) and (A.13) over our January 1990–December 2016 sample delivers \( \Lambda_0 = [0, 0, 0, 0.4304] \) and the fourth row of \( \Lambda_1 = [-52.6804, 27.6676, -1.1615, 0] \). The elements of the first three rows are all equal to zero.

Figure A.1 plots the bond yields on bonds of maturities 1 month, 3 months, 1 year, 3 years, 5 years, and 10 years implied by this simple model. The model matches the time series of the short rate perfectly, by construction, but fails to fit the data as the bond maturity becomes longer. The slope of the nominal term structure is close to zero, and in fact slightly downward sloping. For example, annualized average 1-month yields are 2.80% in data and model, 5-year yields are 3.70% in data and 2.78% in model, 10-year yields are 4.71% in data and 2.71% in model, and 30-year yields are 5.26% in data and 2.64% in the model. The volatility of model-implied yields is 50% too low. This happens because the simple model implies a nominal bond risk premium that is constant at zero. The model fails to capture compensation for inflation and real rate risk, while the data
display non-zero levels and time variation in these risk premia.

Figure A.1: Dynamics of the Nominal Term Structure of Interest Rates in the Simple Model

The figure plots the observed and model-implied 1-, 3-, 12-, 36-, 60-, and 120-month nominal bond yields.

Figure A.2 shows the equity risk premium, the expected excess return, in the left panel and the price-dividend ratio in the right panels. The dynamics of the risk premia in the data are dictated by the VAR. The equity risk premium only varies with the short rate and the price-dividend ratio in the simple VAR, while it additionally varied with the CP factor in the full model. The model matches the observed equity risk premium implied by the VAR exactly through the choices of $\Lambda_0$ and $\Lambda_1$. The right panel shows the observed price-dividend ratio and the model-implied ratio constructed from dividend strips as explained above. In contrast with the full model, there are no additional market prices of risk available to improve the fit for the price-dividend ratio. While the deviations appear small, they are eight times as large as in the benchmark model: the sum of squared de-
viations between model-implied and observed pd ratio is 776.8 in the simple model and 95.3 in the benchmark model.

Figure A.2: Equity Risk Premia and Price-Dividend Ratios in the Simple Model

The figure plots the observed and model-implied equity risk premium on the overall stock market, REIT market, and infrastructure sector, in the left panels, as well as the price-dividend ratio in the right panels.

Figure A.3 plots the time series for prices of nominal zero-coupon bonds and dividend strips on the overall stock market, for maturities of one-year, five-years, and ten-years. These prices behave differently as in the benchmark model. Furthermore, we note that the term structure of dividend strip risk premia is completely flat in the simple model. Stock market risk at different horizons is priced the same (equal to the overall equity risk premium), just like bond market risk at different horizons is priced the same (equal to zero).

With the market prices of risk in hand, we can form the time series of the log nominal SDF $m_t^S$ by equation (10). Figure A.4 plots the annual log SDF time series, obtained by summing the log monthly SDF across the months in the year. The figure also shows the corresponding log SDF which obtains under the assumption of constant market prices of risk. It sets $\Lambda_1 = 0_{4 \times 4}$. The SDF is highly volatile. Its volatility, which has the interpretation as the maximum Sharpe ratio, is 2.87. Even with constant market prices of risk (and therefore constant equity risk premia), the SDF has a volatility of 2.37. Using such volatile time series to discount cash-flows can lead to unstable and hard-to-interpret results. Our approach avoids this by using prices of bond and stock strips, which are conditional expectations. Such expectations average out the SDF realizations in next period’s states of
Figure A.3: Zero Coupon Bond Prices and Dividend Strip Prices in Simple Model

The figure plots the model-implied prices on zero-coupon Treasury bonds in the first panel, and price-dividend ratios for dividend strips on the overall stock market in the second panel, for maturities of 12 months, 60 months, and 120 months.

Figure A.4: Log Stochastic Discount Factor in Simple Model

The solid line plots the model-implied log annual stochastic discount factor, obtained by summing the monthly log SDF over the months within the year. The dashed line plots the log annual SDF that results when market prices of risk are constant: $\Lambda_1 = 0.4 \times 0.4$. 
B.3 Korteweg-Nagel Details

They propose:

\[ m_{t+1} = a - b r_{t+1}^m, \]

whereby the coefficients \( a \) and \( b \) are chosen so that the Euler equation \( 1 = E[M_{t+1} R_{t+1}] \) holds for the public equity market portfolio and the risk-free asset return. More specifically, they estimate \( a = 0.088 \) and \( b = 2.65 \) using a GMM estimator:

\[
\min_{a,b} \left( \frac{1}{N} \sum_i u_i(a,b) \right)' W \left( \frac{1}{N} \sum_i u_i(a,b) \right)
\]

where

\[ u_i(a,b) = \sum_{j=1}^J M_{t+h(j)}(a,b)[X_{if,t+h(j)}, X_{im,t+h(j)}]. \]

\( N \) is the number of funds, and \( W \) is a \( 2 \times 2 \) identity matrix. The T-bill benchmark fund cash-flow, \( X_{if} \), and the market return benchmark cash-flow, \( X_{im} \), are the cash-flows on a T-bill and stock market investment, respectively, that mimic the timing and magnitude of the private equity fund \( i \)'s cash-flows. The \( t + h(j) \) are the dates on which the private equity fund pays out cash-flow \( j = 1, \ldots, J \). Date \( t \) is the date of the first cash-flow into the fund, so that \( h(1) = 0 \). For each of the two benchmark funds, the inflows are identical in size and magnitude as the inflows into the PE fund. If PE fund \( i \) makes a payout at \( t + h(j) \), the benchmark funds also make a payout. That payout consists of two components. The first component is the return on the benchmark since the last cash-flow date. The second component is a return of principal, according to a preset formula which returns a fraction of the capital which is larger, the longer ago the previous cash-flow was.

A special case of this model is the public market equivalent of Kaplan and Schoar (2005), which sets \( a = 0 \) and \( b = 1 \). This is essentially the log utility model. The simple PME model is rejected by Korteweg and Nagel (2016), in favor of their generalized PME model.

There are several key differences between our method and that of Korteweg and Nagel (2016). First, we do not use SDF realizations to discount fund cash-flows. Rather, we use bond prices and dividend strip prices, which are conditional expectations. Realized SDFs are highly volatile. Second, the KN approach does not take into account heterogeneity in the amount of systematic risk of the funds. All private equity funds are assumed to have a 50-50 allocation to the stock and bond benchmark funds. Our model allows for
different funds to have different stock and bond exposure. Third, the KN approach uses a preset capital return policy which is not tailored to the fund in question. For example, a fund may be making a modest distribution in year 5, say 10%, and a large distribution in year 10 (90%). Under the KN assumption, the public market equivalent fund would sell 50% in year 5 and the other 50% in year 10. There clearly is a mismatch between the risk exposure of the public market equivalent fund and that of the private equity fund. In other words, the KN approach does not take into the account the magnitude of the fund distributions, only their timing. Fourth, we use additional risk factors beyond those considered in KN.

To study just the importance of the last assumption, we can redo our calculations using a much simplified state vector that only contains the short rate, inflation, and the stock market return. This model has constant risk premia.

C  Shock-exposure and Shock-price Elasticities

Borovička and Hansen (2014) provide a dynamic value decomposition, the asset pricing counterparts to impulse response functions, which let a researcher study how a shock to an asset’s cash-flow today affects future cash-flow dynamics as well as the prices of risk that pertain to these future cash-flows. What results is a set of shock-exposure elasticities that measure the quantities of risk resulting from an initial impulse at various investment horizons, and a set of shock-price elasticities that measure how much the investor needs to be compensated currently for each unit of future risk exposure at those various investment horizons. We now apply their analysis to our VAR setting.

Recall that the underlying state vector dynamics are described by:

$$z_{t+1} = \Psi z_t + \Sigma z \epsilon_{t+1}$$

The log cash-flow growth rates on stocks, REITs, and infrastructure stocks are described implicitly by the VAR since it contains both log returns and log price-dividend ratios for each of these assets. The log real dividend growth rate on an asset $$i \in \{m, reit, infra\}$$ is given by:

$$\log(D_{t+1}^i) - \log(D_t^i) = \Delta d_{t+1}^i = A_0^i + A_1^i z_t + A_2^i \epsilon_{t+1},$$

where $$A_0^i = \mu_m, A_1^i = e'_{divi} \Psi, \text{ and } A_2^i = e'_{divi} \Sigma z.$$
Denote the cash-flow process \( Y_t = D_t \). Its increments in logs can we written as:

\[
y_{t+1} - y_t = \Gamma_0 + \Gamma_1 z_t + z_t' \Gamma_3 z_t + \Psi_0 \varepsilon_{t+1} + z_t' \Psi_1 \varepsilon_{t+1} \tag{A.14}
\]

with coefficients \( \Gamma_0 = A_0^i \), \( \Gamma_1 = A_1^i \), \( \Gamma_3 = 0 \), \( \Psi_0 = A_2^i \), and \( \Psi_1 = 0 \).

The one-period log real SDF, which is the log change in the real pricing kernel \( S_t \), is a quadratic function of the state:

\[
\log(S_{t+1}) - \log(S_t) = m_{t+1} = B_0 + B_1 z_t + B_2 \varepsilon_{t+1} + z_t' B_3 z_t + z_t' B_4 \varepsilon_{t+1}
\]

where \( B_0 = -y_0^S(1) + \pi_0 - \frac{1}{2} \Lambda_0' \Lambda_0, B_1 = -e_y' \Psi - \Lambda_0' \Lambda_1, B_2 = -\Lambda_0' + e_\pi' \Sigma^z, B_3 = -\frac{1}{2} \Lambda_1' \Lambda_1, \) and \( B_4 = -\Lambda_1' \).

We are interested in the product \( Y_t = S_t D_t \). Its increments in logs can be written as in equation \((A.14)\), with coefficients \( \Gamma_0 = A_0^i + B_0, \Gamma_1 = A_1^i + B_1, \Gamma_3 = B_3, \Psi_0 = A_2^i + B_2, \) and \( \Psi_1 = B_4 \).

Starting from a state \( z_0 = z \) at time 0, consider a shock at time 1 to a linear combination of state variables, \( \alpha_{1}^j \varepsilon_1 \). The shock elasticity \( \varepsilon(z,t) \) quantifies the date-\( t \) impact:

\[
\varepsilon(z,t) = \alpha_{1}^j (I - 2\Psi_{2,i})^{-1} (\Psi_{0,i} + \Psi_{1,i} z)
\]

where the \( \Psi \) matrices solve the recursions:

\[
\begin{align*}
\Psi_{0,j+1} &= \hat{\Gamma}_{1,j} \Sigma^{1/2} + \Psi_0 \\
\Psi_{1,j+1} &= 2 \Psi' \hat{\Gamma}_{3,j} \Sigma^{1/2} + \Psi_1 \\
\Psi_{2,j+1} &= \left( \Sigma^{1/2} \right)' \hat{\Gamma}_{3,j} \Sigma^{1/2}
\end{align*}
\]

The \( \hat{\Gamma} \) and \( \bar{\Gamma} \) coefficients follow the recursions:

\[
\begin{align*}
\hat{\Gamma}_{0,j+1} &= \hat{\Gamma}_{0,j} + \Gamma_0 \\
\hat{\Gamma}_{1,j+1} &= \hat{\Gamma}_{1,j} \Psi + \Gamma_1 \\
\hat{\Gamma}_{3,j+1} &= \Psi' \hat{\Gamma}_{3,j} \Psi + \Gamma_3 \\
\bar{\Gamma}_{0,j+1} &= \bar{\Gamma}_{0,j+1} - \frac{1}{2} \log \left( |I - 2\Psi_{2,j+1}| \right) + \frac{1}{2} \Psi_{0,j+1} \left( I - 2\Psi_{2,j+1} \right)^{-1} \Psi_{0,j+1} \\
\bar{\Gamma}_{1,j+1} &= \bar{\Gamma}_{1,j+1} + \Psi_{0,j+1} \left( I - 2\Psi_{2,j+1} \right)^{-1} \Psi_{1,j+1} \\
\bar{\Gamma}_{3,j+1} &= \bar{\Gamma}_{3,j+1} + \frac{1}{2} \Psi_{1,j+1} \left( I - 2\Psi_{2,j+1} \right)^{-1} \Psi_{1,j+1}
\end{align*}
\]
starting from $\hat{\Gamma}_{0,0} = 0$, $\hat{\Gamma}_{1,0} = 0_{1 \times N}$, $\hat{\Gamma}_{2,0} = 0_{N \times N}$, and where $I$ is the $N \times N$ identity matrix.

Let $\epsilon_g(z, t)$ be the shock-exposure elasticity (cash-flows $Y = D$) and $\epsilon_{sg}(z, t)$ the shock-value elasticity, then the shock-price elasticity $\epsilon_p(z, t)$ is given by

$$\epsilon_p(z, t) = \epsilon_g(z, t) - \epsilon_{sg}(z, t).$$

In an exponentially affine framework like ours, the shock price elasticity can also directly be derived by setting $Y_t = S_t^{-1}$ or $y_{t+1} - y_t = -m_{t+1}$, with coefficients in equation (A.14) equal to $\Gamma_0 = -B_0$, $\Gamma_1 = -B_1$, $\Gamma_3 = -B_3$, $\Psi_0 = -B_2$, and $\Psi_1 = -B_4$.

The shock-price elasticity quantifies implied market compensation for horizon-specific risk exposures. In our case, these risk compensations are extracted from a rich menu of observed asset prices matched by a reduced form model, rather than by constructing a structural asset pricing model. The horizon-dependent risk prices are the multi-period impulse responses for the cumulative stochastic discount factor process.
D Additional Results

Figure A.5: cash-flows by Vintage

(a) Fund of Funds

(b) Debt Funds

(c) Restructuring
Figure A.6: One-Factor Fund Exposures for Other Categories

Panel A: Fund of Funds

Panel B: Debt Funds

Panel C: Restructuring
Figure A.7: Cross Sectional Residual Standard Deviation by Age

(a) Buyout

(b) Venture Capital

(c) Real Estate

(d) Infrastructure

(e) Fund of Funds

(f) Debt Funds

(g) Restructuring
Figure A.8: One Factor Risk-Adjusted Profits by Category

**Panel A: Fund of Funds**

Histogram of Fund-Level Profit Relative to Replicating Portfolio
Avg Profit is: -0.009

Average Profit by Vintage Year

**Panel B: Debt Funds**

Histogram of Fund-Level Profit Relative to Replicating Portfolio
Avg Profit is: 0.005

Average Profit by Vintage Year

**Panel C: Restructuring**

Histogram of Fund-Level Profit Relative to Replicating Portfolio
Avg Profit is: 0.007

Average Profit by Vintage Year
Figure A.9: One Factor Expected Return

Panel A: Fund of Funds

Expected Return by Horizon and Risk Exposure

Panel B: Debt Funds

Expected Return by Horizon and Risk Exposure

Panel C: Restructuring

Expected Return by Horizon and Risk Exposure

Avg Expected Return is: 0.123
Avg Expected Return is: 0.073
Avg Expected Return is: 0.053
Figure A.10: Four-Factor Fund Exposures for Other Categories

Panel A: Fund of Funds

Panel B: Debt Funds

Panel C: Restructuring
Figure A.11: Four Factor Risk-Adjusted Profits by Category

Panel A: Fund of Funds

Panel B: Debt Funds

Panel C: Restructuring
Figure A.12: Four Factor Expected Return

Panel A: Fund of Funds

Panel B: Debt Funds

Panel C: Restructuring