The Value of Performance Signals Under Contracting Constraints*

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January 17, 2017

Abstract

This paper studies the value of additional performance signals in compensation and financing contracts under contracting constraints, such as limited liability, monotonicity, or upper bounds to pay or incentives. We show that informative signals may have no value for contracting, because the payment cannot be adjusted to reflect the signal realization – contrary to the informativeness principle, which was derived assuming no contracting constraints. We derive necessary and sufficient conditions for a signal to have value under such constraints. Our results have implications for pay-for-luck, option repricing, performance-based vesting, performance-sensitive debt, and the conditions under which a principal should invest in costly monitoring.

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Executive contracts are typically based on multiple signals of performance, and the use of multi-signal contracts is increasing over time. For example, with performance-based equity, the number of securities granted typically depends on performance relative to a threshold (or set of thresholds). The proportion of large U.S. firms that pay their executives with performance-vesting equity has risen from 20% in 1998 to 70% in 2012 (Bettis et al. (2016)). Moreover, 86% of such grants employ at least one accounting threshold, and so their value depends on factors other than the stock price – the standard "output" measure for executive contracts. The survey of Murphy (2013) reports that companies use a variety of financial and non-financial performance measures when determining CEO bonuses. Additional performance signals are also used in financing contracts. Manso, Strulovici, and Tchistyi (2010) document that 40% of loans have performance pricing provisions, i.e. the coupon rate depends on signals such as the firm's credit rating, leverage, and solvency ratios. Thus, the payment to investors depends on factors other than cash flow – the standard "output" measure for financing contracts.

The main theoretical justification for including additional performance measures in a contract is Holmstrom's (1979) informativeness principle. This principle states that any signal should be included in a contract if it provides incremental information about the agent's performance, over and above the information already conveyed in output. Indeed, Murphy (2013) writes that the "informativeness principle was widely embraced by many academics who used it as the theoretical justification for ... performance measures used in CEO contracts." However, real-life contracts appear to *violate* the principle. Even though some contracts are based on signals other than output, they are not based on *every* potentially informative signal.

The informativeness principle was derived assuming no contracting constraints. However, contracting constraints are an important feature in real life. The most common is limited liability: limited liability of equity applies to financing contracts between entrepreneurs and investors; in compensation contracts, the salary paid by a firm to a manager cannot be negative. In addition, in compensation contracts, regulation and notions of fairness often constrain realized pay. For example, the European Union limits banker bonuses to twice the level of salary; in March 2016, Israel removed tax deductibility from banker realized pay that exceeds 35 times the salary of the lowest-paid worker in the institution (or 2.5 million shekels, if this is lower); the leader of the UK Labour Party proposed a maximum wage; in November 2016, the UK government's Green Paper proposed that company pay policies stipulate a cap on realized pay; and in 2013, French Prime Minister Jean-Marc Ayrault proposed extending the pay cap on executives in state-owned firms to non-state-owned firms. Turning to soft constraints, "outrage constraints" (Bebchuk and Fried (2004)) may prevent companies from paying their executives above a certain level. Constraints can apply to the level of incentives (the sensitivity of realized pay to output). The European Union Shareholder Rights Directive stipulates that stock-based compensation should generally not exceed 50% of total variable pay, limiting the sensitivity of pay to the stock price. The financial crisis has led many commentators to propose limits to equity-based pay to reduce risk-taking incentives, and some shareholder proposals aim to cap equity awards.[1] Thus, to apply the informativeness principle to many real-life settings, we must first study whether it holds under contracting constraints, and if necessary extend it.

This paper derives necessary and sufficient conditions for a signal to have value under contracting constraints, thus shedding light on the conditions under which contracts should incorporate additional performance signals. We first consider the standard framework of risk neutrality and limited liability on the manager, originally analyzed by Innes (1990) and widely used in a number of settings (e.g. Biais et al. (2010), Clementi and Hopenhayn (2006), DeMarzo and Fishman (2007a, 2007b), DeMarzo and Sannikov (2006), and the textbook of Tirole (2006)). Similar to Innes (1990), we consider up to two additional constraints. First, the firm also exhibits limited liability. If the distribution of outputs satisfies the monotone likelihood ratio property, then high (low) outputs signal high (low) effort and the optimal contract is "live-or-die" – the manager receives zero if output falls below a threshold q^*, and the entire output if it exceeds it. A similar result holds if the payment is bounded not by the level of output but instead by a maximum set by regulation or social constraints. Second, a monotonicity (or "free disposal") constraint requires the firm's payoff to be non-decreasing in output, and so the manager cannot gain more than one-for-one with an increase in output. The optimal contract is then an option on output: the manager receives zero if output falls below a different threshold q^{**}, and the residual $q - q^{**}$ if output exceeds it. A similar result holds if the sensitivity of pay to output is capped not at one, but a different level due to regulation or social constraints.

Under either constraint, the only non-trivial dimension of the contract that the firm must decide is the threshold q^* (q^{**}) – under the optimal contract, the payment below the threshold is automatically zero, and the payment above the threshold is automatically the entire output or the residual. Thus, an additional signal of performance will only be included in the contract if it affects the threshold, i.e. it is optimal for the firm to vary the threshold according to the signal realization – the firm will not wish to use it to change any other dimension of the contract. Under the "live-or-die" contract, changing the threshold q^* alters the payment (from 0 to q or vice-versa) only in a local neighborhood around q^*. As a result, a signal is only useful if it affects the likelihood ratio that output equals q^*, i.e. is informative about whether output

[1] Ertimur, Ferri, and Muslu (2011) discuss a 2004 shareholder proposal at Motorola to cap equity grants at $1 million, and a 2004 proposal at Eastman Kodak to scrap equity grants.

equalling q^* is the outcome of high or low effort. If the signal suggests the manager has worked (shirked), the firm decreases (increases) the threshold. Under the option contract, changing the threshold q^{**} alters the payment for all $q \geq q^{**}$. Thus, a signal is only useful if it affects the likelihood ratio that output exceeds q^{**} – i.e. is informative about whether output exceeding q^{**} is the outcome of high or low effort.

In both cases, the contract depends on a signal s if and only if it is informative about effort at the threshold $q^* \, (q^{**})$, i.e. at an intermediate output level. Signals that are informative about effort only above and/or below the threshold are of no value, because the payment is bounded by either a limited liability or monotonicity constraint. As a result, a signal can be informative almost everywhere (i.e. at all output levels except the threshold) yet still have zero value.

We then extend the model to risk aversion. Now, the contract takes a more general form: under an upper bound on payments, the manager still receives zero if output is below a threshold, but does not necessarily receive the entire output if it exceeds it. Thus, the upper bound does not bind for all high output levels. As a result, the firm can make use of the signal at output levels above the threshold (not just at the threshold), as long as the upper bound does not bind at these output levels. Thus, the necessary and sufficient conditions for a signal to have value are weaker under risk aversion.

Quite separate from extending the informativeness principle, this model also generates the first set of sufficient conditions – limited liability, log utility, and a linear likelihood ratio – for options to be the optimal contract when the agent is risk-averse. Even though options are commonly granted, the only existing justification in a moral hazard model to our knowledge is Innes (1990), which requires the agent to be risk-neutral. Unlike in the risk-neutral model where the manager is the residual claimant for $q > q^{**}$, so that the number of options is fixed at 1, under risk aversion it need not be.

The results have a number of implications. Our main theoretical implication is that the informativeness principle needs to be modified under contracting constraints: a signal has value if and only if it is informative about effort at an output level for which constraints do not bind, rather than at any output level. A second theoretical implication is that the value of information is non-monotonic in output. Thus, the firm should only invest in additional signals on manager performance at moderate output realizations. If output is low, the manager is fired anyway; if output is high, he is the residual claimant anyway. Thus, in neither case are additional signals valuable.

Moving to applied implications, our stronger conditions for a signal to have value under contracting constraints can potentially explain why real-life contracts do not depend on as many signals as the original informativeness principle suggests they should.[2] For example, executive contracts typically do not depend on the firm's recovery rate in bankruptcy, the outcome of litigation against the firm, and citations of major patents. Relatedly, Bebchuk and Fried (2004) argue that the common practice of paying managers for luck, i.e. not filtering out industry shocks, is leading evidence that CEO pay results from rent extraction rather than optimal contracting. However, since the informativeness principle does not automatically apply under limited liability, these practices are not necessarily suboptimal. If a firm suffers a catastrophe, the manager is typically fired anyway, regardless of whether it was due to bad luck (e.g. poor industry performance) or shirking, and so cannot be punished further. However, the model does suggest that pay-for-luck is suboptimal at moderate output realizations.

The results also have implications for the design of option compensation, where "output" is now the stock price. They suggest that option repricing (which, empirically, nearly always involves a lowering of the strike price) can be justified if prompted by positive signals of CEO effort. This result implies that the practice of lowering the strike price upon poor performance is generally inefficient. However, it also gives conditions under which repricing can be optimal, contrary to conventional wisdom that it necessarily results from rent extraction.

In addition, our model provides conditions under which the number of options granted to the manager should depend on additional signals, as in the case of performance-based vesting. Despite its popularity, we are unaware of any theories that study under what conditions performance-based vesting is optimal, and what performance signals should be used. Simple intuition may suggest that the number of options that vest should depend on a signal if it provides incremental information about effort over and above that contained in the stock price. Instead, we show that it should depend on a signal if and only if the signal affects the rate at which the informativeness of the stock price changes with the level of stock price. This is because the number of options that vest is only one component of the compensation contract, and even a signal which is incrementally informative need not affect this component. For example, bad macroeconomic conditions are not individually informative about effort if they are outside the manager's control. However, if effort affects the stock price in booms more than in recessions, the number of options should be higher in the former. This result also suggests that signals that trigger vesting need not be adjusted for "luck".

Conversely, even if a signal (say, revenues) is informative about effort, it should not affect

[2] Salanié (1997, p128-129) writes that "the sufficient statistic theorem indicates that the optimal wage schedule should depend on all signals that may bring information on the action chosen by the agent(...). This prediction does not accord well with experience; real-life contracts appear (...) to depend on a small number of variables only".

vesting if it does not affect how the informativeness of the stock price changes with the stock price. Even if it does, it may be optimal to grant the manager more options (albeit with higher strike prices) upon *low* revenues, if the likelihood ratio is more sensitive to the stock price when revenues are low – the universal practice of vesting being triggered by beating a threshold is not predicted by theory.

In addition to compensation, the risk-neutral model can also be applied to a financing setting, in which case the optimal contract is debt (Innes (1990)). In theory, the promised debt repayment could depend on many signals, but in practice it often does not. Our results suggest that this practice may be optimal – for example, a signal that suggests that bankruptcy was due to poor effort by the borrower, rather than bad luck, does not affect the repayment since the borrower receives zero in bankruptcy anyway. They also give conditions under which the repayment should depend on additional signals, as with performance-sensitive debt – if and only if it is informative about effort at intermediate output realizations.

This paper is related to both theoretical and applied literatures. Starting with the former, Gjesdal (1982), Amershi and Hughes (1989), Kim (1995), and Chaigneau, Edmans, and Gottlieb (2016b) extend the original Holmstrom (1979) informativeness principle, but not to settings with contracting constraints. Chaigneau, Edmans, and Gottlieb (2016a) study the effect on the optimal contract of increasing the precision of a given signal, but do not study the introduction of additional signals and thus have implications for performance-sensitive debt or performance-vesting options; they also do not allow for risk aversion. Moving to the latter, Dittmann, Maug, and Zhang (2011) quantify the effect on pay and firm value of various restrictions on CEO pay – restrictions on ex-post payments, ex-ante expected pay, and specific components of pay. Their calibration differs from our optimal contracting approach. Dittmann, Maug, and Spalt (2013) calibrate the cost savings from incorporating peer performance in executive contracts and Johnson and Tian (2000) compare the incentives provided by indexed and non-indexed options. The model of Manso, Strulovici, and Tchistyi (2010) offers an explanation for performance-sensitive debt based on adverse selection; ours is based on moral hazard. They also provide empirical evidence for performance-sensitive debt, as do Asquith, Beatty, and Weber (2005) and Adam and Streitz (2016). Bettis et al. (2010, 2016) are empirical studies of the frequency, value, and characteristics of performance-vesting equity.

1 The Model

We consider a principal (firm) and an agent (manager). The manager is protected by limited liability and has zero reservation utility. He exerts unobservable effort of $e \in \{0, 1\}$, where

$e = 0$ ("low effort") costs the manager 0, and $e = 1$ ("high effort") costs $C > 0$. In this section, we assume that both the manager and firm are risk-neutral as then contracting constraints (rather than risk sharing considerations) drive the contract, and so this is a natural framework to study the value of a signal under contracting constraints. Section 2 will extend the model to risk aversion and a continuum of effort levels.

Effort affects the probability distribution of output, which is distributed over an interval $q \in [0, \bar{q}]$ where $\bar{q}$ may be $+\infty$, and of an additional signal $s \in \{s_1, ..., s_S\}$.[3] Both output and the signal are contractible. We refer to an output/signal realization (q, s) as a "state" and assume that the distribution of (q, s) conditional on any e has full support.[4]

Conditional on effort e and signal s, output q is distributed according to the probability density function ("PDF"):

$$f(q|e, s) := \begin{cases} \pi_s(q) & \text{if } e = 1 \\ p_s(q) & \text{if } e = 0 \end{cases} .$$

The marginal distribution of the signal is represented by $\phi_{e'}^{s'} := \Pr(s = s'|e = e') > 0$. The joint distribution of (q, s) conditional on effort, denoted $f(q, s|e)$, is determined by their product. The marginal distribution of output is given by

$$f(q|e) = \sum_s \phi_e^s f(q|e, s) . \tag{1}$$

Let

$$LR_s(q) := \frac{\phi_1^s \pi_s(q)}{\phi_0^s p_s(q)} \tag{2}$$

denote the likelihood ratio associated with output q and signal s. We assume that the output distribution satisfies the strict monotone likelihood ratio property ("MLRP"): $LR_s(q)$ is strictly increasing in q for all s.

The firm has full bargaining power and offers the manager a vector of payments $\{w_s(q)\}$ conditional on the state. We assume that the incremental gain from effort $E[q|e = 1] - E[q|e = 0]$ is sufficiently higher than the cost of effort C that it is optimal for the firm to implement high

[3] A discrete signal space avoids measurability issues but is unimportant for our results. With a continuum of outputs and without limited liability on the principal, existence of an optimal contract is typically an issue. The contract cannot involve the principal paying only in the state with the highest likelihood ratio (as with discrete outputs) since this is a set of measure zero, so it must involve her paying in a neighborhood around that state. Without limited liability, the principal can generically improve on the contract by concentrating the payment in a smaller neighborhood, in which case an optimal contract fails to exist.

[4] The results are robust to a relaxation of this assumption, except that the optimal contract might not be unique. There could exist other optimal contracts that differ on a set of outputs that occur with probability zero.

effort (otherwise, the optimal contract would trivially involve a constant payment of zero). The firm thus solves the following program:

$$\min_{w_s(q)} \sum_s \int_0^{\bar{q}} w_s\left(q\right) \phi_1^s \pi_s\left(q\right) dq \tag{3}$$

$$s.t. \sum_s \int_0^{\bar{q}} w_s\left(q\right) \phi_1^s \pi_s\left(q\right) dq - C \geq 0 \tag{4}$$

$$\sum_s \int_0^{\bar{q}} w_s\left(q\right) \left[\phi_1^s \pi_s\left(q\right) - \phi_0^s p_s\left(q\right)\right] dq \geq C \tag{5}$$

$$w_s\left(q\right) \geq 0 \quad \forall q, s. \tag{6}$$

it minimizes the expected payment (3) subject to the manager's individual rationality constraint ("IR") (4), the incentive compatibility constraint ("IC") (5), and the limited liability constraint ("LL") (6). The IC (5) and LL (6) imply that the IR (4) is automatically satisfied, and so we ignore it in the analysis that follows.

1.1 Upper Bound on Payments

In this subsection, in addition to limited liability, we assume that there is a maximum payment to the manager, which can be output-dependent and is denoted $\overline{w}(q)$:

$$0 \leq w_s\left(q\right) \leq \overline{w}(q). \tag{7}$$

We assume that $\overline{w}(q)$ is nondecreasing in q. The primary application is $\overline{w}(q) = q$, i.e., limited liability on the firm. We consider the more general upper bound $\overline{w}(q)$ to allow the model to capture other contracting constraints; for example, a finite $\overline{w}(q)$ independent of q represents a cap on ex-post payments. To ensure that high effort is implementable, we assume:

$$\int_{q_0^s}^{\bar{q}} \overline{w}(q) \left[\phi_1^s \pi_s(q) - \phi_0^s p_s(q)\right] dq > C, \tag{8}$$

where, for each s, q_0^s is implicitly defined by $\phi_1^s \pi_s(q_0^s) = \phi_0^s p_s(q_0^s)$; if $\phi_1^s \pi_s(q) > \phi_0^s p_s(q)$ for all q, set $q_0^s = 0$, and if $\phi_1^s \pi_s(q) < \phi_0^s p_s(q)$ for all q, set $q_0^s = \bar{q}$. q_0^s exists and is unique by MLRP. Without (8), the firm would implement low effort and the optimal contract would trivially involve a zero payment.

This upper bound on payments is not necessary for our results; Appendix B considers the case in which there are no additional contracting constraints and shows that, like in this section, informative signals may have zero value. However, in the absence of an upper bound, the optimal contract typically involves a very large payment in the highest likelihood ratio state, which would vastly exceed total output and violate the firm's limited liability constraint, and zero payments in all other states. We thus consider an upper bound to achieve more realistic contracts.

Similar to Innes (1990), the solution involves paying the minimum amount possible (zero) when the likelihood ratio is below a threshold κ, and the maximum amount possible when it exceeds it. The threshold κ is chosen so that the IC binds (existence is shown in Appendix A); if more than one such threshold exists, we choose the largest one:

$$\kappa := \sup \left\{ \hat{\kappa} : \sum_s \int_{LR_s(q)>\hat{\kappa}} \overline{w}(q) \left[\phi_1^s \pi_s(q) - \phi_0^s p_s(q) \right] dq = C \right\}. \tag{9}$$

By MLRP, for each signal realization, the threshold for the likelihood ratio translates into a threshold for output. Lemma 1 characterizes the optimal contract:

Lemma 1 *The optimal contract with agent limited liability and an upper bound on payments is*

$$w_s(q) = \begin{cases} 0 & \text{if } q < q_s^*(\kappa) \\ \overline{w}(q) & \text{if } q > q_s^*(\kappa) \end{cases}, \tag{10}$$

where

$$q_s^*(\kappa) := \begin{cases} 0 & \text{if } LR_s(0) > \kappa \\ \bar{q} & \text{if } LR_s(\bar{q}) < \kappa \\ LR_s^{-1}(\kappa) & \text{if } LR_s(0) \leq \kappa \leq LR_s(\bar{q}) \end{cases} \tag{11}$$

and κ is determined by (9).

Lemma 1 yields a "live or die" contract: the manager receives the maximum payment $\overline{w}(q)$ if output exceeds a threshold q_s^* and zero otherwise. For a given signal realization s, the threshold output level q_s^* is chosen so that the likelihood ratio at this output level equals κ.[5]

In general, the threshold will depend on the signal realization s, and so the optimal contract is contingent upon both output and the signal. Proposition 1 shows that the contract is independent of s if and only if, for every s, the output q_s^* associated with a likelihood ratio of κ is the same, i.e. $q_s^* = q^*$, and so the firm optimally sets the same threshold q^*.

[5] For some signal realizations, it is possible that this threshold output level is a corner solution, in which case the manager either always receives the maximum or always receives zero; if all thresholds are interior, then $q_s^* = LR_s^{-1}(\kappa)$ for all s.

Proposition 1 *The optimal contract with agent limited liability and an upper bound on payments is independent of the signal if and only if*

$$LR_s(q^*) = LR_{s'}(q^*) = \kappa \ \ \forall s, s', \tag{12}$$

where κ is determined by (9).

Proposition 1 shows that the presence of contracting constraints requires us to refine the informativeness principle. Intuitively, q^* is the threshold that would be chosen in the absence of the additional signal s. A signal has positive value if and only if it affects the likelihood ratio at q^* – rather than in general – as only at q^* does the firm have freedom to change the contract, by making q^* depend on the signal. When $q_s^* = q^*$ – i.e. the firm would choose not to make the threshold depend on the signal – a signal that only affects the likelihood ratios at $q \neq q^*$ has zero value because the firm cannot make use of it. It cannot change the contract for $q < q^*$ because it is already paying zero, nor for $q > q^*$ because it is already paying the maximum. As a result, additional signals about effort are only valuable for intermediate output levels, not at tail output realizations. Note the requirement for informativeness in the "tails" does not mean that our result only applies to extreme signals. Any output realization above or below the threshold is a "tail" realization. Instead, our result implies that signals that shift probability weight either only above and/or below the threshold should not enter the optimal contract.

In sum, if output q is a sufficient statistic for effort e given (q, s), the signal s has zero value. However, even if q is not a sufficient statistic, s still has zero value if it is informative about effort only in states at which contracting constraints bind. In turn, contracting constraints bind everywhere except for at the threshold q_s^*.

1.2 Upper Bound on Incentives

In this subsection, in addition to agent limited liability, we assume that – rather than an upper bound on payments – there is an upper bound on the sensitivity of pay to performance:

$$w_s(q + \delta\epsilon) - w_s(q) \leq \epsilon, \ \forall \ \epsilon > 0. \tag{13}$$

Constraint (13) states that, for a dollar increase in output, payment to the agent can increase by at most $\delta \leq 1$ dollars. We consider values of δ such that an incentive compatible contract exists, by assuming:

$$\delta \left\{ \mathbb{E}[q|e=1] - \mathbb{E}[q|e=0] \right\} > C. \tag{14}$$

The primary application is a monotonicity constraint, as in Innes (1990), which corresponds to $\delta = 1$.[6] In this case, the manager cannot gain more than one-for-one with an increase in q. Innes (1990) justifies this constraint on two grounds. First, if it were violated, the manager would secretly borrow on his own account to increase output, since he would gain more from his contract than the amount he would have to repay. Second, if it were violated, the firm's payoff would fall with output over some region. Thus, it would exercise its control rights to "burn" output, raising its payoff. We generalize the upper bound on pay-performance sensitivity to a general $\delta = 1$ to allow the model to capture other constraints on incentives, e.g. maximum level of equity awards, or a maximum payment that increases with output (since higher performance makes higher pay more socially acceptable.)

Let $\overline{LR}_s(q) := \frac{\phi_1^s \int_q^{\bar{q}} \pi_s(z)dz}{\phi_0^s \int_q^{\bar{q}} p_s(z)dz} = \frac{\Pr(\tilde{q} \geq q, \tilde{s} = s|e=1)}{\Pr(\tilde{q} \geq q, \tilde{s} = s|e=0)}$ denote the likelihood ratio associated with the event $(\tilde{q} \geq q, \tilde{s} = s)$, which is strictly increasing by MLRP.[7] For each fixed κ and signal realization s, construct the threshold "strike price" as follows:

$$q_s^{**}(\kappa) := \begin{cases} 0 & \text{if } \overline{LR_s}(0) > \kappa \\ \bar{q} & \text{if } \overline{LR_s}(\bar{q}) < \kappa \\ \overline{LR_s}^{-1}(\kappa) & \text{if } \overline{LR_s}(0) \leq \kappa \leq \overline{LR_s}(\bar{q}) \end{cases}. \tag{15}$$

The threshold for the likelihood ratio κ is chosen so that the IC binds (existence is shown in Appendix A); if more than one such threshold exists, we choose the largest one:

$$\kappa := \sup\left\{\hat{\kappa} : \sum_s \int_{\overline{LR}_s(q) > \hat{\kappa}} \delta(q - q_s^{**}(\hat{\kappa}))\left[\phi_1^s \pi_s(q) - \phi_0^s p_s(q)\right] dq = C\right\} \in (0, \bar{q}). \tag{16}$$

The optimal contract given by Lemma 2 below:

[6] Under the primary applications of $\delta = 1$ (the upper bound on incentives is a monotonicity constraint) and $\overline{w}(q) = q$ (the upper bound on pay arises from principal limited liability), then it does not matter whether we retain the upper bound on pay when introducing the upper bound on incentives, since the principal's limited liability constraint never binds in the presence of monotonicity.

[7] We have:

$$\frac{d}{dq}\left\{\frac{\phi_1^s \int_q^{\bar{q}} \pi_s(z)dz}{\phi_0^s \int_q^{\bar{q}} p_s(z)dz}\right\} = \frac{\phi_1^s}{\phi_0^s} \frac{-\pi_s(q)\int_q^{\bar{q}} p_s(z)dz + p_s(q)\int_q^{\bar{q}} \pi_s(z)dz}{\left(\int_q^{\bar{q}} p_s(z)dz\right)^2},$$

which is positive if and only if

$$\frac{\pi_s(q)}{p_s(q)} < \frac{\int_q^{\bar{q}} \pi_s(z)dz}{\int_q^{\bar{q}} p_s(z)dz} \Leftrightarrow \int_q^{\bar{q}} \frac{\pi_s(z)}{\pi_s(q)}dz > \int_q^{\bar{q}} \frac{p_s(z)}{p_s(q)}dz \Leftrightarrow \int_q^{\bar{q}} \left[\frac{\pi_s(z)}{\pi_s(q)} - \frac{p_s(z)}{p_s(q)}\right] dz > 0,$$

which is satisfied because, for any $z > q$, MLRP guarantees that $\frac{\pi_s(z)}{p_s(z)} > \frac{\pi_s(q)}{p_s(q)}$.

Lemma 2 *The optimal contract with an upper bound on incentives is* $w_s(q) = \delta \max\{q - q_s^{**}(\kappa), 0\}$, *where* $q_s^{**}(\kappa)$ *and* κ *are determined by (15) and (16).*

Lemma 2 yields an option contract: if output exceeds q_s^{**}, the manager receives a proportion δ of the residual $q - q_s^{**}$, rather than the maximum payment as in Section 1.1. Proposition 2 gives a necessary and sufficient condition under which the strike price does not depend on the signal realization (i.e. $q_s^{**} = q^{**}\ \forall\, s$).

Proposition 2 *The optimal contract with an upper bound on incentives is independent of the signal if and only if*

$$\overline{LR}_s\left(q^{**}\right) = \overline{LR}_{s'}\left(q^{**}\right) = \kappa \ \ \forall s, s', \tag{17}$$

where κ *is determined by (16).*

Proposition 2 shows that the contract is independent of the signal if and only if the likelihood ratio that $q > q^{**}$ is always κ, regardless of s. Then, the firm optimally sets the same threshold q^{**}.

Note that the likelihood ratios in Propositions 1 and 2 concern different events. With a maximum payment in addition to limited liability, (Proposition 1), the firm pays zero for outputs below a threshold q^* and the maximum payment if it exceeds q^*, and so it can only adjust the payment by changing q^*. Doing so only affects the payment in a local neighborhood around q^* (i.e. changes it from 0 to q or vice-versa). As a result, a signal is only useful if it affects the likelihood ratio at a single point $q = q^*$ – i.e. provides information on whether $q = q^*$ is more likely to have resulted from working or shirking. If signal realization s' suggests that the manager has worked, the firm increases the payment from 0 to q by reducing the threshold to $q_{s'}^* < q^*$. If it suggests that he has shirked, the firm reduces the payment from q to 0 by increasing the threshold to $q_{s'}^* > q^*$.

With an upper bound on incentives (Proposition 2), the manager is paid $\delta(q - q^{**})$ if output exceeds q^{**}. Thus, if the firm uses the signal to change the strike price q^{**}, this alters the payment not only at $q = q^{**}$ (as in Proposition 1) but for all $q \geq q^{**}$; it cannot change the payment at specific output levels in isolation as this would violate the upper bound on incentives. Thus, a signal has value if it affects the likelihood ratio over a whole range $q \geq q^{**}$ – i.e. provides information on whether $q \geq q^{**}$ is more likely to have resulted from working or shirking. Any signal that shifts probability mass from below to above the threshold (or vice-versa) is valuable, as it affects the likelihood that output exceeds the threshold. For example, consider $q^{**} = 5$ and a monotonicity constraint ($\delta = 1$). The likelihood ratio is higher for $q = 7$ than $q = 3$, and so (in the absence of a signal), the manager receives 2 if $q = 7$ and 0 if $q = 3$.

If the event $(q \geq 5, s = s')$ is more informative about effort than the event $(q \geq 5, s = s'')$, i.e., observing that output is above the threshold is more informative about effort when the signal is s' than when it is s'', then the firm will optimally increase the payment when the signal is s' compared to when it is s''. To preserve monotonicity, this is achieved by varying the threshold across signals realizations, by setting a lower threshold for s' than for s'': $q^{**}_{s'} < q^{**}_{s''}$.

However, any signal that only redistributes mass below the threshold so that it stays below the threshold, or only redistributes mass above the threshold so that it stays above the threshold, has no value. Continuing the earlier example, if the event $(q \geq 7, s = s')$ is more informative about effort than the event $(q \geq 7, s = s'')$, but the event $(q \geq 5, s = s')$ is not more informative about effort than the event $(q \geq 5, s = s'')$, then the firm would like to increase the payment for $(q \geq 7, s = s')$. However, such a change would violate monotonicity, and so the firm cannot use the signal.

Despite the difference in the relevant likelihood ratios, Propositions 1 and 2 both establish similar conditions for a signal to have value. In both cases, the firm's only degree of freedom is the threshold q^* or q^{**}. With an upper bound on payments in addition to limited liability, changing q^* only has local effects, and so condition (12) depends on the likelihood ratio associated with $q = q^*$. With an upper bound on incentives, changing the strike price q^{**} affects payments at all higher outputs, and so condition (17) depends on the likelihood associated with $q \geq q^*$.

The above result has a number of applications for compensation contracts. First, it identifies the settings in which boards should invest in additional signals of manager performance, for instance through monitoring. A signal that shifts mass locally is only useful at intermediate output levels, not tail outputs, as only then will it affect the payment.[8] For example, in risk management, a "smoking gun" indicates that a bad event is due to poor performance (e.g. excessive risk-taking) rather than bad luck, but the bad event will likely lead to firing anyway. For instance, investors only noticed that Enron was adopting misleading accounting practices when it was already going bankrupt. Relatedly, the threshold output can be interpreted as a performance target below which the manager is fired. Signals are then only useful if they affect this target. If performance were very low, the manager would be fired anyway; if performance were very high, he would receive the entire output (or residual output) anyway.

Second, it implies that pay-for-luck (i.e. not obtaining signals to verify whether an output level was due to effort or luck) need not be suboptimal if it occurs at tail output realizations.

[8] If the monotonicity constraint is imposed, a signal that shifts mass from (say) 0 to $\overline{q}$ has value, even though it does not shift mass at intermediate output levels. However, a signal that shifts mass locally only has value at intermediate outputs.

In reality, instances of "pay for luck" typically concern very good or very bad outcomes – for example, Bertrand and Mullainathan (2001) consider how CEO pay varies with spikes and troughs in the oil price, and Jenter and Kanaan (2015) find that peer-group performance does not affect CEO firing decisions – but additional signals are only valuable for moderate outcomes.

Third, for stock options, it provides conditions under which the strike price should depend on additional signals, which can be implemented via option indexing or option repricing. Brenner, Sundaram, and Yermack (2000) find empirically that repricing nearly always involves a lowering of the strike price, and follows poor stock price performance (both absolute and industry-adjusted). Our model suggests that a reduction in the strike price should be prompted by positive, rather than negative, signals of effort, suggesting that such practices are suboptimal.[9] However, the model provides conditions under which repricing is optimal under the optimal contract, suggesting that it is not universally inefficient, contrary to concerns (e.g. Bebchuk and Fried (2004)) that it represents rewards for failure. In these cases, repricing should be part of the contract negotiated at $t = 0$, not the outcome of a renegotiation at $t = 1$.

The result also has implications for debt contracts. Innes (1990, footnote 2) notes that the model of risk neutrality and limited liability can be interpreted in two ways. First, the firm offers a compensation contract to the manager, as in the above exposition. Second, the manager is an entrepreneur who raises financing from the firm, an investor, which is the exposition in Innes (1990). The optimal contract is debt, and so a signal has no value in determining the repayment schedule, which is automatically the entire firm value if performance is poor, and the entire promised repayment (the face value of debt plus interest) if performance is good. It has value if and only if it affects the promised repayment. In theory, this amount could depend on many signals, but in practice it is often signal-independent. Proposition 2 potentially rationalizes this practice – even if signals are informative about effort, they should not enter the contract if they are only informative about effort in the tails. A signal which suggests that bankruptcy was due to poor effort by the borrower, rather than bad luck, does not affect the repayment since the borrower receives zero in bankruptcy anyway.

In addition, Proposition 2 provides conditions under which the repayment *should* depend on additional signals, as in performance-sensitive debt, where the promised repayment is higher upon negative signals of borrower performance. This is the case if and only if the signal affects the probability that performance exceeds the threshold under high effort relative to low

[9] Acharya, John, and Sundaram (2000) also study the repricing of options theoretically. In their model, repricing is not undertaken to make use of additional informative signals, but instead to maintain effort incentives when options fall out of the money.

effort. For example, if macroeconomic conditions affect the probability that output exceeds the threshold for both high and low effort by the same proportion, then the repayment should be independent of macroeconomic conditions.

2 Continuous Effort and Risk Aversion

This section studies the necessary and sufficient conditions for a signal to add value under risk aversion and contracting constraints. We also generalize the model to a continuous effort decision, but retain previous assumptions unless otherwise specified. Effort is now given by $e \in \mathbb{R}_+$.

Let $F(q|e,s)$ and $f(q|e,s)$ denote the cumulative distribution function ("CDF") and PDF of q conditional on e and s. We assume that, for each s, $F(\cdot|\cdot,s)$ is twice continuously differentiable (with respect to q and e). We continue to assume MLRP, which here entails $\frac{d}{dq}\left\{\frac{f_e(q|e,s)}{f(q|e,s)}\right\} > 0$, where $f_e(q|e,s)$ denotes the first derivative of the PDF with respect to e. We assume that the marginal distribution of the signal ϕ_e^s is differentiable with respect to e.

The manager's utility of money is given by a strictly increasing, weakly concave, twice differentiable function u. The manager has positive outside wealth $\bar{W}$ and reservation utility $\overline{u}$.[10] His cost of effort $C(e)$ is a twice differentiable, strictly increasing, and strictly convex function. Thus, given a contract $w_s(q)$ and an effort level e, his objective function is $\mathbb{E}[u(\bar{W} + w_s(q))|e] - C(e)$.

As in the first stage of Grossman and Hart (1983), the firm induces a given effort level $\hat{e}$. It chooses a function $w_s(\cdot)$, for each possible value of the signal s, to solve the following problem:

$$\min_{w_s(q)} \sum_s \phi_{\hat{e}}^s \int_0^{\overline{q}} w_s(q)\, f(q|\hat{e},s)dq \tag{18}$$

subject to

$$\sum_s \phi_{\hat{e}}^s \int_0^{\overline{q}} u(\bar{W} + w_s(q)) f(q|\hat{e},s)dq - C(\hat{e}) \geq \overline{u}, \tag{19}$$

$$\hat{e} \in \arg\max_e \sum_s \phi_e^s \int_0^{\overline{q}} u(\bar{W} + w_s(q)) f(q|e,s)dq - C(e), \tag{20}$$

$$w_s(q) \in [0, \overline{w}(q)]. \tag{21}$$

[10] With risk neutrality (Section 1) we assumed zero reservation utility, so that solving the incentive problem is costly to the principal as it involves paying the agent rents (i.e. a slack IR). With risk aversion, solving the incentive problem is costly for the principal even if the agent does not receive rents (i.e. the IR binds), since the principal must pay a premium for the risk the agent bears from receiving incentive compensation.

We study two cases. In the first case, only the manager is subject to limited liability: $\overline{w}(q) = +\infty$ for all q.[11] In the second case, there is also an upper bound $\overline{w}(q)$ on payments, as in Section 1.1.

Following Holmstrom (1979), Shavell (1979) and the subsequent literature on the informativeness principle (e.g. Gjesdal (1982), Kim (1995)), we assume that the first-order approach ("FOA") is valid; see Chaigneau, Edmans, and Gottlieb (2016b) for the informativeness principle without the FOA. We can thus replace the IC in (20) by the following equation: [-3em]1em

$$\frac{d}{de}\sum_s \phi_{\hat{e}}^s \int_0^{\overline{q}} u(\bar{W} + w_s(q)) f(q|\hat{e}, s) dq = C'(\hat{e}) \tag{22}$$

$$\Leftrightarrow \sum_s \left[\frac{d\phi_{\hat{e}}^s}{de}\int_0^{\overline{q}} u(\bar{W} + w_s(q)) f(q|\hat{e}, s) dq + \phi_{\hat{e}}^s \int_0^{\overline{q}} u(\bar{W} + w_s(q)) f_e(q|\hat{e}, s) dq\right] = C'(\hat{e}) \tag{23}$$

The optimal contract is given by Lemma 3 below.

Lemma 3 *Suppose an optimal contract exists and the first-order approach is valid. Let λ and μ denote the nonnegative Lagrange multipliers associated with the participation constraint in (19) and the incentive constraint in (20), respectively. With limited liability on the manager, the optimal contract is:*

$$w_s(q) = \max\left\{u'^{-1}\left(1/\left(\lambda + \mu\left[\frac{d\phi_{\bar{e}}^s/de}{\phi_{\hat{e}}^s} + \frac{f_e(q|\hat{e}, s)}{f(q|\hat{e}, s)}\right]\right)\right), 0\right\}. \tag{24}$$

With a maximum payment in addition to limited liability, the optimal contract is:

$$w_s(q) = \max\left\{\min\left\{u'^{-1}\left(1/\left(\lambda + \mu\left[\frac{d\phi_{\hat{e}}^s/de}{\phi_{\hat{e}}^s} + \frac{f_e(q|\hat{e}, s)}{f(q|\hat{e}, s)}\right]\right)\right), \overline{w}(q)\right\}, 0\right\} \tag{25}$$

Without the signal s, the likelihood ratio at a given value of q can be written as

$$LR(q) := \frac{f_e(q|\hat{e})}{f(q|\hat{e})}.$$

[11] In the risk-neutral model of Section 1 we also considered an upper bound on incentives, because it was necessary to obtain realistic contracts. Under risk aversion, realistic contracts can be obtained without such a bound (see, e.g., Proposition 4).

With the signal s, we define the likelihood ratio as

$$LR_s(q) := \frac{f_e(q,s|\hat{e})}{f(q,s|\hat{e})} = \frac{d\phi_{\hat{e}}^s/de}{\phi_{\hat{e}}^s} + \frac{f_e(q|\hat{e},s)}{f(q|\hat{e},s)}. \tag{26}$$

With limited liability on the manager, for each fixed κ and signal realization s, construct the threshold above which the payment is strictly positive as follows:

$$q_s^{***}(\kappa) := \begin{cases} 0 & \text{if } LR_s(0) > \kappa \\ \overline{w}(q) & \text{if } LR_s(\bar{q}) < \kappa \\ LR_s^{-1}(\kappa) & \text{if } LR_s(0) \leq \kappa \leq LR_s(\bar{q}) \end{cases} \tag{27}$$

The threshold likelihood ratio κ is chosen so that the IC binds for effort $\hat{e}$; if more than one such threshold exists, we choose the largest one:

$$\begin{aligned} \kappa := \quad & \sup\left\{\hat{\kappa} : \sum_s \left[\int_{LR_s(q)\leq\hat{\kappa}} u(\bar{W})\left[\frac{d\phi_{\hat{e}}^s}{de} f(q|\hat{e},s) + \phi_{\hat{e}}^s f_e(q|\hat{e},s)\right] dq \right.\right. \\ & \left.\left. + \int_{LR_s(q)>\hat{\kappa}} u(\bar{W} + w_s(q))\left[\frac{d\phi_{\hat{e}}^s}{de} f(q|\hat{e},s) + \phi_{\hat{e}}^s f_e(q|\hat{e},s)\right] dq\right] = C'(\hat{e})\right\} \end{aligned} \tag{28}$$

The contract in equation (24) is monotonic (via MLRP) and also continuous, since the likelihood ratio is continuous: its numerator and denominator are continuously differentiable with respect to q. However, its shape (e.g. whether it is concave, convex, or linear above q_s^{***}) depends on the shape of the utility function and likelihood ratio.

With a maximum payment in addition to limited liability, for each realization of s, define $\mathcal{M}_s$ as the set of values of q such that, with the contract described in (25):

$$w_s(q) = u'^{-1}\left(1/\left(\lambda + \mu\left[\frac{d\phi_{\hat{e}}^s/de}{\phi_{\hat{e}}^s} + \frac{f_e(q|\hat{e},s)}{f(q|\hat{e},s)}\right]\right)\right). \tag{29}$$

Intuitively, $\mathcal{M}_s$ is the set of output levels for which neither constraint on contracting binds.

The optimal contract is given by Proposition 3 below.

Proposition 3 *(i) With limited liability on the manager, the optimal contract is independent of the signal if and only if* $LR_s(q) = LR_{s'}(q)\ \forall s, s', q \geq q^{***} := \min_s\{q_s^{***}\}$.

(ii) With a maximum payment in addition to limited liability, the optimal contract is independent of the signal if and only if $LR_s(q) = LR_{s'}(q)\ \forall s, s', q \in \bigcup_s \mathcal{M}_s$.

The intuition is as follows. In both the binary and continuous effort cases, a signal has no value if and only if it does not affect the likelihood ratio. This likelihood ratio is $\frac{f_e(q,s|\hat{e})}{f(q,s|\hat{e})}$ with continuous effort and $\frac{\phi_1^s \pi_s(q)}{\phi_0^s p_s(q)}$ with binary effort. With risk neutrality, the relevant likelihood ratio is at a single intermediate output level, below which the manager receives zero and above which he receives the entire output. Here, the relevant likelihood ratio is at a range of output realizations ($q \geq q^{***}$ or $q \in \bigcup_s \mathcal{M}_s$). The intuition is as follows. A signal is only valuable at output levels where contracting constraints do not bind. With risk neutrality and a maximum payment in addition to limited liability (Section 1.1), for a given realization of s, contracting constraints bind everywhere except for at the threshold q_s^{***}. However, with risk aversion, there are many output levels ($q \geq q_s^{***}$ or $q \in \mathcal{M}_s$) where contracting constraints do not bind, and so the conditions for a signal to have value are weaker. In particular, while the manager receives zero below a threshold q_s^{***}, he does not automatically receive the full output above q_s^{***}. Thus, with limited liability on the manager only, the firm can change the payment in response to the signal for any $q > q_s^{***}$; with a maximum payment in addition to limited liability it can do so for any $q \in \mathcal{M}_s$.

Using the decomposition of the likelihood ratio in (26), the contract depends on the signal if it affects either $\frac{d\phi_{\hat{e}}^s/de}{\phi_{\hat{e}}^s}$, or $\frac{f_e(q|\hat{e},s)}{f(q|\hat{e},s)}$, or both (as long as it does not affect both in such a way that they cancel out so $\frac{d\phi_{\hat{e}}^s/de}{\phi_{\hat{e}}^s} + \frac{f_e(q|\hat{e},s)}{f(q|\hat{e},s)}$ is independent of s).[12] Thus, a signal can have value for two reasons. First, it is individually informative about a local deviation in effort from the equilibrium effort level $\hat{e}$, i.e. $\frac{d\phi_{\hat{e}}^s/de}{\phi_{\hat{e}}^s}$ depends on s. Second, it affects the informativeness of output, i.e. $\frac{f_e(q|\hat{e},s)}{f(q|\hat{e},s)}$ depends on s. For example, if the signal is a measure of macroeconomic conditions, low output during a boom (high signal) may be more informative about a deviation in effort than low output during a recession (low signal). In this case, the signal will be used in the contract, even if it is uninformative about effort (i.e. $\frac{d\phi_{\hat{e}}^s/de}{\phi_{\hat{e}}^s}$ is independent of s) because the manager's effort does not affect the probability of a recession.

We now move to applications of Proposition 3. In addition to the general implications, shared with Section 1, that a signal is only valuable if it is informative about effort at output levels for which contracting constraints do not bind, we can also derive more specific implications for *how* a valuable signal should be incorporated in the optimal contract where we can solve for it. Proposition 4 provides sufficient conditions for the optimal contract to be an option: limited liability on the manager, log utility, and a likelihood ratio that is linear in output ($\frac{f_e(q|\hat{e},s)}{f(q|\hat{e},s)} = a_s + b_s q$ so that $LR_s(q) = \frac{d\phi_{\hat{e}}^s/de}{\phi_{\hat{e}}^s} + a_s + b_s q$), as with the Normaland Gamma

[12]In Section 1, the likelihood ratio was multiplicative rather than additive (equation (2)) since effort was binary.

distributions. For each realization s of the signal, we define $\check{q}_s$ by

$$\check{q}_s := \begin{cases} 0 & \text{if } LR_s(0) > 0 \\ \bar{q} & \text{if } LR_s(\bar{q}) < 0 \\ LR_s^{-1}(0) & \text{if } LR_s(0) < 0 < LR_s(\bar{q}) \end{cases} \tag{30}$$

Note that for each s, $\check{q}_s$ is unique due to MLRP.

Proposition 4 *(i) With limited liability on the manager, a likelihood ratio linear in q, and log utility ($u(w) = \ln w$), the optimal contract can be written as*

$$w_s(q) = n_s^* \max\{q - q_s^{***}, 0\}, \tag{31}$$

with $n_s^* \geq 0\ \forall s$.

(ii) The number of options n_s^* received ex-post by the manager is independent of the signal if and only if $\frac{d}{dq}\frac{f_e(q|\hat{e},s)}{f(q|\hat{e},s)}$ is independent of s.

(iii) The strike price q_s^{***} is independent of the signal if the IR is nonbinding and $\check{q}_s$ is independent of s, or if $\frac{d}{dq}\frac{f_e(q|\hat{e},s)}{f(q|\hat{e},s)}$ and $\frac{d\phi_{\hat{e}}^s/de}{\phi_{\hat{e}}^s} + \frac{f_e(0|\hat{e},s)}{f(0|\hat{e},s)}$ are independent of s.

In Proposition 4, the manager has n_s^* options with exercise price q_s^{***}. With log utility, $u'^{-1}(1/x) = x$. Thus, with a linear likelihood ratio $LR_s(q) = \frac{d\phi_{\hat{e}}^s/de}{\phi_{\hat{e}}^s} + a_s + b_s q$, log utility yields $w_s(q) = u'^{-1}(1/LR_s(q)) = \frac{d\phi_{\hat{e}}^s/de}{\phi_{\hat{e}}^s} + a_s + b_s q$, i.e., the contract is linear in q. To our knowledge, part (i) of Proposition 4 provides the first set of sufficient conditions for options to be the optimal contract when the manager is risk-averse, in contrast to Innes (1990) where he is risk-neutral.[13]

While Proposition 3 studied the conditions under which a signal affects any dimension of the contract, part (ii) of Proposition 4 studies the conditions under which a signal affects specifically the number of options. Proposition 3 showed that a signal has value if it affects any component of the likelihood ratio: either $\frac{d\phi_{\hat{e}}^s/de}{\phi_{\hat{e}}^s}$ (i.e. is individually informative about effort) or $a_s + b_s q$ (i.e. affects the informativeness of output for effort). Such a signal will, in general, alter the Lagrange multiplier μ and thus scale up or down the number of options $n_s^* = \mu b_s$ received across all signals s. However, the number of options actually received ex post may

[13] Jewitt, Kadan, and Swinkels (2008) show that the contract is "option-like" with risk aversion and agent limited liability, in that incentives are zero for low output and positive for high output, but do not identify conditions under which the increasing portion of the contract is linear. Hemmer, Kim, and Verrecchia (1999) identify a linear likelihood ratio and log utility as leading to the contract having a linear portion, but did not combine them with limited liability to obtain an option contract.

still not depend on the actual signal realization. This will only arise if b_s, rather than any other component of the likelihood ratio, depends on s – i.e. the signal realization affects the rate at which the informativeness of the stock price changes with the level of stock price. The intuition is as follows. As in any principal-agent model, pay is increasing in the likelihood ratio, and so the sensitivity of pay to output (here, the number of options $n_s^* = \mu b_s$) depends on the sensitivity of the likelihood ratio to output, $\frac{dLR_s(q)}{dq} = b_s$. If the likelihood ratio increases faster with output when s is high (i.e. if $s > s'$ implies $b_s > b_{s'}$), the contract should be steeper and more options should be granted (i.e. $n_s^* > n_{s'}^*$).

Thus, continuing the earlier example, even if effort does not affect the probability of a recession ($\frac{d\phi_{\hat{e}}^s/de}{\phi_{\hat{e}}^s}$ is independent of s, a signal of economic conditions), it is optimal for more options to vest in bad (good) times if the likelihood ratio is more sensitive to the stock price – i.e. effort has a greater effect on the stock price – in bad (good) times. To our knowledge, this is the first theoretical justification of the conditions under which performance-based vesting is optimal.

Note that it may be efficient for more options to vest upon *low* signals. This result implies that the existing practice, of vesting always being triggered by good performance, may not be optimal. However, it echoes the theoretical prediction of Edmans, Gabaix, Sadzik, and Sannikov (2012) who show that more equity should be granted when firm value is low, and the empirical finding of Core and Larcker (2002) who show that more equity is granted upon poor performance (although they study stock rather than options).[14] Conversely, even if a signal (say, revenues) is informative about effort (i.e. $\frac{d\phi_{\hat{e}}^s/de}{\phi_{\hat{e}}^s}$ depends on revenues s), it should not affect vesting if it does not affect the sensitivity of the likelihood ratio to the stock price. The likelihood ratio given s could be a vertical translation of the likelihood ratio given s', but if both are as sensitive to q then there should not be any performance-based vesting.

Part (iii) identifies two sets of conditions under which the strike price is independent of the signal. The first is that the IR is slack and that the likelihood ratio equals zero at the same q for all s. If the IR is slack, the contract is determined by the IC. The number of options then depends on the sensitivity of the likelihood ratio to output, and the strike price is chosen such that the payment is sensitive to performance (i.e., is nonzero) if and only if the likelihood ratio is positive. If the likelihood ratio turns positive at the same value of q for all s, i.e., if $\check{q}_s$ is independent of s, then the strike price is independent of s. For example, improvements in stock price efficiency or the informativeness of accounting information systems may affect the slope of the likelihood ratio, but not the level at which the likelihood ratio is zero. The

[14] Note that, if a low signal leads to more options vesting, it may also lead to their strike price increasing, so that the agent is not better off by generating a low signal.

second condition is that the likelihood ratio is independent of the signal for any q. A sufficient condition is that both the slope $\left(\frac{d}{dq}\frac{f_e(q|\hat{e},s)}{f(q|\hat{e},s)}\right)$ and level of the likelihood ratio $\left(\frac{d\phi^s_{\hat{e}}/de}{\phi^s_{\hat{e}}} + \frac{f_e(0|\hat{e},s)}{f(0|\hat{e},s)}\right)$ are independent of s.

3 Conclusion

This paper shows that the informativeness principle must be modified in the presence of contracting constraints, in turn allowing us to understand whether and what conditions real-life compensation and financing contracts should depend on performance signals in addition to output. Specifically, a signal is valuable if and only if it is informative about effort at output levels at which contracting constraints do not bind, rather than in general. We derive necessary and sufficient conditions for a signal to have value under various contracting constraints. Starting with risk neutrality and bilateral limited liability (or an upper bound on payments), a signal is valuable if and only if it is informative about effort at a single intermediate output. If there is also a monotonicity constraint, or an upper bound on output-based incentives, a signal is valuable if and only if it provides information on whether beating the target performance level is more likely to have resulted from working or shirking. Likewise, under risk aversion, a signal is valuable if and only if it is informative about effort at outputs for which constraints on contracting do not bind.

In addition to the theoretical contribution of new conditions for a signal to have value in the presence of contracting constraints, our results have a number of implications for real-life contracts. Starting with compensation contracts, our results offer a potential explanation as to why both pay and the firing decision do not depend on many potentially informative signals, why it may not be optimal to filter out luck, when options should be repriced, and whether options should have performance-based vesting conditions. In particular, performance-based vesting is not necessarily optimal even if a signal is incrementally informative about effort; instead, it must affect the rate at which the informativeness of output changes with the level of output. Moving to financing contracts, they suggest whether and under what conditions debt should be performance-sensitive.

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A Proofs

Proof of Lemma 1. The firm's program is:

$$\min_{\{w_s(q)\}} \sum_s \int_0^{\bar{q}} w_s(q)\, \phi_1^s \pi_s(q)\, dq$$

subject to

$$\begin{aligned} &0 \leq w_s(q) \leq \overline{w}(q) \ \ \forall q \in [0, \bar{q}], \\ &\sum_s \int_0^{\bar{q}} w_s(q) \left[\phi_1^s \pi_s(q) - \phi_0^s p_s(q)\right] dq \geq C. \end{aligned}$$

This is an infinite-dimensional linear program, which has the following first-order conditions:

$$w_s(q) = \begin{Bmatrix} \overline{w}(q) \\ 0 \end{Bmatrix} \text{ if } \phi_1^s \pi_s(q) - \mu \left[\phi_1^s \pi_s(q) - \phi_0^s p_s(q)\right] \begin{Bmatrix} > \\ < \end{Bmatrix} 0, \tag{32}$$

for all s (where μ is the Lagrange multiplier associated with the IC), as well as the IC, which must bind:

$$\sum_s \int_{LR_s(q) \geq \frac{\mu}{\mu-1}} \overline{w}(q) \left[\phi_1^s \pi_s(q) - \phi_0^s p_s(q)\right] dq = C. \tag{33}$$

Letting $\kappa := \frac{\mu}{\mu-1}$ and using (32), it follows that $w_s(q) = \overline{w}(q)$ if $LR_s(q) > \kappa$, and $w_s(q) = 0$ if $LR_s(q) < \kappa$. Moreover, equation (33) becomes:

$$\sum_s \int_{LR_s(q) > \kappa} \overline{w}(q)[\phi_1^s \pi_s(q) - \phi_0^s p_s(q)] dq = C. \tag{34}$$

We first show that the set of contracts satisfying these necessary conditions is not empty. Since each value of κ fully characterizes a contract through equations (10) and (11), it suffices to show that there exists a κ that solves (34). The left-hand side ("LHS") of (34) converges to $\int_{q_0^s}^{\bar{q}} \overline{w}(q) \left[\phi_1^s \pi_s(q) - \phi_0^s p_s(q)\right] dq$ as $\kappa \searrow 1$. From (8), this exceeds C. Moreover, it converges to $0 < C$ as $\kappa \nearrow +\infty$. Therefore, by the Intermediate Value Theorem, there exists κ satisfying (34).

Notice that κ orders all contracts that satisfy the necessary optimality conditions: by MLRP, a higher threshold for the likelihood ratio means that the firm pays (weakly) less in each state. Thus, if (34) has multiple solutions, the optimum is the contract associated with the highest κ, as defined in equation (9). ■

Proof of Proposition 1. Let $q^*_{s_1} = ... = q^*_{s_S} = q^*$. There are two possibilities: an interior solution $q^* \in (0, \bar{q})$ and a boundary solution $q^* \in \{0, \bar{q}\}$. Using the conditions from Lemma 1 for an interior solution establishes the result stated in the proposition.

We next verify that the solution cannot be at the boundary. For a boundary solution we need either $LR_s(0) > \kappa$ or $LR_s(\bar{q}) < \kappa \; \forall \; s$, where κ is determined by equation (9). In the first case, the firm always receives zero, which violates (9) since we have assumed that high effort is optimal for the principal (which means that the firm can get a strictly positive payoff from implementing it). In the second case, the manager always receives zero, violating equation (9) again as the IC is not satisfied. ■

Proof of Lemma 2. The proof is divided into two parts:

Step 1. Conditional on each signal realization, the optimal contract is an option.

This part adapts the argument from Matthews (2001) to show that the optimal contract gives the manager δ options with payoff $\max\{q - q_s, 0\}$ for some strike price q_s. Let $w_s(q)$ be a contract satisfying the LL, monotonicity, and the IC. Notice that there exists a unique option contract with the same expected payment conditional on each signal realization. In other words, for each s, there exists a unique q_s that solves

$$\int_0^{\bar{q}} \delta \max\{q - q_s, 0\}\pi_s(q)dq = \int_0^{\bar{q}} w_s(q)\pi_s(q)dq. \tag{35}$$

Suppose $w_s(q) \neq \delta \max\{q - q_s, 0\}$ in a set of states with positive measure. We claim that the manager's incentives to shirk are higher under $w_s(q)$ than with the option contract:

$$\int_0^{\bar{q}} w_s(q)p_s(q)dq > \int_0^{\bar{q}} \delta \max\{q - q_s, 0\}p_s(q)dq.$$

Let $v_s(q) := w_s(q) - \delta \max\{q - q_s, 0\}$. Since $v_s(q) \neq 0$ with positive probability and it has mean zero, it must be strictly positive and strictly negative in sets of states with positive probability. Moreover, because $w_s(q)$ satisfies the LL and monotonicity, there exists $k \in (0, \bar{q})$ such that $v_s(q) \geq 0$ if $q \leq k$ and $v_s(q) \leq 0$ if $q \geq k$. Then,

$$\begin{aligned}
0 &= \textstyle\int_0^{\bar{q}} v_s\left(q\right)\pi_s\left(q\right)dq \\
&= \textstyle\int_0^{\bar{q}} v_s\left(q\right)\frac{\pi_s(q)}{p_s(q)}p_s(q)dq \\
&= \textstyle\int_0^{k} v_s\left(q\right)\frac{\pi_s(q)}{p_s(q)}p_s(q)dq + \int_k^{\bar{q}} v_s\left(q\right)\frac{\pi_s(q)}{p_s(q)}p_s(q)dq \\
&< \textstyle\int_0^{k} v_s\left(q\right)\frac{\pi_s(k)}{p_s(k)}p_s(q)dq + \int_k^{\bar{q}} v_s\left(q\right)\frac{\pi_s(k)}{p_s(k)}p_s(q)dq \\
&= \textstyle\frac{\pi_s(k)}{p_s(k)}\int_0^{\bar{q}} v_s\left(q\right)p_s(q)dq,
\end{aligned} \tag{36}$$

where the first line uses the fact that $v_s(q)$ has mean zero under high effort; the second multiplies and divides by $p_s(q)$, the third splits the integral between the positive and negative values of $v_s(q)$; the fourth uses MLRP and the fact that the terms in the first integral are positive whereas the ones in the second integral are negative; and the last line regroups the integrals.

Thus, conditional on each signal realization s, shirking gives the manager a higher payment with the original contract than with the option. Moreover, both contracts pay the same expected amount when the manager exerts effort. We have therefore shown that substituting a non-option contract with an option allows the firm to relax the IC. Since the IC must bind at the optimum, this establishes that the original contract cannot be optimal.

Step 2. Determining the optimal strike prices.

Since any option contract satisfies the LL and monotonicity, the firm's program becomes:

$$\min_{\{q_s\}_{s=1,\ldots,S}} \sum_s \int_{q_s}^{\bar{q}} \delta\left(q - q_s\right) \phi_1^s \pi_s\left(q\right) dq. \tag{37}$$

subject to

$$\sum_s \int_{q_s}^{\bar{q}} \delta\left(q - q_s\right) \left[\phi_1^s \pi_s\left(q\right) - \phi_0^s p_s\left(q\right)\right] dq \geq C. \tag{38}$$

The necessary first-order conditions associated with this program are equation (15) and the binding IC

$$\sum_s \int_{\overline{LR}_s(q)>\kappa} \delta(q - q_s^{**}(\kappa)) \left[\phi_1^s \pi_s\left(q\right) - \phi_0^s p_s\left(q\right)\right] dq = C, \tag{39}$$

where $\kappa := \frac{\lambda}{\lambda - 1}$ and λ is the Lagrange multiplier associated with the IC.

The remainder of the proof follows the same steps as the proof of Lemma 1. Each κ determines $q_s^{**}(\kappa)$ according to equation (15). From the Intermediate Value Theorem, there exists κ that solves equation (39): the LHS of (39) evaluated at $\kappa = 0$ exceeds C (by equation (14)) and it converges to $0 < C$ as $\kappa \to \infty$. Moreover, the firm's profits are ordered by κ: by MLRP, higher thresholds are associated with higher strike prices, which are cheaper. Thus, the best contract among all contracts that satisfy the necessary optimality conditions is the one associated with the largest κ, yielding (16). ■

Proof of Proposition 2. Analogous to the proof of Proposition 1. ■

Proof of Lemma 3. For now we ignore the LL constraint(s) in (21). Denoting by λ and μ the Lagrange multipliers associated respectively with (19) and (23), the first-order condition

("FOC") with respect to $w_s(q)$ in the program in (18), (19), and (23) is:

$$\phi_{\hat{e}}^s f(q|\hat{e},s) - \lambda \phi_{\hat{e}}^s u'(\bar{W} + w_s(q)) f(q|\hat{e},s) - \mu u'(\bar{W} + w_s(q)) \left[\frac{d\phi_{\hat{e}}^s}{de} f(q|\hat{e},s) + \phi_{\hat{e}}^s f_e(q|\hat{e},s) \right] = 0$$

$$\Leftrightarrow \quad \frac{1}{u'(\bar{W} + w_s(q))} = \lambda + \mu \left[\frac{d\phi_{\hat{e}}^s/de}{\phi_{\hat{e}}^s} + \frac{f_e(q|\hat{e},s)}{f(q|\hat{e},s)} \right]. \tag{40}$$

With limited liability on the manager only, we have $\underline{m}(q) = \bar{W}$ and $\overline{m}(q) = \infty$, using the notations in Jewitt, Kadan, and Swinkels (2008). Using the FOC in (40), the same reasoning as in Proposition 1 in Jewitt, Kadan, and Swinkels (2008) applies for any given signal realization s, so that the optimal contract for a given s is defined implicitly by:

$$\frac{1}{u'(\bar{W} + w_s(q))} = \begin{cases} \lambda + \mu \left[\frac{d\phi_{\hat{e}}^s/de}{\phi_{\hat{e}}^s} + \frac{f_e(q|\hat{e},s)}{f(q|\hat{e},s)} \right] & \text{if } \lambda + \mu \left[\frac{d\phi_{\hat{e}}^s/de}{\phi_{\hat{e}}^s} + \frac{f_e(q|\hat{e},s)}{f(q|\hat{e},s)} \right] \geq \frac{1}{u'(\bar{W})}, \\ \frac{1}{u'(\bar{W})} & \text{if } \lambda + \mu \left[\frac{d\phi_{\hat{e}}^s/de}{\phi_{\hat{e}}^s} + \frac{f_e(q|\hat{e},s)}{f(q|\hat{e},s)} \right] < \frac{1}{u'(\bar{W})}, \end{cases} \tag{41}$$

with $\lambda \geq 0$ and $\mu > 0$. Equation (41) can be rewritten as (24).

With an upper bound on payments, we have $\underline{m}(q) = \bar{W}$ and $\overline{m}(q) = \overline{w}(q) + \bar{W}$. The optimal contract for a given value s of the signal is defined implicitly by:

$$\frac{1}{u'(\bar{W} + w_s(q))} = \begin{cases} \frac{1}{u'(\bar{W}+\overline{w}(q))} & \text{if } \frac{1}{u'(\bar{W}+\overline{w}(q))} < \lambda + \mu \left[\frac{d\phi_{\hat{e}}^s/de}{\phi_{\hat{e}}^s} + \frac{f_e(q|\hat{e},s)}{f(q|\hat{e},s)} \right], \\ \lambda + \mu \left[\frac{d\phi_{\hat{e}}^s/de}{\phi_{\hat{e}}^s} + \frac{f_e(q|\hat{e},s)}{f(q|\hat{e},s)} \right] & \text{if } \frac{1}{u'(\bar{W})} \leq \lambda + \mu \left[\frac{d\phi_{\hat{e}}^s/de}{\phi_{\hat{e}}^s} + \frac{f_e(q|\hat{e},s)}{f(q|\hat{e},s)} \right] \leq \frac{1}{u'(\bar{W}+\overline{w}(q))}, \\ \frac{1}{u'(\bar{W})} & \text{if } \lambda + \mu \left[\frac{d\phi_{\hat{e}}^s/de}{\phi_{\hat{e}}^s} + \frac{f_e(q|\hat{e},s)}{f(q|\hat{e},s)} \right] < \frac{1}{u'(\bar{W})}, \end{cases} \tag{42}$$

with $\lambda \geq 0$ and $\mu > 0$. Equation (42) can be rewritten as (25).

Proof of Proposition 3. With limited liability on the manager, and for each s, the optimal contract described in (24) depends on $LR_s(q) = \frac{d\phi_{\hat{e}}^s/de}{\phi_{\hat{e}}^s} + \frac{f_e(q|\hat{e},s)}{f(q|\hat{e},s)}$ for $q \geq q_s^{***}$, while it is independent of s for $q < q_s^{***}$. If $LR_s(q_s^{***}) = LR_{s'}(q_{s'}^{***})\ \forall s, s', q > \min_s\{q_s^{***}\}$, MLRP and the definition of q_s^{***} imply that $q_s^{***} = q_{s'}^{***}\ \forall s, s'$, i.e., there exists q^{***} such that $q_s^{***} = q^{***}\ \forall s$. Therefore, if $LR_s(q)$ does not depend on s for any $q \geq \min_s\{q_s^{***}\}$, then the payment is independent of s, otherwise it depends on s for some output realizations.

With a maximum payment in addition to limited liability, and for each s, the optimal contract described in (25) depends on $LR_s(q)$ for $q \in \mathcal{M}_s$, while it is independent of s for $q \notin \mathcal{M}_s$. Therefore, if $LR_s(q)$ does not depend on s for any $q \in \bigcup_s \mathcal{M}_s$, then the payment is independent of s, otherwise it depends on s for some output realizations.

Proof of Proposition 4. (i) A linear likelihood ratio can be written as $\frac{f_e(q|\hat{e},s)}{f(q|\hat{e},s)} := a_s + b_s q$

with $b_s > 0 \; \forall s$ due to MLRP. With log utility, $u'^{-1}(w) = \frac{1}{w}$. Thus, (24) becomes

$$w_s(q) = \max\left\{\lambda + \mu\left[\frac{d\phi_{\hat{e}}^s/de}{\phi_{\hat{e}}^s} + \frac{f_e(q|\hat{e},s)}{f(q|\hat{e},s)}\right], 0\right\}. \tag{43}$$

Letting $\bar{a}_s := \lambda + \mu\left[\frac{d\phi_{\hat{e}}^s/de}{\phi_{\hat{e}}^s} + a_s\right]$ and $n_s^* := \mu b_s$, we have $\lambda + \mu\left[\frac{d\phi_{\hat{e}}^s/de}{\phi_{\hat{e}}^s} + \frac{f_e(q|\hat{e},s)}{f(q|\hat{e},s)}\right] = \bar{a}_s + n_s^* q$. Equation (43) can then be rewritten as

$$w_s(q) = \max\left\{\bar{a}_s + n_s^* q, 0\right\}. \tag{44}$$

Letting $q_s^{***} := -\frac{\bar{a}_s}{n_s^*}$, equation (44) can be rewritten as

$$w_s(q) = \max\left\{n_s^*\left(q - q_s^{***}\right), 0\right\} = n_s^* \max\left\{q - q_s^{***}, 0\right\}. \tag{45}$$

The contract $w_s(q)$ can be implemented by giving n_s^* options with exercise price q_s^{***} to the manager.

(ii) The number of options received by the manager for a given realization of s is $n_s^* = \mu b_s$. In addition, $\frac{d}{dq}\frac{f_e(q|\hat{e},s)}{f(q|\hat{e},s)} = b_s$. Therefore, n_s^* is independent of s if and only if $\frac{d}{dq}\frac{f_e(q|\hat{e},s)}{f(q|\hat{e},s)}$ is independent of s.

(iii) The exercise price q_s^{***} is

$$q_s^{***} = -\frac{\bar{a}_s}{n_s^*} = -\frac{\frac{\lambda}{\mu} + \frac{d\phi_{\hat{e}}^s/de}{\phi_{\hat{e}}^s} + a_s}{b_s}.$$

If $\frac{d}{dq}\frac{f_e(q|\hat{e},s)}{f(q|\hat{e},s)}$ is independent of s so that n_s^* is independent of s (cf. part (ii)), then q_s^{***} is independent of s if and only if $\bar{a}_s$ is independent of s. Setting $q = 0$ in $\frac{f_e(q|\hat{e},s)}{f(q|\hat{e},s)} := a_s + b_s q$ gives $a_s = \frac{f_e(0|\hat{e},s)}{f(0|\hat{e},s)}$. Given the definition of $\bar{a}_s$ above, it follows that $\bar{a}_s$ is independent of s if and only if $\frac{d\phi_{\hat{e}}^s/de}{\phi_{\hat{e}}^s} = -\frac{f_e(0|\hat{e},s)}{f(0|\hat{e},s)} \; \forall s$.

In addition, if the participation constraint is nonbinding so that $\lambda = 0$, we have

$$w_s(q) = \max\left\{\mu\left(\frac{d\phi_{\hat{e}}^s/de}{\phi_{\hat{e}}^s} + a_s + b_s q\right), 0\right\} = \mu \max\left\{LR_s(q), 0\right\}.$$

By construction, the strike price q_s^{***} is such that $w_s(q) > 0$ if and only if $q > q_s^{***}$. Therefore, q_s^{***} is independent of s if and only if $LR_s(q) = 0$ at the same value of q for all s, i.e., $\check{q}_s$ (which is by definition such that $LR_s(\check{q}_s) = 0$) is independent of s.

B Limited Liability on Manager Only

This Appendix considers the core model of Section 1 but without any upper bound on payments or incentives, i.e. the only constraint is limited liability on the agent. With a continuum of outputs and without limited liability on the principal, existence of an optimal contract is typically an issue.[15] We thus here assume a discrete output distribution $q \in \{q_1, ..., q_Q\}$. Let $\pi_{q,s}$ and $p_{q,s}$ denote the joint probabilities of (q,s) conditional on high and low efforts, respectively (whereas $\pi_s(q)$ and $p_s(q)$ refer to marginal distributions in the core model). To simplify the exposition, we assume full support ($\pi_{q,s} > 0$ and $p_{q,s} > 0$), although this is not needed for our results.

The firm solves the following program:

$$\min_{w_{q,s}} \sum_{q,s} \pi_{q,s} w_{q,s} \tag{46}$$

$$s.t. \sum_{q,s} \pi_{q,s} w_{q,s} - C \geq 0 \tag{47}$$

$$\sum_{q,s} (\pi_{q,s} - p_{q,s}) w_{q,s} \geq C \tag{48}$$

$$w_{q,s} \geq 0 \quad \forall q, s. \tag{49}$$

As in Section 1, the IC and the manager's LL guarantee that the IR holds.

A signal is valuable if including it in the contract (in addition to output) reduces the firm's cost of implementing $e = 1$. Lemma 4 below states that a signal is valuable if and only if it is informative about effort (i.e. affects the likelihood ratio) in states where the payment is strictly positive. (All proofs are in Appendix A.)

Lemma 4 *Let $\{w_{q,s}\}$ be an optimal contract for implementing $e = 1$ with $w_{q,s} > 0$ and $w_{q,s'} > 0$ for some q, s, and s'. Then, $w_{q,s} = w_{q,s'}$ only if $\frac{\pi_{q,s}}{p_{q,s}} = \frac{\pi_{q,s'}}{p_{q,s'}}$.*

Proof of Lemma 4. Fix a vector of payments that satisfy the IC, and consider the following perturbation:

$$w'_{q,s} = w_{q,s} + \frac{\epsilon}{\pi_{q,s} - p_{q,s}}, \quad \text{and} \quad w'_{q,s'} = w_{q,s'} - \frac{\epsilon}{\pi_{q,s'} - p_{q,s'}}.$$

[15] Under discrete outputs, the optimal contract involves the principal paying only in the state with the highest likelihood ratio. However, with continuous outputs, this is a set of measure zero, so the contract must involve her paying in a neighborhood around that state. Without limited liability, the principal can generically improve on the contract by concentrating the payment in a smaller neighborhood, in which case an optimal contract fails to exist.

This perturbation keeps the incremental benefit from effort constant and therefore preserves the IC. The LL continues to hold for $\epsilon > 0$ if $w_{q,s'} > 0$, and for $\epsilon < 0$ if $w_{q,s} > 0$. The expected payment (46) increases by:

$$\left(\frac{\pi_{q,s}}{\pi_{q,s} - p_{q,s}} - \frac{\pi_{q,s'}}{\pi_{q,s'} - p_{q,s'}} \right) \epsilon. \tag{50}$$

If the original contract entails $w_{q,s} = w_{q,s'} > 0$ (i.e., a strictly positive payment for output q that does not depend on whether the signal is s or s'), then such a perturbation would satisfy both the IC and LL. Thus, for this contract to be optimal, such a perturbation cannot reduce the expected payment. The term in (50) must be non-positive for all ϵ small enough:

$$\frac{\pi_{q,s}}{\pi_{q,s} - p_{q,s}} = \frac{\pi_{q,s'}}{\pi_{q,s'} - p_{q,s'}},$$

which yields $\frac{\pi_{q,s}}{p_{q,s}} = \frac{\pi_{q,s'}}{p_{q,s'}}$. ■

Lemma 5 states that the payment is strictly positive only in states that maximize the likelihood ratio.

Lemma 5 *Let $\{w_{q,s}\}$ be an optimal contract for implementing $e = 1$. If $\frac{\pi_{q,s}}{p_{q,s}} < \max_{(q',s')} \left\{ \frac{\pi_{q',s'}}{p_{q',s'}} \right\}$, then $w_{q,s} = 0$.*

Proof of Lemma 5. Consider the following perturbation, which, as in the proof of Lemma 4, keeps the incremental benefit from effort constant, thereby preserving the IC:

$$w'_{q,s} = w_{q,s} + \frac{\epsilon}{\pi_{q,s} - p_{q,s}}, \quad \text{and} \quad w'_{q',s'} = w_{q',s'} - \frac{\epsilon}{\pi_{q',s'} - p_{q',s'}}.$$

LL continues to hold for $\epsilon > 0$ if $w_{q',s'} > 0$ and for $\epsilon < 0$ if $w_{q,s} > 0$. The expected payment (46) increases by:

$$\left(\frac{\pi_{q,s}}{\pi_{q,s} - p_{q,s}} - \frac{\pi_{q',s'}}{\pi_{q',s'} - p_{q',s'}} \right) \epsilon. \tag{51}$$

Let $(q, s) \in \arg\max_{(q'',s'')} \left\{ \frac{\pi_{q'',s''}}{p_{q'',s''}} \right\}$ denote a state with the highest likelihood ratio and consider a state (q', s') that does not have the highest likelihood ratio:

$$\frac{\pi_{q',s'}}{p_{q',s'}} < \frac{\pi_{q,s}}{p_{q,s}}. \tag{52}$$

From (52), the term inside the parentheses in (51) is strictly negative. Thus, the firm can reduce the expected payment by selecting $\epsilon > 0$ small enough, which does not violate the

LL when $w_{q',s'} > 0$. As a result, the solution entails zero payments in all states that do not maximize the likelihood ratio. ■

Combining these results yields Proposition 5, which states that a signal is valuable if and only if it is informative about effort in states with the highest likelihood ratio:

Proposition 5 *A signal has positive value if and only if,* $\forall\ (q,s) \in \arg\max_{(q',s')}\{\frac{\pi_{q',s'}}{p_{q',s'}}\}$*, there exists* $\hat{s}$ *such that* $\frac{\pi_{q,s}}{p_{q,s}} \neq \frac{\pi_{q,\hat{s}}}{p_{q,\hat{s}}}$.

The presence of limited liability requires us to refine the informativeness principle. A signal has positive value if and only if it affects the likelihood ratio at the output level with the maximum likelihood ratio – rather than in general – as only at this output level is the payment positive. In this case, the firm can improve on the contract by making the payment at this output level contingent upon the signal – increase it at the signal where (q,s) has the highest likelihood ratio and decrease it to zero at other signal realizations. In contrast, a signal is not useful if it changes the likelihood ratio only for output levels at which the likelihood ratio is not maximized. Since the payment is zero to begin with, the firm cannot decrease it upon a low signal.

In sum, if output q is a sufficient statistic for effort e given (q,s), the signal s has zero value. However, even if q is not a sufficient statistic, s still has zero value if it is informative about effort only in states at which the likelihood ratio is not maximized.

Example 1 below illustrates the result from Proposition 5:

Example 1 *Consider* $q \in \{0,1\}$, $s \in \{L,H\}$, *and the following conditional probabilities:*

	$e=1$		$e=0$		*Likelihood Ratio*	
	$q=0$	$q=1$	$q=0$	$q=1$	$q=0$	$q=1$
$s=H$	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{3}$	2
$s=L$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{2}$	2

By Lemma 2, the optimal contract pays only in states $(1,H)$ *and* $(1,L)$*, where the likelihood ratio is maximized. Since the likelihood ratios are equal at these two states, any payments that satisfy the IC with equality generate the same payoff to the firm:*

$$\frac{w_{1,H}}{4} + \frac{w_{1,L}}{8} = C.$$

One solution is to pay a payment that does not depend on the signal:

$$w_{1,H} = w_{1,L} = \frac{8}{3}C.$$

Note, however, that q is not a sufficient statistic for e given (q, s) because the likelihood ratios at states $(0, L)$ and $(0, H)$ are different.[16]

Remark 1 shows that the driver of our results is limited liability and not risk neutrality. With risk aversion, it remains the case that a signal has no value if it is informative about effort only at output levels for which limited liability binds.

Remark 1 *Assume that the manager's utility of money is a strictly increasing, weakly concave function u. The firm's program is:*

$$\min_{w_{q,s}} \sum_{q,s} \pi_{q,s} w_{q,s} \tag{53}$$

subject to

$$\sum_{q,s} \pi_{q,s} u\left(w_{q,s}\right) - C \geq \overline{u} \tag{54}$$

$$\sum_{q,s} \left(\pi_{q,s} - p_{q,s}\right) u\left(w_{q,s}\right) \geq C \tag{55}$$

$$w_{q,s} \geq 0 \tag{56}$$

To ensure that limited liability binds, we let $u(0) \geq \overline{u}$ so that IR (54) does not bind.[17]

Let $(w_{q,s})$ be a contract that satisfies the IC and LL and suppose $w_{q,s} > 0$ for a state (q, s) with $\pi_{q,s} \leq p_{q,s}$. Replacing the payment in this state by $w_{q,s} = 0$ is feasible (it relaxes the IC and preserves LL) and strictly decreases the expected payment (53). Thus, any optimal contract

[16] It is straightforward to generalize this example to more than two outputs. To see this, let $q \in \{1, ..., Q\}$, $\pi_{N,H} = \alpha$, $\pi_{N,L} = \beta$, $p_{N,H} = \frac{\alpha}{2}$, $p_{N,L} = \frac{\beta}{2}$, and $\frac{\pi_{q,s}}{p_{q,s}} < 2$ for all $q \neq N$ and all s. Note that q is not a sufficient statistic for e given (q, s) as long as the likelihood ratio is not constant: $\frac{\pi_{q,H}}{p_{q,H}} \neq \frac{\pi_{q,L}}{p_{q,L}}$ for some q. As before, the optimal contract pays zero in all states except the ones with the highest likelihood ratios: (N, H) and (N, L). Moreover, any wage in these states that satisfies the IC with equality is optimal. In particular, paying $w_{N,H} = w_{N,L} = \frac{2C}{\alpha+\beta}$, $w_{q,H} = w_{q,L} = 0$ for $q \neq N$ is optimal.

[17] To see that IR is slack, use IC (55) and LL (56) to write:

$$\sum_{q,s} \pi_{q,s} u\left(w_{q,s}\right) - C \underbrace{\geq}_{\text{IC}} \sum_{q,s} p_{q,s} u\left(w_{q,s}\right) \underbrace{\geq}_{\text{LL}} u(0) \geq \overline{u}.$$

has $w_{q,s} = 0$ *whenever* $\pi_{q,s} \leq p_{q,s}$*. Suppose that* $\frac{\pi_{q,s}}{p_{q,s}} = \phi(q)$ *(the likelihood ratio is independent of* s *for* (q, s)*) whenever* $\pi_{q,s} > p_{q,s}$*. Then, the signal* s *has zero value.*

Since the payment is zero whenever $\pi_{q,s} \leq p_{q,s}$*, a signal that is only informative about effort in such states has no value, as limited liability binds and so the firm cannot use the signal to modify the contract.*